4.04 *Quadratic Inequalities*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

Before introducing quadratic inequalities, remember the concepts of **absolute value equations** and **inequalities**, and return again to the **Trichotomy Axiom**. There are **three** ways to compare $|X|$ to a positive number such as 4: $|X|=4$, $|X|<4$, and $|X|>4$. By trial and error, you can see the solution to $|X|=4$ consists of the two numbers $X=-4$ and $|X|=4$. Also by trial and error, you can see that $|X|<4$ consists of all values of X **between** -4 and 4, that is $-4<X<4$, or in interval notation: $(-4,4)$. What remains for the solution to $|X|>4$ are two separate intervals in which $X>4$ or $X<-4$, or in interval notation: $(-\infty, -4)\cup(4,\infty)$ (see below).

. . . -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 . . .

	Betweenness	Two points	Extremes
	$\lvert X\rvert < 4$	$\lvert X\rvert = 4$	$\lvert X\rvert > 4$
	$-4 < X < 4$	$X = 4$ or $X = -4$	$X > 4$ or $X < -4$
Int.Notation:	$(-4, 4)$		$(-\infty, -4) \cup (4, \infty)$

In the more general cases, where $c > 0$,

Betweenness	Two points	Extremes
$\lvert aX+b\rvert < c$	$\lvert aX+b\rvert = c$	$\lvert aX+b\rvert > c$
$-c < aX+b < c$	$aX+b = c$ or $aX+b = -c$	$aX+b > c$ or $aX+b < -c$

Continuing with the idea of "trichotomy," there are three conditions needed to guarantee each of these cases:

Betweenness	Two points	Extremes
1. Absolute value	**1.** Absolute value	**1.** Absolute value
2. $<$ or $\le$	**2.** $=$	**2.** $>$ or $\ge$
3. $c > 0$	**3.** $c > 0$	**3.** $c > 0$

EXAMPLE 1

$$|2X + 5| = 7$$
$$\begin{array}{rclcrcl} 2X + 5 &=& 7 & \text{or} & 2X + 5 &=& -7 \\ 2X &=& 2 & & 2X &=& -12 \\ X &=& 1 & \text{or} & X &=& -6 \end{array}$$

EXAMPLE 2

$|2X + 5| < 7$ (BETWEENNESS)

$$\begin{array}{rcccl} -7 &<& 2X + 5 &<& 7 \\ \underline{-5} & & \underline{-5} & & \underline{-5} \\ -12 &<& 2X &<& 2 \\ -6 &<& X &<& 1 \end{array}$$

$(-6, 1)$

EXAMPLE 3

$|2X + 5| \ge 7$ (EXTREMES)

$$\begin{array}{rclcrcl} 2X + 5 &\ge& 7 & \textbf{or} & 2X + 5 &\le& -7 \\ \underline{-5} & & \underline{-5} & & \underline{-5} & & \underline{-5} \\ 2X &\ge& 2 & & 2X &\le& -12 \\ X &\ge& 1 & \textbf{or} & X &\le& -6 \end{array}$$

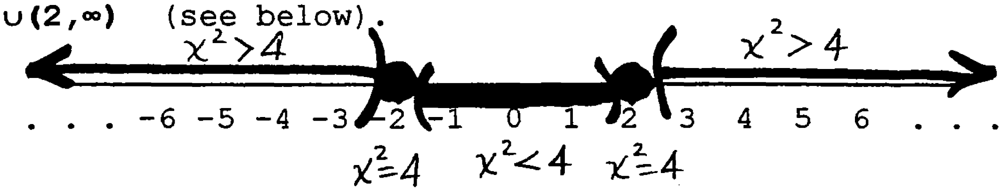
$(-\infty, -6] \cup [1, \infty)$

In a previous section, quadratic equations were solved by factoring, by completing the square, and by the quadratic formula. Usually, although not always, there were two solutions, as there were with absolute value equations. Now, having solved **absolute value equations and inequalities** with **endpoints, betweenness, and extremes,** wouldn't it be nice if **quadratic equations and inequalities** could somehow be **"endpoints, betweenness, and extremes"** with similar patterns? The following is almost too good to be true, but it is . . .

There are **three** ways to compare X^2 to a number such as 4:

$$X^2 = 4, \; X^2 < 4, \text{ and } X^2 > 4.$$

You can see the solution to $X^2 = 4$ consists of the two numbers $X=-2$ and $X=2$. By trial and error, you can see that $X^2 < 4$ consists of all values of X **between** -2 and 2, that is $-2<X<2$, or in interval notation: $(-2,2)$. What remains for the solution to $X^2 > 4$ are two separate intervals in which $X>2$ or $X<-2$, or in interval notation: $(-\infty, -2)\cup(2,\infty)$ (see below).



	Betweenness	Two points	Extremes
	$X^2 < 4$	$X^2 = 4$	$X^2 > 4$
	$-2 < X < 2$	$X = 2$ or $X = -2$	$X > 2$ or $X < -2$
Interval Notation	$(-2, 2)$		$(-\infty, -2) \cup (2, \infty)$

In the more general cases, where $a > 0$ and there are **two real endpoints,**

BETWEENNESS	TWO POINTS	EXTREMES
$aX^2+bX+c < 0$	$aX^2+bX+c = 0$	$aX^2+bX+c > 0$

BETWEENNESS	**TWO POINTS**	**EXTREMES**
$aX^2+bX+c < 0$	$aX^2+bX+c = 0$	$aX^2+bX+c > 0$

Continuing with the idea of "trichotomy," there are three conditions needed to guarantee each of these cases:

Betweenness	Two points	Extremes
1. Quadratic (X^2)	1. Quadratic (X^2)	1. Quadratic (X^2)
2. $<$ or $\le$	2. $=$	2. $>$ or $\ge$
3. $a > 0$	3. $a > 0$	3. $a > 0$

EXAMPLE 1 (See section 1.08)

$$\mathbf{X^2 + 2X - 8 = 0}$$
$$(X + 4)(X - 2) = 0$$
$$X = -4 \text{ or } X = 2$$

EXAMPLE 2

$\mathbf{X^2 + 2X - 8 < 0}$ (BETWEENNESS)

$X^2 + 2X - 8 = 0$ (Endpoints!)

$(X + 4)(X - 2) = 0$

$X = -4$ or $X = 2$

$-4 < X < 2$

$\mathbf{(-4, 2)}$

EXAMPLE 3

$\mathbf{X^2 + 2X - 8 > 0}$ (EXTREMES)

$X^2 + 2X - 8 = 0$ (Endpoints!)

$(X + 4)(X - 2) = 0$

$X = -4$ or $X = 2$

$X > 2$ **or** $X < -4$

$\mathbf{(-\infty, -4) \cup (2, \infty)}$

What happens if a<0? For example, $\mathbf{-X^2 + 4X + 5 > 0}$. No problem! Just divide both sides of the inequality by **(-1)**, then don't forget to reverse the direction of the inequality sign!

EXAMPLE 4

$\mathbf{-X^2 + 4X + 5 > 0}$

$X^2 - 4X - 5 < 0$ (BETWEENNESS)

$X^2 - 4X - 5 = 0$ (Endpoints!)

$(X - 5)(X + 1) = 0$

$X = 5$ or $X = -1$

$-1 < X < 5$

$\mathbf{(-1, 5)}$

EXAMPLE 5

$\mathbf{-X^2 + 4X + 5 \le 0.}$

$X^2 - 4X - 5 \ge 0$ (EXTREMES)

$X^2 - 4X - 5 = 0$ (Endpoints!)

$(X - 5)(X + 1) = 0$

$X = 5$ or $X = -1$

$X \ge 5$ **or** $X \le -1$

$\mathbf{(-\infty, -1] \cup [5, \infty)}$

EXERCISES. Solve for X. Give answers in interval notation.

1. $X^2 - 3X - 4 < 0$

2. $X^2 - 3X < 0$

3. $X^2 - 2X - 8 \le 0$

4. $X^2 - 12X + 35 \le 0$

5. $X^2 - 3X - 4 > 0$

6. $X^2 - 3X > 0$

7. $X^2 - 2X - 8 \ge 0$

8. $X^2 - 12X + 35 \ge 0$

9. $X^2 - 5X < 6$

10. $X^2 - 6X > 16$

11. $X^2 - 6X \geq 16$

12. $X^2 - 5X \leq 6$

13. $- X^2 + X + 6 \leq 0$

14. $- X^2 - 3X < 0$

15. $- X^2 + 6X > 0$

16. $6X - X^2 > 5$

17. $2(2 - X^2) \le 7X$

18. $2X(4 - X) > 15 - 5X$

What happens if the quadratic equation does not factor? Again, no problem! Just find the two endpoints by quadratic formula or completing the square (if there are two real endpoints!). Then from the "±" in the quadratic formula, the "-" will be the left endpoint, and the "+" will be the right endpoint, as illustrated in the next examples.

EXAMPLE 6

$X^2 + 2X - 5 \le 0$ (BETWEENNESS)

$X^2 + 2X - 5 = 0$ (Endpoints!)

(Does not factor, so use quadratic formula or complete square.)

a=1 b=2 c=-5

$$X = \frac{-2 \pm\sqrt{2^2-4(1)(-5)}}{2(1)}$$

$$X = \frac{-2 \pm \sqrt{24}}{2}$$

$$X = \frac{-2 \pm 2\sqrt{6}}{2}$$

$$X = \frac{2(-1 \pm \sqrt{6})}{2}$$

$$X = -1 \pm \sqrt{6}$$

Betweenness means:

$-1 - \sqrt{6} \le X \le -1 + \sqrt{6}$

$[-1 - \sqrt{6}, -1 + \sqrt{6}]$

EXAMPLE 7

$X^2 + 2X - 5 > 0$ (EXTREMES)

$X^2 + 2X - 5 = 0$ (Endpoints!)

a=1 b=2 c=-5

$$X = \frac{-2 \pm\sqrt{2^2-4(1)(-5)}}{2(1)}$$

$$X = \frac{-2 \pm \sqrt{24}}{2}$$

$$X = \frac{-2 \pm 2\sqrt{6}}{2}$$

$$X = \frac{2(-1 \pm \sqrt{6})}{2}$$

$$X = -1 \pm \sqrt{6}$$

Extremes means:

$(-\infty, -1 - \sqrt{6}) \cup (-1 + \sqrt{6}, \infty)$

Finally, what happens if there are not two real endpoints? Perhaps there is only one endpoint, or perhaps, because the endpoints are complex numbers, there are no endpoints. Think about it! If there are no real endpoints or only one endpoint, what could possibly be "betweenness"? Answer: Nothing; empty set. Now picture "extremes" from only one endpoint, or from no endpoints. What would be "extremes"? Answer: Everything; the entire numberline, sometimes including or sometimes excluding the endpoint itself as the particular problem warrants.

EXAMPLE 8

$\mathbf{X^2 + 2X + 5 < 0}$ (BETWEENNESS)

$X^2 + 2X + 5 = 0$ (Endpoints!)

EXAMPLE 9

$\mathbf{X^2 + 2X + 5 > 0}$ (EXTREMES)

$X^2 + 2X + 5 = 0$ (Endpoints!)

(Again, does not factor, so quad formula or complete the square!)

Example 8:

a=1 b=2 c=5

$$X = \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)}$$

$$X = \frac{-2 \pm \sqrt{-16}}{2}$$

$$X = \frac{-2 \pm 4i}{2}$$

$$X = -1 \pm 2i$$

No real endpoints

No Solution -- ϕ

Example 9:

a=1 b=2 c=5

$$X = \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)}$$

$$X = \frac{-2 \pm \sqrt{-16}}{2}$$

$$X = \frac{-2 \pm 4i}{2}$$

$$X = -1 \pm 2i$$

No real endpoints

All Real X -- $(-\infty, \infty)$

EXAMPLE 10

$\mathbf{X^2 + 2X + 1 < 0}$ (BETWEENNESS)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Betweenness, nothing--not even the endpoint!

No Solution!

EXAMPLE 11

$\mathbf{X^2 + 2X + 1 \geq 0}$ (EXTREMES)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Extremes, include endpoint!

All Reals $(-\infty, \infty)$

EXAMPLE 12

$\mathbf{X^2 + 2X + 1 \le 0}$ (BETWEENNESS)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Betweenness, including the endpoint:

Solution: $X = -1$ only.

EXAMPLE 13

$\mathbf{X^2 + 2X + 1 > 0}$ (EXTREMES)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Extremes, no endpoint:

$\mathbf{(-\infty, -1) \cup (-1, \infty)}$

or **All real $X \ne -1$**

EXERCISES. Solve for X. Give answers in interval notation:

19. $X^2 + 4X + 2 < 0$

20. $X^2 + 4X + 2 > 0$

21. $X^2 - 6X + 4 \ge 0$

22. $X^2 - 6X + 4 \le 0$

23. $X^2 + 4X + 6 < 0$

24. $X^2 + 4X + 6 \ge 0$

25. $X^2 - 2X + 8 > 0$

26. $X^2 - 2X + 8 \le 0$

27. $X^2 + 4X + 4 > 0$

28. $X^2 + 4X + 4 < 0$

29. $X^2 - 10X + 25 \leq 0$

30. $X^2 - 10X + 25 \geq 0$

31. $X^2 - 8X \geq -16$

32. $X^2 + 9 > 6X$

33. $X^2 + 4 \geq 0$

34. $X^2 + 4X \geq 0$

35. $X^2 - 4 > 0$

36. $X^2 + 4 < 0$

37. $4 - X^2 \geq 0$

38. $X^2 - 4X \leq 0$

39. $X(6 - X) \leq -27$

40. $X(8 - X) > 16$

41. $X(6 - X) \leq -9$

42. $X(6 - X) \leq 9$

ANSWERS 4.04

p.335-341:

1. $(-1,4)$; **2.** $(0,3)$; **3.** $[-2,4]$; **4.** $[5,7]$; **5.** $(-\infty,-1)\cup(4,\infty)$;

6. $(-\infty,0)\cup(3,\infty)$; **7.** $(-\infty,-2]\cup[4,\infty)$; **8.** $(-\infty,5]\cup[7,\infty)$;

9. $(-1,6)$; **10.** $(-\infty,-2)\cup(8,\infty)$; **11.** $(-\infty,-2]\cup[8,\infty)$; **12.** $[-1,6]$;

13. $(-\infty,-2]\cup[3,\infty)$; **14.** $(-\infty,-3)\cup(0,\infty)$; **15.** $(0,6)$; **16.** $(1,5)$;

17. $(-\infty,-4]\cup[\frac{1}{2},\infty)$; **18.** $(3/2,5)$; **19.** $(-2-\sqrt{2},-2+\sqrt{2})$;

20. $(-\infty,-2-\sqrt{2})\cup(-2+\sqrt{2},\infty)$; **21.** $(-\infty,3-\sqrt{5}]\cup[3+\sqrt{5},\infty)$;

22. $[3-\sqrt{5},3+\sqrt{5}]$; **23.** ϕ; **24.** $(-\infty,\infty)$; **25.** $(-\infty,\infty)$; **26.** ϕ;

27. All $X\neq-2$ or $(-\infty,-2)\cup(-2,\infty)$; **28.** $\emptyset$; **29.** $X=5$; **30.** $(-\infty,\infty)$;

31. $(-\infty,\infty)$; **32.** All $X\neq3$ or $(-\infty,3)\cup(3,\infty)$; **33.** $(-\infty,\infty)$;

34. $(-\infty,-4]\cup[0,\infty)$; **35.** $(-\infty,-2)\cup(2,\infty)$; **36.** ϕ; **37.** $[-2,2]$;

38. $[0,4]$; **39.** $(-\infty,-3]\cup[9,\infty)$; **40.** ϕ;

41. $(-\infty,3-3\sqrt{2}]\cup[3+3\sqrt{2},\infty)$; **42.** $(-\infty,\infty)$.

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