1.11 *Inequalities and Properties of Inequalities*
Linear, Absolute Value, and Quadratic

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[NOTE: This section is presented in Chapter 1 as a review of inequalities and the properties of inequalities, specifically the solving of linear inequalities, absolute value inequalities, and quadratic inequalities. More advanced inequalities, such as polynomial inequalities and rational (fractional) inequalities, will be introduced later (Section 3.06) after graphing. As an instructional alternative, this section may be postponed for now and placed before Section 3.06, where graphical methods may be appropriate. Either way, it may be advisable to solve these exercises and those in Section 3.06 using the graphics calculator.]

As an introduction to inequalities, consider again the properties of equations from section 1.08. Which, if any, of these properties of equations apply also to inequalities?

ADDITION PROPERTY FOR EQUATIONS

If $a=b$, then $a + c = b + c$
If $a=b$, then $a - c = b - c$.

The same number may be added (or subtracted) from both sides of an equation.

I. ADDITION PROPERTY FOR INEQUALITIES

If $a < b$, then $a + c < b + c$
If $a < b$, then $a - c < b - c$
If $a > b$, then $a + c > b + c$
If $a > b$, then $a - c > b - c$.

The addition property for inequalities is basically the same as it is for equations--the same number may be added (or subtracted) from both sides of an inequality, and the inequality remains the same. Of course, this property is also valid for "≤" and "≥."

MULTIPLICATION PROPERTY FOR EQUATIONS If a=b, then ac = bc
If a=b and c≠0, then a/c = b/c.
Both sides of an equation may be multiplied or divided by the same non-zero number.

II. **MULTIPLICATION PROPERTY FOR INEQUALITIES must be given in two distinct parts:**

A. If a < b and c > 0, then ac < bc.
If a > b and c > 0, then ac > bc.
If a ≤ b and c > 0, then ac ≤ bc.
If a ≥ b and c > 0, then ac ≥ bc.

This means that if both sides of an inequality are multiplied (or divided) by a positive number, then the inequality sign remains the same.

B. If a < b and c < 0, then ac > bc.
If a > b and c < 0, then ac < bc.
If a ≤ b and c < 0, then ac ≥ bc.
If a ≥ b and c < 0, then ac ≤ bc.

This means that if both sides of an inequality are multiplied (or divided) by a negative number, then the inequality sign must be reversed.

REFLEXIVE PROPERTY FOR EQUATIONS: a = a. Any number is equal to itself. **Clearly, a number is not less than or greater than itself. Therefore, there is NO reflexive property for inequalities.**

SYMMETRIC PROPERTY FOR EQUATIONS: If a = b, then b = a. The order in which the equality is given does not matter. For example, you can say "X=4" or "4=X", the meaning is the same--the value of X is 4. **Clearly, if a < b, then it is not true that b < a. There is NO symmetric property for inequalities.**

TRANSITIVE PROPERTY FOR EQUATIONS: If a = b and b = c, then a = c. The word "trans" means "across." If you can get from point "a" to "b", and then from "b" to "c", then you can get from "a" across "b" to "c."

III. **TRANSITIVE PROPERTY FOR INEQUALITIES.**

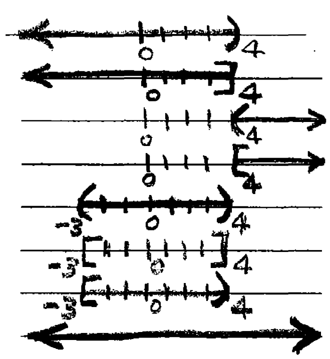
If a < b and b < c, then a < c.
If a > b and b > c, then a > c.

The transitive property is also valid for "≤" and "≥."

LINEAR INEQUALITIES

In describing equations and inequalities, the **Trichotomy Axiom** and **interval notation** will be helpful. Consider the variable X and any number, for example, 4. As the word "tri" means "three," according to the Trichotomy Axiom, there are <u>three</u> ways to compare the variable X to the number 4: **X=4**, **X<4**, **or** **X>4**.

