

1.01 Real Numbers, Properties

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

Number Systems

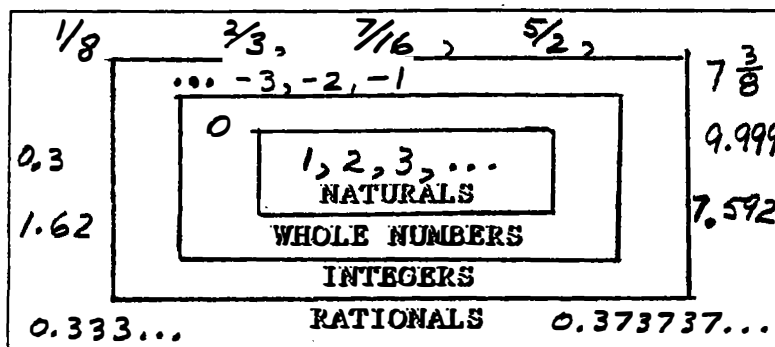
In any study of mathematics, there must be **number systems** and **operations** (such as addition, subtraction, multiplication, and division). It seems only "natural" to begin this study with the natural number system. The **natural numbers**, also known as the **counting numbers**, is the set of numbers that would be used for counting $\{1, 2, 3, 4 \dots\}$ (three dots mean "and so on"). When the number 0 is included, this set $\{0, 1, 2, 3, 4 \dots\}$ is the set of **whole numbers**. With the advent of credit, it became necessary to have negative numbers. The set of **integers** is defined to be the set of all whole numbers and their negatives:

$\{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4 \dots\}$

SYSTEMS OF NUMBERS	
NATURAL NUMBERS:	1, 2, 3, 4, . . .
WHOLE NUMBERS:	0, 1, 2, 3, 4, . . .
INTEGERS:	. . . -4, -3, -2, -1, 0, 1, 2, 3, 4, . . .

You probably noticed that fractional and decimal numbers are not included in any of the sets mentioned thus far. The set of **rational numbers**, from the root word **ratio**, is the set of all numbers that can be expressed as a ratio of two integers (assuming of course that **division by zero is undefined**). When one integer is divided by another integer, the result can be expressed as a fraction, or it can be divided out to express it in decimal form. When two integers are divided, the result will either come out even (called a **terminating decimal**) or there will be a repeating pattern of numbers in the quotient (called a **repeating decimal**). The fractions, $1/2 = .5$, $3/8 = .375$, $9/5 = 1.8$ result in terminating decimals, but $1/3 = .333\dots$, $2/9 = .222\dots$, $4/11 = .363636\dots$, and $2/7 = .285714285714\dots$ result in repeating decimals.

As you probably noticed, each set of numbers so far is developed or built upon the previous set of numbers. This makes each previous set of numbers a subset (i.e., a set contained within a set) of each succeeding set of numbers. These illustrations that have been used are called Venn Diagrams, named after the mathematician John Venn (1880).



Having built up to the set of rational numbers, there are some numbers that are not rational--that is, these numbers cannot be expressed as a ratio of integers. For example, the solution to the equation $x^2 = 2$ is $\pm\sqrt{2}$. Another example of a number that cannot be expressed as a ratio of integers is the number π , whose value is approximately (but not exactly!) $22/7$ or 3.14 . It can be proven that the actual value of π will never terminate, and it will never repeat a pattern. The set of all numbers like $\sqrt{2}$, $-\sqrt{5}$, and π that never terminate and never repeat a pattern form the set of irrational numbers. These two sets, the rational and the irrational numbers, are said to be disjoint sets, which means that they have no members in common. The intersection of the sets (i.e., what is common to both sets) is the empty set, denoted \emptyset or $\{ \}$. Notice that $\{\emptyset\}$ is not acceptable notation for the empty set--this is not empty because it has a "0" in it! The set of all rational and irrational numbers combined (the union of the two sets) is the real number system \mathbf{R} . [Note: in math, this is the real thing!]

There are two operations that are not allowed in the real number system:

- I. Division by zero,
- II. Square roots (in general, "even" roots) of negative numbers.

