2.10 Polynomial and Synthetic Division

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

You probably remember from your previous algebra background that there are two different types of polynomial division problems, which are performed by entirely different methods: **division by by a monomial** and **division by** a polynomial. When **dividing by** a **monomial,** such as $\frac{a+b+c}{d}$, simply break the fraction into separate fractions: $\frac{a+b+c}{d} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d}$, and, if possible, reduce each fraction.

EXAMPLE 1: $\frac{4X^3 - 8X^2 + 2X - 6}{4X^2}$

SOLUTION: Break the fraction down into four separate fractions:

$$= \frac{4X^3}{4X^2} - \frac{8X^2}{4X^2} + \frac{2X}{4X^2} - \frac{6}{4X^2}$$

$$= X - 2 + \frac{1}{2X} - \frac{3}{2X^2}$$ **(Reduce each fraction!)**

EXERCISES: Divide. Reduce all fractions.

1. $\frac{X^2 - 9X + 12}{6X}$

2. $\frac{X^2 + 10X - 15}{5X}$

3. $\frac{15X^2 - 12X}{6X}$

4. $\frac{30X^3 + 20X^2 - 15}{15X}$

5. $\dfrac{4X^3 + 12X^2 - 6X}{4X^2}$

6. $\dfrac{6X^3 - 8X^2 - 12X}{8X^2}$

7. $\dfrac{9X^3 - 12X - 18}{-12X}$

8. $\dfrac{-9X^3 - 6X^2 + 12}{-3X^2}$

9. $\dfrac{-18X^4 - 12X^2 + 16}{-12X^2}$

10. $\dfrac{16X^4 + 18X^3 - 24X}{-12X^2}$

When the denominator is a polynomial (two or more terms), the method of splitting up into separate fractions does not work. It will be necessary to use the method of **long division** (or perhaps the shortcut called **synthetic division.**) Consider the following examples in four steps, the first from arithmetic, the second from algebra. Notice that step 3 from arithmetic "subtract" becomes step 3 in algebra "change the signs and add." This is done to avoid sign errors that are likely to occur when subtracting negatives. Compare and contrast the two examples that follow. See also the example on **page 233.**

EXAMPLE 2: LONG DIVISION (ARITHMETIC)

STEP 1: **Divide first into first**

$$\begin{array}{r} 1 \\ 63\;\overline{)\,821} \end{array}$$

STEP 2: **Multiply** 1 **times** 63

$$\begin{array}{r} 1 \\ 63\;\overline{)\,821} \\ 63 \end{array}$$

STEP 3: **Subtract 63 from 82**

$$\begin{array}{r} 1 \\ 63\;\overline{)\,821} \\ -\underline{63} \\ 19 \end{array}$$

STEP 4: **Bring down the** 1

$$\begin{array}{r} 1 \\ 63\;\overline{)\,821} \\ -\underline{63} \\ 191 \end{array}$$

Repeat the process:
STEP 1: divide 63 **into** 191
STEP 2: multiply 3 **times** 63
STEP 3: subtract. Remainder = 2

$$\begin{array}{r} 13 \;\; \text{r. } 2 \\ 63\;\overline{)\,821} \\ -\underline{63} \\ 191 \\ -\underline{189} \\ 2 \end{array}$$

EXAMPLE 3: LONG DIVISION (ALGEBRAIC)

STEP 1: **Divide first into first**

$$\begin{array}{r} X \phantom{{}+8} \\ X+2\;\overline{)\,X^2+5X+8} \end{array}$$

STEP 2: **Multiply** X **times** X+2

$$\begin{array}{r} X \phantom{{}+8} \\ X+2\;\overline{)\,X^2+5X+8} \\ X^2+2X\phantom{{}+8} \end{array}$$

STEP 3: **Change the signs and add**

$$\begin{array}{r} X \phantom{{}+8} \\ X+2\;\overline{)\,X^2+5X+8} \\ \underline{X^2+2X}\phantom{{}+8} \\ 3X\phantom{{}+8} \end{array}$$

STEP 4: **Bring down the** 8

$$\begin{array}{r} X \phantom{{}+8} \\ X+2\;\overline{)\,X^2+5X+8} \\ \underline{X^2+2X}\phantom{{}+8} \\ 3X+8 \end{array}$$

Repeat the process:
STEP 1: divide X **into** 3X
STEP 2: multiply 3 **times** (X + 2)
STEP 3: subtract. Remainder = 2

$$\begin{array}{r} X+3 \\ X+2\;\overline{)\,X^2+5X+8} \\ \underline{X^2+2X}\phantom{{}+8} \\ 3X+8 \\ \underline{3X+6} \\ 2 \end{array}$$

ANSWER: $X+3+\frac{2}{X+2}$

Complete the following exercises according to the indicated steps.

