5.02 *Slope/Distance/Midpoint Formulas*

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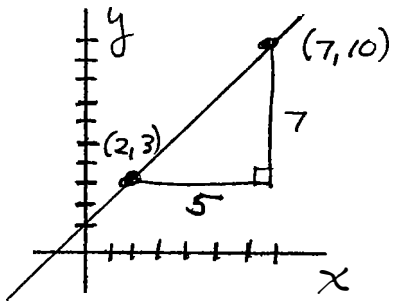
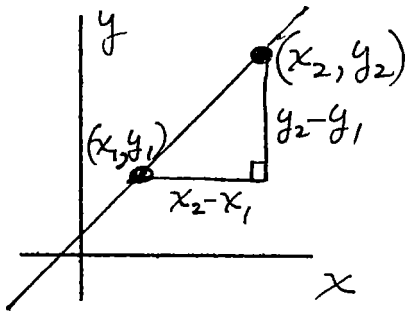
ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

In the last section, you learned that the slope of a line describes the "steepness" of a line. It is the **rise over the run**, the vertical divided by the horizontal distance between any two points on a given line. In this section, a formula will be developed for the **slope between two given points.** In addition, similar formulas will be developed for the **distance** and the **midpoint** between two points.

In order to understand the slope formula between two points, consider the points **(2,3) and (7,10).** From the graph at the right you can see that the **rise (vertical distance) is 7** and the **run (horizontal distance) is 5.** What did you do to get the 7 and the 5? You can probably see that you subtracted the Y values to get 7, and you subtracted the X values to get 5. Then the slope between these points is **m=7/5.** In the general case, consider the points with **subscripts (X_1,Y_1) and (X_2,Y_2).** Using the same procedure (subtraction), you can see that the **rise is Y_2-Y_1** and the **run is X_2-X_1.** Therefore, the formula for slope is:





$$m = \frac{rise}{run} = \frac{Y_2 - Y_1}{X_2 - X_1}$$

NOTE: Be sure the Y difference is in the numerator, and the X difference is in the denominator.

Use the slope formula to find the slope between the points:

	(X_1,Y_1) and (X_2,Y_2)		(X_1,Y_1) and (X_2,Y_2)
1a)	(2, 3) and (5, 15)	b)	(5, 15) and (2, 3)
	$m = \frac{Y_2-Y_1}{X_2-X_1} =$ _____		$m = \frac{Y_2-Y_1}{X_2-X_1} =$ _____
	= _____		= _____

2a) (6, 1) and (4, -2) b) (4, -2) and (6, 1)

Did you notice from these exercises that the order of the points does not matter? However, as mentioned before, be certain the Y difference is in the numerator and the X difference is in the denominator. Also, be sure to be consistent with the order of subtraction.

3. (-2, 4) and (6, 8)

4. (4, -2) and (6, 8)

5. (4, -2) and (6, -8)

6. (1, -3) and (-1, -5)

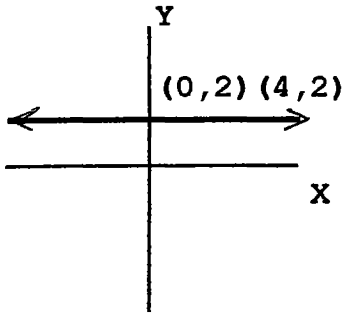
7. (-2, 2) and (4, -10)

8. (-2, 6) and (10, 4)

9. (-6, -12) and (4, -8)

10. (-4, -16) and (-16, -4)

11. Consider the points on the horizontal line Y = 2.

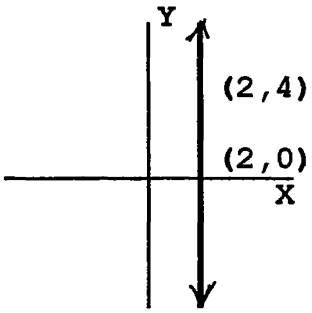


Find the slope between the indicated points.

rise = _____; run = _____

m = _____

12. Consider the points on the vertical line X = 2



Find the slope between the indicated points.

rise = _____; run = _____

m = _____

13. The slope of any horizontal line is a)_____, since zero divided by any number is b) _____.

14. The slope of any vertical line is a)_____, since any number divided by zero is b) _____.

Find the slope of each of the following lines:

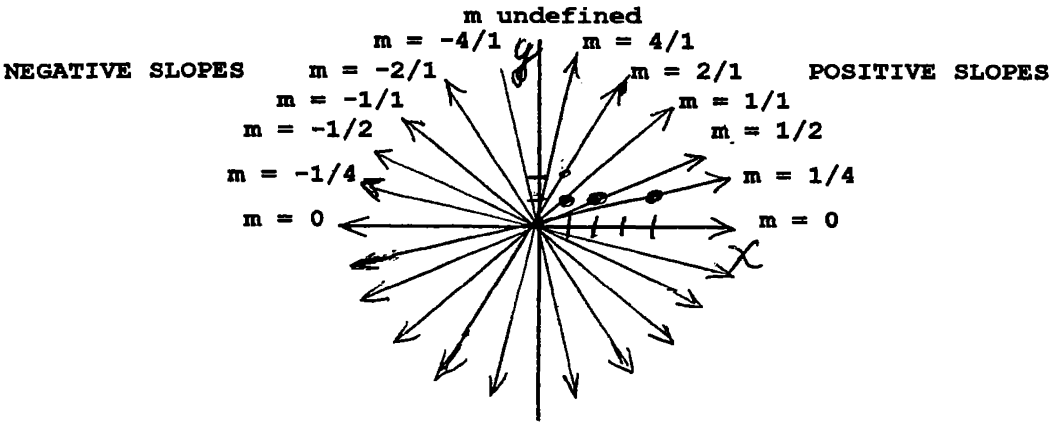
15. Y = 6
m = _____

16. Y = -6
m = _____

17. X = -6
m = _____

18. X = 6
m = _____

SLOPE SUMMARY



It is fairly obvious that if two lines are parallel, then they are going the same direction, so they have the same slope. What happens with slopes of perpendicular ($\perp$) lines is not so evident. You have probably noticed that any line that leans to the right (increasing from left to right) has a positive slope, while any line that leans to the left (decreasing from left to right) has a negative slope. If two lines are perpendicular, then one must lean to the left while the other leans to the right. Moreover, if one of the lines has a "steep" slope, then the other will have a "shallow" slope. This means that if one slope is positive, then the other must be negative, and if one slope is a "large" number, then the other must be a "small" number. Putting this together, if two lines are perpendicular, then one slope must be the **negative reciprocal** of the other slope. This is summarized as follows:

SLOPES OF PARALLEL AND PERPENDICULAR LINES

Parallel lines--same slope

Perpendicular ($\perp$) lines--slope is negative reciprocal*

***except for vertical/horizontal lines**

EXERCISES: Given the slope "m" of a line,

a) find the slope of a line parallel to the given line;

b) find the slope of a line perpendicular to the given line.

		a) m (parallel)	b) m (perpendicular)
19.	$m = 3$	a) _____	b) _____
20.	$m = \frac{3}{4}$	a) _____	b) _____
21.	$m = -3$	a) _____	b) _____
22.	$m = -\frac{3}{2}$	a) _____	b) _____
23.	$m = -1$	a) _____	b) _____
24.	$m = \frac{1}{7}$	a) _____	b) _____
25.	$m = 0$	a) _____	b) _____

In 26 - 31, find the slopes of the line segments L_1 and L_2. Determine if the line segments are parallel, perpendicular, or neither.

26. L_1: (4, 7) (8, -1);
L_2: (1, 6) (-5, 18)

$m_1 = \frac{-1-7}{8-4}$ $m_2 = \frac{18-6}{-5-1}$

$=$ _____ $=$ _____

$=$ _____ $=$ _____

Lines are _____

27. L_1: (-1, 7) (-6, 0);
L_2: (-1, 7) (13, -3)

$m_1 = \frac{0-7}{-6+1}$ $m_2 =$ _____

$=$ _____ $=$ _____

$=$ _____ $=$ _____

Lines are _____

28. L_1: (4,7) (6,1);
L_2: (1,6) (-2,-3)

$m_1 =$ $m_2 =$

Lines are _____

29. L_1: (4,7) (6,-1);
L_2: (-3,-3) (9,-6)

$m_1 =$ $m_2 =$

Lines are _____

30. L_1: (4,7) (6,-1);
L_2: (-3,-6) (9,-3)

$m_1 =$ $m_2 =$

Lines are _____

31. L_1: (-4,-7) (6,-1);
L_2: (12,-8) (-28,-32)

$m_1 =$ $m_2 =$

Lines are _____

DISTANCE FORMULA

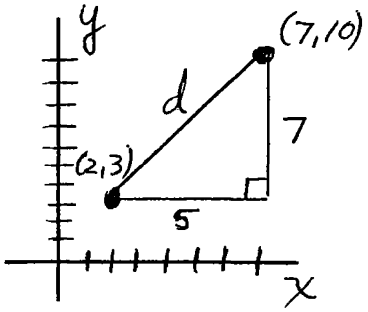
Finding the distance between two points requires the use of the **Theorem of Pythagoras.**

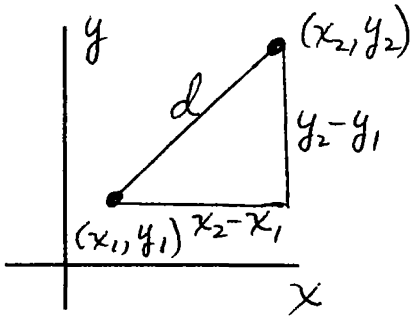
THEOREM OF PYTHAGORAS

In any right triangle, where "a" and "b" are **legs,** and "c" is the **hypotenuse,**

$$a^2 + b^2 = c^2.$$

The concept of distance is similar to **slope between two given points** and may be illustrated with the same diagram as before. Consider again the points **(2,3) and (7,10).** From the graph at the right you can see that there is a **right triangle whose legs are 5 (horizontal distance) and 7 (vertical distance).** What did you do to get the 5 and 7? Again, you can see that you subtracted the X and Y values respectively. Then the distance between these points is the **hypotentuse of the right triangle.** According to **Theorem of Pythagoras,** $d^2 = 5^2 + 7^2$, $d = \sqrt{5^2 + 7^2} = \sqrt{74}$.





In the general case, with points (X_1,Y_1) and (X_2,Y_2), the legs of the right triangle are X_2-X_1 and Y_2-Y_1. Therefore, the distance formula is:

$$d = \sqrt{(X\ diff)^2 + (Y\ diff)^2} = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$

WARNING: This does not simplify to $d = X_2-X_1 + Y_2-Y_1$.

In words, the distance formula says to find the **X difference** (distance or difference between the X-coordinates) and the **Y difference** (distance or difference between the Y-coordinates), **square these differences** (which means you do not need to worry about the signs as you did with the slope formula), **sum the squares**, and **take the square root.** (With the slope formula, remember to be careful of the signs!)

In the following exercises, find the distance between the points. Simplify the radicals, and also express as a decimal to nearest hundredth.

1. (3, 4) and (8, -2)

X diff = 5; Y diff = 6

$d = \sqrt{5^2 + 6^2}$

$d = \sqrt{\quad + \quad}$

$d = \sqrt{\quad} =$ _____

2. (3, 4) and (-2, 6)

X diff = 5; Y diff = 2

$d = \sqrt{\quad + \quad}$

$d = \sqrt{\quad + \quad}$

$d = \sqrt{\quad} =$ _____

3. (1, 2) and (-3, 5)

X diff = _____; Y diff = _____

$d = \sqrt{\quad + \quad}$

$d = \sqrt{\quad + \quad}$

$d = \sqrt{\quad} =$ _____

4. (2, -4) and (-10, -9)

X diff = _____; Y diff = _____

$d = \sqrt{\quad}$

$d = \sqrt{\quad}$

$d = \sqrt{\quad} =$ _____

5. (-7, -4) and (1, 2)

X diff = _____; Y diff = _____

$d = \sqrt{\quad + \quad}$

$d = \sqrt{\quad + \quad}$

$d = \sqrt{\quad} =$ _____

6. (8, -2) and (-1, 10)

X diff = _____; Y diff = _____

$d = \sqrt{\quad}$

$d = \sqrt{\quad}$

$d = \sqrt{\quad} =$ _____

7. **(8, -2) and (2, -4)**

X diff = _____; Y diff = _____

$d = \sqrt{}$

$d = \sqrt{}$

$d = \sqrt{} = \_\_\_\_ \sqrt{}$

$= \_\_\_\_\_$

8. **(-6, 2) and (-3, 8)**

X diff = _____; Y diff = _____

$d = \sqrt{}$

$d = \sqrt{}$

$d = \sqrt{} = \_\_\_\_ \sqrt{}$

$= \_\_\_\_\_$

9. **(8, -6) and (4, 2)**

10. **(5, -2) and (3, -6)**

MIDPOINT

To find the midpoint of two points is like averaging test grades. Suppose you have two tests in math and two tests in biology. How do you find the average of your grades? Since the math and biology grades are independent of one another, it makes no sense to average them all together. Rather, you average the math grades by adding them together and dividing by 2; then you add the biology grades together and divide by 2. You have two separate averages. In like manner, when finding the midpoint of the two points (X_1,Y_1) **and** (X_2,Y_2), you **add the X-coordinates** and **divide by 2**, then you **add the Y-coordinates** and **divide by 2.** The following midpoint formula is the result:

$$MIDPOINT = \left(\frac{X_1 + X_2}{2}, \frac{Y_1 + Y_2}{2}\right)$$

REMEMBER: Midpoint--You add the coordinates together.
Slope and distance formulas--Subtract the coordinates.

In 11 - 16, find the midpoint of the two given points:

11. (2, 14) and (10, 8)

12. (3, -9) and (15, 17)

13. (-3, 9) and (-15, 17)

14. (-13, 10) and (15, -4)

15. (-8, -13) and (15, -4)

16. (-12, 4) and (0, -21)

GETTING IT ALL TOGETHER

In each of the following, find the slope, distance, and midpoints.

17. (-2, 5) and (3, -5)

$$m = \frac{Y_2 - Y_1}{X_2 - X_1} \qquad d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \qquad midpt = \left(\frac{X_1 + X_2}{2}, \frac{Y_1 + Y_2}{2}\right)$$

18. (-4, 3) and (-6, 7)

19. (-3, 5) and (5, 2)

20. (8, -2) and (-2, 8)

21. (-3, 5) and (5, -1)

22. (-6, 8) and (-2, 2)

23. (0,6) and (6,0)

24. (0, 4) and (6, -4)

ANSWERS 5.02

p.387-390:

1. 4; **2.** 3/2; **3.** 1/2; **4.** 5; **5.** -3; **6.** 1; **7.** -2; **8.** -1/6;
9. 2/5; **10.** -1; **11.** 0; **12.** Undef; **13.** 0; **14.** Undef;
15. 0; **16.** 0; **17.** Undef; **18.** Undef; **19a)** 3, **b)** -1/3;
20a) 3/4, **b)** -4/3; **21a)** -3, **b)** 1/3; **22a)** -3/2, **b)** 2/3;
23a) -1, **b)** 1; **24a)** 1/7, **b)** -7; **25a)** 0, **b)** Undef;
26. $m_1=m_2=-2$, Parallel; **27.** $m_1=7/5$, $m_2=-5/7$, $\perp$;
28. $m_1=-3$, $m_2=3$, Neither; **29.** $m_1=-4$, $m_2=-1/4$, Neither;
30. $m_1=-4$, $m_2=1/4$, $\perp$; **31.** $m_1=m_2=3/5$, parallel.

p.392-395:

1. $\sqrt{61}$ or 7.81; **2.** $\sqrt{29}$ or 5.39; **3.** 5; **4.** 13; **5.** 10;

6. 15; **7.** $2\sqrt{10}$ or 6.32; **8.** $3\sqrt{5}$ or 6.71; **9.** $4\sqrt{5}$ or 8.94;

10. $2\sqrt{5}$ or 4.47; **11.** (6,11); **12.** (9,4); **13.** (-9,13)

14. (1,3); **15.** (7/2,-17/2); **16.** (-6,-17/2)

17. m=-2, d= $5\sqrt{5}$ or 11.18, mid=(1/2,0);
18. m=-2, d= $2\sqrt{5}$ or 4.47, mid=(-5,5);
19. m=--3/8, d= $\sqrt{73}$ or 8.54, mid=(1,7/2);
20. m=-1, d= $10\sqrt{2}$ or 14.14, mid=(3,3);
21. m=-3/4, d= 10 , mid=(1,2);
22. m=-3/2, d= $2\sqrt{13}$ or 7.21, mid=(-4,5);
23. m=-1, d= $6\sqrt{2}$ or 8.49, mid=(3,3);
24. m=-4/3, d= 10 , mid=(3,0);

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