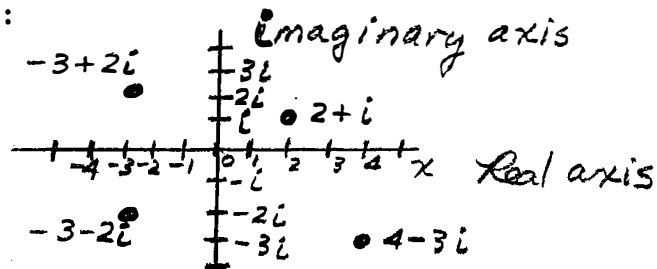
While square roots of negative numbers are not defined in the real number system, the possibility of taking square roots of negative numbers leads to the definition of **imaginary numbers**:

DEFINITION: $i = \sqrt{-1}$
 $i^2 = -1$

It follows that $\sqrt{-a} = i\sqrt{a}$ where "a" represents any real number.

Notice that this definition $i = \sqrt{-1}$ applies only to square roots.

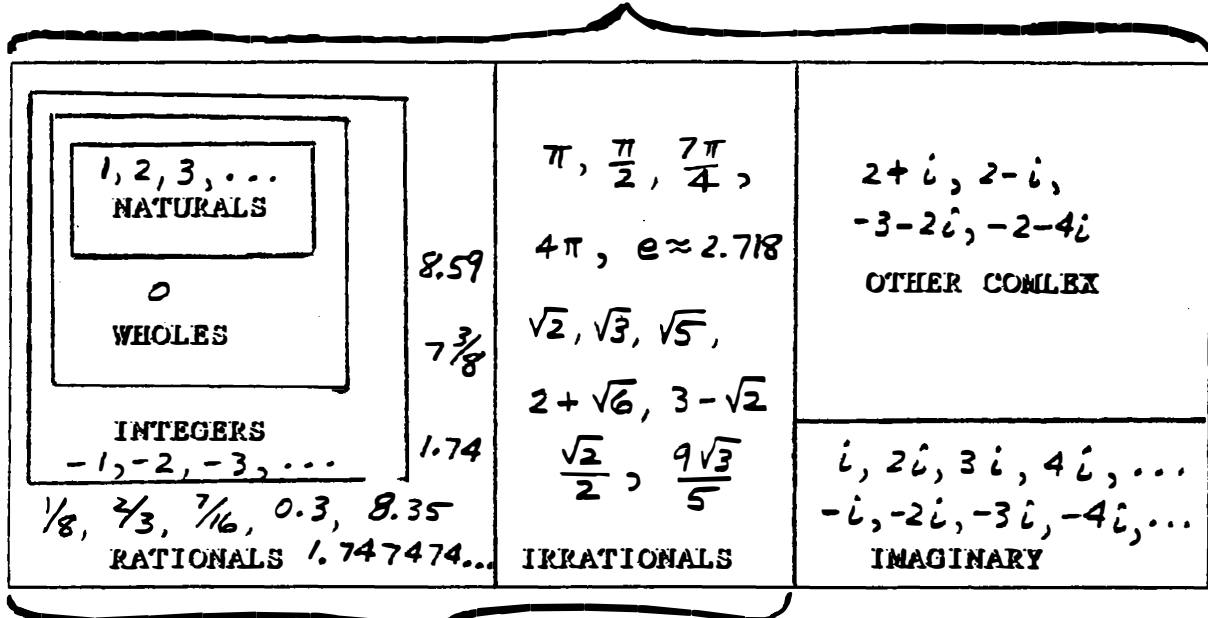
Any odd root of -1 is -1. The fourth root, sixth root, eighth root, or any other even root of -1 is not i. These will be discussed later (perhaps much later!). Obviously, these numbers are **un-real**. If the **real numbers** can be graphed on a numberline (like the X-axis), then the **imaginary numbers** can be graphed on a numberline (like the Y-axis) that is perpendicular to the **real axis** as shown below:



The final number system to be considered is the **complex number system**. The complex numbers consist of any combination of **real** and **imaginary numbers**. For example, if **a** and **b** are any **real numbers**, then the expression represented by $Z = a + bi$ is said to be a **complex number**. It is not that this is "complex" in the sense of being "complicated" (it is not!). Rather, it is complex in that it consists of inter-connected or interwoven parts, as a B-complex vitamin. If the **real numbers** are contained on the **X-axis** and the **imaginary numbers** are on the **Y-axis**, then the **complex numbers** cover the entire **XY-plane**. Realize that, for example, the **real number** $X = 6$ can be written as $Z = 6 + 0i$. Since every **real number** X can be written in the form $Z = X + 0i$, the **real numbers** are actually a **subset** of the **complex numbers**. **SEE SECTION 3.06--COMPLEX NUMBERS.**

The following is a summary of all the number systems: natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, imaginary numbers, and complex numbers.

COMPLEX NUMBERS



R E A L S

EXERCISES:

For each number, list the number systems of which it is a member:

- N = Natural Numbers
- W = Whole Numbers
- I = Integers
- Ra = Rationals (Save R for Reals)
- Ir = Irrationals
- R = Reals
- Im = Imaginary
- C = Complex

- Examples: 4 -- N, W, I, Ra, R, C
- $\sqrt{5}$ -- Ir, R, C
- $2i$ -- Im, C
1. 3 -- _____
 2. -3 -- _____
 3. π -- _____
 4. 7.5 -- _____

- 5. -5.65 -- _____
- 6. 2π -- _____
- 7. $8i$ -- _____
- 8. $3 + 8i$ -- _____
- 9. -8.0 -- _____
- 10. 3.14 -- _____
- 11. $\frac{2}{5}$ -- _____
- 12. $\frac{-5}{2}$ -- _____
- 13. $\frac{25}{5}$ -- _____
- 14. $\frac{-25}{5}$ -- _____
- 15. $\frac{22}{7}$ -- _____
- 16. $-\frac{25}{10}$ -- _____
- 17. $\sqrt{47}$ -- _____
- 18. $-\sqrt{47}$ -- _____
- 19. $-\sqrt{49}$ -- _____
- 20. $\sqrt{25}$ -- _____
- 21. $\sqrt{-25}$ -- _____
- 22. $\sqrt{125}$ -- _____
- 23. $\frac{0}{5}$ -- _____
- 24. $-\frac{20}{5}$ -- _____

25. $\frac{5}{0}$ -- _____
26. $\frac{2}{25}$ -- _____
27. $\frac{10}{2} - \sqrt{25}$ -- _____
28. $\frac{10}{2} + \sqrt{25}$ -- _____
29. $\frac{\sqrt{25}}{5}$ -- _____
30. $\sqrt{\frac{25}{5}}$ -- _____
31. $\frac{\sqrt{125}}{5}$ -- _____
32. $\sqrt{\frac{25}{9}}$ -- _____
33. $\frac{\sqrt[3]{125}}{5}$ -- _____
34. $\frac{\sqrt[3]{125}}{5} - 1$ -- _____
35. $\sqrt[3]{12}$ -- _____
36. $\sqrt[3]{\frac{125}{5}}$ -- _____
37. $\frac{\sqrt[3]{-125}}{5}$ -- _____
38. $\frac{\sqrt[3]{125}}{10}$ -- _____

PROPERTIES OF REAL NUMBERS

There are several properties involving addition and/or multiplication, most of which are widely known and perhaps taken for granted. Rather than take these properties for granted and use them haphazardly, it is better to know and use these properties by name. Recognizing and naming these properties is the objective of this lesson.

- I. **CLOSURE PROPERTY.** The word **closure** comes from the word "**closed**", as the expression from union jargon "**closed shop**." The union expression "**closed shop**" means that all employees must be hired from within the union. You do not go outside of the union to hire employees. In the same way, whenever you **add** or **multiply** two **real numbers**, you will not go out of the real number system to get the sum or product.
- A. **Closure Property for Addition:** If "**a**" and "**b**" are real numbers, then "**a + b**" is also a real number.
- B. **Closure Property for Multiplication:** If "**a**" and "**b**" are real numbers, then "**a·b**" is also a real number.

Examples: There is closure for multiplication for the set of integers because the product of two integers is always an integer.

There is closure for addition for the set of integers because the sum of two integers is always an integer.

There is closure for multiplication of odd integers because the product of two odd integers is always an odd integer.

There is not closure for addition of odd integers because the sum of two odd integers is not always an odd integer. (In fact, in this case, the sum of two odd integers is never an odd integer!)

