3.05 *Fractional Exponents*

Dr. Robert J. Rapalje

ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

A lesson in fractional exponents, properly done, is a deep and theoretical one. Such explanations are readily available in other textbooks. However, the goal here is to provide a rationale for the procedure, an easy-to-understand, non-rigorous explanation, and a step-by-step method of evaluating such expressions, with or without calculators.

First, what does it mean to have a fraction for an exponent? For example, what does it mean to have $9^{1/2}$? Begin by setting up a pattern of powers of **9**:

$$\begin{aligned} 9^2 &= 81 \\ 9^1 &= 9 \\ 9^{1/2} &= ???? \\ 9^0 &= 1 \\ 9^{-1} &= 1/9 \end{aligned}$$

From this you can see that whatever it is, the value of $9^{1/2}$ is between 1 and 9. Since you don't know the value of $9^{1/2}$, let $9^{1/2} = Y$ where Y is some unknown number, and square both sides of the equation to get:

$$9^{\frac{1}{2}} = Y$$

$$\left[9^{\frac{1}{2}} \right]^2 = Y^2 \qquad \text{(Multiplying exponents!)}$$

$$9 = Y^2$$

$$Y = \pm\sqrt{9} = \pm 3$$

From this it looks as if there are two answers to $9^{1/2}$, since Y=3 or Y=-3. However, the pattern above indicates that the value of $9^{1/2}$ should be between 1 and 9, so the value of $9^{1/2}$ must be 3 (and <u>not</u> -3). More importantly, the process involved to find the value of $9^{1/2}$ is "**SQUARE ROOT!**" From this it is clear that "a number raised to the one half power" simply means "square root".

It follows that if $X^{\frac{1}{2}} = \sqrt{X}$, then $X^{\frac{1}{3}} = \sqrt[3]{X}$, $X^{\frac{1}{4}} = \sqrt[4]{X}$, and in general, $X^{\frac{1}{n}} = \sqrt[n]{X}$. **(See the summary box on the next page.)**

EXERCISES.

1. $16^{\frac{1}{2}} = \sqrt{16}$ $= \_\_\_\_\_$

2. $16^{\frac{1}{4}} = \sqrt[4]{16}$ $= \_\_\_\_\_$

3. $125^{\frac{1}{3}} = \sqrt[3]{125}$ $= \_\_\_\_\_$

4. $4^{\frac{1}{2}} = \_\_\_\_\_$ $= \_\_\_\_\_$

5. $27^{\frac{1}{3}} = \_\_\_\_\_$ $= \_\_\_\_\_$

6. $32^{\frac{1}{5}} = \_\_\_\_\_$ $= \_\_\_\_\_$

7. $144^{\frac{1}{2}} = \_\_\_\_\_$ $= \_\_\_\_\_$

8. $8^{\frac{1}{3}} = \_\_\_\_\_$ $= \_\_\_\_\_$

9. $27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2$ $= (\sqrt[3]{27})^2$ $= \_\_\_\_\_$ $= \_\_\_\_\_$

10. $9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3$ $= (\sqrt{9})^3$ $= \_\_\_\_\_$ $= \_\_\_\_\_$

11. $16^{\frac{3}{2}} = \_\_\_\_\_$ $= \_\_\_\_\_$ $= \_\_\_\_\_$ $= \_\_\_\_\_$

12. $125^{\frac{2}{3}} = \_\_\_\_\_$ $= \_\_\_\_\_$ $= \_\_\_\_\_$ $= \_\_\_\_\_$

In general, $X^{\frac{m}{n}} = (\sqrt[n]{X})^m$ or $X^{\frac{m}{n}} = \sqrt[n]{X^m}$.

For fractional exponents, the denominator of the fraction is always the index of the radical, and the numerator of the fraction is always the power to which it is raised. This is summarized in the following definition:

Fractional Exponents

$$X^{\frac{1}{n}} = \sqrt[n]{X}$$

$$X^{\frac{m}{n}} = (\sqrt[n]{X})^m$$

$$X^{\frac{m}{n}} = \sqrt[n]{X^m}$$

13. $32^{\frac{4}{5}} =$ _____ $=$ _____ $=$ _____ $=$ _____

14. $16^{\frac{3}{4}} =$ _____ $=$ _____ $=$ _____ $=$ _____

15. $8^{\frac{2}{3}} =$ _____ $=$ _____ $=$ _____ $=$ _____

16. $81^{\frac{3}{4}} =$ _____ $=$ _____ $=$ _____ $=$ _____

17. $32^{-\frac{4}{5}} = (\sqrt[5]{32})^{-4}$ $= (\quad)^{-4}$ $= \frac{1}{(\quad)}$ $=$ _____

18. $9^{-\frac{3}{2}} =$ _____ $=$ _____ $=$ _____ $=$ _____

19. $8^{-\frac{2}{3}} =$ _____ $=$ _____ $=$ _____ $=$ _____

20. $8^{-\frac{4}{3}} =$ _____ $=$ _____ $=$ _____ $=$ _____

21. $16^{-\frac{1}{2}} =$ _____ $=$ _____ $=$ _____

22. $16^{-\frac{1}{4}} =$ _____ $=$ _____ $=$ _____

23. $25^{-\frac{3}{2}} =$ _____ $=$ _____ $=$ _____

24. $32^{-\frac{3}{5}} =$ _____ $=$ _____ $=$ _____

Now that you have learned to work with fractional exponents from the basic definition, you may be interested in applying the calculator to this type of problem. Of course, what you can do varies from calculator to calculator, and you may not be able to do everything that follows. Generally speaking, if you have **"raising to the power"** and **"parentheses"**, you will be able to use your calculator. If your calculator has the **"a b/c"** function, and if it can convert from **decimals to fractions**, it will be even better.

EXAMPLE 1: Use your calculator to find $16^{\frac{3}{4}}$. (SEE #14 ABOVE!)

METHOD 1: If your calculator has the **"a b/c"** button, then enter **"16", "Y^X", "3", "a b/c", "4", "="**. ANSWER = 8.

METHOD 2: If your calculator has parentheses, then you will enter **"16", "Y^X", "(", "3", "÷", "4", ")", "="**. ANSWER = 8.

METHOD 3: Write the exponent 3/4 as a decimal 0.75 (you knew that didn't you?). Then type **"16", "Y^X", ".75", "="**. ANSWER = 8. (There are two problems with this method. Frequently you do not know the decimal equivalent of the fractional exponent. Secondly, sometimes the decimal equivalent does not come out even--there could be a roundoff error.)

EXAMPLE 2: Use your calculator to find $8^{-\frac{2}{3}}$. (SEE #19 ABOVE!)

METHOD 1: If your calculator has the **"a b/c"** button, then enter **"8", "Y^X", "2", "+/-", "a b/c", "3", "="**. ANSWER = 0.25. Of course 0.25 = 1/4.

METHOD 2: If your calculator has parentheses, then you will enter **"8", "Y^X", "(", "2", "+/-", "÷", "3", ")", "="**. ANSWER = 0.25 and **0.25 = 1/4**.

METHOD 3: Write the exponent 2/3 as a decimal with lots of 6s, like 0.666666667. Then type **"8", "Y^X", ".666666667", "+/-", "="**. ANSWER=0.25, but only if you have enough 6s to avoid a roundoff error. **Try using ".66" as the exponent. Does this work??** ANSWER = 0.25349 – WAY OFF!!

EXAMPLE 3: **Use your calculator to find** $32^{-\frac{4}{5}}$. (SEE #17 ABOVE!)

METHOD 1: If your calculator has the "**a b/c**" button, then enter "**32**", "Y^X", "**4**", "**+/-**", "**a b/c**", "**5**", "**=**". ANSWER = 0.0625. Do you recognize 0.0625? Probably not. Then, if your calculator will convert decimals to fractions, this is a good place to try it out. It may be as simple as entering "**F↔D**", or "**FRAC**". Then again, some calculators will not convert from decimals to fractions. At any rate, **.0625 = 1/16.**

METHOD 2: If your calculator has parentheses, then you will enter "**32**", "Y^X", "**(**", "**4**", "**+/-**", "**÷**", "**5**", "**)**", "**=**". ANSWER = 0.0625 and your calculator may or may not be able to convert from decimal to fraction **0.0625 = 1/16.**

METHOD 3: Write the exponent 4/5 as a decimal 0.8. Then type "**32**", "Y^X", "**.8**", "**+/-**", "**=**". ANSWER=0.0625.

EXERCISES. Use a calculator to find the value. Find fractional values if possible.

25. $25^{\frac{1}{2}}$ = _____ 26. $64^{\frac{1}{3}}$ = _____ 27. $25^{-\frac{1}{2}}$ = _____ 28. $64^{-\frac{1}{3}}$ = _____

29. $32^{\frac{4}{5}}$ = _____ 30. $64^{\frac{2}{3}}$ = _____ 31. $16^{\frac{3}{2}}$ = _____ 32. $81^{\frac{3}{4}}$ = _____

33. $32^{-\frac{4}{5}}$ = _____ 34. $64^{-\frac{2}{3}}$ = _____ 35. $16^{-\frac{3}{2}}$ = _____ 36. $81^{-\frac{3}{4}}$ = _____

37. $25^{-\frac{3}{2}}$ = _____ 38. $49^{-\frac{1}{2}}$ = _____ 39. $27^{-\frac{4}{3}}$ = _____ 40. $100^{-\frac{3}{2}}$ = _____

NEGATIVE BASE NUMBERS

What happens if the base number is negative. In the next exercises, look for two distinctly different situations to develop.

Remember that: $\sqrt[3]{-1} = -1$ **because** $(-1)^3 = -1$,

$\sqrt[3]{-8} = -2$ **because** $(-2)^3 = -8$,

$\sqrt[3]{-27} = -3$ **because** $(-3)^3 = -27$,

$\sqrt[3]{-64} = -4$ **because** $(-4)^3 = -64$.

HOWEVER: $\sqrt{-1} = (\quad)$ **because** $(\quad)^2 = -1$ **(?????)**

$\sqrt[4]{-1} = (\quad)$ **because** $(\quad)^4 = -1$ **(?????)**

$\sqrt[6]{-1} = (\quad)$ **because** $(\quad)^6 = -1$ **(?????)**

The problem here is this, **"What number can you square or raise to any <u>even</u> power and get a negative number?? Answer: NO REAL SOLUTION.** In the next section, the topic of **complex numbers** will be studied, and at that time it will be possible to solve square roots of negative numbers using a special definition of **"i"** (which stands for **"imaginary numbers"**, where $\sqrt{-1} = i$). Until then, whenever you have an even root of a negative number, the acceptable answer is **"NO REAL SOLUTION."**

$$\sqrt[3]{-1} = -1; \quad \sqrt[5]{-1} = -1; \quad \sqrt[7]{-1} = -1; \quad \sqrt[odd]{-1} = -1;$$

$$\sqrt{-1}, \quad \sqrt[4]{-1}, \quad \sqrt[even]{-1} \quad \textit{NO REAL SOLUTION}$$

EXERCISES:

41. $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$

$= $ ______

42. $(-27)^{\frac{1}{3}} = $ ______

$= $ ______

43. $(-125)^{\frac{1}{3}} = $ ______

$= $ ______

44. $(-8)^{\frac{2}{3}} = (\sqrt[3]{-8})^2$

$= $ ______

45. $(-27)^{\frac{2}{3}} = $ ______

$= $ ______

46. $(-125)^{\frac{2}{3}} = $ ______

$= $ ______

47. $(-32)^{\frac{1}{5}} = $ ______

$= $ ______

48. $(-32)^{\frac{2}{5}} = $ ______

$= $ ______

49. $(-32)^{\frac{3}{5}} = $ ______

$= $ ______

50. $(-32)^{\frac{4}{5}} = $ ______

$= $ ______

51. $(-16)^{\frac{1}{2}} = $ ______

$= $ ______

52. $(-9)^{\frac{1}{2}} = $ ______

$= $ ______

53. $(-16)^{\frac{1}{4}} = $ ______

$= $ ______

54. $(-16)^{\frac{3}{4}} = $ ______

$= $ ______

55. $(-8)^{\frac{5}{3}} = $ ______

$= $ ______

Is there a difference between $(-25)^{\frac{1}{2}}$ and $-25^{\frac{1}{2}}$? ________

56. $(-25)^{\frac{1}{2}} = \sqrt{}$
$= \_\_\_\_$

57. $-25^{\frac{1}{2}} = -\sqrt{}$
$= \_\_\_\_$

58. $-16^{\frac{1}{2}} = \_\_\_\_$
$= \_\_\_\_$

Is there a difference between $(-27)^{\frac{2}{3}}$ and $-27^{\frac{2}{3}}$? ________

59. $(-27)^{\frac{2}{3}} = (\sqrt[3]{})^2$
$= \_\_\_\_$
$= \_\_\_\_$

60. $-27^{\frac{2}{3}} = -(\sqrt[3]{})^2$
$= \_\_\_\_$
$= \_\_\_\_$

61. $-8^{\frac{2}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

62. $(-8)^{\frac{2}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

63. $(-125)^{\frac{2}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

64. $-125^{\frac{2}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

65. $(-8)^{-\frac{1}{3}} = (\sqrt[3]{-8})^{-1}$
$= \_\_\_\_$
$= \_\_\_\_$

66. $(-27)^{-\frac{1}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

67. $(-125)^{-\frac{1}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

68. $(-8)^{-\frac{2}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

69. $(-27)^{-\frac{2}{3}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

70. $(-64)^{-\frac{1}{2}} = \_\_\_\_$
$= \_\_\_\_$
$= \_\_\_\_$

ANSWERS 3.05

p. 268–274:

1. 4; **2.** 2; **3.** 5; **4.** 2; **5.** 3; **6.** 2; **7.** 12; **8.** 2; **9.** 9;
10. 27; **11.** 64; **12.** 25; **13.** 16; **14.** 8; **15.** 4; **16.** 27;
17. 1/16; **18.** 1/27; **19.** 1/4; **20.** 1/16; **21.** 1/4; **22.**1/2;
23. 1/125; **24.** 1/8; **25.** 5; **26.** 4; **27.** 0.2 or 1/5;
28. 0.25 or 1/4; **29.** 16; **30.** 16; **31.** 64; **32.** 27;
33. 0.0625 or 1/16; **34.** 0.0625 or 1/16; **35.** 0.015625 or 1/64; **36.** 0.037037037... or 1/27; **37.** 0.008 or 1/125; **38.** 0.142857142847... or 1/7;
39. 0.012345679012345679... or 1/81; **40.** 0.001 or 1/1000; **41.** -2; **42.** -3; **43.** -5; **44.** 4; **45.** 9;
46. 25; **47.** -2; **48.** 4; **49.** -8; **50.** 16; **51.** No Solution;
52. No Solution; **53.** No Solution; **54.** No Solution;
55. -32; Yes **56.** No Solution; **57.** -5; **58.** -4; Yes **59.**9;
60. -9; **61.** -4; **62.** 4; **63.** 25; **64.** -25; **65.** -1/2;
66. -1/3; **67.** -1/5; **68.** 1/4; **69.** 1/9; **70.** No Solution.

Dr. Robert J. Rapalje

More FREE help available from my website at www.mathinlivingcolor.com

ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE