1.01 *Real Numbers, Properties*

Number Systems

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

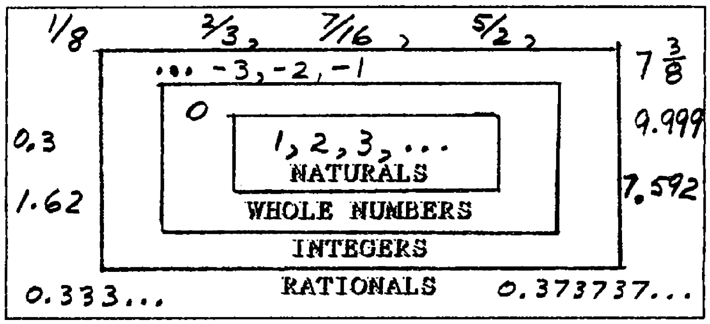
In any **study of mathematics, there must be number systems** and **operations** (such as addition, subtraction, multiplication, and division). It seems only "natural" to begin this study with the natural number system. The **natural numbers**, also known as the **counting numbers**, is the set of numbers that would be used for counting {1, 2, 3, 4 . . .} (three dots mean "and so on"). When the number 0 is included, this set {0, 1, 2, 3, 4 . . .} is the set of **whole numbers**. With the advent of credit, it became necessary to have negative numbers. The set of **integers** is defined to be the set of all whole numbers and their negatives:

{. . .-4, -3, -2, -1, 0, 1, 2, 3, 4 . . .} .

SYSTEMS OF NUMBERS	
NATURAL NUMBERS:	**1, 2, 3, 4, . . .**
WHOLE NUMBERS:	**0, 1, 2, 3, 4, . . .**
INTEGERS:	**. . . -4, -3, -2, -1, 0, 1, 2, 3, 4, . . .**

You probably noticed that fractional and decimal numbers are not included in any of the sets mentioned thus far. The set of **rational numbers**, from the root word **ratio**, is the set of all numbers that can be expressed as a ratio of two integers (assuming of course that **division by zero is undefined**). When one integer is divided by another integer, the result can be expressed as a fraction, or it can be divided out to express it in decimal form. When two integers are divided, the result will either come out even (called a **terminating decimal**) or there will be a repeating pattern of numbers in the quotient (called a **repeating decimal**). The fractions, 1/2 = .5, 3/8 = .375, 9/5 = 1.8 result in terminating decimals, but 1/3 = .333..., 2/9 = .222..., 4/11 = .363636..., and 2/7= .285714285714... result in repeating decimals.

As you probably noticed, each set of numbers so far is developed or built upon the previous set of numbers. This makes each previous set of numbers a **subset** (i.e., a set contained within a set) of each succeeding set of numbers. These illustrations that have been used are called **Venn Diagrams**, named after the mathematician John Venn (1880).



Having built up to the set of rational numbers, there are some numbers that are not rational--that is, these numbers cannot be expressed as a ratio of integers. For example, the solution to the equation $x^2 = 2$ is $\pm\sqrt{2}$. Another example of a number that cannot be expressed as a ratio of integers is the number π, whose value is approximately (but not exactly!) 22/7 or 3.14. It can be proven that the actual value of π will never terminate, and it will never repeat a pattern. The set of all numbers like $\sqrt{2}$, $-\sqrt{5}$, and π that never terminate and never repeat a pattern form the set of **irrational numbers**. These two sets, the rational and the irrational numbers, are said to be **disjoint sets**, which means that they have no members in common. The **intersection** of the sets (i.e., what is common to both sets) is the **empty set**, denoted $\emptyset$ or { }. Notice that $\{\emptyset\}$ is <u>not</u> acceptable notation for the empty set--this is <u>not</u> empty because it has a "$\emptyset$" in it! The set of all rational and irrational numbers combined (the **union** of the two sets) is the **real number system** $\mathbb{R}$. [Note: in math, this is the <u>real thing</u>!]

There are two operations that are not allowed in the real number system:

I. **Division by zero,**

II. **Square roots (in general, "even" roots) of negative numbers.**

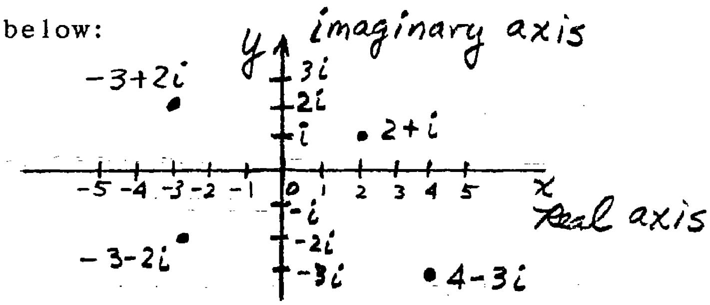
While square roots of negative numbers are not defined in the real number system, the possibility of taking square roots of negative numbers leads to the definition of **imaginary numbers:**

DEFINITION: $i = \sqrt{-1}$

$i^2 = -1$

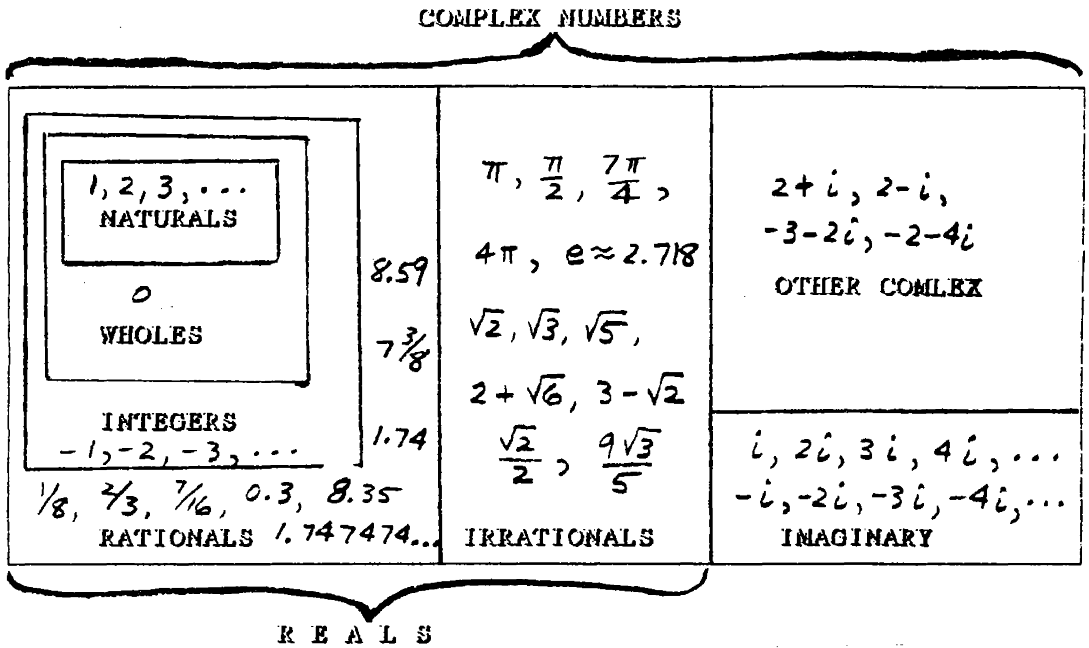
It follows that $\sqrt{-a} = i\sqrt{a}$ where "a" represents any real number.

Notice that this definition $i = \sqrt{-1}$ applies only to square roots. **Any odd root of -1 is -1. The fourth root, sixth root, eighth root,** or any other **even root of -1** is not i. These will be discussed later (perhaps much later!). Obviously, these numbers are **un-real** (joke!). If the **real numbers** can be graphed on a numberline (like the X-axis), then the **imaginary numbers** can be graphed on a numberline (like the Y-axis) that is perpendicular to the **real axis** as shown below:



The final number system to be considered is the **complex number system.** The complex numbers consist of any combination of **real** and **imaginary numbers.** For example, if a and b are any **real numbers,** then the expression represented by Z = **a + bi** is said to be a **complex number.** It is not that this is "complex" in the sense of being "complicated" (it is not!). Rather, it is complex in that it consists of inter-connected or interwoven parts, as a B-complex vitamin. If the **real numbers** are contained on the X-axis and the **imaginary numbers** are on the Y-axis, then the **complex numbers** cover the entire **XY-plane.** Realize that, for example, the **real number** X = **6** can be written as Z = **6 + 0i**. Since every **real number** X can be written in the form Z = **X + 0i**, the **real numbers** are actually a subset of the **complex numbers.** SEE SECTION 1.07 —COMPLEX NUMBERS.

The following is a summary of all the number systems: natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, imaginary numbers, and complex numbers.



EXERCISES:

For each number, list the number systems of which it is a member:

N = Natural Numbers
W = Whole Numbers
I = Integers
Ra = Rationals (Save R for Reals)
Ir = Irrationals
R = Reals
Im = Imaginary
C = Complex

Examples: **4** -- **N, W, I, Ra, R, C**
$\sqrt{5}$ -- **Ir, R, C**
2i -- **Im, C**

1. 3 -- ______________________
2. -3 -- ______________________
3. π -- ______________________
4. 7.5 -- ______________________

5. -5.65 -- ____________________

6. 2π -- ____________________

7. 8i -- ____________________

8. 3 + 8i -- ____________________

9. -8.0 -- ____________________

10. 3.14 -- ____________________

11. $\frac{2}{5}$ -- ____________________

12. $\frac{-5}{2}$ -- ____________________

13. $\frac{25}{5}$ -- ____________________

14. $\frac{-25}{5}$ -- ____________________

15. $\frac{22}{7}$ -- ____________________

16. $-\frac{25}{10}$ -- ____________________

17. $\sqrt{47}$ -- ____________________

18. $-\sqrt{47}$ -- ____________________

19. $-\sqrt{49}$ -- ____________________

20. $\sqrt{25}$ -- ____________________

21. $\sqrt{-25}$ -- ____________________

22. $\sqrt{125}$ -- ____________________

23. $\frac{0}{5}$ -- ____________________

24. $-\frac{20}{5}$ -- ____________________

25. $\frac{5}{0}$ -- ____________________

26. $\frac{2}{25}$ -- ____________________

27. $\frac{10}{2} - \sqrt{25}$ -- ____________________

28. $\frac{10}{2} + \sqrt{25}$ -- ____________________

29. $\frac{\sqrt{25}}{5}$ -- ____________________

30. $\sqrt{\frac{25}{5}}$ -- ____________________

31. $\frac{\sqrt{125}}{5}$ -- ____________________

32. $\sqrt{\frac{25}{9}}$ -- ____________________

33. $\frac{\sqrt[3]{125}}{5}$ -- ____________________

34. $\frac{\sqrt[3]{125}}{5} - 1$ -- ____________________

35. $\sqrt[3]{12}$ -- ____________________

36. $\sqrt[3]{\frac{125}{5}}$ -- ____________________

37. $\frac{\sqrt[3]{-125}}{5}$ -- ____________________

38. $\frac{\sqrt[3]{125}}{10}$ -- ____________________

PROPERTIES OF REAL NUMBERS

There are several properties involving addition and/or multiplication, most of which are widely known and perhaps taken for granted. Rather than take these properties for granted and use them haphazardly, it is better to know and use these properties by name. Recognizing and naming these properties is the objective of this lesson.

I. **CLOSURE PROPERTY.** The word **closure** comes from the word **"closed"**, as the expression from union jargon **"closed shop."** The union expression **"closed shop"** means that all employees must be hired from within the union. You do not go outside of the union to hire employees. In the same way, whenever you **add** or **multiply** two **real numbers**, you will not go out of the real number system to get the sum or product.

A. **Closure Property for Addition: If "a" and "b" are real numbers, then "a + b" is also a real number.**

B. **Closure Property for Multiplication: If "a" and "b" are real numbers, then "a·b" is also a real number.**

Examples: There is closure for multiplication for the set of integers because the product of two integers is always an integer.

There is closure for addition for the set of integers because the sum of two integers is always an integer.

There is closure for multiplication of odd integers because the product of two odd integers is always an odd integer.

There is not closure for addition of odd integers because the sum of two odd integers is not always an odd integer. (In fact, in this case, the sum of two odd integers is never an odd integer!)

Is there closure for addition of natural numbers?

Is there closure for multiplication of natural numbers?

Is there closure for subtraction of natural numbers?

Is there closure for subtraction of integers?

Is there closure for multiplication of irrationals?

[ANSWERS: Yes, Yes, No, Yes, No.]

II. **COMMUTATIVE PROPERTY.** When adding or multiplying two real numbers, **the order may be changed without affecting the result.** (This is like **commuting** to and from work or school.)

A. **Commutative Property for Addition: a + b = b + a.**

Examples:

$$4 + 6 = 6 + 4$$
$$8\cdot(4 + 6) = 8\cdot(6 + 4)$$
$$2 + (4 + 6) = 2 + (6 + 4)$$
$$2 + (4 + 6) = (4 + 6) + 2$$
$$(5X + 3Y)\cdot 14 = (3Y + 5X)\cdot 14$$

B. **Commutative Property for Multiplication: a·b = b·a**

Examples:

$$4\cdot 6 = 6\cdot 4$$
$$8\cdot(4 + 6) = (4 + 6)\cdot 8$$
$$2\cdot(4\cdot 6) = (6\cdot 4)\cdot 2$$
$$2\cdot(4\cdot 6) = 2\cdot(4\cdot 6)$$
$$(5X + 3Y)\cdot 14 = 14\cdot(5X + 3Y)$$

Note: Subtraction and division are not commutative. Why not?

III. **ASSOCIATIVE PROPERTY.** The way in which real numbers are **associated** (by means of parentheses or other symbols of grouping) may be changed without affecting the result. The associative property is frequently used to "reorganize" a problem so it can be simplified. Notice that the order does not change.

A. **Associative Property for Addition:**

$$\mathbf{a + (b + c) = (a + b) + c}$$

Examples:

$$2 + (4 + 6) = (2 + 4) + 6$$
$$(8 + 9) + 1 = 8 + (9 + 1)$$
$$3X + (7X + 10) = (3X + 7X) + 10$$

B. **Associative Property for Multiplication:**

$$\mathbf{a\cdot(b\cdot c) = (a\cdot b)\cdot c}$$

Examples:

$$2\cdot(4\cdot 6) = (2\cdot 4)\cdot 6$$
$$(8\cdot 9)\cdot 1 = 8\cdot(9\cdot 1)$$
$$3X\cdot(7X\cdot 10) = (3X\cdot 7X)\cdot 10$$

Give an example to show that subtraction and division are not associative.

IV. DISTRIBUTIVE PROPERTY. $a\cdot(b + c) = a\cdot b + a\cdot c$

Technically, this property is called the distributive property for multiplication over addition. Sometimes it is called the distributive property for multiplication or, even better, the distributive property. In this property, a multiplier **distributes** over a sum or difference. If this equation is written "backwards," such as the equation:

$$a\cdot b + a\cdot c = a\cdot(b + c)$$

then this is the process of **factoring the common factor.** Factoring the common factor is accomplished by using the **distributive property.**

Examples: $5\cdot(X + 8) = 5\cdot X + 5\cdot 8$ or $5\cdot(X + 8) = 5X + 40$

$8\cdot(Y - 9) = 8\cdot Y - 8\cdot 9$ or $8\cdot(Y - 9) = 8Y - 72$

$7\cdot(3X + 5Y - 6) = 21X + 35Y - 42$

$8X + 24Y = 8\cdot(X + 3Y)$

$48X^2 - 32XY = 16X(3X - 2Y)$

V. IDENTITY PROPERTY. The identity property involves keeping something **identically the same** or preserving the **identity** of a quantity when adding or multiplying.

A. Identity Property for Addition, Additive Identity, or Zero Property for Addition:

For every X in the set, X + 0 = X and 0 + X = X.

When **0** is added to any number **X**, the result is <u>identically</u> the same as the original number **X**. This is why it is called the **identity property for addition.**

Examples: $5 + 0 = 5$ or $0 + 5 = 5$

$b + 0 = b$ or $0 + b = b$

$(3X - 4Y) + 0 = (3X - 4Y)$ or $0 + (3X - 4Y) = (3X - 4Y)$

B. Identity Property for Multiplication, Multiplicative Identity, or Unity Property:

For every X in the set, X·1 = X and 1·X = X.

When any number **X** is multiplied by **1** the result is **identically** the same as the original number.

Examples: $5\cdot 1 = 5$ or $1\cdot 5 = 5$

$b\cdot 1 = b$ or $1\cdot b = b$

$(3X - 4Y)\cdot 1 = (3X - 4Y)$ or $1\cdot(3X - 4Y) = (3X - 4Y)$

VI. INVERSE PROPERTY. The inverse properties are as simple as "wrapping" and "unwrapping" presents! When you are given a wrapped present, you must <u>un</u>wrap it to get to the present. Likewise, the inverse properties are used to "undo" an operation. If **4** has been added, then adding a **(-4)** will "undo" the **4**. If there has been a multiplication by **4**, then multiplication by **¼** will "undo" the operation. In general, you would **inverse** (undo) an **X** with a **(-X)** for addition, and with $\frac{1}{X}$ for multiplication.

A. Inverse Property for Addition or Additive Inverse Property:

$$X + (-X) = 0.$$

Every number **X** in the set has an inverse **(-X)** such that the sum of the number and its inverse is the identity **0**.

Examples:

$$5 + (-5) = 0$$

$$(-56) + 56 = 0$$

$$6b + (-6b) = 0$$

B. Inverse Property for Multiplication or Multiplicative Inverse Property:

$$X \cdot \frac{-1}{X} = 1$$

For every number **X** in the set (except zero!) there is an inverse such that the product of the number and its inverse is the identity **1**.

Examples: $5 \cdot \frac{1}{5} = 1$ $\qquad \frac{1}{4} \cdot 4 = 1$

$$(-6) \cdot (\frac{-1}{6}) = 1 \qquad (-\frac{3}{4}) \cdot (-\frac{4}{3}) = 1$$

Zero has no multiplicative inverse. Why not?

Notice that the **identity properties** always involve **addition of 0** or **multiplication times 1** and the result is always **"identically the same."** For the **inverse properties**, you always end up with the **identity** number: **0** for **addition,** or **1** for **multiplication.**

SUMMARY OF NUMBER PROPERTIES

Closure--the resulting sum or product is always within the set.

Commutative--always involves a change in the order.

Associative--always involves a regrouping or re-association of the numbers in parentheses.

Distributive--always involves a product with parentheses on one side, distributed to each term on the other side of the equation.

Identity--always involves a **0** for addition or **1** for multiplication, ending up with <u>identically</u> the same thing you started with.

Inverse--always ends up with the identity element, **0** for addition or **1** for multiplication.

EXERCISES.

In each of the following, answer "Yes" or "No."
If the answer is "No", then give an example to show that it is not.

Is there closure for:

1. addition of integers? **1.** ____________

2. multiplication of integers? **2.** ____________

3. addition of even integers? **3.** ____________

4. multiplication of even integers? **4.** ____________

5. addition of odd integers? **5.** ____________

6. multiplication of odd integers? **6.** ____________

7. addition of positive integers? **7.** ____________

8. multiplication of positive integers? **8.** ____________

9. addition of negative integers? **9.** ____________

10. multiplication of negative integers? **10.** ____________

In each of the following, give the complete name of the property:

1. $50\cdot(2\cdot98) = 50\cdot(98\cdot2)$ **1.** ______________

What happened? Changed order multiplying 2 and 98

2. $50\cdot(2\cdot98) = (2\cdot98)\cdot50$ **2.** ______________

What happened? ______________

3. $50\cdot(2\cdot98) = (50\cdot2)\cdot98$ **3.** ______________

What happened? Re-association of numbers. Order did not change

4. $50+(2+98) = 50+(98+2)$ **4.** ______________

What happened? ______________

5. $50+(2+98) = (2+98)+50$ **5.** ______________

What happened? ______________

6. $50+(2+98) = (50+2)+98$ **6.** ______________

What happened? ______________

7. $50\cdot(2+98) = 50\cdot(98+2)$ **7.** ______________

What happened? ______________

8. $50\cdot(2+98) = (2+98)\cdot50$ **8.** ______________

What happened? ______________

9. $50\cdot(2+98) = (50\cdot2)+(50\cdot98)$ **9.** ______________

What happened? ______________

10. $(50\cdot2)+(50\cdot98) = 50\cdot(2+98)$ **10.** ______________

What happened? ______________

11. $50\cdot(1\cdot98) = 50\cdot98$ **11.** ______________

What happened? ______________

12. $50\cdot(98\cdot1) = 50\cdot98$ **12.** ______________

What happened? ______________

13. $50 \cdot [98+(-98)] = 50 \cdot 0$ 13. ______________________

What happened? ______________________

14. $50 \cdot [98+(-98)] = 50 \cdot 98+50 . (-98)$ 14. ______________________

What happened? ______________________

15. $50 \cdot [98+(-98)] = 50 \cdot [(-98)+98]$ 15. ______________________

What happened? ______________________

16. $50 \cdot (0+98) = 50 \cdot (98+0)$ 16. ______________________

What happened? ______________________

17. $50 \cdot (0+98) = 50 \cdot 98$ 17. ______________________

What happened? ______________________

18. $50 \cdot (0+98) = 50 \cdot 0+50 \cdot 98$ 18. ______________________

What happened? ______________________

19. $50 \cdot (2 \cdot \frac{1}{2}) = 50 \cdot 1$ 19. ______________________

What happened? ______________________

20. $50 \cdot (2 \cdot \frac{1}{2}) = 50 \cdot (\frac{1}{2} \cdot 2)$ 20. ______________________

What happened? ______________________

21. $50 \cdot (2 \cdot \frac{1}{2}) = (50 \cdot 2) \cdot \frac{1}{2}$ 21. ______________________

What happened? ______________________

22. $50 \cdot [2+(-2)] = 50 \cdot 0$ 22. ______________________

What happened? ______________________

23. $50 \cdot [(-2)+2] = 50 \cdot [2+(-2)]$ 23. ______________________

What happened? ______________________

24. $50 \cdot (\frac{1}{4} \cdot 4) = 50 \cdot 1$ 24. ______________________

What happened? ______________________

ANSWERS 1.01

p. 4-6: **1.** N,W,I,Ra,R,C; **2.** I,Ra,R,C; **3.** Ir,R,C; **4.** Ra,R,C; **5.** Ra,R,C; **6.** Ir,R,C; **7.** Im,C; **8.** C; **9.** I,Ra,R,C; **10.** Ra,R,C; **11.** Ra,R,C; **12.** Ra,R,C; **13.** N,W,I,Ra,R,C; **14.** I,Ra,R,C; **15.** Ra,R,C; **16.** Ra,R,C; **17.** Ir,R,C; **18.** Ir,R,C; **19.** I,Ra,R,C; **20.** N,W,I,Ra,R,C; **21.** Im,C; **22.** Ir,R,C; **23.** W,I,Ra,R,C; **24.** I,Ra,R,C; **25.** None; **26.** Ra,R,C; **27.** W,I,Ra,R,C; **28.** N,W,I,Ra,R,C; **29.** N,W,I,Ra,R,C; **30.** Ir,R,C; **31.** Ir,R,C; **32.** Ra,R,C; **33.** N,W,I,Ra,R,C; **34.** W,I,Ra,R,C; **35.** Ir,R,C; **36.** Ir,R,C; **37.** I,Ra,R,C; **38.** Ra,R,C;

p. 11: **1.** Yes; **2.** Yes; **3.** Yes; **4.** Yes; **5.** No, $(3+3)$; **6.** Yes; **7.** Yes; **8.** Yes; **9.** Yes; **10.** No, $(-3)\cdot(-3)$;

p.12-13:

1. Commutative for mult; **2.** Commutative for mult;
3. Associative for mult; **4.** Commutative for add;
5. Commutative for add; **6.** Associative for add;
7. Commutative for add; **8.** Commutative for mult;
9. Distributive; **10.** Distributive;
11. Identity for mult; **12.** Identity for mult;
13. Inverse for add; **14.** Distributive;
15. Commutative for add; **16.** Commutative for add;
17. Identity for add; **18.** Distributive;
19. Inverse for mult; **20.** Commutative for mult;
21. Associative for mult; **22.** Inverse for add;
23. Commutative for add; **24.** Inverse for mult.

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