Is there closure for addition of natural numbers?

Is there closure for multiplication of natural numbers?

Is there closure for subtraction of natural numbers?

Is there closure for subtraction of integers?

Is there closure for multiplication of irrationals?

[ANSWERS: Yes, Yes, No, Yes, No.]

II. **COMMUTATIVE PROPERTY.** When adding or multiplying two real numbers, the order may be changed without affecting the result. (This is like commuting back and forth from work or school.)

A. **Commutative Property for Addition:** $a + b = b + a$.

Examples: $4 + 6 = 6 + 4$
 $8 \cdot (4 + 6) = 8 \cdot (6 + 4)$
 $2 + (4 + 6) = 2 + (6 + 4)$
 $2 + (4 + 6) = (4 + 6) + 2$
 $(5X + 3Y) \cdot 14 = (3Y + 5X) \cdot 14$

B. **Commutative Property for Multiplication:** $a \cdot b = b \cdot a$

Examples: $4 \cdot 6 = 6 \cdot 4$
 $8 \cdot (4 + 6) = (4 + 6) \cdot 8$
 $2 \cdot (4 \cdot 6) = 2 \cdot (6 \cdot 4)$
 $2 \cdot (4 \cdot 6) = (4 \cdot 6) \cdot 2$
 $(5X + 3Y) \cdot 14 = 14 \cdot (5X + 3Y)$

Note: Subtraction and division are not commutative. Why not?

III. **ASSOCIATIVE PROPERTY.** The way in which real numbers are associated (by means of parentheses or other symbols of grouping) may be changed without affecting the result. The associative property is frequently used to "reorganize" a problem so it can be simplified. Notice that the order does not change.

A. **Associative Property for Addition:**

$a + (b + c) = (a + b) + c$
Examples: $2 + (4 + 6) = (2 + 4) + 6$
 $(8 + 9) + 1 = 8 + (9 + 1)$
 $3X + (7X + 10) = (3X + 7X) + 10$

B. **Associative Property for Multiplication:**

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Examples: $2 \cdot (4 \cdot 6) = (2 \cdot 4) \cdot 6$
 $(8 \cdot 9) \cdot 1 = 8 \cdot (9 \cdot 1)$
 $3X \cdot (7X \cdot 10) = (3X \cdot 7X) \cdot 10$

Give an example to show that subtraction and division are not associative.

IV. DISTRIBUTIVE PROPERTY. $a \cdot (b + c) = a \cdot b + a \cdot c$

Technically, this property is called the distributive property for multiplication over addition. Sometimes it is called the distributive property for multiplication or, even better, the distributive property. In this property, a multiplier **distributes** over a sum or difference. If this equation is written "backwards," such as the equation:

$$a \cdot b + a \cdot c = a \cdot (b + c)$$

then this is the process of **factoring the common factor**. Factoring the common factor is accomplished by using the **distributive property**.

Examples: $5 \cdot (X + 8) = 5 \cdot X + 5 \cdot 8$ or $5 \cdot (X + 8) = 5X + 40$
 $8 \cdot (Y - 9) = 8 \cdot Y - 8 \cdot 9$ or $8 \cdot (Y - 9) = 8Y - 72$
 $7 \cdot (3X + 5Y - 6) = 21X + 35Y - 42$
 $8X + 24Y = 8 \cdot (X + 3Y)$
 $48X^2 - 32XY = 16X(3X - 2Y)$

V. IDENTITY PROPERTY. The identity property involves keeping something **identically the same** or preserving the **identity** of a quantity when adding or multiplying.

A. Identity Property for Addition, Additive Identity, or Zero Property for Addition:

For every X in the set, $X + 0 = X$ and $0 + X = X$.

When 0 is added to any number X, the result is identically the same as the original number X. This is why it is called the **identity property for addition**.

Examples: $5 + 0 = 5$ or $0 + 5 = 5$
 $b + 0 = b$ or $0 + b = b$
 $(3X - 4Y) + 0 = (3X - 4Y)$ or $0 + (3X - 4Y) = (3X - 4Y)$

B. Identity Property for Multiplication, Multiplicative Identity, or Unity Property:

For every X in the set, $X \cdot 1 = X$ and $1 \cdot X = X$.