The first category **X=4** consists of just one point. The second category **X<4** represents an entire interval of numbers on the numberline to the left of the number 4. Likewise, the third category **X>4** represents an entire interval of numbers on the numberline to the right of the number 4. Since in each of these intervals there are infinitely many numbers, it is helpful to describe the interval with **interval notation**. Interval notation is always given from **left to right**, with brackets **"["** or **"]"** indicating **included endpoints**, and parentheses **"("** or **")"** indicating that **endpoints are not included.**

Variable Notation	Interval Notation	Graph on Numberline
X < 4	(-∞, 4)	
X ≤ 4	(-∞, 4]	
X > 4	(4, ∞)	
X ≥ 4	[4, ∞)	
-3 < X < 4	(-3, 4)	
-3 ≤ X ≤ 4	[-3, 4]	
-3 ≤ X < 4	[-3, 4)	
All Reals	(-∞, ∞)	

The tradition has always been to consider -∞ and ∞ non-inclusive, since infinity is not something that can be "contained" or "included." Remember that X < 4 is equivalent to 4 > X. Also, realize that inequalities, like equations, can be **conditional, identities, or contradictions.**

In the following examples, solve the inequalities. Give answers in interval notation:

EXAMPLE 1

$$3X - 12 \geq 5X - 20$$
$$-2X \geq -8$$
$$X \leq 4$$
$$(-\infty, 4]$$

EXAMPLE 2

$$5X - 12 < 5X + 20$$
$$-12 < +20$$
True for all X
$$(-\infty, \infty)$$

EXAMPLE 3

$$-5 \leq \frac{2X + 3}{3} \leq 7$$
$$-15 \leq 2X + 3 \leq 21$$
$$-18 \leq 2X \leq 18$$
$$-9 \leq X \leq 9$$
$$[-9, 9]$$

EXAMPLE 4

$$-1 < \frac{3 - 2X}{3} \leq 5$$
$$-3 < 3 - 2X \leq 15$$
$$-6 < -2X \leq 12$$
$$3 > X \geq -6$$
$$[-6, 3)$$

EXERCISES: Solve for X. Give answers in interval notation.

1. $5 - 3(X - 4) \leq 2(X - 4)$

2. $-3(X + 4) > 4(2X + 3) - 5X$

3. $2X - 8(7 - X) > 5(2X - 4)$

4. $-8 - 3(X - 4) < 3(6 - X)$

5. $2X - 8 < 2(2 - 5X) + 12X$

6. $-2(2 + 3X) \geq 3(5 - X) + 8$

7. $-7 \leq \frac{3 - 2X}{3} < 5$

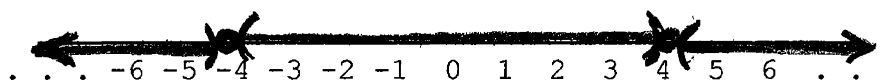
8. $-5 < \frac{2X + 3}{3} < -1$

9. $-3 \leq \frac{-3X + 6}{2} \leq 6$

10. $-9 < \frac{3 - 3X}{2} \leq 6$

ABSOLUTE VALUE INEQUALITIES

To introduce **absolute value equations** and **inequalities**, return again to the **Trichotomy Axiom.** There are **three** ways to compare **$|X|$** to a positive number such as 4: **$|X|=4$, $|X|<4$**, and **$|X|>4$.** By trial and error, you can see the solution to **$|X|=4$** consists of the two numbers **$X=-4$** and **$|X|=4$.** Also by trial and error, you can see that **$|X|<4$** consists of all values of X **between -4** and **4**, that is **$-4<X<4$**, or in interval notation: **$(-4,4)$.** What remains for the solution to **$|X|>4$** are two separate intervals in which **$X>4$** or **$X<-4$**, or in interval notation: **$(-\infty, -4)\cup(4,\infty)$** (see below).



	Betweenness	Two points	Extremes
	$\lvert X\rvert < 4$	**$\lvert X\rvert = 4$**	**$\lvert X\rvert > 4$**
	$-4 < X < 4$	**$X = 4$ or $X = -4$**	**$X > 4$ or $X < -4$**
Int.Notation:	**$(-4, 4)$**		**$(-\infty, -4) \cup (4, \infty)$**

In the more general cases, where **$c > 0$**,

Betweenness	Two points	Extremes
$\lvert aX+b\rvert < c$	**$\lvert aX+b\rvert = c$**	**$\lvert aX+b\rvert > c$**
$-c < aX+b < c$	**$aX+b = c$ or $aX+b = -c$**	**$aX+b > c$ or $aX+b < -c$**

Continuing with the idea of "trichotomy," there are three conditions needed to guarantee each of these cases:

Betweenness	Two points	Extremes
1. Absolute value	1. Absolute value	1. Absolute value
2. $<$ or $\le$	2. $=$	2. $>$ or $\ge$
3. $c > 0$	3. $c > 0$	3. $c > 0$

EXAMPLE 1 (See section 1.08)

$$|2X + 5| = 7$$

$2X + 5 = 7$ or $2X + 5 = -7$

$2X = 2$ $\quad$ $2X = -12$

$X = 1$ or $X = -6$

EXAMPLE 2

$|2X + 5| < 7$ (BETWEENNESS)

$$\begin{array}{rcccl} -7 < & 2X & + 5 & < 7 \\ -5 & & -5 & -5 \\ \hline -12 < & 2X & & < 2 \end{array}$$

$-6 < X < 1$

(-6, 1)

EXAMPLE 3

$|2X + 5| \geq 7$ (EXTREMES)

$2X + 5 \geq 7$ **or** $2X + 5 \leq -7$

$\quad - 5 \;\; -5 \qquad\qquad - 5 \;\; -5$

$2X \geq 2 \qquad\qquad 2X \leq -12$

$X \geq 1$ **or** $X \leq -6$

$(-\infty, -6] \cup [1, \infty)$

EXAMPLE 4 (See section 1.08)

$|2X + 5| = -7$ Absolute value cannot be negative.
No Solution!

EXAMPLE 5

$|2X + 5| < -7$ (BETWEENNESS? **No!**)

No Solution!

EXAMPLE 6

$|2X + 5| > -7$ (EXTREMES? **No!**)

Absolute value is always > -7.
Solution is all real X: **$(-\infty, \infty)$**

EXERCISES. Solve for X. Give answers in interval notation.

1. $|X + 4| < 8$
2. $|X - 4| < 8$
3. $|2X - 4| \leq 8$
4. $|2X + 4| \leq 8$
5. $|2X - 4| < -8$
6. $|2X + 4| \leq -8$

7. $|X + 4| > 8$

8. $|X - 4| > 8$

9. $|2X - 4| \ge 8$

10. $|2X + 4| \ge 8$

11. $|2X - 4| > -8$

12. $|2X + 4| \ge -8$

13. $|4 - X| < 8$

14. $|4 - 2X| \le 8$

15. $|4 - 2X| \ge 8$

16. $|4 - X| \le 8$

17. $|6 - 2X| < 8$

18. $|6 - 2X| > -8$

19. $|6 - 2X| > 8$

20. $|8 - 2X| \geq 8$

21. $\left|\frac{2X - 2}{3}\right| < 4$

22. $\left|\frac{2X - 2}{3}\right| > 4$

23. $\left|\frac{3X + 2}{2}\right| \geq 4$

24. $\left|\frac{3X + 2}{2}\right| \leq 4$

25. $\left|\frac{6 - X}{3}\right| < 2$

26. $\left|\frac{6 - X}{3}\right| < -2$

27. $\left|\frac{6 - X}{3}\right| \geq 2$

28. $\left|\frac{2X - 2}{3}\right| \leq 0$

29. $\left|\frac{2X - 2}{3}\right| > 0$

30. $\left|\frac{2X - 2}{3}\right| \geq 0$

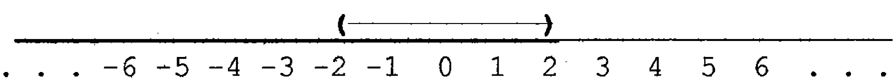
QUADRATIC INEQUALITIES

In section 1.08, quadratic equations were solved by factoring, by completing the square, and by the quadratic formula. Usually, although not always, there were two solutions, as there were with absolute value equations. Now, having solved **absolute value equations and inequalities** with **endpoints, betweenness, and extremes**, wouldn't it be nice if **quadratic equations and inequalities** could somehow be **"endpoints, betweenness, and extremes"** with similar patterns? The following is almost too good to be true, but it is . . .

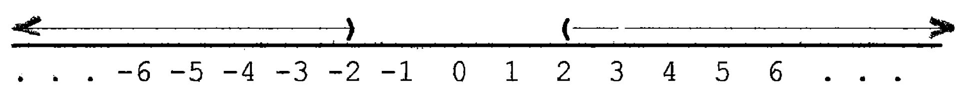
There are **three** ways to compare X^2 to a number such as 4:

$$X^2 = 4, \quad X^2 < 4, \text{ and } X^2 > 4.$$

First, you can see the solution to $X^2 = 4$ consists of the two numbers **X=-2** and **X=2**. **Second**, by trial and error, you can see that $X^2 < 4$ consists of all values of X **between -2** and **2**, that is, $-2<X<2$, or in interval notation **(-2,2)** (see numberline below):



Third, what remains on the numberline for the solution to $X^2 > 4$ are two separate intervals in which $X>2$ or $X<-2$, or in interval notation: $(-\infty, -2) \cup (2, \infty)$ (see numberline below).



In summary, the three cases are as follows:

	Betweenness	**Two points**	**Extremes**
	$X^2 < 4$	$X^2 = 4$	$X^2 > 4$
	$-2 < X < 2$	$X = 2$ or $X = -2$	$X > 2$ or $X < -2$
Interval Notation	$(-2, 2)$		$(-\infty, -2) \cup (2, \infty)$

In the general case, where **$a > 0$** and there are **two real endpoints**,

BETWEENNESS	**TWO POINTS**	**EXTREMES**
$aX^2+bX+c < 0$	$aX^2+bX+c = 0$	$aX^2+bX+c > 0$

Continuing with the idea of "trichotomy," there are three conditions needed to guarantee each of these cases:

	Betweenness		Two points		Extremes
1.	Quadratic (X^2)	1.	Quadratic (X^2)	1.	Quadratic (X^2)
2.	$<$ or $\le$	2.	$=$	2.	$>$ or $\ge$
3.	$a > 0$	3.	$a > 0$	3.	$a > 0$

EXAMPLE 1 (See section 1.08)

$$X^2 + 2X - 8 = 0$$
$$(X + 4)(X - 2) = 0$$
$$X = -4 \text{ or } X = 2$$

EXAMPLE 2

$X^2 + 2X - 8 < 0$ (BETWEENNESS)

$X^2 + 2X - 8 = 0$ (Endpoints!)

$(X + 4)(X - 2) = 0$

$X = -4$ or $X = 2$

$-4 < X < 2$

$(-4, 2)$

EXAMPLE 3

$X^2 + 2X - 8 > 0$ (EXTREMES)

$X^2 + 2X - 8 = 0$ (Endpoints!)

$(X + 4)(X - 2) = 0$

$X = -4$ or $X = 2$

$X > 2$ **or** $X < -4$

$(-\infty, -4) \cup (2, \infty)$

What happens if $a<0$? For example, **$-X^2 + 4X + 5 > 0$.**
No problem! Just divide both sides of the inequality by **(-1)**, then don't forget to reverse the direction of the inequality sign!

EXAMPLE 4

$-X^2 + 4X + 5 > 0$

$X^2 - 4X - 5 < 0$ (BETWEENNESS)

$X^2 - 4X - 5 = 0$ (Endpoints!)

$(X - 5)(X + 1) = 0$

$X = 5$ or $X = -1$

$-1 < X < 5$

$(-1, 5)$

EXAMPLE 5

$-X^2 + 4X + 5 \le 0.$

$X^2 - 4X - 5 \ge 0$ (EXTREMES)

$X^2 - 4X - 5 = 0$ (Endpoints!)

$(X - 5)(X + 1) = 0$

$X = 5$ or $X = -1$

$X \ge 5$ **or** $X \le -1$

$(-\infty, -1] \cup [5, \infty)$

EXERCISES. Solve for X. Give answers in interval notation.

1. $X^2 - 3X - 4 < 0$

2. $X^2 - 3X < 0$

3. $X^2 - 2X - 8 \leq 0$

4. $X^2 - 12X + 35 \leq 0$

5. $X^2 - 3X - 4 > 0$

6. $X^2 - 3X > 0$

7. $X^2 - 2X - 8 \geq 0$

8. $X^2 - 12X + 35 \geq 0$

9. $X^2 - 5X < 6$

10. $X^2 - 6X > 16$

11. $X^2 - 6X \geq 16$

12. $X^2 - 5X \leq 6$

13. $-X^2 + X + 6 \leq 0$

14. $-X^2 - 3X < 0$

15. $-X^2 + 6X > 0$

16. $6X - X^2 > 5$

17. $\mathbf{2(2 - X^2) \le 7X}$

18. $\mathbf{2X(4 - X) > 15 - 5X}$

What happens if the quadratic equation does not factor? Again, no problem! Just find the two endpoints by quadratic formula or completing the square (if there are two real endpoints!). Then from the "±" in the quadratic formula, the "-" will be the left endpoint, and the "+" will be the right endpoint, as illustrated in the next examples.

EXAMPLE 6

$\mathbf{X^2 + 2X - 5 \le 0}$ (BETWEENNESS)

$X^2 + 2X - 5 = 0$ (Endpoints!)

EXAMPLE 7

$\mathbf{X^2 + 2X - 5 > 0}$ (EXTREMES)

$X^2 + 2X - 5 = 0$ (Endpoints!)

(Does not factor, so use quadratic formula or complete square.)

Example 6:

$$X^2 + 2X + 1 = 6$$
$$(X + 1)^2 = 6$$
$$X + 1 = \pm \sqrt{6}$$
$$X = -1 \pm \sqrt{6}$$

Betweenness means:

$-1 - \sqrt{6} \le X \le -1 + \sqrt{6}$

$\mathbf{[-1 - \sqrt{6}, -1 + \sqrt{6}]}$

Example 7:

$$X^2 + 2X + 1 = 6$$
$$(X + 1)^2 = 6$$
$$X + 1 = \pm \sqrt{6}$$
$$X = -1 \pm \sqrt{6}$$

Extremes means:

$\mathbf{(-\infty, -1 - \sqrt{6}) \cup (-1 + \sqrt{6}, \infty)}$

Finally, what happens if there are not two real endpoints? Perhaps there is only one endpoint, or perhaps, because the endpoints are complex numbers, there are no endpoints. Think about it! If there are no real endpoints or only one endpoint, what could possibly be "betweenness"? Answer: Nothing; empty set. Now picture "extremes" from only one endpoint, or from no endpoints. What would be "extremes"? Answer: Everything; the entire numberline, sometimes including or sometimes excluding the endpoint itself as the particular problem warrants.

EXAMPLE 8

$X^2 + 2X + 5 < 0$ (BETWEENNESS)

$X^2 + 2X + 5 = 0$ (Endpoints!)

(Again, does not factor, so quad formula or complete the square!)

$X^2 + 2X + 1 = -4$

$(X + 1)^2 = -4$

$X + 1 = \pm 2i$

No real endpoints

No Solution -- Φ

EXAMPLE 9

$X^2 + 2X + 5 > 0$ (EXTREMES)

$X^2 + 2X + 5 = 0$ (Endpoints!)

$X^2 + 2X + 1 = -4$

$(X + 1)^2 = -4$

$X + 1 = \pm 2i$

No real endpoints

All Real X -- $(-\infty, \infty)$

EXAMPLE 10

$X^2 + 2X + 1 < 0$ (BETWEENNESS)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Betweenness, nothing--not even the endpoint!

No Solution!

EXAMPLE 11

$X^2 + 2X + 1 \geq 0$ (EXTREMES)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Extremes, include endpoint!

All Reals $(-\infty, \infty)$

EXAMPLE 12

$X^2 + 2X + 1 \leq 0$ (BETWEENNESS)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Betweenness, including the endpoint:

Solution: $X = -1$ only.

EXAMPLE 13

$X^2 + 2X + 1 > 0$ (EXTREMES)

$X^2 + 2X + 1 = 0$ (Endpoints!)

$(X + 1)^2 = 0$

$X = -1$ (Only 1 endpoint!)

Extremes, no endpoint:

$(-\infty, -1) \cup (-1, \infty)$

or **All real $X \neq -1$**

EXERCISES. Solve for X. Give answers in interval notation:

19. $X^2 + 4X + 2 < 0$

20. $X^2 + 4X + 2 > 0$

21. $X^2 - 6X + 4 \geq 0$

22. $X^2 - 6X + 4 \leq 0$

23. $X^2 + 4X + 6 < 0$

24. $X^2 + 4X + 6 \geq 0$

25. $X^2 - 2X + 8 > 0$

26. $X^2 - 2X + 8 \leq 0$

27. $X^2 + 4X + 4 > 0$

28. $X^2 + 4X + 4 < 0$

29. $X^2 - 10X + 25 \le 0$

30. $X^2 - 10X + 25 \ge 0$

31. $X^2 - 8X \ge -16$

32. $X^2 + 9 > 6X$

33. $X^2 + 4 \ge 0$

34. $X^2 + 4X \ge 0$

35. $X^2 - 4 > 0$

36. $X^2 + 4 < 0$

37. $4 - X^2 \geq 0$

38. $X^2 - 4X \leq 0$

39. $X(6 - X) \leq -27$

40. $X(8 - X) > 16$

41. $X(6 - X) \leq -9$

42. $X(6 - X) \leq 9$

23. $4, -1, \frac{3\pm3\sqrt{5}}{2}$; **24.** $-8, 2, -3\pm3\sqrt{2}$; **25.** $-9, 1, -4\pm4\sqrt{2}$.

ANSWERS 1.11

p. 158-159:

1. $[5,\infty)$; **2.** $(-\infty, -4)$; **3.** ϕ; **4.** $(-\infty,\infty)$; **5.** $(-\infty,\infty)$;

6. $(-\infty,-9]$; **7.** $(-6, 12]$; **8.** $(-9,-3)$; **9.** $[-2,4]$; **10.** $[-3,7)$.

p. 161-163:

1. $(-12,4)$; **2.** $(-4,12)$; **3.** $[-2,6]$; **4.** $[-6,2]$; **5.** ϕ; **6.** ϕ;

7. $(-\infty,-12)\cup(4,\infty)$; **8.** $(-\infty,-4)\cup(12,\infty)$; **9.** $(-\infty,-2]\cup[6,\infty)$;

10. $(-\infty,-6]\cup[2,\infty)$; **11.** $(-\infty,\infty)$; **12.** $(-\infty,\infty)$; **13.** $(-4,12)$;

14. $[-2,6]$; **15.** $(-\infty,-2]\cup[6,\infty)$; **16.** $[-4,12]$; **17.** $(-1,7)$;

18. $(-\infty,\infty)$; **19.** $(-\infty,-1)\cup(7,\infty)$; **20.** $(-\infty,0]\cup[8,\infty)$; **21.** $(-5,7)$;

22. $(-\infty,-5)\cup(7,\infty)$; **23.** $(-\infty,-10/3]\cup[2,\infty)$; **24.** $[-10/3,2]$;

25. $(0,12)$; **26.** ϕ; **27.** $(-\infty,0]\cup[12,\infty)$; **28.** $X=1$;

29. All $X\neq1$ or $(-\infty,1)\cup(1,\infty)$; **30.** $(-\infty,\infty)$.

ANSWERS 1.11

p.166-172:

1. $(-1,4)$; 2. $(0,3]$; 3. $[-2,4]$; 4. $[5,7]$; 5. $(-\infty,-1)\cup(4,\infty)$;
6. $(-\infty,0)\cup(3,\infty)$; 7. $(-\infty,-2]\cup[4,\infty)$; 8. $(-\infty,5]\cup[7,\infty)$;
9. $(-1,6)$; 10. $(-\infty,-2)\cup(8,\infty)$; 11. $(-\infty,-2]\cup[8,\infty)$; 12. $[-1,6]$;
13. $(-\infty,-2]\cup[3,\infty)$; 14. $(-\infty,-3)\cup(0,\infty)$; 15. $(0,6)$; 16. $(1,5)$;
17. $(-\infty,-4]\cup[\frac{1}{2},\infty)$; 18. $(3/2,5)$; 19. $(-2-\sqrt{2},-2+\sqrt{2})$;
20. $(-\infty,-2-\sqrt{2})\cup(-2+\sqrt{2},\infty)$; 21. $(-\infty,3-\sqrt{5}]\cup[3+\sqrt{5},\infty)$;
22. $[3-\sqrt{5},3+\sqrt{5}]$; 23. Φ 24. $(-\infty,\infty)$; 25. $(-\infty,\infty)$; 26. Φ
27. All $x\neq-2$ or $(-\infty,-2)\cup(-2,\infty)$; 28. c; 29. $x=5$; 30. $(-\infty,\infty)$;
31. $(-\infty,\infty)$; 32. All $x\neq3$ or $(-\infty,3)\cup(3,\infty)$; 33. $(-\infty,\infty)$;
34. $(-\infty,-4]\cup[0,\infty)$; 35. $(-\infty,-2)\cup(2,\infty)$; 36. Φ 37. $[-2,2]$;
38. $[0,4]$; 39. $(-\infty,-3]\cup[9,\infty)$; 40. Φ
41. $(-\infty,3-3\sqrt{2}]\cup[3+3\sqrt{2},\infty)$; 42. $(-\infty,\infty)$.