2.08 *Algebra of Functions*
Piecewise Functions

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

In the last section, the concepts of **functions** and **functional notation** were introduced. In this section, the use of functional notation continues with the operations of addition, subtraction, multiplication, division, and composition (by substitution) of functions. Remember that the notation **f(X)**, which is read **"f of X"**, is used to name a second variable. Instead of writing **"Y = 3X + 2"**, we sometimes write **"f(X) = 3X + 2"** or **"g(X) = 3X + 2"** or **"Y(X) = 3X + 2"**. Any letter may be used. This notation says that **f** is a **function** of **X**, or that **f** can be expressed in terms of **X**.

Complete the following exercises:

1. $f(X) = -X^2 + 3X + 4$

a) f(0) = ________
b) f(2) = ________ = ________
c) f(-3) = ________ = ________
d) f(π) = ________
e) f(*) = ________
f) f(junk) = ________
g) f(5X) = ________ = ________
h) f(5X-3) = -(   )²+3(   )+ 4 = = =
i) f(3-4X) = = = =

2. $g(X) = -2X^2 - 4X + 10$

a) g(0) = ________
b) g(2) = ________ = ________
c) g(-4) = ________ = ________
d) g(π) = ________
e) g(*) = ________
f) g(junk) = ________
g) g(5X) = ________ = ________
h) g(5X-3) = = = =
i) g(3-4X) = = = =

3. f(X) = 3X+2 and g(X) = -2X+5

a) f[g(X)] = 3(   ) + 2
=
=

b) g[f(X)] = -2(   ) + 5
=
=

c) f[f(X)] = 3(   ) + 2
=
=

d) g[g(X)] =
=
=

4. f(X) = 2X-3 and g(X) = -3X+2

a) f[g(X)] =
=
=

b) g[f(X)] =
=
=

c) f[f(X)] =
=
=

d) g[g(X)] =
=
=

5. $f(X) = \frac{3X-2}{4X}$ and $g(X) = 5X+2$

a) $f[g(X)] = \frac{3(\quad) - 2}{4(\quad)}$
=
=

b) $g[f(X)] = 5(\quad) + 2$
=
=

6. $f(X) = \frac{2X+3}{4X}$ and $g(X) = 5X-2$

a) f[g(X)] =
=
=

b) g[f(X)] =
=
=

7. $f(X) = \frac{2X+3}{4X}$ and $g(X) = X^2-3X+6$

a) f[g(X)] =

=

=

b) g[f(X)] =

=

=

=

c) f[f(X)] =

=

=

=

8. $f(X) = \frac{3X-2}{4X}$ and $g(X) = 5X^2-2X$

a) f[g(X)] =

=

=

b) g[f(X)] =

=

=

=

c) f[f(X)] =

=

=

=

DEFINITIONS:

$(f + g)(X) = f(X) + g(X)$, for all X in the domain of f and g.

$(f - g)(X) = f(X) - g(X)$, for all X in the domain of f and g.

$(f \cdot g)(X) = f(X) \cdot g(X)$, for all X in the domain of f and g.

$(f/g)(X) = f(X)/g(X)$, for all X in the domain of f and g, $g(X) \neq 0$.

$(f \circ g)(X) = f[g(X)]$, for all X in the domain of g, and all g(X) in the domain of f.

$(g \circ f)(X) = g[f(X)]$, for all X in the domain of f, and all f(X) in the domain of g.

The last notations, $f \circ g = f[g(X)]$ and $g \circ f = g[f(X)]$, are called **composite functions** or **composition of functions.**

9. Let $f(X) = 5X + 2$ and $g(X) = X^2 - 2X - 6$

$f(2)$ = ________ $g(2)$ = ________

a) $(f + g)(2) = f(2) + g(2)$
= _____ + _____
= _____

b) $(f - g)(2) = f(2) - g(2)$
= _____ - _____
= _____

c) $(f \cdot g)(2) = f(2) \cdot g(2)$
= _____ · _____
= _____

d) $(f/g)(2) = \frac{f(2)}{g(2)}$
=
=

e) $(f \circ g)(2) = f[g(2)]$
= f[ ]
= ________
= ________

f) $(g \circ f)(2) = g[f(2)]$
= g[ ]
= ________
= ________

g) $(f \circ f)(2) = f[f(2)]$
= f[ ]
= ________
= ________

h) $(g \circ g)(2) = g[g(2)]$
= g[ ]
= ________
= ________

10. Let $f(X) = 2X - 3$ and $g(X) = X^2 - 4X + 3$

$f(5) =$ ________ $g(\quad) =$ ________

a) $(f + g)(5) =$

b) $(f - g)(5) =$

c) $(f \cdot g)(5) =$

d) $(f/g)(5) =$

e) $(f \circ g)(5) =$

f) $(g \circ f)(5) =$

g) $(f \circ f)(5) =$

h) $(g \circ g)(5) =$

11. Let $f(X) = 5X + 6$ and $g(X) = X^2 + 2X - 3$

a) $(f + g)(-3) =$

b) $(f - g)(-3) =$

c) $(f \cdot g)(-3) =$

d) $(f/g)(-3) =$

e) $(f \circ g)(-3) =$

f) $(g \circ f)(-3) =$

g) $(f \circ f)(-3) =$

h) $(g \circ g)(-3) =$

12. Let $f(X) = \frac{3X-4}{X}$ and $g(X) = X^2 + 4X - 8$

$f(-4) =$
$ =$

$g(-4) =$
$ =$

a) $(f + g)(-4) =$

b) $(f - g)(-4) =$

c) $(f \cdot g)(-4) =$

d) $(f/g)(-4) =$

e) $(f \circ g)(-4) =$

f) $(g \circ f)(-4) =$

g) $(g \circ g)(-4) =$

h) $(f \circ f)(-4) =$

13. Let $f(X) = \frac{3X-2}{X}$ and $g(X) = X^2 + 4X - 5$

a) $(f + g)(0) =$

b) $(f - g)(-2) =$

c) $(f \cdot g)(1) =$

d) $(f/g)(1) =$

e) $(f \circ g)(1) =$

f) $(g \circ f)(1) =$

It is possible to use functional notation with a set of points instead of a formula as we have been using. For example, consider:

f(X) = { (1,3), (2,5), (3,0), (4,-2), (5,8) }

This example means that if **X=1**, then **f(X)** is **3**;
if **X=2**, then **f(X)** is **5**;
if **X=3**, then **f(X)** is **0**;
if **X=4**, then **f(X)** is **-2**;
and if **X=5**, then **f(X)** is **8**.

If X is any other value than those given, then f(X) has no assigned value and therefore is undefined. In this example, the **domain (set of all X values) is {1,2,3,4,5}** and the **range (set of all Y values) is {-2,0,3,5,8}**. [Notice that set braces are required!]

WARNING: The following exercises are so simple you may not understand them. If so, please get help quickly.

14. Let f(X) = {(0,6), (1,-2), (2,4), (3,1), (4,2), (5,3)}
g(X) = {(0,1), (1,3), (2,8), (3,5), (4,0), (5,2)}

a) f(0) = _____ b) f(3) = _____ c) g(4) = _____

d) f(5) + g(2) = _____+_____
= __________

e) g(3) - f(1) = __________
= __________

f) (f + g)(3) = f(3) + g(3)
= __________
= __________

g) (f·g)(2) = __________
= __________
= __________

h) (f/g)(2) = __________

i) (f/g)(4) = __________
= __________

j) (f∘g)(4) = f[g(4)]
= f[ ]
= __________

k) (g∘f)(4) = __________
= __________
= __________

l) (f∘g)(5) =

m) (g∘g)(5) =

n) (f∘g)(2) =

o) (f∘f)(1) =

PIECEWISE FUNCTIONS

Frequently, functions are defined in "pieces." For example, a function may be defined by one equation if X is greater than or equal to zero, while the same function is defined by a different equation if X is less than zero. Here is an example:

$$f(X) = \begin{cases} X^2 & \text{if } X \geq 0 \quad \text{(call this category 1)} \\ X - 3 & \text{if } X < 0 \quad \text{(call this category 2)} \end{cases}$$

In this example, to find **f(5)**, you first ask, "Is the value of **X=5** in **category 1** or **category 2**?" Since, 5 is **category 1** (because the value of X is greater than or equal to zero), you must use $f(X)=X^2$ to calculate $f(5) = 5^2 = 25$. To find **f(-5)**, since **X=-5** is in **category 2**, you must use **f(X)=X-3** to calculate **f(-5) = -5-3 = -8.** Notice that the "category" is determined only by the value of X that is given to you.

EXERCISES:

1. $$f(X) = \begin{cases} X^2 & \text{if } X \geq 0 \quad \text{(call this category 1)} \\ X - 3 & \text{if } X < 0 \quad \text{(call this category 2)} \end{cases}$$

a) f(2) = _____ (category 1) **b) f(-2)** = _____ (category 2)
= _____ = _____

c) f(-8) = _____ (category ____) **d) f(8)** = _____ (category ____)
= _____ = _____

e) f(0) = _____ (category ____) **f) f(-25)** = _____ (category ____)

2. $$f(X) = \begin{cases} 2X - 7 & \text{if } X \geq 0 \quad \text{(call this category 1)} \\ 3X + 2 & \text{if } X < 0 \quad \text{(call this category 2)} \end{cases}$$

a) f(2) = _____ (category ____) **b) f(-2)** = _____ (category ____)
= _____ = _____

c) f(-8) = _____ (category ____) **d) f(8)** = _____ (category ____)
= _____ = _____

e) f(0) = _____ (category ____) **f) f(-25)** = _____ (category ____)
= _____

3.

$$f(X) = \begin{cases} 2X & \text{if } X \le -3 \quad \text{(call this \textbf{category 1})} \\ 3X & \text{if } -3 < X < 2 \quad \text{(call this \textbf{category 2})} \\ 4X & \text{if } X \ge 2 \quad \text{(call this \textbf{category 3})} \end{cases}$$

a) f(2) = _____ (category 3) **b) f(-2) =** _____ (category 2)

= _____ = _____

c) f(-8) = _____ (category ____) **d) f(8) =** _____ (category ____)

= _____ = _____

e) f(0) = _____ (category ____) **f) f(-25) =** _____ (category ____)

= _____

4.

$$f(X) = \begin{cases} X^2 & \text{if } X \le -2 \quad \text{(call this \textbf{category 1})} \\ 3 & \text{if } -2 < X \le 0 \quad \text{(call this \textbf{category 2})} \\ -2X & \text{if } X > 0 \quad \text{(call this \textbf{category 3})} \end{cases}$$

a) f(2) = _____ (category 3) **b) f(-2) =** _____ (category 1)

= _____ = _____

c) f(-8) = _____ (category ____) **d) f(8) =** _____ (category ____)

= _____ = _____

e) f(0) = _____ (category ____) **f) f(-13) =** _____ (category ____)

= _____

5.

$$f(X) = \begin{cases} -X+5 & \text{if } X < -3 \quad \text{(call this \textbf{category 1})} \\ X^2 & \text{if } -3 \le X \le 2 \quad \text{(call this \textbf{category 2})} \\ \sqrt{X} & \text{if } X > 2 \quad \text{(call this \textbf{category 3})} \end{cases}$$

a) f(2) = **b) f(-2) =**

c) f(-8) = **d) f(8) =**

e) f(0) = **f) f(-3) =**

6.

$$f(X) = \begin{cases} X + 5 & \text{if } X \le -3 \\ 3X - 6 & \text{if } -3 < X < 2 \\ -4X + 8 & \text{if } X \ge 2 \end{cases}$$

a) f(2) =

b) f(-2) =

c) f(-3) =

d) f(8) =

e) f(0) =

f) f(-10) =

7.

$$f(X) = \begin{cases} X^2 + 1 & \text{if } X < -3 \\ 6 & \text{if } -3 \le X \le 2 \\ -X + 5 & \text{if } X > 2 \end{cases}$$

a) f(2) =

b) f(6) =

c) f(-3) =

d) f(16) =

e) f(0) =

f) f(-7) =

8.

$$f(X) = \begin{cases} -X^2 + 5 & \text{if } X < -3 \\ 6 - 5X & \text{if } -3 \le X < 2 \\ -3 - 2X & \text{if } X \ge 2 \end{cases}$$

a) f(2) =

b) f(-3) =

c) f(-5) =

d) f(8) =

e) f(0) =

f) f(-8) =

ANSWERS 2.08

p. 296-302:

1a) 4; b) 6; c) -14; d) $-\pi^2+3\pi+4$; e) $-*^2+3*+4$; f) $-(\text{Junk})^2+3(\text{Junk})+4$; g) $-25X^2+15X+4$; h) $-25X^2+45X-14$; i) $-16X^2+12X+4$; **2**a) 10; b) -6; c) -6; d) $-2\pi^2-4\pi+10$; e) $-2*^2-4*+10$; f) $-2(\text{Junk})^2-4(\text{Junk})+10$; g) $-50X^2-20X+10$; h) $-50X^2+40X+4$; i) $-32X^2+64X-20$; **3**a) $-6X+17$; b) $-6X+1$; c) $9X+8$; d) $4X-5$; **4**a) $-6X+1$; b) $-6X+11$; c) $4X-9$; d) $9X-4$

5a) $\dfrac{15X+4}{4(5X+2)}$; b) $\dfrac{23X-10}{4X}$; **6**a) $\dfrac{10X-1}{4(5X-2)}$; b) $\dfrac{2X+15}{4X}$;

7a) $\dfrac{2X^2-6X+15}{4(X^2-3X+6)}$; b) $\dfrac{76X^2-24X+9}{16X^2}$; c) $\dfrac{8X+3}{2(2X+3)}$;

8a) $\dfrac{15X^2-6X-2}{4X(5X-2)}$; b) $\dfrac{21X^2-44X+20}{16X^2}$; c) $\dfrac{X-6}{4(3X-2)}$;

9. f(2)=12; g(2)=-6; a) 6; b) 18; c) -72; d) -2; e) -28; f) 114; g) 62; h) 42; **10**. f(5)=7; g(5)=8; a) 15; b) -1; c) 56; d) 7/8; e) 13; f) 24; g) 11; h) 35; **11**. f(-3)=-9; g(-3)=0; a) -9; b) -9; c) 0; d) Undefined; e) 6; f) 60; g) -39; h) -3; **12**. f(-4)=4; g(-4)=-8; a) -4; b) 12; c) -32; d) -1/2; e) 7/2; f) 24; g) 24; h) 2; **13**a) Undefined; b) 13; c) 0; d) Undefined; e) Undefined; f) 0; **14**a) 6; b) 1; c) 0; d) 11; e) 7; f) 6; g) 32; h) 1/2; i) Undefined; j) 6; k) 8; l) 4; m) 8; n) Undef; o) Undefined.

p. 303-305:

1a) 4; b) -5; c) -11; d) 64; e) 0; f) -28; **2**a) -3; b) -4; c) -22; d) 9; e) -7; f) -73; **3**a) 8; b) -6; c) -16; d) 32; e) 0; f) -50; **4**a) -4; b) 4; c) 64; d) -16; e) 3; f) 169; **5**a) 4; b) 4; c) 13; d) $2\sqrt{2}$; e) 0; f) 9; **6**a) 0; b) -12; c) 2; d) -24; e) -6; f) -5; **7**a) 6; b) -1; c) 6; d) -11; e) 6; f) 50; **8**a) -7; b) 21; c) -20; d) -19; e) 6; f) -59.

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