1. $X + 5 \overline{)X^2 + 6X + 1}$

STEP 1: Divide X into X^2
STEP 2: Multiply X times (X + 5)
STEP 3: Change signs and add
STEP 4: Bring down 1 and repeat

STEP 1: Divide X into X
STEP 2: Multiply 1 times (X + 5)
STEP 3: Change signs and add
Remainder = -4

2. $X - 2 \overline{)3X^2 - 4X - 10}$

STEP 1: Divide X into $3X^2$
STEP 2: Multiply 3X times (X-2)
STEP 3: Change signs and add
STEP 4: Bring down -10 and repeat

STEP 1: Divide X into 2X
STEP 2: Multiply 2 times (X - 2)
STEP 3: Change signs and add
Remainder = -6

3. $X - 4 \overline{)3X^2 + 4X - 10}$

STEP 1: Divide X into $3X^2$
STEP 2: Multiply 3X times (X - 4)
STEP 3: Change signs and add
STEP 4: Bring down -10 and repeat

STEP 1: Divide X into (16X)
STEP 2: Multiply 16 times (X - 4)
STEP 3: Change signs and add
Remainder = 54

4. $X + 2 \overline{)3X^2 + 4X - 10}$

STEP 1: Divide X into $3X^2$
STEP 2: Multiply 3X times (X + 2)
STEP 3: Change signs and add
STEP 4: Bring down -10 and repeat

STEP 1: Divide X into (-2X)
STEP 2: Multiply -2 times (X + 2)
STEP 3: Change signs and add
Remainder = -6

5. $2X + 3 \overline{) 6X^2 - 7X - 9}$

6. $2X - 3 \overline{) 6X^2 - 5X - 9}$

7.

$2X+3 \overline{) 8X^3 + 10X^2 + 11X + 15}$

Step 1: Divide 2X into $8X^3$
Step 2: Multiply $4X^2$ (2X + 3)
Step 3: Change signs and add
Step 4: Bring down +11X + 15
Step 1: Divide 2X into $(-2X^2)$
Step 2: Multiply -X (2X + 3)
Step 3: Change signs and add
Step 4: Bring down + 15
Step 1: Divide 2X into (14X)
Step 2: Multiply 7 (2X + 3)
Step 3: Change signs and add
Remainder = -6

8. $\dfrac{8X^3 - 6X^3 - 15X + 9}{2X - 3}$

Solution: Rewrite as $2X-3\,\overline{)\,8X^3 - 6X^2 - 15X + 9}$

There are two additional thoughts: First, you must be sure the terms are **arranged in descending** (or ascending) **powers of the variable,** and second, **if there are terms missing, insert place-holder zeros or spaces for the missing terms.**

9. $\dfrac{-7X - 2X^3 + 12X^4 - 2}{3X + 4}$

10. $\dfrac{-X^3 + 12X^4 - 29X^2 - 4}{3X - 4}$

The shortcut of **synthetic division** may be used if the **divisor** is of the form **X - a** (of course this includes **X + a**). In the method of synthetic division, as with regular long division, be sure the terms are in descending order, with zeros as placeholders for missing terms. Then write the coefficients of the numerator.

For the example $\dfrac{X^4 + 5X^3 + 6X - 2}{X + 4}$, write the coefficients:

1 5 0 6 -2 . To write the divisor **X + 4** in the form **X - a**, let **a = -4**. In other words, if the divisor is **X + 4**, you perform synthetic division using **-4**. If the divisor is **X - 4**, you perform synthetic division using **+4**. After writing the coefficients as illustrated below, bring down the first coefficient, then repeatedly multiply, add, multiply, add, multiply, add, etc. until you run out of numbers.

Bring down "1"

-4	1	5	0	6	-2
	↓				
	1				

Mult "1" times "-4"

-4	1	5	0	6	-2
	↓	-4			
	1				

Add "5" and "-4"

-4	1	5	0	6	-2
	↓	-4			
	1	1			

Mult "1" times "-4"

-4	1	5	0	6	-2
	↓	-4	-4		
	1	1			

Add "0" and "-4"

-4	1	5	0	6	-2
	↓	-4	-4		
	1	1	-4		

Mult "-4" times "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & \\ \hline & 1 & 1 & -4 & & \end{array}$$

Add "6" and "16"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & \\ \hline & 1 & 1 & -4 & 22 & \end{array}$$

Mult "22" times "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & -88 \\ \hline & 1 & 1 & -4 & 22 & \end{array}$$

Add "-2" and "-88"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & -88 \\ \hline & 1 & 1 & -4 & 22 & -90 \end{array}$$

The numbers obtained on the last line above are the coefficients of the quotient, with the last number being the remainder. The exponent of the first term will always be one less than the degree of the numerator. Therefore, the answer is $1X^3 + 1X^2 - 4X + 22 \quad r. -90$.

