1.03 *Polynomials*

Recall from previous lessons that when algebraic expressions are added (or subtracted) they are called **terms**, while expressions that are multiplied are called **factors**. An algebraic expression that contains only **one term** is called a **monomial.** If the expression has **two terms** is called a **binomial**, and if there are **three terms** it is a **trinomial.** A **polynomial** is an algebraic expression consisting of more than one term. A polynomial may consist of numbers and variables, where the numerical part of a given term is called the **coefficient.** If there is only one variable in the polynomial, such as **X**, then it is called a **polynomial in X.** The **degree** of a polynomial in one variable is the highest exponent of the variable. If there is more than one variable in the polynomial, then the **degree** is the highest "sum of the exponents" of the variables of a given term.

Frequently polynomials can be **simplified** by combining like terms; sometimes they can be **factored.** Polynomials can be added, subtracted, multiplied **(expanded)**, or divided. Since addition and subtraction of polynomials is little more than combining like terms, and division of polynomials is saved for Chapter 3, this section will involve only the **multiplication (expansion) of polynomial expressions.** The next section is the **factoring of polynomial expressions.** Chapter 3 includes **division of polynomial expressions, the solution of polynomial equations** are **solved**, and **the graphing of polynomial functions.** Notice that **polynomial expressions** are not equations, and therefore cannot be "solved." This chapter involves only **polynomial expressions.**

Summary

Type	Examples	Action to be taken
Polynomial Expression	$3X + 5(X-8)$ $(2X - 3)^2$ $X^4 - 5X^2 + 4$	**Simplify, expand, or factor the expression.**
Polynomial Equation	$3X + 5(X-8) = 0$ $(2X - 3)^2 = 0$ $X^4 - 5X^2 + 4 = 0$	**Solve (or find the roots of) the equation.**
Polynomial Function	$Y = 3X + 5$ $Y = (2X - 3)^2$ $Y = X^4 - 5X^2 + 4$	**Graph the function.**

General Form

Polynomial Expression $\quad a_nX^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \ldots a_0$

Polynomial Equation $\quad a_nX^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \ldots a_0 = 0$

Polynomial Function $\quad Y = a_nX^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \ldots a_0.$

This explanation will begin with a review of products: **monomial times monomial, monomial times binomial, binomial times binomial, binomial times trinomial, and trinomial times trinomial.** The basic property that underlies these products is the **distributive property for multiplication** (products) **over addition** (two or more terms). Also, the law of exponents about products ("when you multiply, you add exponents") is used.

EXAMPLES:

1. **Monomial times monomial:** $\quad 4X^2 \cdot 6X^3 = 24X^5$
   (by law of exponents)

2. **Monomial times binomial:** $\quad 4X^2 \cdot (6X^3 + 5X^2) = 24X^5 + 20X^4$
   (by distributive property)

3. **Binomial times binomial:**
   (F OI L): $\quad (3X + 5) \cdot (4X + 7) = 12X^2 + 41X + 35$
   (This is actually the distributive property applied twice!)

4. **Binomial times trinomial:** $(3X + 5)\cdot(4X^2 + 7X + 2) =$

3X times trinomial: $12X^3 + 21X^2 + 6X$

5 times trinomial: $\underline{\qquad 20X^2 + 35X + 10}$

Combine like terms: $= \mathbf{12X^3 + 41X^2 + 41X + 10}$

5. **Trinomial times trinomial:** $(3X^2 + 5X + 9)\cdot(4X^2 + 7X + 2) =$

$3X^2$ times trinomial: $12X^4 + 21X^3 + 6X^2$

5X times trinomial: $20X^3 + 35X^2 + 10X$

9 times trinomial: $\underline{\qquad 36X^2 + 63X + 18}$

Combine like terms: $= \mathbf{12X^4 + 41X^3 + 77X^2 + 73X + 18}$

In the exercises that follow, these products will be followed by exercises that reach to a higher level of abstraction. Hopefully, the "one step" format will help you understand easily.

EXERCISES:

1. **(X + 4)(X + 3)** 2. **(Y + 4)(Y + 3)** 3. **[$ + 4][$ + 3]**

4. **[π + 4][π + 3]** 5. **[(Junk) + 4][(Junk) + 3]**

6. **[(Junk) + 6][(Junk) + 7]** 7. **[(Junk) + 4][(Junk) - 3]**

8. **[(Junk) - 4][(Junk) + 3]** 9. **[(Junk) - 6][(Junk) - 7]**

Consider the problem: **[(X + Y) + 4)][(X + Y) + 3]**. This product can be treated as a **"FOIL"** problem, or a **" product of trinomials."**

F L

Either **[(X + Y) + 4)][(X + Y) + 3]**

I

F OI L

$$= (X + Y)^2 + 7(X + Y) + 12$$

$$= X^2 + 2XY + Y^2 + 7X + 7Y + 12.$$

or **(X + Y + 4)(X + Y + 3)**

$$\begin{array}{llll} = X^2 + XY + 3X & & & \\ \quad XY & + Y^2 + 3Y & & \\ \quad 4X & +4Y + 12 & & \\ \hline \end{array}$$

$$= X^2 + 2XY + 7X + Y^2 + 7Y + 12.$$

Of these two methods, the first is the preferred method.

In 10 - 25, expand the polynomials completely:

10. [(X + Y) - 4)][(X + Y) - 3] **11. [(X + Y) - 4)][(X + Y) + 7]**

12. [(2X + Y) + 4)][(2X + Y) - 7] **13. [(X - 3Y) - 8)][(X - 3Y) - 6]**

14. [(2X-3Y) + 8)][(2X-3Y) - 4] **15. [(3X-2Y) - 8)][(3X-2Y) + 4]**

In 16-21, remember that **[(X+Y) + 4)]^2 = [(X+Y) + 4)][(X+Y) + 4].**

16. [(X + Y) + 4)]^2 **17. [(X + Y) - 4)]^2**

18. $[(3X - 5Y) + 4)]^2$

19. $[(2X - 5Y) - 3]^2$

20. $[(5X - 2Y) - 8)]^2$

21. $[(5X + 4Y) - 6]^2$

22. $[(3X-5Y) - 4)][(3X-5Y) + 4]$

23. $[(2X-5Y) - 3][(2X-5Y) + 3]$

24. $[(2X+7Y) - 5)][(2X+7Y) + 5]$

25. $[(3X+8Y) - 7][(3X+8Y) + 7]$

26. $$\begin{aligned}(X + Y)^3 &= (X + Y)\ (X + Y)\ (X + Y)\\ &= (X + Y)\ (X^2 + 2XY + Y^2)\\ &=\\ &=\end{aligned}$$

Now consider the problems $(X + Y)^3$, $(X - Y)^3$, $(X + Y)^4$, $(X - Y)^4$, etc. These are **binomials** raised to a **power**. The general case, $(X + Y)^n$, is called the **Binomial Theorem** or **Binomial Expansion**.

The following pattern can easily be developed:

	$(X + Y)^0$ =	1
	$(X + Y)^1$ =	$1X + 1Y$
	$(X + Y)^2$ =	$1X^2 + 2XY + 1Y^2$
	$(X + Y)^3$ =	$1X^3 + 3X^2Y + 3XY^2 + 1Y^3$
1.	$(X + Y)^4$ =	$1X^4 + \bigcirc x^3y + \bigcirc x^2y^2 + \bigcirc xy^3 + 1Y^4$
2.	$(X + Y)^5$ =	$\bigcirc x^5 + \bigcirc x^4y + \bigcirc x^3y^2 + \bigcirc x^2y^3 + \bigcirc xy^4 + \bigcirc y^5$

This is called **Pascal's Triangle**, named after the French mathematician **Blaise Pascal** (1623-1662). Notice the pattern of **1**s going down both sides of the "triangle." This pattern of **1**s continues as the triangle continues for higher powers of $(X + Y)^n$. The **numbers** inside the triangle can be obtained from the previous row, by adding the two numbers that are circled above the number. For examples, **1 + 1 = 2; 1 + 2 = 3;** and **2 + 1 = 3.** Now, complete the numbers in the $(X + Y)^4$ row: **1 + 3 =** ____; **3 + 3 =** ____; and **3 + 1 =** ____. Now, notice that moving from left to right on the "triangle," the power of X begins with the power of the binomial, and it decreases with each term, left to right. At the same time, the power of Y begins with "no Ys" in the first term, and, moving left to right, the Y power increases until it reaches the power of the binomial in the last term of the expansion. If you have not done so, complete the pattern of numbers above for $(X + Y)^5$.

Now complete the following, using the triangle above:

3. $(X + Y)^6 = \bigcirc x^6 + \bigcirc x^5y + \bigcirc x^4y^2 + \bigcirc x^3y^3 + \bigcirc x^2y^4 + \bigcirc xy^5 + \bigcirc y^6$

4. $(X + Y)^7 =$ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____

5. $(X + Y)^8 =$ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____ $\bigcirc$____

What do you think would be the effect of having $(X - Y)^n$? Answer: Instead of having **Y raised to a variety of powers**, you will now have **(-Y) raised to a variety of powers.** The **net effect of the $(X - Y)^n$ is that the signs will alternate.** For examples, $(X - Y)^2 = X^2 - 2XY + Y^2$; $(X - Y)^3 = X^3 - 3X^2Y + 3XY^2 - Y^3$.

Now, complete the following:

6. $(X - Y)^4 =$ ____ ____ ____ ____ ____

7. $(X - Y)^5 =$ ____ ____ ____ ____ ____ ____

8. $(X - Y)^6 =$ ____ ____ ____ ____ ____ ____ ____

The effect of having $(2X + 5Y)^3$ is that in the formula already developed for $(X + Y)^3$, the **X must be replaced by 2X** and the **Y must be replaced by 5Y**, as follows:

$$(X + Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$$

9. Now, $(2X + 5Y)^3 = (2X)^3 + 3(2X)^2(5Y) + 3(2X)(5Y)^2 + (5Y)^3$

$=$

$=$

10. $(2X - 5Y)^3 =$

11. $(3X - 5Y)^3 =$

12. $(3X + 5Y)^3 =$

13. $(2X + 3Y)^4 =$

14. $(2X - 3Y)^4 =$

15. $(2X - Y)^5 =$

16. $(2X - Y)^6 =$

ANSWERS 1.03

p. 25 - 27: 1. $x^2 + 7x + 12$ 2. $y^2 + 7y + 12$ 3. $\theta^3\ 7\theta + 12$

4. $\pi^2 + 7\pi + 12$ 5. $([illegible])^2 + 7([illegible]) + 12$

6. $([illegible])^2 + 13([illegible]) + 42$ 7. $([illegible])^2 + ([illegible]) - 12$

8. $([illegible])^2 - ([illegible]) - 12$ 9. $([illegible])^2 - 13\,[illegible] + 42$

10. $x^2 + 2xy + y^2 - 7x - 7y + 12$ 11. $x^2 + 2xy + y^2 + 3x + 3y - 28$

12. $4x^2 + 4xy + y^2 - 6x - 3y - 28$ 13. $x^2 - 6xy + 9y^2 - 14x + 42y + 48$

14. $4x^2 - 12xy + 9y^2 + 8x - 12y - 32$ 15. $9x^2 - 12xy + 4y^2 - 12x + 8y - 32$

16. $x^2 + 2xy + y^2 + 8x + 8y + 16$ 17. $x^2 + 2xy + y^2 - 8x - 8y + 16$

18. $9x^2 - 30xy + 25y^2 + 24x - 40y + 16$ 19. $4x^2 - 20xy + 25y^2 - 12x + 30y + 9$

20. $25x^2 - 20xy + 4y^2 - 80x + 32y + 64$ 21. $25x^2 + 40xy + 16y^2 - 60x - 48y + 36$

22. $9x^2 - 30xy + 25y^2 - 16$ 23. $4x^2 - 20xy + 25y^2 - 9$

24. $4x^2 + 28xy + 49y^2 - 25$ 25. $9x^2 + 48xy + 64y^2 - 49$

26. $x^3 + 3x^2y + 3xy^2 + y^3$

03 p 28–30:

1. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
2. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
3. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$
4. $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$
5. $x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$
6. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
7. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$
8. $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$
9. $8x^3 + 60x^2y + 150xy^2 + 125y^3$
10. $8x^3 - 60x^2y + 150xy^2 - 125y^3$
11. $27x^3 - 135x^2y + 225xy^2 - 125y^3$
12. $27x^3 + 135x^2y + 225xy^2 + 125y^3$
13. $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$
14. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$
15. $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$
16. $64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6$