3.06 *Complex Numbers*

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As a quick review of radicals, answer each of the following using real numbers:

1. $\sqrt{25}$ = _____ because ($)^2 = 25$.

2. $\sqrt[3]{27}$ = _____ because ($)^3 = 27$.

3. $\sqrt[3]{-27}$ = _____ because ($)^3 = -27$.

4. $\sqrt[5]{-32}$ = _____ because ($)^5 = -32$.

5. $\sqrt{-25}$ = _____ because ($)^2 = -25$.

Notice that there is no real solution for $\sqrt{-25}$, since no real number squared is negative. Likewise, the square root, fourth root, sixth root, any <u>even</u> root of any negative number is undefined and has no solution in the real number system.

However, another set of numbers, the **complex numbers**, will be explained now involving the letter **"i"** (for **imaginary**) which is defined follows:

DEFINITIONS

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$\sqrt{-a} = i\sqrt{a}$$

NOTE: $\sqrt{-1} = i$, *but* $\sqrt[3]{-1} = -1$

Now use these definitions to answer the questions using the "i-notation." These are called **imaginary numbers** because they are **"unreal"**!!

6. $\sqrt{-25} = \sqrt{25} \cdot \sqrt{-1}$ $= \_\_\_\_\_\_$

7. $\sqrt{-36} = \sqrt{} \cdot \sqrt{}$ $= \_\_\_\_\_\_$

8. $\sqrt{-169} = \sqrt{} \cdot \sqrt{}$ $= \_\_\_\_\_\_$

9. $\sqrt{-20} = \sqrt{4} \cdot \sqrt{5} \cdot \sqrt{-1}$

$= \_\_\_\_\_\_$ or $2\,i\sqrt{5}$

10. $\sqrt{-40} = \sqrt{} \cdot \sqrt{} \cdot \sqrt{}$

$= \_\_\_\_\_\_$

11. $\sqrt{-18} = \sqrt{} \cdot \sqrt{} \cdot \sqrt{}$

$= \_\_\_\_\_\_$

12. $\sqrt{-72} = \sqrt{} \cdot \sqrt{} \cdot \sqrt{}$

$= \_\_\_\_\_\_$

Remember that the $\sqrt{-1} = i$ definition applies __only__ to __square roots__!

Remember that $\sqrt[3]{-1} = -1$, $\sqrt[5]{-1} = -1$, $\sqrt[7]{-1} = -1$, any **odd root of -1** is **-1.** Higher __**even**__ **roots** such as $\sqrt[4]{-1}$, $\sqrt[6]{-1}$, $\sqrt[8]{-1}$ are still not defined until higher math courses.

13. $\sqrt[3]{-8} = \_\_\_\_\_\_$

14. $\sqrt[3]{-64} = \_\_\_\_\_\_$

15. $\sqrt{-64} = \_\_\_\_\_\_$

16. $\sqrt{-9} = \_\_\_\_\_\_$ $= \_\_\_\_\_\_$

17. $\sqrt[3]{-125} = \_\_\_\_\_\_$ $= \_\_\_\_\_\_$

18. $\sqrt{-144} = \_\_\_\_\_\_$ $= \_\_\_\_\_\_$

19. $\sqrt{-125} = \sqrt{} \cdot \sqrt{} \cdot \sqrt{}$

$= \_\_\_\_\_\_$

20. $\sqrt[3]{-54} = \sqrt[3]{} \cdot \sqrt[3]{} \cdot \sqrt[3]{}$

$= \_\_\_\_\_\_$

21. $\sqrt{-54}$ = _______ 22. $\sqrt[3]{-40}$ = _______ 23. $\sqrt[4]{-16}$ = _______

= _______ = _______ = _______

24. $\sqrt{-200}$ = _______ 25. $\sqrt{-72}$ = _______ 26. $\sqrt[3]{-72}$ = _______

27. $\sqrt[3]{-80}$ = _______ 28. $\sqrt{-80}$ = _______ 29. $\sqrt[4]{-80}$ = _______

30. $\sqrt{-32}$ = _______ 31. $\sqrt[3]{-32}$ = _______ 32. $\sqrt[4]{-32}$ = _______

33. $\sqrt[5]{-32}$ = _______ 34. $\sqrt[7]{-128}$ = _______ 35. $\sqrt[3]{-144}$ = _______

As you probably noticed in these exercises, for any positive "a", $\sqrt{-a} = \sqrt{a}\,i$ or $i\sqrt{a}$, and $\sqrt[3]{-a} = -\sqrt[3]{a}$. Notice that when writing $\sqrt{a}\,i$, the **i** is <u>not</u> inside the radical. To avoid confusion, it is usually written " $i\sqrt{a}$ " instead of " $\sqrt{a}\,i$." Notice also that for any positive "a", $\sqrt[3]{-a}$ is a negative but nontheless a <u>**real number**</u> located on the real number line. However, $\sqrt{-a}$ is an <u>**imaginary number**</u>, which may be graphed on a number line that is perpendicular to the real numberline as illustrated and explained in **Section 1.01** (see pages **3** and **4**). Finally, a **complex number is any combination of real and imaginary numbers combined in the form of Z = a + bi, where "a" and "b" are real numbers.** Remember that these are not "complex" in the sense of being "complicated" (they are not!). They are complex in the sense of "inter-connected" parts (real and imaginary), as a B-complex vitamin. Moreover, realize that **every real number is a complex number,** since, for example, the **real number 6** can be written **6 + 0i,** and **every imaginary number is a complex number,** since, for example, **6i** can be written **0 + 6i.**

When adding, subtracting, multiplying, or dividing radicals involving negatives, simplify as before. For addition and subtraction problems, just combine like terms. For multiplication, remember that $i \cdot i = i^2$ and $i^2 = -1$.

Simplify as indicated:

1. $\sqrt{-9} + \sqrt{-4}$
   $= 3i + 2i$
   $=$ ________

2. $\sqrt{-16} + \sqrt{-1}$
   $=$ ____ $+$ ____
   $=$ ____________

3. $\sqrt{-16} - \sqrt{-81}$
   $=$ ____ $-$ ____
   $=$ ____________

4. $\sqrt{-25} - \sqrt{-121}$

5. $-\sqrt{-144} - \sqrt{-25}$

6. $-\sqrt{-81} + \sqrt{-4}$

7. $\sqrt{144} + \sqrt{-49}$
   $=$ ____________
   (Does not combine!)

8. $\sqrt{100} - \sqrt{-64}$
   $=$ ____________

9. $-\sqrt{-4} + \sqrt{-36}$
   $=$ ____________
   $=$ ____________

10. $\sqrt{36} + \sqrt{-36}$

= ______________

11. $\sqrt[3]{-27} + \sqrt{-81}$

= ______________

12. $\sqrt[3]{64} - \sqrt{-9}$

= ______________

13. $\sqrt[3]{-8} - \sqrt{-4}$

= ______________

14. $\sqrt[3]{-125} + \sqrt{-144}$

= ______________

15. $\sqrt[3]{-64} - \sqrt{-64}$

= ______________

16. $\sqrt[3]{-40} + \sqrt{-40}$

$= \sqrt[3]{-8}\,\sqrt[3]{5} + \sqrt{-4}\sqrt{10}$

= ______________

17. $\sqrt[3]{-125} - \sqrt{-27}$

= ______________

= ______________

18. $\sqrt[3]{-72} + \sqrt{-72}$

= ______________

= ______________

19. $\sqrt{-4} \cdot \sqrt{-9}$

= 2i · 3i

= 6 i^2 [$i^2 = -1$]

= 6 (   ) = ________

20. $\sqrt{-9} \cdot \sqrt{-16}$

= ______________

= ______________

= ______________

21. $\sqrt{-4} \cdot \sqrt{-25}$

= ______________

= ______________

= ______________

Notice that the multiplication rule for radicals $\sqrt{a} \cdot \sqrt{b} = \sqrt{a\,b}$ does <u>not</u> apply when "a" and "b" are both negative numbers.

In #19 it would be <u>incorrect</u> to say $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{(-4)\,(-9)}$

$= \sqrt{36} = 6$ <u>WRONG</u>!!

The correct way is to change to "i" notation, where $i^2 = -1$.

$\sqrt{-4} \cdot \sqrt{-9} = 2i \cdot 3i$

$= 6\,i^2 = -6$ <u>CORRECT</u>!

RULE: Whenever there are negatives in square roots, convert immediately to "i" notation using the formula

$$\sqrt{-a} = i\sqrt{a}$$

22. $\sqrt{-16} \cdot \sqrt{-25}$

23. $\sqrt{-4} \cdot \sqrt{-1}$

24. $\sqrt{-36} \cdot \sqrt{-25}$

25. $\sqrt{-8} \cdot \sqrt{-2}$

$= i\sqrt{8} \cdot i\sqrt{2}$

$= i^2\sqrt{16}$

= ____________

26. $\sqrt{-12} \cdot \sqrt{-3}$

27. $\sqrt{-30} \cdot \sqrt{-2}$

28. $\dfrac{\sqrt{-8}}{\sqrt{-2}} = \dfrac{i\sqrt{8}}{i\sqrt{2}}$

$= \sqrt{\dfrac{8}{2}}$

$= \sqrt{4}$ = ________

29. $\dfrac{\sqrt{-48}}{\sqrt{-3}}$

30. $\dfrac{\sqrt{-48}}{\sqrt{-2}}$

31. $\dfrac{\sqrt{-8}}{\sqrt{2}}$

32. $\dfrac{\sqrt{-48}}{\sqrt{3}}$

33. $\dfrac{\sqrt{-48}}{\sqrt{2}}$

In the next exercises, there is an "**i**" that needs to be eliminated from the denominator. For **monomial denominators** this is done by **multiplying numerator and denominator by "i"** as illustrated in **#34.**

34. $\dfrac{\sqrt{8}}{\sqrt{-2}} = \dfrac{\sqrt{8}}{i\sqrt{2}} \cdot \dfrac{i}{i}$

$= \dfrac{i\sqrt{4}}{-1}$

$=$ ________

35. $\dfrac{\sqrt{48}}{\sqrt{-3}}$

36. $\dfrac{\sqrt{48}}{\sqrt{-2}}$

In general, a **complex number** is a number that can be written in the form "**a + bi**", where "**a**" and "**b**" are real numbers, and "**i**" represents $\sqrt{-1}$ Notice in the exercises that follow, that addition, subtraction, multiplication, and division of complex numbers in the form **a + bi** always results in a complex number in the form **a + bi**. These are known as the **closure properties for complex numbers.**

EXERCISES. Express each answer in the form "a + bi."

Addition and subtraction is just a matter of combining like terms:

1. (-6 + 2i) + (8 - 6i)

= -6 + 8 + 2i - 6i

= ________________

2. (-6 + 2i) - (8 - 6i)

= -6 - 8 + 2i + 6i

= ________________

3. (-8 - 5i) + (6 - 9i)

= ________________

4. (-8 - 5i) - (6 - 9i)

= ________________

= ________________

Multiplication involves the distributive property (or F OI L) and remembering that $i^2 = -1$.

