1.07 *Complex Numbers*

Dr. Robert J. Rapalje

ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

There are two operations that are not defined in the set of real numbers:

I. Division by zero,

II. Square roots (or even roots) of negative numbers.

The possibility of taking square roots of negative numbers leads to the definition of **imaginary numbers**:

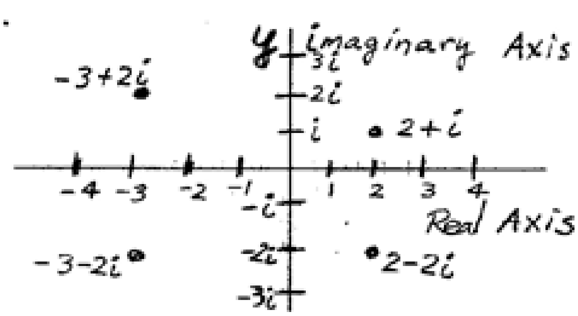
DEFINITION: $i = \sqrt{-1}$

$i^2 = -1$

It follows that $\sqrt{-a} = i\sqrt{a}$ where "a" represents any real number.

Notice that this definition $i = \sqrt{-1}$ applies only to square roots. **Any odd root of -1 is -1.** Finding the **fourth root, sixth root, eighth root**, or any other **even root of -1** is a much more complicated process, and the answer is not simply "**i**."

If the **real numbers** can be graphed on a numberline (like the X-axis), then the **imaginary numbers** can be graphed on a numberline (like the Y-axis) that is perpendicular to the **real axis** as shown below. More generally, **complex numbers** consist of any combination of real and imaginary numbers. For example, if **a** and **b** are any **real numbers**, then the expression represented by **Z = a + bi** is said to be a **complex number**. It is not that this is "complex" in the sense of being "complicated" (it is not!). Rather, it is complex in that it consists of inter-connected or interwoven parts, as a B-complex vitamin. If the **real numbers** are contained on the **X-axis** and the **imaginary numbers** are on the **Y-axis**, then the **complex numbers** cover the entire **XY-plane**.



Realize that, for example, the real number **X = 6** can be written as **Z = 6 + 0i**. Since every **real number X** can be written in the form **Z = X + 0i**, the **real numbers** are actually a **subset** of the **complex number system.**

The **conjugate** of the complex number **a + bi** is defined to be **a - bi**. That is, the conjugate of a complex number has the negative of the imaginary part.

QUICK EXERCISES:

For each of the following complex numbers, give the conjugate.

	1. 3+4i	2. 3-4i	3. -3-4i	4. 4i-3	5. 6i	6. -3i	7. 3	8. -3
conj:	______	______	______	______	______	______	______	______

SUMMARY

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$\sqrt{-a} = i\sqrt{a}$$

NOTE: $\sqrt{-1} = i$, *but* $\sqrt[3]{-1} = -1$

EXAMPLES. Express each answer in the form "a + bi."

1. $\sqrt[3]{-72} + \sqrt{-72}$

$= \sqrt[3]{-8}\sqrt[3]{9} + \sqrt{-36}\sqrt{2}$

$= -2\sqrt[3]{9} + 6i\sqrt{2}$

2. $\sqrt{-4}\sqrt{-9}$

$= 2i \cdot 3i$

$= 6i^2$

$= -6$

$\underline{\text{NOT}}$ $\sqrt{(-4)(-9)}$

$= \sqrt{36}$

$= 6$ WRONG!!

3. $-3i(4 - 2i)$

$= -12i + 6i^2$

$= -6 - 12i$

4. $(2 - 3i)(4 + i)$ (FOIL)

$= 8 + 2i - 12i - 3i^2$

$= 8 - 10i + 3$

$= 11 - 10i$

5. $(3 - 5i)^2$

$= 9 - 30i + 25i^2$

$= 9 - 30i + 25(-1)$

$= -16 - 30i$

6. $2i\underbrace{(2 - 3i)(4 + i)}$

$= 2i(8 + 2i - 12i - 3i^2)$

$= 2i(8 - 10i + 3)$

$= 2i(11 - 10i)$

$= 22i - 20i^2 = (-1)$

$= 20 + 22i$

In #7, note binomial denominator. To rationalize, multiply numer. and denom. by "$4 - i$"

7. $\dfrac{(5 + 14i)(4 - i)}{(4 + i)(4 - i)}$

$= \dfrac{20 - 5i + 56i - 14i^2}{16 - \cancel{4i} + \cancel{4i} - i^2}$

$= \dfrac{20 + 51i + 14}{16 + 1}$

$= \dfrac{34 + 51i}{17}$

$= \dfrac{34}{17} + \dfrac{51i}{17} = 2 + 3i$

$$\boxed{\begin{array}{l}\text{NOTE: See \#4 p.98.}\\ \text{If } (2+3i)(4+i) = 5 + 14i,\\ \text{then } \dfrac{5+14i}{4+i} = 2 + 3i\end{array}}$$

In #8, note monomial denominator. To rationalize, multiply numer. and denom. by "i"

8. $\dfrac{(3 - 10i)}{5i} \dfrac{i}{i}$

$= \dfrac{3i - 10i^2}{5i^2}$

$= \dfrac{3i + 10}{-5}$

$= -2 - \frac{3}{5} i$

EXERCISES. Express each answer in the form of "a + bi."

1. $\sqrt[3]{-8} + \sqrt{-64}$

$= \_\_\_\_ + \_\_\_\_$

2. $\sqrt{-81} + \sqrt[3]{-125}$

$= \_\_\_\_ + \_\_\_\_$

$= \_\_\_\_\_\_\_\_$

3. $-\sqrt{-4} + \sqrt{-100}$

$= -\_\_\_\_ + \_\_\_\_$

$= \_\_\_\_\_\_\_\_$

4. $-\sqrt[3]{-27} + \sqrt{-27}$

$= -(\quad) + \sqrt{-9}\,\sqrt{3}$

$=$

5. $\sqrt[3]{-40} + \sqrt{-40}$

$= \sqrt[3]{-8}\sqrt[3]{(\quad)} + \sqrt{-4}\sqrt{(\quad)}$

$=$

6. $\sqrt[3]{-54} + \sqrt{-54}$

7. $\sqrt[3]{-250} + \sqrt{-250}$

8. $\sqrt[3]{-48} + \sqrt{-48}$

9. $\sqrt{-9} \cdot \sqrt{-16}$

$= \_\_\_\_ \circ \_\_\_\_$

$= \_\_\_\_\ i^2$

$= \_\_\_\_$

10. $\sqrt{-4} \cdot \sqrt{-25}$

11. $\sqrt{-18} \cdot \sqrt{-10}$

12. $\sqrt{-12} \cdot \sqrt{-30}$

Graphing Calculator

Operations with complex numbers can be easily performed with a graphing calculator by identifying the real and imaginary parts of the numbers. Begin by locating the parentheses keys and the i key located above the [.] key.

To enter a complex number, type: [real part] + [imaginary part] [2nd] [.]

Example 9. Enter the complex number $2 + 3i$.

Example 10. Enter the complex number $2 - 3i$.

Notice that this is the minus key!

Example 11. Enter the complex number i .

Example 12. Enter the complex number $-3i$.

Notice that this is the negative key!

NOT minus!

Now, operations with most complex numbers are as simple as typing them into the calculator.

In 13 - 80, use the graphing calculator to perform the indicated operations. Express your answers in "a+bi" form.

13. i^3 = ______ 14. i^4 = ______ 15. i^5 = ______

16. i^6 = ______ 17. i^7 = ______ 18. i^8 = ______

19. i^9 = ______ 20. i^{10} = ______ 21. i^{12} = ______

22. i^{20} = ______ 23. i^{21} = ______ 24. i^{103} = ______

Example 13. (See Example 3). $-3i(4 - 2i)$

Solution: [(] [(-)] [3] [2nd] [•] [)] [(] [–] [2] [2nd] [•] [)] [ENTER]

Negative key Minus key

(Notice that the multiplication sign is not needed.)

The calculator shows $6 - 12i$

Example 14. (See Example 4). $(2 - 3i)(4 + i)$

Solution: [(] [2] [–] [3] [2nd] [•] [)] [(] [4] [+] [2nd] [•] [)] [ENTER]

The calculator shows $11 - 10i$

Example 15. (See Example 5). $(3 - 5i)^2$

Solution: [(] [3] [–] [5] [2nd] [•] [)] [^] [2] [ENTER]

The calculator shows

Example 16. (See Example 6). $2i(2 - 3i)(4 + i)$

Solution: [2] [2nd] [•] [(] [2] [–] [3] [2nd] [•] [)] [(] [4] [+] [2nd] [•] [)] [ENTER]

The calculator shows $20 + 22i$

NOTE: As a shortcut, you might have taken the answer to the previous Example times $(2i)$.

Perform the indicated operations and express in the form "a + bi."

25. (-6+2i) + (8-6i)

26. (7-8i) + (-20-6i)

27. (-6+2i) - (8-6i)

28. (7-8i) - (-20-6i)

29. 3i(2+i)

30. -3i(4-2i)

31. (4+i)(2+3i)

32. (8-i)(6+2i)

33. (3+2i)(2-3i)

34. (2-3i)(3-2i)

35. (2-3i)(2+3i)

36. (3-5i)(3+5i)

37. $(2-3i)^2$

38. $(3+2i)^2$

Before solving #39, recall from #31 that (4+i)(2+3i)=(5+14i). What do you think you would get if you divide the answer 5+14i by one of its factors 2+3i? Of course you would get the other factor 4+i.

Example 17. (See Example 7). $\dfrac{11 - 10i}{2 - 3i}$ Be careful to use ( )!

Solution: [(] [11] [-] [10] [2nd] [.] [)] [÷] [(] [2] [-] [3] [2nd] [.] [)] [ENTER]

The calculator shows 4+i

Example 18. (See Example 8). $\dfrac{3 - 10i}{5i}$ Don't forget parentheses for numerator and denom!

Solution: [(] [3] [-] [10] [2nd] [.] [)] [÷] [(] [5] [2nd] [.] [)] $\boxed{\text{ENTER}}$

The calculator shows $-2 - .6i$, which means $-2 - \frac{3}{5}i$

The calculator can convert decimals to fractions: $\boxed{\text{MATH}}$ $\boxed{\text{FRAC}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$.

39. $\dfrac{5 + 14i}{2 + 3i}$

40. $\dfrac{5 + 14i}{4 + i}$

41. $\dfrac{25 + 5i}{8 - i}$

42. $\dfrac{23 - 11i}{3 - i}$

43. $\dfrac{16 + 30i}{-3 + 5i}$

44. $\dfrac{13 + 13i}{3 - 2i}$

45. $\dfrac{4 - 3i}{2 - i}$

46. $\dfrac{2 + 3i}{1 - 3i}$

When dividing by a monomial complex number, remember that
you need parentheses around the numerator AND denominator.

47. $\dfrac{-8 + 5i}{2i}$

48. $\dfrac{3 - 11i}{i}$

49. $\dfrac{6 + 3i}{-2i}$

50. $\dfrac{25 - 10i}{-5i}$

51. $-4i(2-3i)(2+4i)$

52. $-3i(3-2i)(5-i)$

53. $\dfrac{(5 + 14i)\cdot(2 - i)}{4 + i}$

54. $\dfrac{(5 + i)\cdot(5 + 5i)}{8 - i}$

55. $(1+i)^2$

56. $(1-i)^2$

57. $(1+i)^3$

58. $(1-i)^3$

59. $(1+i)^4$

60. $(1-i)^4$

61. $(1+i)^6$

62. $(1-i)^6$

63. $(1+i)^{12}$

64. $(1-i)^{12}$

65. $(1+i)^{18}$

66. $(1-i)^{18}$

Sometimes the answers do not come out even, in which case it is helpful to express the decimal answers in fractional form.

Example 19. $\dfrac{3 + i}{2 + i}$

Solution: [(] [3] [+] [2nd] [.] [)] [÷] [(] [2] [+] [2nd] [.] [)] $\boxed{\text{ENTER}}$

The calculator shows 1.4 − .2i

This can easily be converted to fractional form, by using the [MATH] [FRAC] [ENTER]. The calculator then gives

$$\frac{7}{5} - \frac{1}{5}i.$$

As the next example illustrates, the decimals and fractions can get worse--much worse! Nevertheless, the calculator handles them nicely!

Example 20. $\dfrac{17 - 32i}{14 + 25i}$

Solution: [(] [17] [-] [32] [2nd] [i] [)] [÷] [(] [14] [+] [25] [2nd] [i] [)] [ENTER]

The calculator shows -.684531059683 - 1.0...i too long to express.

However, [MATH] [FRAC] converts to the fraction

$$-\frac{562}{821} - \frac{873}{821}i.$$

In 67 - 80, perform the indicated operations and convert to fractional form.

67. $\dfrac{2 + i}{3 + 4i}$

68. $\dfrac{2 + i}{4 + 3i}$

69. $\dfrac{5 - 7i}{6 - 3i}$

70. $\dfrac{8 - 3i}{8 + 6i}$

71. $\dfrac{17 + 32i}{14 + 25i}$

72. $\dfrac{-7 + 8i}{6 + 19i}$

73. $\dfrac{(-5 + 8i)\cdot(6 - 9i)}{7 - 5i}$

74. $\dfrac{(7 - 8i)\cdot(6 - 4i)}{5 + 2i}$

In 75 - 80, don't forget parentheses around the denominators.

75. $\dfrac{72 + 13i}{(2 + i)\cdot(3 + 2i)}$

76. $\dfrac{72 + 13i}{(2 + i)\cdot(3 - 2i)}$

77. $\dfrac{7 - 5i}{(-5 + 8i)\cdot(6 - 9i)}$

78. $\dfrac{5 + 2i}{(7 - 8i)\cdot(6 - 4i)}$

79. $\dfrac{(-4 - i)\cdot(2 + 3i)}{(3 - 4i)\cdot(6 + 8i)}$

80. $\dfrac{(6 - i)\cdot(7 + 8i)}{(7 - 8i)\cdot(4 + 5i)}$

ANSWERS 1.07

p. 98:

1. 3-4i; **2.** 3+4i; **3.** -3+4i; **4.** -4i-3; **5.** -6i; **6.** 3i;
7. 3 or 3+0i; **8.** -3 or -3+0i.

p. 100-108:

1. -2+8i; **2.** -5+9i; **3.** 8i; **4.** $3+3i\sqrt{3}$; **5.** $-2\sqrt[3]{5}+2i\sqrt{10}$
6. $-3\sqrt[3]{2}+3i\sqrt{6}$ **7.** $-5\sqrt[3]{2}+5i\sqrt{10}$ **8.** $-2\sqrt[3]{6}+4i\sqrt{3}$ **9.** -12; **10.** -10;
11. $-6\sqrt{5}$; **12.** $-6\sqrt{10}$; **13.** -i; **14.** 1; **15.** i; **16.** -1; **17.** -i;
18. 1; **19.** i; **20.** -1; **21.** 1; **22.** 1; **23.** i; **24.** -i; **25.** 2-4i;
26. -13-14i; **27.** -14+8i; **28.** 27-2i; **29.** -3+6i; **30.** -6-12i;
31. 5+14i; **32.** 50+10i; **33.** 12-5i; **34.** -13i; **35.** 13; **36.** 34;
37. -5-12i; **38.** 5+12i; **39.** 4+i; **40.** 2+3i; **41.** 3+i; **42.** 8-i;
43. 3-5i; **44.** 1+5i; **45.** 11/5 - 2/5 i; **46.** -7/10 + 9/10 i;
47. 5/2 + 4i; **48.** -11-3i; **49.** -3/2 + 3i; **50.** 2+5i; **51.** 8-64i;
52. -39-39i; **53.** 7+4i; **54.** 2+4i; **55.** 2i; **56.** -2i; **57.** -2+2i;
58. -2-2i; **59.** -4; **60.** -4; **61.** -8i; **62.** 8i; **63.** -64;
64. 64; **65.** 512i; **66.** -512i; **67.** 2/5 - 1/5i; **68.** 11/25-2/25i; **69.** 17/15 - 3/5i;
70. 23/50 - 18/25i; **71.** 1038/821+ 23/821i; **72.** 110/397 + 181/397 i;
73. -171/74 + 861/74 i; **74.** -102/29 - 400/29 i; **75.** 379/65 - 452/65 i;
76. 563/65 + 176/65 i; **77.** -19/1157 - 287/3471 i; **78.** -51/2938 + 100/1469 i;
79. -1/10 - 7/25 i; **80.** 3523/4633 + 2638/4633 i.

Dr. Robert J. Rapalje