When any number X is multiplied by 1 the result is **identically** the same as the original number.

Examples: $5 \cdot 1 = 5$ or $1 \cdot 5 = 5$
 $b \cdot 1 = b$ or $1 \cdot b = b$
 $(3X - 4Y) \cdot 1 = (3X - 4Y)$ or $1 \cdot (3X - 4Y) = (3X - 4Y)$

VI. INVERSE PROPERTY. The inverse properties are as simple as "wrapping" and "unwrapping" presents! When you are given a wrapped present, you must unwrap it to get to the present. Likewise, the inverse properties are used to "undo" an operation. If 4 has been added, then adding a (-4) will "undo" the 4. If there has been a multiplication by 4, then multiplication by $\frac{1}{4}$ will "undo" the operation. In general, you would **inverse** (undo) an **X** with a **(-X)** for addition, and with $\frac{1}{X}$ for multiplication.

A. Inverse Property for Addition or Additive Inverse Property:

$$X + (-X) = 0.$$

Every number **X** in the set has an inverse **(-X)** such that the sum of the number and its inverse is the identity element 0.

Examples:

$$5 + (-5) = 0$$

$$(-56) + 56 = 0$$

$$6b + (-6b) = 0$$

B. Inverse Property for Multiplication or Multiplicative Inverse Property:

$$X \cdot \frac{1}{X} = 1$$

For every number **X** in the set (except zero!) there is an inverse such that the product of the number and its inverse is the identity element 1.

Examples:

$$5 \cdot \frac{1}{5} = 1 \qquad \frac{1}{4} \cdot 4 = 1$$

$$(-6) \cdot \left(\frac{-1}{6}\right) = 1 \qquad \left(-\frac{3}{4}\right) \cdot \left(-\frac{4}{3}\right) = 1$$

Zero has no multiplicative inverse. Why not?

Notice that the **identity properties** always involve **addition of 0** or **multiplication times 1** and the result is always **"identically the same."** For the **inverse properties**, you always end up with the **identity number: 0 for addition, or 1 for multiplication.**

SUMMARY OF NUMBER PROPERTIES

Closure--the resulting sum or product is always within the set.

Commutative--always involves a change in the order.

Associative--always involves a regrouping or re-association of the numbers in parentheses.

Distributive--always involves a product with parentheses on one side, distributed to each term on the other side of the equation.

Identity--always involves a 0 for addition or 1 for multiplication, ending up with identically the same thing you started with.

Inverse--always ends up with the identity element, 0 for addition or 1 for multiplication.

EXERCISES.

In each of the following, answer "Yes" or "No."

If the answer is "No", then give an example to show that it is not.

Is there closure for:

- | | |
|--|-----------|
| 1. addition of integers? | 1. _____ |
| 2. multiplication of integers? | 2. _____ |
| 3. addition of even integers? | 3. _____ |
| 4. multiplication of even integers? | 4. _____ |
| 5. addition of odd integers? | 5. _____ |
| 6. multiplication of odd integers? | 6. _____ |
| 7. addition of positive integers? | 7. _____ |
| 8. multiplication of positive integers? | 8. _____ |
| 9. addition of negative integers? | 9. _____ |
| 10. multiplication of negative integers? | 10. _____ |

In each of the following, give the complete name of the property:

1. $50 \cdot (2 \cdot 98) = 50 \cdot (98 \cdot 2)$ 1. _____

What happened? Changed order multiplying 2 and 98

2. $50 \cdot (2 \cdot 98) = (2 \cdot 98) \cdot 50$ 2. _____

What happened? _____

3. $50 \cdot (2 \cdot 98) = (50 \cdot 2) \cdot 98$ 3. _____

What happened? Re-association of numbers. Order did not change

4. $50 + (2 + 98) = 50 + (98 + 2)$ 4. _____

What happened? _____

5. $50 + (2 + 98) = (2 + 98) + 50$ 5. _____

What happened? _____

6. $50 + (2 + 98) = (50 + 2) + 98$ 6. _____

What happened? _____

7. $50 \cdot (2 + 98) = 50 \cdot (98 + 2)$ 7. _____

What happened? _____

8. $50 \cdot (2 + 98) = (2 + 98) \cdot 50$ 8. _____

What happened? _____

9. $50 \cdot (2 + 98) = (50 \cdot 2) + (50 \cdot 98)$ 9. _____

What happened? _____

10. $(50 \cdot 2) + (50 \cdot 98) = 50 \cdot (2 + 98)$ 10. _____

What happened? _____

11. $50 \cdot (1 \cdot 98) = 50 \cdot 98$ 11. _____

What happened? _____

12. $50 \cdot (98 \cdot 1) = 50 \cdot 98$ 12. _____

What happened? _____

13. $50 \cdot [98 + (-98)] = 50 \cdot 0$ 13. _____

What happened? _____

14. $50 \cdot [98 + (-98)] = 50 \cdot 98 + 50 \cdot (-98)$ 14. _____

What happened? _____

15. $50 \cdot [98 + (-98)] = 50 \cdot [(-98) + 98]$ 15. _____

What happened? _____

16. $50 \cdot (0 + 98) = 50 \cdot (98 + 0)$ 16. _____

What happened? _____

17. $50 \cdot (0 + 98) = 50 \cdot 98$ 17. _____

What happened? _____

18. $50 \cdot (0 + 98) = 50 \cdot 0 + 50 \cdot 98$ 18. _____

What happened? _____

19. $50 \cdot (2 \cdot \frac{1}{2}) = 50 \cdot 1$ 19. _____

What happened? _____

20. $50 \cdot (2 \cdot \frac{1}{2}) = 50 \cdot (\frac{1}{2} \cdot 2)$ 20. _____

What happened? _____

21. $50 \cdot (2 \cdot \frac{1}{2}) = (50 \cdot 2) \cdot \frac{1}{2}$ 21. _____

What happened? _____

22. $50 \cdot [2 + (-2)] = 50 \cdot 0$ 22. _____

What happened? _____

23. $50 \cdot [(-2) + 2] = 50 \cdot [2 + (-2)]$ 23. _____

What happened? _____

24. $50 \cdot (\frac{1}{4} \cdot 4) = 50 \cdot 1$ 24. _____

What happened? _____

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CHAPTER ONE

ANSWERS 1.01

- p. 4-6: 1. N,W,I,Ra,R,C; 2. I,Ra,R,C; 3. Ir,R,C; 4. Ra,R,C;
5. Ra,R,C; 6. Ir,R,C; 7. Im,C; 8. C; 9. I,Ra,R,C;
10. Ra,R,C; 11. Ra,R,C; 12. Ra,R,C; 13. N,W,I,Ra,R,C;
14. I,Ra,R,C; 15. Ra,R,C; 16. Ra,R,C; 17. Ir,R,C;
18. Ir,R,C; 19. I,Ra,R,C; 20. N,W,I,Ra,R,C; 21. Im,C;
22. Ir,R,C; 23. W,I,Ra,R,C; 24. I,Ra,R,C; 25. None;
26. Ra,R,C; 27. W,I,Ra,R,C; 28. N,W,I,Ra,R,C;
29. N,W,I,Ra,R,C; 30. Ir,R,C; 31. Ir,R,C; 32. Ra,R,C;
33. N,W,I,Ra,R,C; 34. W,I,Ra,R,C; 35. Ir,R,C; 36. Ir,R,C;
37. I,Ra,R,C; 38. Ra,R,C;
- p. 11: 1. Yes; 2. Yes; 3. Yes; 4. Yes; 5. No, $(3+3)$; 6. Yes;
7. Yes; 8. Yes; 9. Yes; 10. No, $(-3)\cdot(-3)$;
- p.12-13: 1. Commutative for mult; 2. Commutative for mult;
3. Associative for mult; 4. Commutative for add;
5. Commutative for add; 6. Associative for add;
7. Commutative for add; 8. Commutative for mult;
9. Distributive; 10. Distributive;
11. Identity for mult; 12. Identity for mult;
13. Inverse for add; 14. Distributive;
15. Commutative for add; 16. Commutative for add;
17. Identity for add; 18. Distributive;
19. Inverse for mult; 20. Commutative for mult;
21. Associative for mult; 22. Inverse for add;
23. Commutative for add; 24. Inverse for mult.