5. $3i(2 + i)$
$= 6i + 3i^2$
$= 6i + 3(-1)$
= ____________

6. $-3i(4 - 2i)$
= ____________
= ____________
= ____________

7. $(2 + 3i)(4 + i)$
$= 8 + 2i + 12i + 3i^2$
$= 8 + 14i - 3$
= ____________

8. $(8 + i)(2 + i)$

9. $(8 - i)(6 + 2i)$

10. $(8 - i)(6 - 2i)$

11. $(8 - i)(8 + i)$

12. $(3 + 6i)(3 - 6i)$

13. $(3 + 2i)(3 - 2i)$

14. $(8 + i)^2$
$= (8 + i)(8 + i)$

15. $(8 - i)^2$

16. $(3 + 2i)^2$

17. $(3 - 5i)^2$

18. $(5 - 7i)(6 + 2i)$

19. $(5 + 3i)(6 - 5i)$

20. $(8 - 5i)(3 - 2i)$

21. $i(2 - 3i)(2 + 4i)$

$= i(\qquad\qquad)$

$= i(\qquad\qquad)$

$=$

$=$

22. $i(3 + 2i)(5 - i)$

23. $-2i(4 - i)(2 + 3i)$

24. $-3i(2 - i)(3 + 2i)$

When it comes to dividing complex numbers, notice that you are usually dividing a binomial by a binomial. The strategy will be very similar to the one used before to **rationalize binomial denominators** (see **SECTION 3.04**)--***multiply numerator and denominator by the conjugate of the denominator***. Before solving **#25**, look back at **# 7** of this exercise set, in which $(2 + 3i)(4 + i) = 5 + 14i$. What would you expect if you divided the answer $5 + 14i$ by one of its factors $2 + 3i$? Shouldn't you get the other factor $4 + i$? In **#25**, the strategy is to multiply the numerator and denominator by $2 - 3i$.

25. $\dfrac{5 + 14i}{2 + 3i}$

SOLUTION: $\dfrac{5 + 14i}{2 + 3i} \cdot \dfrac{2 - 3i}{2 - 3i}$

$$= \frac{10 - 15i + 28i - 42i^2}{4 - 6i + 6i - 9i^2}$$

$$= \frac{10 + 13i + 42}{4 + 9}$$

$$= \frac{52 + 13i}{13} = \frac{13\,(4 + i)}{13} = \underline{\qquad\qquad}$$

26. $\frac{5 + 14i}{4 + i}$

27. $\frac{15 + 10i}{2 + i}$

28. $\frac{15 + 10i}{8 + i}$

29. $\frac{25 + 5i}{8 - i}$

30. $\frac{23 - 11i}{3 - i}$

31. $\frac{5 + 12i}{3 + 2i}$

32. $\frac{16 + 30i}{-3 + 5i}$

33. $\frac{13 + 13i}{3 - 2i}$

34. $\frac{20 - 22i}{4 - i}$

35. $\frac{3 - 24i}{3 + 2i}$

You probably noticed that the exercises so far have all "come out even." They were "carefully" selected to do so. Real life is usually not so kind. Nevertheless, answers can still be given in the form "a + bi." **Be sure to reduce all answers.**

36. $\frac{(8 + 2i)}{(1 + 3i)} \cdot \frac{(1 - 3i)}{(1 - 3i)}$

$= \frac{8 - 24i + 2i - 6i^2}{1 - 9i^2}$

$= \frac{8 - 22i + 6}{1 + 9}$

$= \frac{14 - 22i}{10}$

$= \frac{14}{10} - \frac{22}{10} i =$ ________

37. $\frac{2 + 3i}{1 - 3i}$

38. $\frac{3 - 5i}{3 + i}$

39. $\frac{4 - 3i}{2 + i}$

40. $\frac{4 - 3i}{2 - i}$

There are two ways to do #41-43. Which way seems easier?

41. $\frac{6 + 3i}{2i}$

$= \frac{(6 + 3i)}{(2i)} \cdot \frac{(i)}{(i)}$

$= \frac{6i + 3i^2}{2i^2}$

$= \frac{-3 + 6i}{-2}$

$=$ ________

$\frac{6 + 3i}{2i}$

$\frac{6}{2i} + \frac{3i}{2i}$

$\frac{6}{2i} \cdot \frac{i}{i} + \frac{3i}{2i}$

$\frac{6i}{-2} + \frac{3}{2}$

____

42. $\frac{-2 + 16i}{2i}$

43. $\frac{3 - 10i}{5i}$

MISCELLANEOUS EXERCISES:

44. $(-12 + 3i) - (8 - 5i)$ 45. $(-2 + 3i) + (-35 - 8i)$

46. $\frac{4 - 3i}{2i}$ 47. $\frac{4 - 3i}{2 + i}$ 48. $\frac{-2 + 16i}{2 + 4i}$

49. $(2 - 3i)(4 + i)$ 50. $(3 - 5i)^2$

51. $i(2 + 6i)(1 - i)$ 52. $2i(2 - 3i)(4 + i)$

53. $(2 - 2i)^2$ 54. $2i(2 - 2i)^2$ 55. $(2 - 2i)^3$

"i" RAISED TO THE POWER

1. $i^3 = i^2 \cdot i$
   $= (-1) \cdot i$
   $=$ ______

2. $i^4 = i^2 \cdot i^2$
   $= (\quad)(\quad)$
   $=$ ______

3. $i^5 = i^4 \cdot i$
   $= (\quad)(\quad)$
   $=$ ______

4. $i^6 = i^4 \cdot i^2$
   $= (\quad)(\quad)$
   $=$ ______

5. $i^7 = i^4 \cdot i^2 \cdot i$
   $= (\quad)(\quad)(\quad)$
   $=$ ______

6. $i^8 = i^4 \cdot i^4$
   $= (\quad)(\quad)$
   $=$ ______

7. $i^9 = i^8 \cdot i$
   $= (\quad)(\quad)$
   $=$ ______

8. $i^{10} = i^8 \cdot i^2$
   $= (\quad)(\quad)$
   $=$ ______

9. $i^{11} = i^8 \cdot i^2 \cdot i$
   $= (\quad)(\quad)(\quad)$
   $=$ ______

10. $i^{12} = i^8 \cdot (\quad)$
   $= (\quad)(\quad)$
   $=$ ______

11. $i^{13} = (\quad) \cdot i$
   $= (\quad)(\quad)$
   $=$ ______

12. $i^{14} =$ ___ $\cdot$ ___
   $=$ ______
   $=$ ______

13. $i^{15} =$
   $= (\quad)(\quad)(\quad)$
   $=$ ______

14. $i^{16} =$ ___ $\cdot$ ___
   $= (\quad)(\quad)$
   $=$ ______

15. $i^{17} =$ ___ $\cdot$ ___ $\cdot$ ___
   $= (\quad)(\quad)(\quad)$
   $=$ ______

16. THERE ARE ONLY FOUR POSSIBLE ANSWERS FOR i TO A POWER:
   TWO THAT ARE REAL: _____ and _____; TWO THAT ARE IMAGINARY: _____ and _____.

17a) i raised to an even power is either _____ or _____.
   b) i raised to an odd power is either _____ or _____.

18a) i raised to a multiple of 4 is ______.
   b) i raised to an even, but not multiple of 4, power is _____.
   c) i raised to one more than a multiple of 4 power is _____.
   d) i raised to three more than a multiple of 4 power is _____.
   e) i raised to one less than a multiple of 4 power is _____.
   ("Three more" is the same as "one less" than a multiple of 4.)

19. Given that n is an integer, 4n is a multiple of 4. Then,
   a) $i^{4n} =$ _____ b) $i^{4n+1} =$ _____ c) $i^{4n+2} =$ _____ d) $i^{4n-1} =$ _____

20. $i^0 =$ ______

21. $i^{-2} = \dfrac{1}{i^2}$

$=$ ________

$=$ ________

22. $i^{-4} = \dfrac{1}{(\quad)}$

$=$ ________

$=$ ________

23. $i^{-6} =$

24. $i^{-8} =$

25. $i^{-10} =$

26. $i^{-1} = \dfrac{1}{i}$

$= \dfrac{1}{i} \cdot \dfrac{i}{i}$

$= \dfrac{i}{i^2} = \dfrac{i}{(-1)}$

$=$ ______

27. $i^{-3} = \dfrac{1}{i^3}$

$= \dfrac{1}{i^3} \cdot \dfrac{i}{i}$

28. $i^{-5} = \dfrac{1}{i^5}$

$=$

GRAPHING CALCULATOR SPECIAL

Many calculators, especially the TI-85 are able to do computations involving complex numbers with parentheses and a comma to separate real and imaginary parts. Try entering the number 3+2i as "(", "3a", ",", "2", ")". You can then calculate using ordinary operations: Try (3 + 2i)/(1 + 3i) from #36, p.285. Enter "(3,2)", "÷", "(1,3)", "ENTER". The answer should appear as a complex number: (1.4, -2.2) which means 1.4 - 2.2i .

If the problem does not come out even (more or less), your calculator should be capable of converting to fractions in order to give the exact values.