In the illustration below, notice the similarity (and amount!) of the work in the long division process illustrated on the left and the synthetic division process on the right:

LONG DIVISION

$$\begin{array}{l} \qquad\quad x^3 + x^2 - 4x + 22 - \frac{90}{x+4} \\ x + 4 \,\big|\, x^4 + 5x^3 + 0x^2 + 6x - 2 \\ \qquad\ \ -\cancel{x^4} + 4x^3 \\ \hline \qquad\qquad\quad \cancel{x^3} + 0x^2 + 6x - 2 \\ \qquad\qquad -\cancel{x^3} + 4x^2 \\ \hline \qquad\qquad\qquad -\cancel{4x^2} + 6x - 2 \\ \qquad\qquad\qquad -\cancel{4x^2} - 16x \\ \hline \qquad\qquad\qquad\qquad 22\cancel{x} - 2 \\ \qquad\qquad\qquad\quad -\cancel{22x} + 88 \\ \hline \qquad\qquad\qquad\qquad\quad -90 \end{array}$$

SYNTHETIC DIVISION

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & -88 \\ \hline & 1 & 1 & -4 & 22 & -90 \end{array}$$

$1x^3 + 1x^2 - 4x + 22 - \frac{90}{x+4}$

Perform the divisions, using synthetic division.

1. $\dfrac{X^3 - 4X^2 + 5X - 19}{X - 4}$

2. $\dfrac{X^3 - 5X^2 - 6X - 2}{X - 3}$

3. $\dfrac{2X^3 + 5X^2 + 6X - 2}{X + 3}$

4. $\dfrac{2X^3 + 5X^2 - 6X + 2}{X + 4}$

5. $\dfrac{5X^2 + X^4 - 6X - 2}{X - 2}$

6. $\dfrac{7X + X^4 - 5X^2 + 2}{X + 1}$

7. $\dfrac{X^5 - 1}{X + 1}$

8. $\dfrac{X^5 - 1}{X - 1}$

ANSWERS 2.10

p.226-227:

1. $\frac{X}{6} - \frac{3}{2} + \frac{2}{X}$; 2. $\frac{X}{5} + 2 - \frac{3}{X}$; 3. $\frac{5X}{2} - 2$; 4. $2X^2 + \frac{4X}{3} - \frac{1}{X}$;

5. $X + 3 - \frac{3}{2X}$; 6. $\frac{3X}{4} - 1 - \frac{3}{2X}$ 7. $-\frac{3X^2}{4} + 1 + \frac{3}{2X}$; 8. $3X + 2 - \frac{4}{X^2}$;

9. $\frac{3X^2}{2} + 1 - \frac{4}{3X^2}$; 10. $-\frac{4X^2}{3} - \frac{3X}{2} + \frac{2}{X}$.

p.229-231:

1. $X + 1 - \frac{4}{X+5}$; 2. $3X + 2 - \frac{6}{X-2}$; 3. $3X + 16 + \frac{54}{X-4}$; 4. $3X - 2 - \frac{6}{X+2}$;

5. $3X - 8 + \frac{15}{2X+3}$; 6. $3X + 2 - \frac{3}{2X-3}$; 7. $4X^2 - X + 7 - \frac{6}{2X+3}$; 8. $4X^2+3X-3$;

9. $4X^3-6X^2+8X-13 + \frac{50}{3X+4}$; 10. $4X^3+5X^2-3X-4 - \frac{20}{3X-4}$.

p.234:

1 . $X^2 + 5 + \frac{1}{X-4}$; 2 . $X^2 - 2X - 12 - \frac{38}{X-3}$; 3 . $2X^2 - X + 9 - \frac{29}{X+3}$;

4 . $2X^2 - 3X + 6 - \frac{22}{X+4}$; 5. $X^3 + 2X^2 + 9X + 12 + \frac{22}{X-2}$;

6. $X^3 - X^2 - 4X + 11 - \frac{9}{X+1}$; 7. $X^4 - X^3 + X^2 - X + 1 - \frac{2}{X+1}$;

8. $X^4 + X^3 + X^2 + X + 1$.

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