5.04 *Systems of Equations (2X2)*

If one equation with two unknowns (X and Y) represents a line graph, then two equations with two unknowns must represent two lines (on the same graph, of course). For example, consider

$$2X + Y = 4$$
$$X - Y = 2$$

Each of these is a straight line, and each line has infinitely many solutions for X and Y. Finding solutions that satisfy both equations simultaneously is the problem of this section.

Think for a moment about two straight lines on the same graph. Will there be intersection points? **There are three possibilities:**

I. a single point of intersection (X,Y);

II. parallel lines (no intersection, no solution);

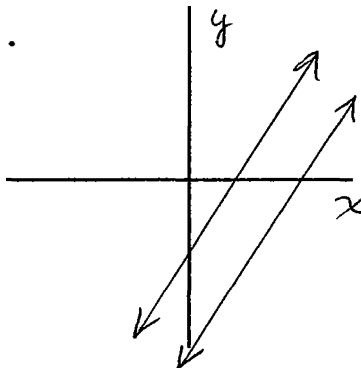**III. same line (the entire line, infinitely many solutions).**

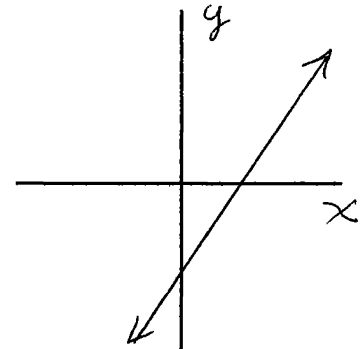These three cases are illustrated below.

I.   **II.**   **III.**
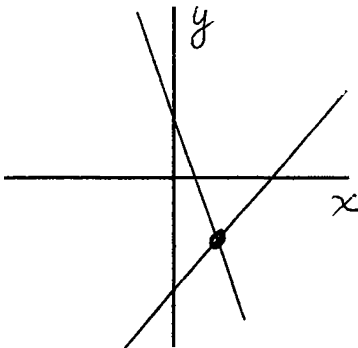
   UNIQUE SOLUTION   **PARALLEL LINES**   **SAME LINE**

The most common methods of solving such systems include the **elimination method** (also known as the "addition" method), the **substitution method,** and the **graphical method.** If both equations are given in **standard form** (i.e., **AX + BY = C**), then usually elimination is the easiest method. If one or more of the equations is given in the **slope-intercept form** (i.e., **Y = mX + b**) or in the form of **X = _____,** then substitution is usually the easiest method. Yet another method involving **determinants** known as **Cramer's Rule** may be introduced at the next level of math.

ELIMINATION METHOD

EXAMPLE 1: Solve for X and Y by the elimination method.

$2X + Y = 4$ NOTICE THAT IF THESE EQUATIONS ARE ADDED TOGETHER,

$\underline{X - Y = 5}$ THE VARIABLE Y SUBTRACTS OUT.

$3X = 9$

$X = 3$ NOW, IF YOU KNOW X, SUBSTITUTE BACK INTO ONE OF THE
EQUATIONS (EITHER WILL DO) AND SOLVE FOR Y.

$2X + Y = 4$

$2(3) + Y = 4$

$6 + Y = 4$

$Y = -2$

Checking the answer is usually easy and worthwhile. Always substitute the values of X and Y into the <u>other</u> equation (the one you did not just use to solve for Y).

Check: $X - Y = 5$

$3 - (-2) = 5 \checkmark$ The solution is (3, -2)

EXAMPLE 2: Solve for X and Y by the elimination method.

$X - 2Y = -4$ IN THIS CASE, ADDING THE EQUATIONS DOES NO GOOD!

$\underline{X - Y = -3}$ REMEMBER, THE OBJECTIVE IS TO ELIMINATE ONE OF THE.

$2X - 3Y = -7$ THE VARIABLES. SO MULTIPLY BOTH SIDES OF ONE OF THE
EQUATIONS BY -1 AND ELIMINATE THE X.

$-X + 2Y = 4$

$\underline{X - Y = -3}$

$Y = 1$ NOW SUBSTITUTE BACK INTO THE FIRST EQUATION

$X - 2Y = -4$ AND SOLVE FOR X.

$X - 2 = -4$ ADD +2 TO BOTH SIDES.

$X = -2$

Check: $X - Y = -3$

$(-2) - (1) = -3 \checkmark$ The solution is (-2, 1)

EXAMPLE 3: Solve for X and Y

$2X + 3Y = -13$ IN ORDER TO ELIMINATE THE Y, YOU WOULD NEED A $+3Y$ AND A $-3Y$

$\underline{3X + Y = 5}$ SO MULTIPLY BOTH SIDES OF THE SECOND EQUATION BY -3.

$2X + 3Y = -13$

$\underline{-9X - 3Y = -15}$ THIS ELIMINATES THE Y.

$-7X = -28$

$X = 4$ SUBSTITUTE BACK INTO THE SECOND EQUATION (IT LOOKS EASIER!)

$3X + Y = 5$ AND SOLVE FOR Y.

$12 + Y = 5$ ADD -12 TO BOTH SIDES.

$Y = -7$ BE SURE TO USE THE <u>OTHER</u> EQUATION (THE FIRST) TO CHECK!

Check: $2X + 3Y = -13$

$8 + (-21) = -13$ √   The solution is $(4, -7)$.

EXERCISES. Solve for X and Y by the eliminaton method.

1. $X + 2Y = 4$
   $-X + 2Y = 8$

2. $3X + 2Y = 6$
   $-X - 2Y = 2$

3. $-X + 3Y = -5$
   $2X + 3Y = -17$

4. $2X - 3Y = -14$
   $2X - Y = -10$

5. $X + 2Y = 8$
   $5X - 6Y = 8$

6. $2X - 3Y = 1$
   $X + Y = 8$

7. $3X + 5Y = 10$
   $X + 2Y = 1$

8. $3X - 2Y = 10$
   $4X - Y = 15$

9. $3X + 2Y = 10$
   $X + 3Y = 8$

10. $3X + 2Y = 38$
    $X + 5Y = 4$

EXAMPLE 4: Solve the system of equations (for X and Y).

$5X + 3Y = 14$ IN THIS EXAMPLE LOOK FOR A COMMON MULTIPLE FOR

$9X + 4Y = 7$ THE X AND Y COEFFICIENTS.

$5X + 3Y = 14$ THE COMMON MULTIPLE OF X COEF. IS 45

$9X + 4Y = 7$ THE COMMON MULTIPLE OF Y COEF. IS 12

Since the numbers will be smaller, it is easier to eliminate the Y terms. Multiply both sides of the first equation by **4** and multiply both sides of the second equation by **-3** to eliminate the Y term.

$20X + 12Y = 56$ MULTIPLY FIRST EQUATION BY 4

$\underline{-27X - 12Y = -21}$ MULTIPLY SECOND EQUATION BY -3

$-7X = 35$

$X = -5$

$5X + 3Y = 14$ SUBSTITUTE X INTO THE FIRST EQUATION AND SOLVE Y.

$5(-5) + 3Y = 14$

$-25 + 3Y = 14$ ADD +25 TO BOTH SIDES

$3Y = 39$

$Y = 13$

Check: $9X + 4Y = 7$

$9(-5) + 4(13) = 7$

$-45 + 52 = 7$ ✓ The solution is (-5,13).

EXERCISES. Solve the systems of equations.

11. $5X + 3Y = 6$ 12. $3X + 5Y = 1$

     $3X + 2Y = 2$ $-2X + 3Y = 12$

13. $3X + 5Y = 2$
     $2X + 3Y = -4$

14. $5X + 2Y = -12$
     $3X - 5Y = -1$

15. $2X - 3Y = -32$
     $3X - 4Y = -36$

16. $12X + 5Y = -24$
     $4X + 3Y = 8$

What happens algebraically when the lines are parallel or the same line? Remember that when the lines are parallel, there are no common points or solutions. Also, remember that when they are the same line, there are infinitely many solutions. In these two cases, when you try to eliminate one of the variables, **both variables are eliminated.** The following examples illustrate this concept, where adding the equations together eliminates both variables.

EXAMPLE 5
$$X - Y = 6$$
$$\underline{-X + Y = 2}$$
$$0 = 8$$

EXAMPLE 6
$$2X - Y = 4$$
$$\underline{-2X + Y = -4}$$
$$0 = 0$$

[Stay tuned for the important conclusion on next page!]

Whenever eliminating one variable "by chance" results in the elimination of <u>both</u> variables, and an impossible statement such as **0 = 8**, or **0 = any non-zero number**, there is **No Solution** possible. This is the case of the **two parallel lines.**

Whenever eliminating one variable "by chance" results in the elimination of <u>both</u> variables <u>and</u> the constants (number terms) as well, then the statement **0 = 0** results. This statement is always true, and this indicates that there are **many solutions.** In fact, this is the case in which the two equations represent the **same line.** The solution is the entire line.

EXERCISES.

17. $4X + 3Y = 5$
     $20X + 15Y = 35$

18. $4X - 3Y = -6$
     $-8X + 6Y = 12$

19. $4X - 7Y = -28$
     $4X + 7Y = -28$

20. $16X - 12Y = 28$
     $4X - 3Y = 0$

SUBSTITUTION METHOD

The substitution method is particularly useful in solving systems of equations in which one or both of the equations is expressed in the form **Y =** ____ or **X =** ____. This method is also very important in non-linear equations, many of which cannot be solved by the elimination method. In linear equations, as the ones presented here, either the **elimination method** or the **substitution method** may be used.

If the equations are both given in standard form, it is easier to solve by the elimination method. If one or both of the equations is in slope-intercept form (or X= ____ form), then the substitution method is usually easier.

EXAMPLE 3: Solve by the Substitution Method.

Since the second equation is in the form **X =** ____, the substitution method is appropriate for this problem.

$$5Y - 3X = 34$$ SINCE THE SECOND EQUATION IS IN THE FORM $X=$____

$$X = 7 - 2Y$$ THE SUBSTITUTION METHOD IS APPROPRIATE FOR THIS PROBLEM.

REWRITE THE FIRST EQUATION:

$$5Y - 3(\quad) = 34$$ AND SUBSTITUTE $7-2Y$ FOR X:

$$5Y - 3(7-2Y) = 34$$ DISTRIBUTE -3

$$5Y - 21 + 6Y = 34$$ COMBINE LIKE TERMS

$$11Y - 21 = 34$$ ADD $+21$

$$11Y = 55$$

$$Y = 5$$ THE SECOND EQUATION $(X=$____$)$ IS THE BEST PLACE to SUBSTITUTE Y AND SOLVE FOR X.

$$X = 7 - 2Y$$
$$X = 7 - 2(5)$$
$$X = 7 - 10$$
$$X = -3$$

Check: $5Y - 3X = 34$ (You must use the <u>other</u> equation!)
$$5(5) - 3(-3) = 34$$
$$25 + 9 = 34$$

25 + 9 = 34
EXERCISES.   Solve by the substitution method.

1. 2X + 3Y = 12

    X = 5 - Y   Substitute (5 - y) for X in the first equation.

    2(   ) + 3Y = 12

    _____ = 12

    _____ = 12

              Y = _____   Substitute into the equation X = 5 - Y

    X = 5 - Y

    X = 5 - (   )

    X = _____   Remember to use the other equation to check

    Check: 2X + 3Y = 12

          2(   ) + 3(   ) = 12

          _____ = ____

2.   Y = 3X - 5

     9X - 2Y = 4   Substitute (3X - 5) for Y in the second equation.

     9X - 2(   ) = 4

     _____ = 4

     _____ = 4

              X = _____   Substitute into the equation Y = 3X - 5

     Y = 3X - 5

     Y = 3(   )- 5

     Y = _____   Remember to use the other equation to check

     Check: 9X - 2Y = 4

          9(   ) - 2(   ) = 4

          _____ = ____

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3. $3X + 5Y = 39$
    $Y = 2X$

4. $Y = X - 2$
    $3X + 5Y = 14$

5. $3X - 5Y = -10$
    $Y = 2X - 5$

6. $3X - 5Y = 8$
    $X = 3Y - 4$

7. $5Y - 3X = 5$
    $X = 2Y + 1$

8. $Y = 4 - X$
    $2X - Y = 11$

9. X = 5Y + 24
    Y = 3X - 2

10. Y = 4X - 25
     X = 3Y - 2

In 11 - 12, to solve by substitution, it will be necessary to solve for one of the variables. Choose the easiest variable to solve.

11.   X - 5Y = 13
     2X + 7Y = -8

     X = _____

     2(   ) + 7Y = -8

12. 7X - 4Y = 40
     Y - X = 2

EXERCISES: In #13 - 24, solve the systems of equations by the "appropriate" method. Indicate if the equations represent parallel lines or the same line.

13. 3X + 7Y = 6
     2X + 3Y = -1

14. -3X + 7Y = 4
      2X - 3Y = -6

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15. $9X - 4Y = 2$
$2X + 5Y = -29$

16. $50X - 9Y = 1$
$7X - 2Y = -8$

17. $2X - 6Y = 12$
$-X + 3Y = -6$

18. $X = 3Y + 18$
$6Y - 2X = 36$

19. $5X - 4Y = 22$
$Y = -4X + 5$

20. $-8X + 6Y = 32$
$X = 2Y + 6$

21. $17X + 8Y = 4$
    $32X + 18Y = -16$

22. $4X - 2Y = -8$
    $2X - Y = -4$

23. $4X - 2Y = 8$
    $Y = 2X + 4$

24. $12Y + 5X = 41$
    $X = 4 - 3Y$

ANSWERS 5.04

p.407-411:

1. $(-2,3)$; 2. $(4,-3)$; 3. $(-4,-3)$; 4. $(-4,2)$; 5. $(4,2)$;
6. $(5,3)$; 7. $(15,-7)$; 8. $(4,1)$; 9. $(2,2)$; 10. $(14,-2)$;
11. $(6,-8)$; 12. $(-3,2)$; 13. $(-26,16)$; 14. $(-2,-1)$;
15. $(20,24)$; 16. $(-7,12)$; 17. No Sol--Parallel lines;
18. Same line--also written $\{(X,Y)\mid 4X-3Y=-6\}$;
19. $(-7,0)$; 20. No Sol--Parallel lines.

p.413-417:

1. $(3,2)$; 2. $(-2,-11)$; 3. $(3,6)$; 4. $(3,1)$; 5. $(5,5)$;
6. $(11,5)$; 7. $(-15,-8)$; 8. $(5,-1)$; 9. $(-1,-5)$;
10. $(7,3)$; 11. $(3,-2)$; 12. $(16,18)$; 13. $(-5,3)$;
14. $(-6,-2)$; 15. $(-2,-5)$; 16. $(2,11)$;
17. Same line--also written $\{(X,Y)\mid 2X-6Y=12\}$;
18. No Sol--Parallel lines; 19. $(2,-3)$; 20. $(-10,-8)$;
21. $(4,-8)$; 22. Same line--also written $\{(X,Y)\mid 2X-Y=-4\}$;
23. No Sol--Parallel lines; 24. $(25,-7)$.