ANSWERS 3.06

p. 275-277:

1. 5; **2.** 3; **3.** -3; **4.** -2; **5.** No real solution; **6.** 5i;
7. 6i; **8.** 13i; **9.** $2i\sqrt{5}$; **10.** $2i\sqrt{10}$; **11.** $3i\sqrt{2}$; **12.** $6i\sqrt{2}$;
13. -2; **14.** -4; **15.** 8i; **16.** 3i; **17.** -5; **18.** 12i; **19.** $5i\sqrt{5}$;
20. $-3\sqrt[3]{2}$; **21.** $3i\sqrt{6}$; **22.** $-2\sqrt[3]{5}$; **23.** Not defined yet;
24. $10i\sqrt{2}$; **25.** $6i\sqrt{2}$; **26.** $-2\sqrt[3]{9}$; **27.** $-2\sqrt[3]{10}$; **28.** $4i\sqrt{5}$;
29. Not defined yet; **30.** $4i\sqrt{2}$; **31.** $-2\sqrt[3]{4}$; **32.** Not defined yet; **33.** -2; **34.** -2; **35.** $-2\sqrt[3]{18}$.

p. 278-281:

1. 5i; **2.** 5i; **3.** -5i; **4.** -6i; **5.** -17i; **6.** -7i; **7.** 12+7i;
8. 10-8i; **9.** 4i; **10.** 6+6i; **11.** -3+9i; **12.** 4-3i; **13.** -2-2i;
14. -5+12i; **15.** -4-8i; **16.** $-2\sqrt[3]{5}+2i\sqrt{10}$; **17.** $-5-3i\sqrt{3}$;
18. $-2\sqrt[3]{9}+6i\sqrt{2}$; **19.** -6; **20.** -12; **21.** -10; **22.** -20; **23.** -2;
24. -30; **25.** -4; **26.** -6; **27.** $-2\sqrt{15}$; **28.** 2; **29.** 4;
30. $2\sqrt{6}$; **31.** 2i; **32.** 4i; **33.** $2i\sqrt{6}$; **34.** -2i; **35.** -4i;
36. $-2i\sqrt{6}$.

p. 281-286:

1. 2-4i; **2.** -14+8i; **3.** -2-14i; **4.** -14+4i; **5.** -3+6i;
6. -6-12i; **7.** 5+14i; **8.** 15+10i; **9.** 50+10i; **10.** 46-22i;
11. 65; **12.** 45; **13.** 13; **14.** 63+16i; **15.** 63-16i; **16.** 5+12i;
17. -16-30i; **18.** 44-32i; **19.** 45-7i; **20.** 14-31i; **21.** -2+16i;
22. -7+17i; **23.** 20-22i; **24.** 3-24i; **25.** 4+i; **26.** 2+3i;
27. 8+i; **28.** 2+i; **29.** 3+i; **30.** 8-i; **31.** 3+2i; **32.** 3-5i;

p. 281-286:

33. $1+5i$; **34.** $6-4i$; **35.** $-3-6i$; **36.** $\frac{7}{5}-\frac{11}{5}i$; **37.** $-\frac{7}{10}+\frac{9}{10}i$;

38. $\frac{2}{5}-\frac{9}{5}i$; **39.** $1-2i$; **40.** $\frac{11}{5}-\frac{2}{5}i$; **41.** $\frac{3}{2}-3i$; **42.** $8+i$;

43. $-2-\frac{3}{5}i$; **44.** $-20+8i$; **45.** $-37-5i$; **46.** $-\frac{3}{2}-2i$;

47. $1-2i$; **48.** $3+2i$; **49.** $11-10i$; **50.** $-16-30i$; **51.** $-4+8i$;
52. $20+22i$; **53.** $-8i$; **54.** 16; **55.** $-16-16i$.

p. 287-288:

1. $-i$; **2.** 1; **3.** i; **4.** -1; **5.** $-i$; **6.** 1; **7.** i; **8.** -1; **9.** $-i$;
10. 1; **11.** i; **12.** -1; **13.** $-i$; **14.** 1; **15.** i;
16. Real: $1, -1$; Imag: $i, -i$; **17a)** $1, -1$; **b)** $i, -i$;
18a) 1, **b)** -1, **c)** i, **d)** $-i$, **e)** $-i$;
19a) 1, **b)** i, **c)** -1, **d)** $-i$; **20.** 1; **21.** -1; **22.** 1; **23.** -1;
24. 1; **25.** -1; **26.** $-i$; **27.** i; **28.** $-i$.

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE