2.09 *Symmetry, Even/Odd, Increasing/Decreasing Functions*

Dr. Robert J. Rapalje

In the study of functions and graphs, you have already looked at a variety of different graphs, such as the parabola, the circle, and many others, which may or not be functions. You can identify which equations (or graphs) represent functions and which do not represent functions. You have studied the concepts of domain and range, which are ways of describing horizontal and vertical limitations of the graph. In addition to these concepts that you have already studied, there are many other definitions that are useful in describing the behavior of functions and graphs. These are especially useful (perhaps "necessary" is a better word!) in higher mathematics. A few of these concepts are presented in this section: **symmetry, even or odd functions, increasing or decreasing functions,** and finally, an informal presentation of **continuity.** Most of these will be presented **first** from the perspective of the **graph**, and **second**ly from the perspective of the **function** or the **equation.**

SYMMETRY

A dictionary definition of "symmetry" is "the exact correspondence of form and constituent configuration on opposite sides of a dividing line or plane or about a center or axis." As this relates to a graph, what it means is that **the graph "behaves" on one side of a line or point in exactly the same way as on the other side.** That is, one side of the graph is the "mirror image" of the other side of the graph. Although symmetry could be defined about any given line or point, there are four (4) main types of symmetry that will be considered here:

symmetry about the Y-axis,
symmetry about the X-axis,
symmetry about the origin, and
symmetry about the line Y = X.

From the perspective of the graph, **symmetry about the Y-axis** means that the portion of the graph to the left of the Y-axis is the mirror image of portion of the graph to the right of the Y-axis. If (X,Y) is a point on the graph, then (-X,Y) will also be a point on the graph. As examples, $\mathbf{Y=X^2}$ (parabola opening upwards) and $\mathbf{Y=-X^2}$ (parabola opening downwards) are **symmetric to the Y-axis**. (See figure the top of the next page.)

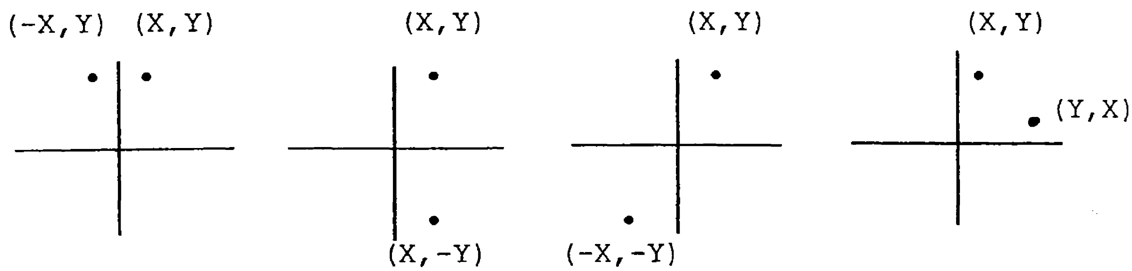
Symmetry about the X-axis means that the portion of the graph that is below the X-axis is the mirror image of the portion of the graph that is above the X-axis. If (X,Y) is a point on the graph, then (X,-Y) will also be a point on the graph. As examples, $\mathbf{X=Y^2}$ (parabola that opens to the right) and $\mathbf{X=-Y^2}$ (parabola opening to the left) are **symmetric to the X-axis**. (See figure on next page.)

Symmetry about the origin (see figure on next page) means that if (X,Y) is a point on the graph, then (-X,-Y) will also be a point on the graph. As examples, $\mathbf{Y=X^3}$ or $\mathbf{Y=-X^3}$ are symmetric to the origin.

Concerning these three symmetries, some graphs will have **no symmetry**, some graphs will have **only one symmetry**, and some graphs (for example, $\mathbf{X^2+Y^2 = 9}$) will have **all three symmetries**. It is NOT possible for a graph or function to have two symmetries, without the third also. Re-stated, **two symmetries implies the third.**

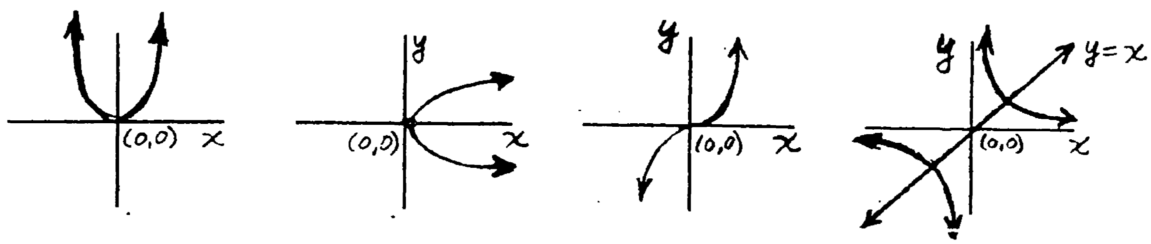
For **symmetry about the line Y=X** (see figure on next page), notice that the graph on one side of the line Y=X (a 45° line) is be the reflection (mirror image) of the graph on the other side. If (X,Y) represents a point on the graph, then (Y,X) [interchanging the variables X and Y] is also a point on the graph. As an example, $Y = \frac{12}{X}$ is symmetric about the line **Y=X**.

Point Symmetry



Y-axis Symmetry | **X-axis Symmetry** | **Origin Symmetry** | **Y = X Symmetry**

Graph Symmetry



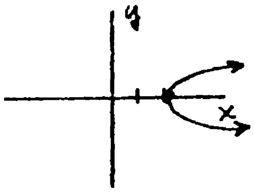
Y-axis Symmetry | **X-axis Symmetry** | **Origin Symmetry** | **Y = X Symmetry** (Also, origin)

Name the point that is symmetric to each of following points:

	1. (3,4)	**2. (0,6)**	**3. (-4,12)**	**4. (-5,-9)**
Y-axis:	**a)** ______	**a)** (0,6)	**a)** ______	**a)** ______
X-axis	**b)** ______	**b)** ______	**b)** (-4,-12)	**b)** ______
Origin	**c)** ______	**c)** ______	**c)** ______	**c)** (5,9)
Y=X	**d)** (4,3)	**d)** ______	**d)** ______	**d)** ______

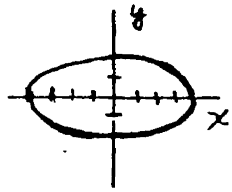
Identify the symmetries for each of the following graphs:

1.



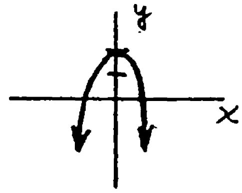
Sym:______________

2.



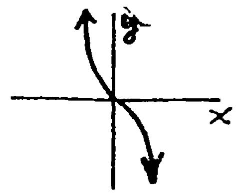
Sym:______________

3.



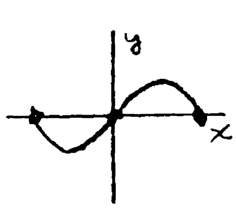
Sym:______________

4.



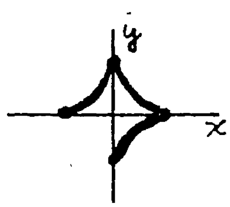
Sym:______________

5.



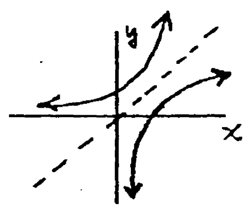
Sym:______________

6.



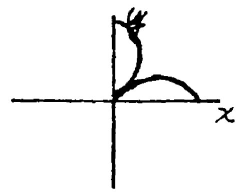
Sym:______________

7.



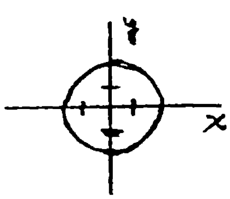
Sym:______________

8.



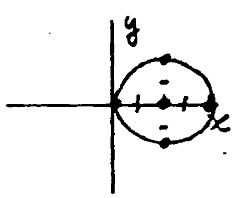
Sym:______________

9.



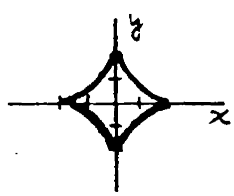
Sym:______________

10.



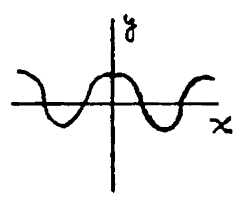
Sym:______________

11.



Sym:______________

12.

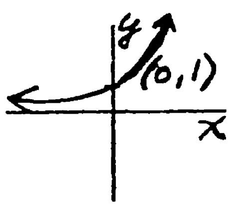


Sym:______________

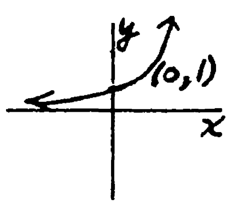
Complete the following so each graph will have symmetry about:

Y-axis	X-axis	Origin	Line Y = X

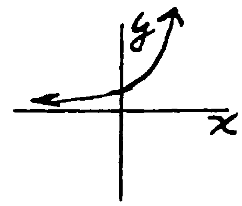
13a)



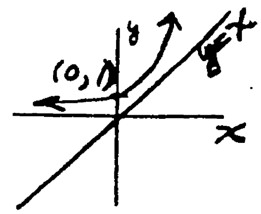
b)



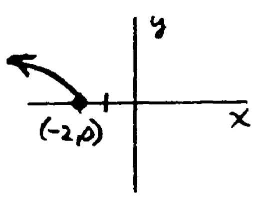
c)



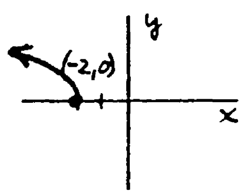
d)



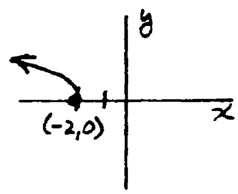
14a)



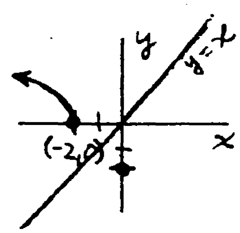
b)



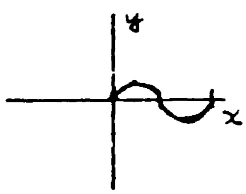
c)



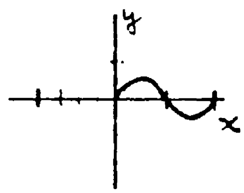
d)



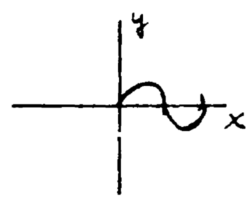
15a)



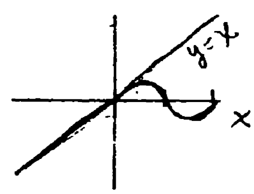
b)



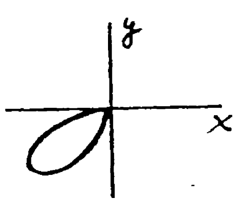
c)



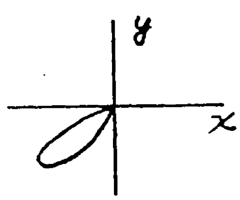
d)



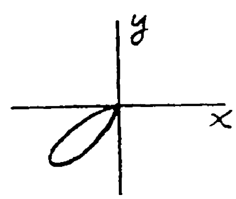
16a)



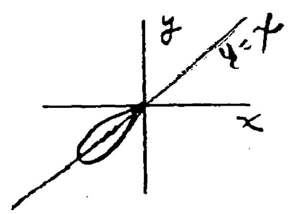
b)



c)

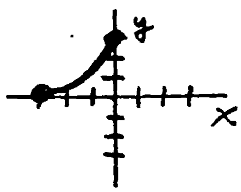


d)

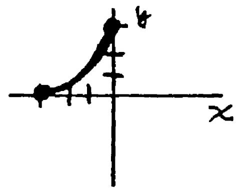


Y-axis	X-axis	Origin	Line Y = X

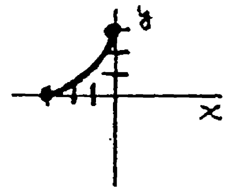
17a)



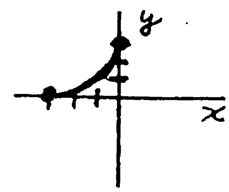
b)



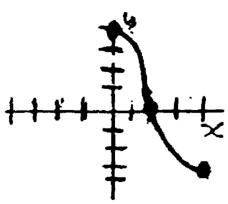
c)



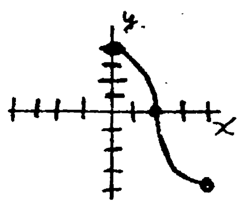
d)



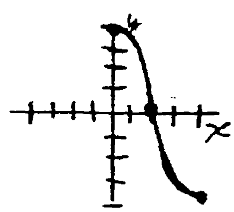
18a)



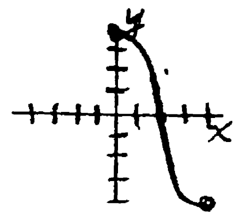
b)



c)

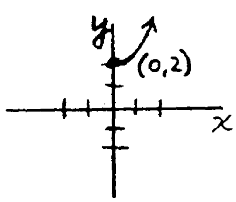


d)

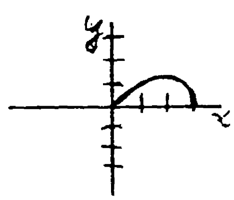


Complete each of the following so each will have symmetry about the X-axis, Y-axis, and (of course!) the origin.

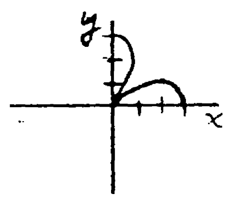
19a)



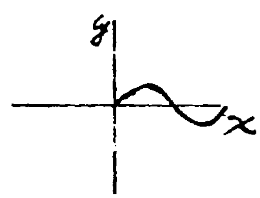
b)



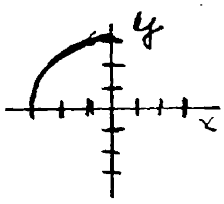
c)



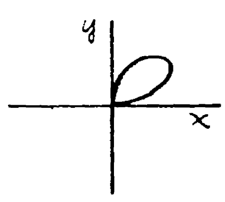
d)



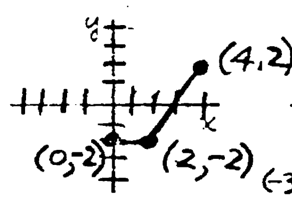
20a)



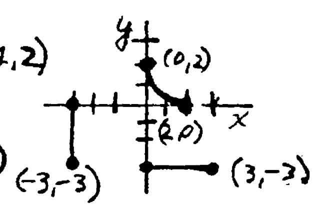
b)



c)



d)



Symmetry can also be determined by looking at the **equation** of the function or relation. **Symmetry about the Y-axis** means that the graph on the left side of the Y-axis (i.e., for negative values of X), is a reflection of the graph on the right side of the Y-axis (i.e., for positive values of X!). Therefore, if the substitution of "(-X)" for each "X" in the equation results in an equation that is equivalent to the original equation, then the graph is symmetric about the Y-axis.

Similarly, **symmetry about the X-axis** means that the graph below the X-axis (i.e., for negative values of Y) is a reflection of the graph above the X-axis (i.e., for positive values of Y). Therefore, the test for symmetry about the X-axis is to substitute "(-Y)" for each "Y" in the equation. If this substitution results in an equation that is equivalent to the original equation, then it is symmetric to the Y-axis.

To test for **symmetry about the origin**, substitute "(-X)" for each "X" and "(-Y)" for each "Y". If the resulting equation is equivalent to the original equation, then there is symmetry about the origin.

To test for **symmetry about the line "Y = X"**, interchange the X and Y variables. If the resulting equation is equivalent to the original equation, then there is symmetry about the line Y = X.

SYMMETRY SUMMARY

1. **Y-axis--replace X with -X**
2. **X-axis--replace Y with -Y**
3. **Origin--replace X with -X, and Y with -Y**
4. **Y = X --interchange the X and Y variables**

EXAMPLE: Show that $Y = X^4 - 3X^2 + 5$ is symmetric about the **Y-axis** and <u>not</u> symmetric about the **X-axis.**

SOLUTION: **Y-axis.** (Substitute -X for X in the original equation.)
$Y = (-X)^4 - 3(-X)^2 + 5$
$Y = X^4 - 3X^2 + 5$ (Same as the original equation.)
Therefore, the graph is symmetric to Y-axis.

X-axis. (Substitute -Y for Y in the original equation.)
$(-Y) = X^4 - 3X^2 + 5$
$Y = -X^4 + 3X^2 - 5$ (Not same as original equation.)
Therefore, the graph is <u>not</u> symmetric to X-axis.

EXAMPLE: Show that $Y = X^3 - 3X$ is symmetric about the **origin** and <u>not</u> symmetric to the line **Y = X.**

SOLUTION: **Origin.** (Substitute -X for X and -Y for Y.)
$(-Y) = (-X)^3 - 3(-X)$
$-Y = -X^3 + 3X$
$Y = X^3 - 3X$ (Same as the original equation.)
Therefore, the graph is symmetric to the origin.

Y = X. (Interchange the X and Y variables.)
$X = Y^3 - 3Y$ (Not same as original equation.)
Therefore, the graph is <u>not</u> symmetric to the line Y = X.

EXERCISES: **In each of the following, perform the indicated symmetry tests.**

1. **Test $Y = X^3$ for symmetry about Y-axis and origin.**

2. **Test $Y = -X^4$ for symmetry about Y-axis and origin.**

3. **Test $X^2-Y^2 = 9$ for symmetry about X-axis and line Y=X.**

4. **Test $X^3+Y^3 = 27$ for symmetry about X-axis and line Y=X.**

Test each of the following equations for symmetry about the Y-axis, the X-axis, and the origin. [Remember that two symmetries automatically implies the third.]

5. $Y = X^5 - X^3$

6. $Y = X^4 + 3X^2 - 6$

7. $Y = X^3 + 6$

8. $Y = X^3 - X^2$

9. $X = Y^2 + 3Y^4$

10. $X = Y^3 - 4Y^5$

11. $X^2 + Y^2 = 9$

12. $X^3 - Y^3 = 27$

13. $Y^2 - X^3 = 64XY$

14. $Y^2 - X^2 = 64$

15. $X^2 + XY + Y^2 = 64$

16. $X^2 - Y^2 = 64$

17a) Is it possible for a "function" to have symmetry about the X-axis? If so, give an example. If not, explain why not.

b) Give symmetry for the function $Y = f(X) = aX^m$, where m is an even integer.

c) Give symmetry for the function $Y = f(X) = bX^n$, where n is an odd integer.

d) Describe the symmetry for the function $Y = f(X) = aX^m + bX^m$, where m is an even and n is an odd integer.

EVEN/ODD FUNCTIONS
INCREASING/DECREASING FUNCTIONS

The study of symmetry in the previous pages of this section involved **relations** that included both **functions** and **non-functions.** The rest of this section will involve **functions only.** In the exercises on symmetry, especially #17, perhaps you noticed that if Y is a function in which the power of X in each term of the function is raised to an **even power**, then the function has **symmetry about the Y-axis.** Likewise, if Y is a function in which each term of the function is raised to an **odd power**, then the function has **symmetry about the origin.** [It is true that *functions* **will not usually have symmetry about the X-axis!**] It is natural to describe these as **even and odd functions** respectively. Question for thought: Does $\mathbf{Y = X^3 + 3}$ represent an odd function? Answer: No, since the equation may be written $\mathbf{Y = X^3 + 3X^0}$, it is neither even nor odd.

The formal definition is much more inclusive, and it goes far beyond simply describing the "powers of X." The formal definition of even and odd functions is as follows:

DEFINITION EVEN/ODD FUNCTIONS

1. An **even function f(X)** is a function such that for every value of X, $\mathbf{f(-X) = f(X)}$ for every X in the domain of f(X). **[Symmetry about the Y-axis!]**

2. An **odd function f(X)** is a function such that for every value of X, $\mathbf{f(-X) = -f(X)}$ for every X in the domain of f(X). **[Symmetry about the origin!]**

Notice that in answering questions about even and odd functions, you always go back to the definition which involves the values of **f(-X)**. Notice that if, for positive values of X, **f(X)** describes the function to the **right of the Y-axis**, then **f(-X)** describes the function to the **left of the Y-axis.** Therefore, **if** f(-X) = f(X), then there is symmetry about the Y-axis, and if f(-X) = -f(X), then there is an *"inverted symmetry" about the Y-axis*, i.e., symmetry about the origin.

In each of the following, determine whether the function is an even function, an odd function, or neither.

1. $f(X) = X^6 + 5X^2$

2. $f(X) = X^5 - 6X$

3. $f(X) = X^6 - 6X$

4. $f(X) = X^5 + 5X^2$

5. $f(X) = 6X$

6. $f(X) = 5X^2 - 7X^4$

7. $f(X) = 3$

8. $f(X) = 6$

9. $f(X) = X^4 + 6$

10. $f(X) = X^3 - 6$

11. $f(X) = X^5 + 5$

12. $f(X) = X^5 + 5X$

Another way of describing the behavior of functions and their graphs is to indicate the intervals on which the functions are **increasing** or **decreasing**. The following is a formal definition of an **increasing** or **decreasing function:**

DEFINITION INCREASING/DECREASING FUNCTIONS

1. A function f(X) is said to be increasing on an open interval (a,b) if $f(X_2) > f(X_1)$ whenever $X_2 > X_1$, for X_1 and X_2 in (a,b).

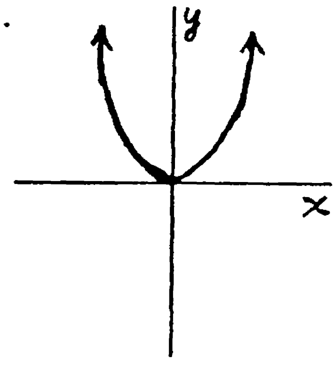
   [That is, the **function values increase as you move from left to right on the graph!**]

2. A function f(X) is said to be decreasing on an open interval (a,b) if $f(X_2) < f(X_1)$ whenever $X_2 > X_1$, for X_1 and X_2 in (a,b).

   [That is, the **function values decrease as you move from left to right on the graph!**]

EXAMPLES: By symmetry, indicate which of the following graphs represent even functions, odd functions, or neither. Give intervals in which the functions are increasing or decreasing (use interval notation).

I.



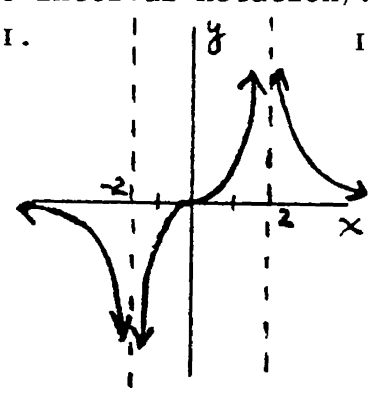
Symmetry? y-axis

Even/Odd/Neither

Incr? $(0, \infty)$

Decr? $(-\infty, 0)$

II.



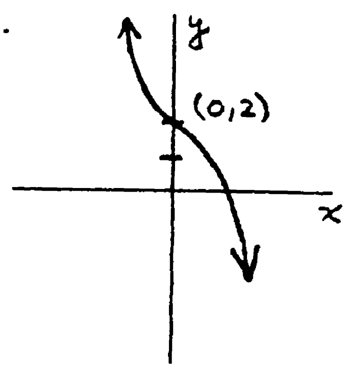
Symmetry? Origin

Even/**Odd**/Neither

Incr? $(-2, 2)$

Decr? $(-\infty, -2)$; $(2, \infty)$

III.



Symmetry? None

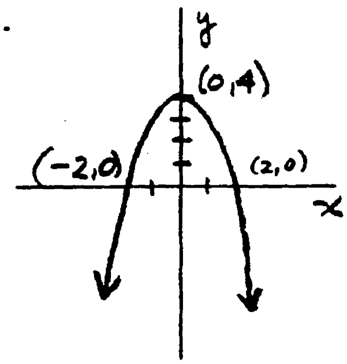
Even/Odd/**Neither**

Incr? $\emptyset$

Decr? $(-\infty, \infty)$

As in the examples on the previous page, determine which of the following graphs represent even functions, odd functions, or neither. Give the intervals in which the functions are increasing or decreasing.

1.



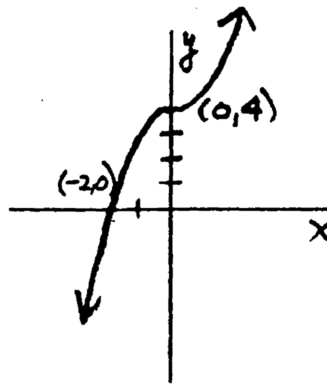
Symmetry? __________

Even/Odd/Neither

Incr? ______________

Decr? ______________

2.



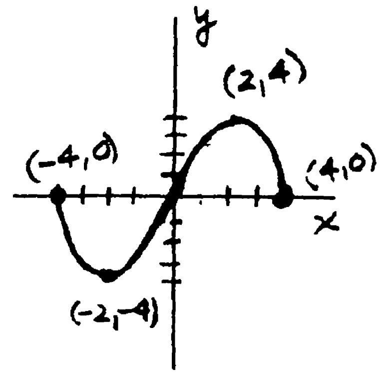
Symmetry? __________

Even/Odd/Neither

Incr? ______________

Decr? ______________

3.



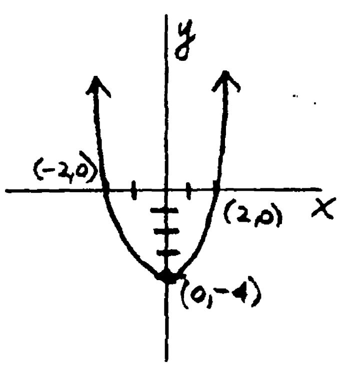
Symmetry? __________

Even/Odd/Neither

Incr? ______________

Decr? ______________

4.



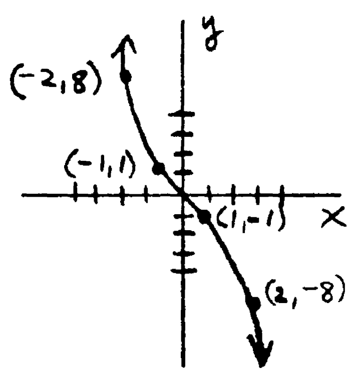
Symmetry? __________

Even/Odd/Neither

Incr? ______________

Decr? ______________

5.



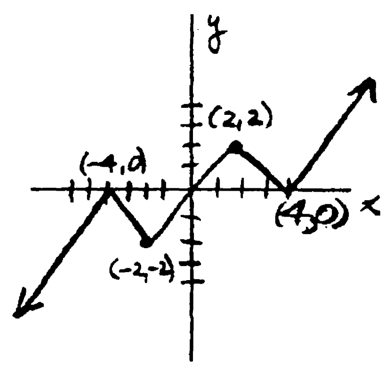
Symmetry? __________

Even/Odd/Neither

Incr? ______________

Decr? ______________

6.



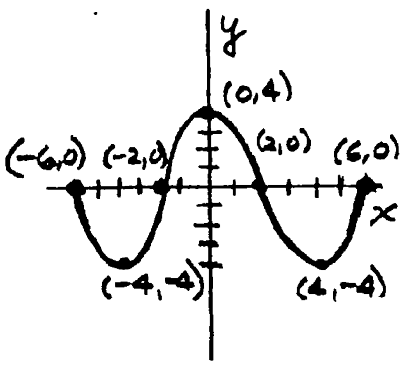
Symmetry? __________

Even/Odd/Neither

Incr? ______________

Decr? ______________

7.



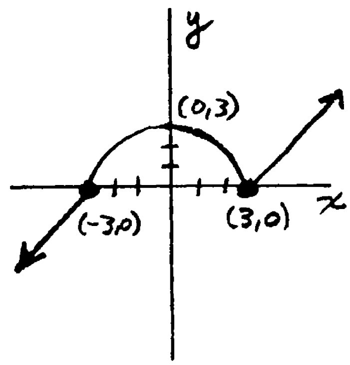
Symmetry? ____________

Even/Odd/Neither

Incr? ____________

Decr? ____________

8.



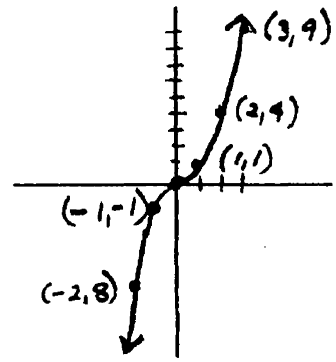
Symmetry? ____________

Even/Odd/Neither

Incr? ____________

Decr? ____________

9.



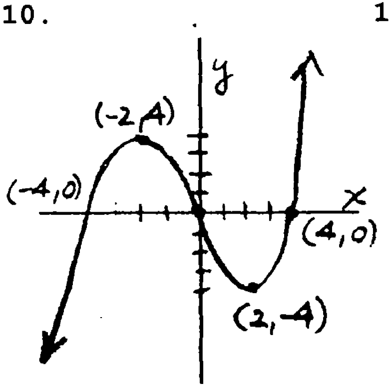
Symmetry? ____________

Even/Odd/Neither

Incr? ____________

Decr? ____________

10.



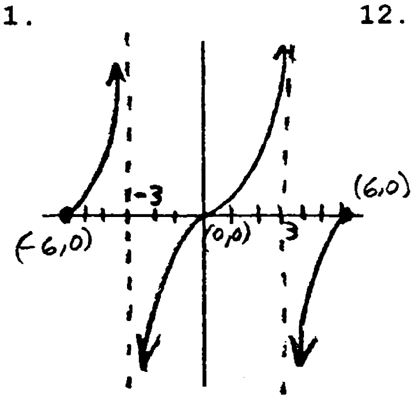
Symmetry? ____________

Even/Odd/Neither

Incr? ____________

Decr? ____________

11.



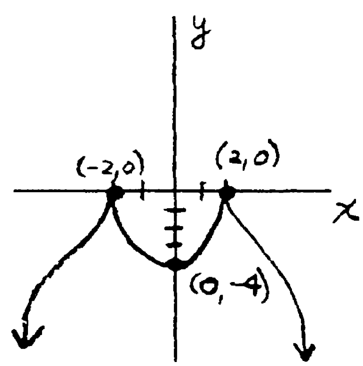
Symmetry? ____________

Even/Odd/Neither

Incr? ____________

Decr? ____________

12.



Symmetry? ____________

Even/Odd/Neither

Incr? ____________

Decr? ____________

CONTINUITY

[Note: The topic of continuity is a deep and theoretical topic from higher mathematics. The presentation here is designed to be a "brief" introduction.]

The final behavior of graphs to be considered here is **continuity**. The graph of a function is said to be **continuous** on an interval (a,b) if the graph can be drawn from "a" to "b" without lifting the pencil (or pen, or chalk, etc.). It follows that a graph is **discontinuous** if there is a "break" or a "jump" in the graph.

SOME DISCONTINUITIES OF FUNCTIONS

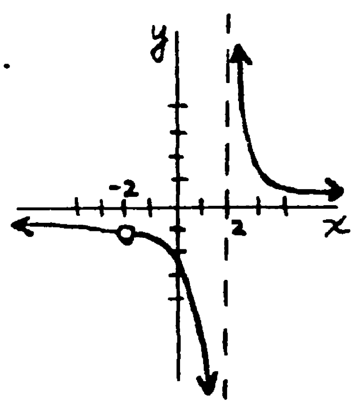
I. Denominator equal to zero.

- **A. Asymptote**
- **B. Hole in graph**

II. Piecewise function (may or may not be discontinuous).

EXAMPLES: In each of the following, indicate values of X for which the function is discontinuous, and indicate the intervals in which the function is continuous. Use interval notation.

I.

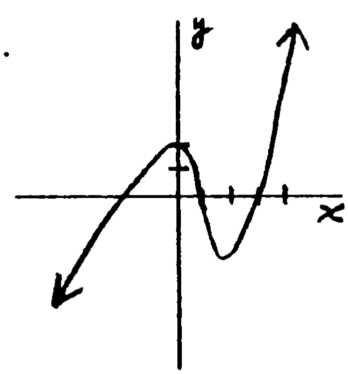


Discont: x = -2, 2

Intervals/Continuity

(-∞,-2) ; (-2,2) ; (2,∞)

II.

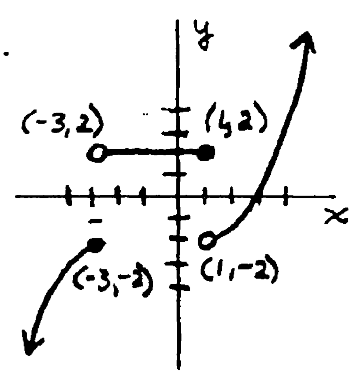


Discont: None

Intervals/Continuity

(-∞,∞)

III.



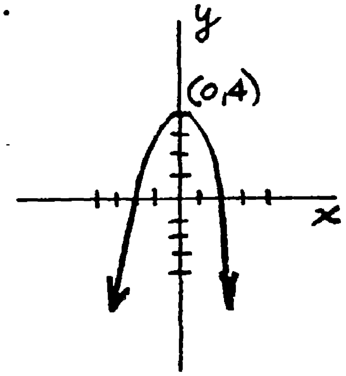
Discont: x = -3, 1

Intervals/Continuity

(-∞,-3] ; (-3,1] ; (1,∞)

As in the examples on the previous page, determine the values of x for which there are discontinuities, and give the intervals for which the functions are continuous.

1.

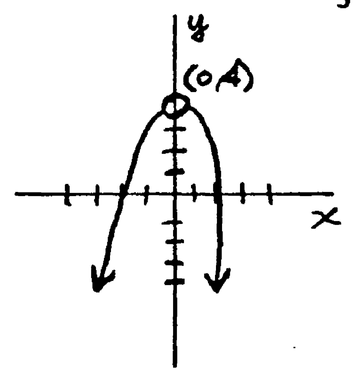


Discont:____________

Intervals/Continuity

____

2.

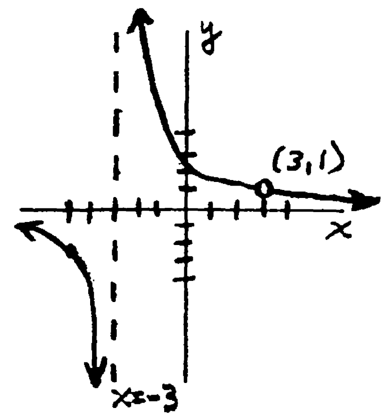


Discont:____________

Intervals/Continuity

____

3.

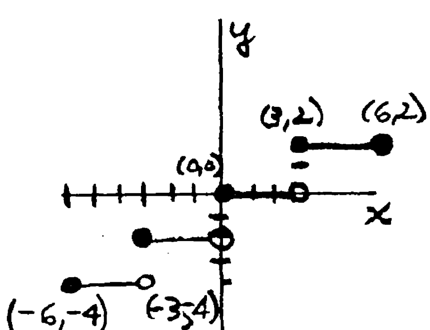


Discont:____________

Intervals/Continuity

____

4.

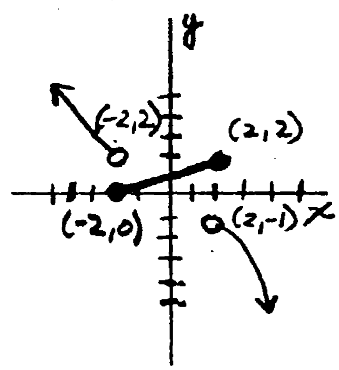


Discont:____________

Intervals/Continuity

____

5.

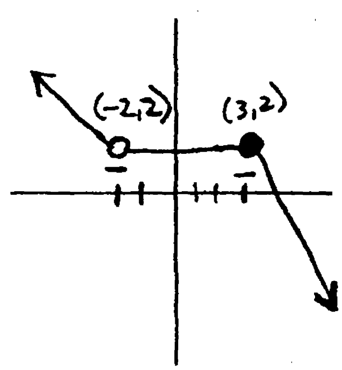


Discont:____________

Intervals/Continuity

____

6.



Discont:____________

Intervals/Continuity

____

ANSWERS 2.09

p.308:

1a) (-3,4); **b)** (3,-4); **c)** (-3,-4); **d)** (4,3);
2a) (0,6); **b)** (0,-6); **c)** (0,-6); **d)** (6,0);
3a) (4,12); **b)** (-4,-12); **c)** (4,-12); **d)** (12,-4);
4a) (5,-9); **b)** (-5,9); **c)** (5,9); **d)** (-9,-5);

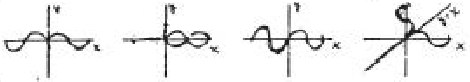
p. 309-311:

1. X-axis; **2.** Y-axis, X-axis, Origin; **3.** Y-axis;
4. Origin; **5.** Origin; **6.** Y=X; **7.** Y=X; **8.** Y=X;
9. X-axis, Y-axis, Origin, Y=X; **10.** X-axis;
11. X-axis, Y-axis, Origin, Y=X; **12.** Y-axis;

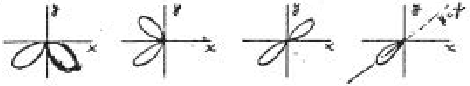
13a) **b)** **c)** **d)**

14.

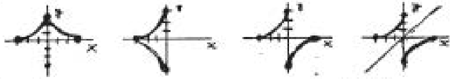
15.



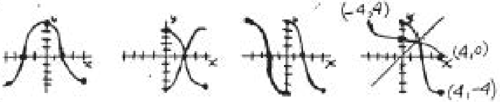
16.



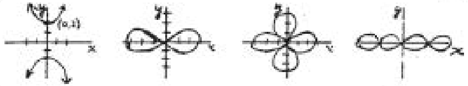
17.



18.



19.



20.

p.313-316:

1. Y-axis no; Origin yes; **2.** Y-axis yes; Orgin no;
3. X-axis yes; Y=X no; **4.** X-axis no; Y=X yes;
5. Origin; **6.** Y-axis; **7.** None; **8.** None; **9.** X-axis;
10. Origin; **11.** Y-axis, X-axis, Origin; **12.** Origin;
13. None; **14.** Y-axis, X-axis, Origin; **15.** Origin;
16. Y-axis, X-axis, Origin; **17a)** At first glance it seems impossible for a function to have symmetry about the X-axis. However, consider the horizontal line **Y=0**.
17b) Y-axis symmetry; **17c)** Symmetry about the origin;
17d) No symmetry.

p. 318:

1. Even; **2.** Odd; **3.** Neither; **4.** Neither; **5.** Odd;
6. Even; **7.** Even; **8.** Even; **9.** Even; **10.** Neither;
11. Neither; **12.** Odd.

p.320-321:

1. Y-axis, even, incr: $(-\infty,0)$, decr: $(0,\infty)$;
2. None, neither, incr: $(-\infty,\infty)$, decr: $\emptyset$;
3. Origin, odd, incr: $(-2,2)$, decr: $(-4,-2)$, $(2,4)$;
4. Y-axis, even, incr: $(0,\infty)$, decr: $(-\infty,0)$;
5. Origin, odd, incr: $\emptyset$; decr: $(-\infty,\infty)$;
6. Origin, odd, incr: $(-\infty,-4)$, $(-2,2)$, $(4,\infty)$; decr: $(-4,-2)$, $(2,4)$
7. Y-axis, even, incr: $(-4,0)$, $(4,6)$, decr: $(-6,-4)$, $(0,4)$;
8. None, neither, incr: $(-\infty,0)$, $(3,\infty)$, decr: $(0,3)$;
9. None, neither, incr: $(-\infty,\infty)$;
10. Origin, odd, incr: $(-\infty,-2)$, $(2,\infty)$, decr: $(-2,2)$;
11. Origin, odd, incr: $(-6,-3)$, $(-3,3)$, $(3,6)$, decr: $\emptyset$;
12. Y-axix, even, incr: $(-\infty,-2)$, $(0,2)$, decr: $(-2,0)$, $(2,\infty)$.

p. 323:

1. Discont: none,
Cont: $(-\infty,\infty)$

2. Discont: X=0,
Cont: $(-\infty,0)$, $(0,\infty)$

3. Discont: X=-3, 3
Cont: $(-\infty,-3)$, $(-3,3)$, $(3,\infty)$

4. Discont: X=-3, 0, 3
Cont: $[-6,-3)$, $[-3,0)$, $[0,3)$, $[3,6)$

5. Discont: X=-2
Cont: $(-\infty,-2)$, $[-2,2]$, $(2,\infty)$

6. Discont: X=-2, 3
Cont: $(-\infty,-2)$, $(-2,\infty)$

Dr. Robert J. Rapalje

ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE