2.04 *The Circle*

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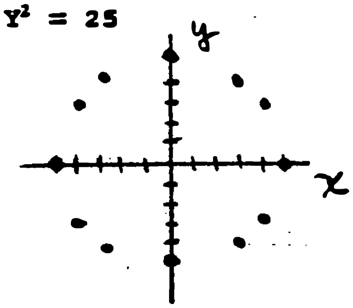
ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

In previous sections of this chapter, you graphed equations involving X^2 or Y^2 (**parabolas!**), but none of the equations in these previous sections has involved both X^2 and Y^2 in the same equation. Equations that include both X^2 and Y^2 terms usually fall into three categories: the **circle** (for example, $\mathbf{X^2 + Y^2 = 16}$), the **ellipse** (for example, $\mathbf{4X^2 + Y^2 = 16}$), and the **hyperbola** (for example, $\mathbf{4X^2 - Y^2 = 16}$). In this section and the next, you will discover ways to identify and graph these types of equations.

Complete the following tables and sketch the graphs for the equations. Remember for example, that if $Y^2 = 9$, then there are two solutions for Y: $Y = \pm 3$. Also, you may need to approximate values when necessary.

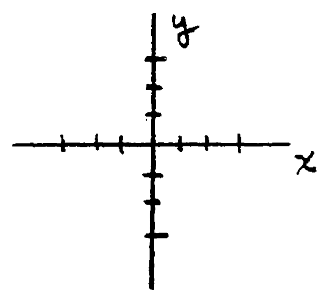
1. $\mathbf{X^2 + Y^2 = 25}$

X	Y
0	±5
3	±
4	
5	
-3	
-4	
-5	



2. $\mathbf{X^2 + Y^2 = 9}$

X	Y
0	
1	$\pm\sqrt{8} \approx 2.8$
3	
-1	
-3	



In 3-8, identify the center and radius of the circles:

3. $\mathbf{X^2 + Y^2 = 4}$

4. $\mathbf{X^2 + Y^2 = 1}$

5. $\mathbf{X^2 + Y^2 = 49}$

6. $\mathbf{X^2 + Y^2 = 12}$

7. $\mathbf{X^2 + Y^2 = 72}$

8. $\mathbf{X^2 + Y^2 = 0.25}$

Recall from the section on parabolas, that an equation such as $Y = (X - h)^2$ is the same shape as the graph $Y = X^2$, but shifted "h" units to the right, and $X = (Y - k)^2$ is the same shape as the graph $X = Y^2$ shifted up "k" units. How do you think the graph of the equation $(X - h)^2 + (Y - k)^2 = r^2$ compares to the graph of $X^2 + Y^2 = r^2$? (Answer: circle of same radius **r**, shifted to the right **h** units and up **k** units.) Formal proof of this result is not difficult.

THEOREM: A circle of radius "r" with center at (h, k) has the equation $(X - h)^2 + (Y - k)^2 = r^2$.

PROOF: Let (h, k) be a fixed point, the center of the circle, and let (X, Y) be any variable point on the circle. According to the distance formula (or the Theorem of Pythagoras) the distance between the points (h, k) and (X, Y) is the radius of the circle "r", where

$$r = \sqrt{(X-h)^2 + (Y-k)^2}.$$

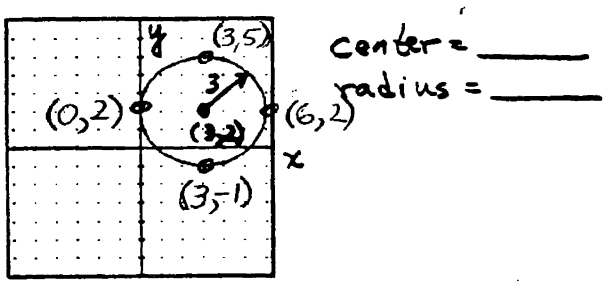
Now, squaring both sides of the equation, you have

$$r^2 = (X-h)^2 + (Y-k)^2.$$

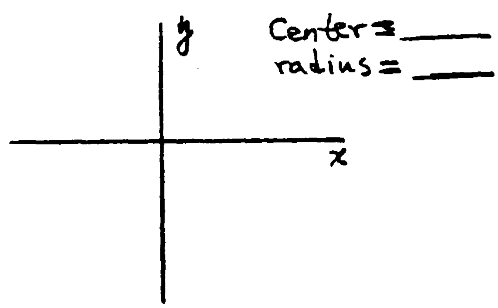
NOTE: For purposes of this text, $(X - h)^2 + (Y - k)^2 = r^2$ will be called **standard form** for the equation of a circle.

In 9-14, find the center and radius, and sketch the graph of the circles. Find the endpoints of vertical and horizontal diameters.

9. $(X-3)^2 + (Y-2)^2 = 9$



10. $(X-3)^2 + (Y+2)^2 = 1$



11. $(X+3)^2 + (Y-2)^2 = 9$

12. $(X+3)^2 + (Y+2)^2 = 25$

13. $X^2 + (Y-2)^2 = 36$

14. $(X+3)^2 + Y^2 = 9$

In the graphing of parabolas, it was often necessary to complete the square to express the equation in standard form to find the vertex. So also in the graphing of circles, it will be necessary to complete the square to express the equation in standard form to find the center and radius.

Consider the equation $(X-3)^2 + (Y+4)^2 = 49$, which represents a circle with **radius 7** and **center** at **(3,-4)**.

The equation:	$(X-3)^2 + (Y+4)^2 = 49$
can be expanded:	$X^2-6X+9 + Y^2+8Y+16 = 49$
and simplified:	$X^2+Y^2 - 6X + 8Y + 25 = 49$
and rewritten as:	$X^2+Y^2 - 6X + 8Y - 24 = 0$

Now suppose the equation had been given in the latter form: $X^2+Y^2 - 6X + 8Y - 24 = 0$. The problem is to reverse the steps (by completing the square) and return to the original form to determine the center and radius. You should organize the problem as follows:

$X^2 + Y^2 - 6X + 8Y - 24 = 0$

$X^2 - 6X + \_\_\_ + Y^2 + 8Y + \_\_\_ = 24 + \_\_\_ + \_\_\_$ Collect X and Y terms. Leave the spaces to complete the square.

[COMPLETING THE SQUARE: HALF AND SQUARE]

$X^2 - 6X + \underline{9} + Y^2 + 8Y + \underline{16} = 24 + \underline{9} + \underline{16}$ Add the appropriate numbers to each side of the equation.

$(X-3)^2 + (Y+4)^2 = 49$

Therefore, the center is at **(3,-4)** and the radius is **7**.

In 15-30, find the center and radius of given circles as in the previous example.

15. $X^2 + Y^2 - 4X + 4Y = 8$

$X^2 - 4X + \quad + Y^2 + 4Y + \quad = 8$

$X^2 - 4X + \_\_\_ + Y^2 + 4Y + \_\_\_ = 8 + \_\_\_ + \_\_\_$

$(\quad)^2 + (\quad)^2 = \_\_\_\_$

Center (,); radius = ____

16. $X^2 + Y^2 - 4X + 4Y = 41$

17. $X^2 + Y^2 + 8X - 6Y = 0$

18. $X^2 + Y^2 + 8X - 6Y = 39$

19. $X^2 + Y^2 - 12X + 8Y - 48 = 0$

20. $X^2 + Y^2 + 4X - 10Y - 7 = 0$

21. $X^2 + Y^2 + 18X - 6Y + 74 = 0$

22. $X^2 + Y^2 - 8X - 6Y + 21 = 0$

23. $X^2 + Y^2 - 10X + 8Y = 9$

24. $X^2 + Y^2 - 14X + 14Y = 0$

25. $X^2 + Y^2 - 3X + 8Y = -12$
[Hint: Fractions necessary]

26. $2X^2 + 2Y^2 + 4X - 3Y = 3$
[Hint: Divide both sides of equation by 2]

27. $2X^2 + 2Y^2 - 10X - 14Y = 13$

28. $4X^2 + 4Y^2 - 20X - 12Y = -25$

29. $4X^2 + 4Y^2 + 12X - 12Y + 9 = 0$

30. $4X^2 + 4Y^2 - 20X + 4Y = -24$

In 31-34, discover by completing the square how these equations differ from the previous exercises. How are the graphs affected?

31. $X^2 + Y^2 - 12X + 8Y + 52 = 0$

32. $X^2 + Y^2 + 4X - 10Y = -33$

33. $X^2 + Y^2 + 18X - 6Y + 99 = 0$

34. $X^2 + Y^2 - 8X - 6Y + 25 = 0$

In 35-46, find the equation of the circle [give standard form]:

35. with center at (4, -2) and radius 8.

36. with center at (-3, 6) and radius 9.

37. with center at (-4, -6) and radius $\frac{3}{4}$.

38. with center at (3, 8) and radius $\sqrt{5}$.

39. with center at (4, -2) and passing through (1, 2).

40. with center at (-3, 6) and passing through (5, 0).

[HINT: use distance formula to find the radius of the circles].

41. with center at (-4, -6) and passing through (0, -2).

42. with center at (8, -3) and passing through (2, 9).

43. **with endpoints of diameter at (0, 0) and (6, 8).**

44. **with endpoints of diameter at (4, 2) and (10, 2).**

[HINT: Use midpoint formula to find the center of the circle, then find the radius by finding the distance from the center to either endpoint.]

45. **with endpoints of diameter at (-4, 6) and (8, -8).**

46. **with endpoints of diameter at (-1, -2) and (-11, 22).**

EXTRA CHALLENGE:

47. **with center at (2, 13) and tangent to (touching) the line 3X + 2Y = 6.**

48. **with center at (16, 5) and tangent to the line 3X + 2Y = 6.**

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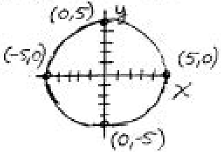
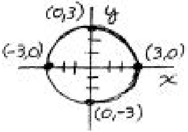
1. $x^2+y^2=25$



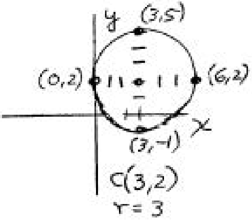
2. $x^2+y^2=9$

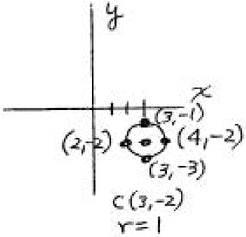


3. $C(0,0)$ $r=2$; 4. $C(0,0)$ $r=1$; 5. $C(0,0)$ $r=7$;

6. $C(0,0)$ $r=2\sqrt{3}$; 7. $C(0,0)$ $r=6\sqrt{2}$; 8. $C(0,0)$ $r=\frac{1}{2}$;

9. $(x-3)^2+(y-2)^2=9$ 10. $(x-3)^2+(y+2)^2=1$





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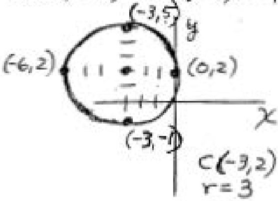
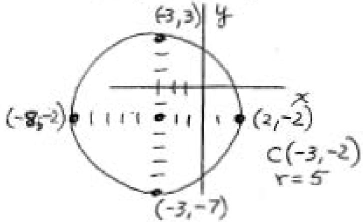
11. $(x+3)^2+(y-2)^2=9$



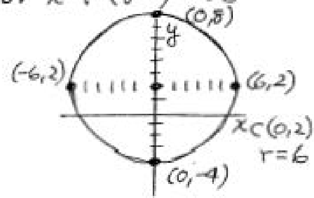
$C(-3,2)$
$r=3$

12. $(x+3)^2+(y+2)^2=25$



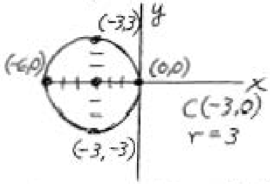
$C(-3,-2)$
$r=5$

13. $x^2+(y-2)^2=36$



$C(0,2)$
$r=6$

14. $(x+3)^2+y^2=9$



$C(-3,0)$
$r=3$

15. $C(2,-2)$ $r=4$; 16. $C(2,-2)$ $r=7$; 17. $C(-4,3)$ $r=5$;
18. $C(-4,3)$ $r=8$; 19. $C(6,-4)$ $r=10$; 20. $C(-2,5)$ $r=6$;
21. $C(-9,3)$ $r=4$; 22. $C(4,3)$ $r=2$; 23. $C(5,-4)$ $r=5\sqrt{2}$;
24. $C(7,-7)$ $r=7\sqrt{2}$; 25. $C(\frac{3}{2},-4)$ $r=\frac{5}{2}$; 26. $C(-1,\frac{3}{4})$ $r=\frac{7}{4}$;
27. $C(\frac{3}{2},\frac{7}{2})$ $r=5$; 28. $C(\frac{5}{2},\frac{3}{2})$ $r=\frac{3}{2}$; 29. $C(-\frac{3}{2},\frac{3}{2})$ $r=\frac{3}{2}$;
30. $C(\frac{1}{2},-\frac{1}{2})$ $r=\frac{\sqrt{2}}{2}$; 31. $(r=0)$ Point $(6,-4)$; 32. No Solution;
33. No Solution; 34. Point $(4,3)$; 35. $(x-4)^2+(y+2)^2=64$;
36. $(x+3)^2+(y-6)^2=81$; 37. $(x+4)^2+(y+6)^2=\frac{9}{16}$; 38. $(x-3)^2+(y-8)^2=5$;
39. $(x-4)^2+(y+2)^2=25$; 40. $(x+3)^2+(y-6)^2=100$; 41. $(x+4)^2+(y+6)^2=32$;
42. $(x-8)^2+(y+3)^2=180$; 43. $(x-3)^2+(y-4)^2=25$; 44. $(x-7)^2+(y-2)^2=9$;
45. $(x-2)^2+(y+1)^2=85$; 46. $(x+6)^2+(y-10)^2=169$;
47. $(x-2)^2+(y-13)^2=52$; 48. $(x-16)^2+(y-5)^2=208$.