2.05 *The Ellipse and Hyperbola*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

In the last section you graphed the equations of circles. Did you notice that in each circle equation, the coefficient of X^2 and Y^2 were always equal? Now in this section, consider what happens if the coefficients of the X^2 and Y^2 terms are not equal. In this section, look for two distinct cases to emerge, depending upon whether the coefficients are the same or of opposite sign.

First consider the equation **#1.** $\mathbf{4X^2 + 9Y^2 = 36.}$ It is convenient in this case to graph the **X** and **Y-intercepts**:

Y-Intercepts

If $X=0$, then $9Y^2=36$

$Y^2= 4$

$Y = \pm 2$

X-Intercepts

If $Y=0$, then $4X^2=36$

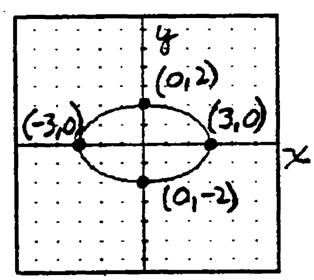
$X^2= 9$

$X = \pm 3$

By graphing the intercepts, you can probably connect the points with a smooth curve and a small amount of imagination to obtain the following graph. If you need additional points, let X=1 or X=2, and use a calculator to find the corresponding value of Y.

X	Y
0	±2
±3	0

#1.

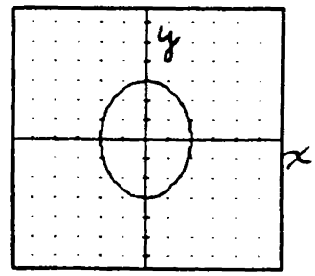


Now, in a similar manner, graph equation **#2.** $\mathbf{9X^2 + 4Y^2 = 36}$.

<u>Y-Intercepts</u>	**<u>X-Intercepts</u>**
If X=0, then __________	If Y=0, then __________
$Y^2 =$ _____	$X^2 =$ _____
$Y =$ _____	$X =$ _____

X	Y

#2.

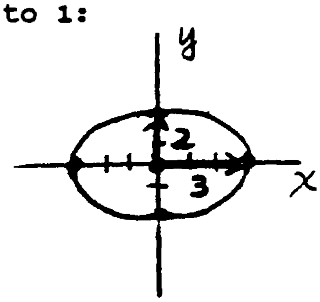


These graphs, in which the coefficients of X^2 and Y^2 differ but are of the same sign, are called **ellipses**. An easier way to graph the ellipse is to write the equation in a form that is equal to 1. In the previous equations **#1.** $\mathbf{4X^2 + 9Y^2 = 36}$, divide both sides by 36, to get the equation in a form equal to **1:**

$$\frac{4X^2}{36} + \frac{9Y^2}{36} = \frac{36}{36}$$

$$\frac{X^2}{9} + \frac{Y^2}{4} = 1$$

$$\frac{X^2}{3^2} + \frac{Y^2}{2^2} = 1$$

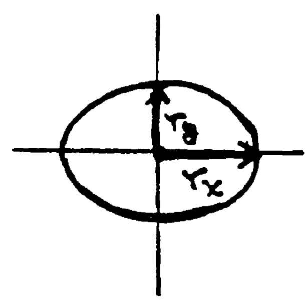


Do you see how the **3** and the **2** in this equation can be used to draw the graph? The ellipse has a "radius" of 3 in the X-direction and a "radius" of 2 in the Y-direction. In the general case

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

from the center at (0,0), a distance of **±a** units is measured in the X direction and **±b** units is measured in the Y direction. This is like an **"X-radius"** and a **"Y-radius"**:

$$\frac{X^2}{r_x^2} + \frac{Y^2}{r_y^2} = 1$$



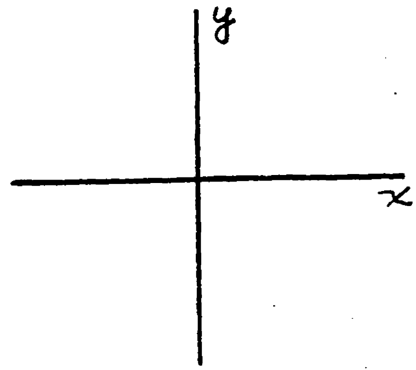
Now, graph equation #2. $9X^2 + 4Y^2 = 36$ by the faster method of setting the equation equal to 1 to find r_x and r_y:

$$\frac{9X^2}{36} + \frac{4Y^2}{36} = \frac{36}{36}$$

$$\frac{X^2}{(\quad)} + \frac{Y^2}{(\quad)} = (\quad)$$

$$\frac{X^2}{(\quad)^2} + \frac{Y^2}{(\quad)^2} = (\quad)$$

$r_x=$____; $r_y=$____



In 3-6, graph the equations:

3. $X^2 + 4Y^2 = 36$

4. $4X^2 + 25Y^2 = 100$

5. $9X^2 + Y^2 = 36$

6. $25X^2 + Y^2 = 25$

In #7, there is a problem! What happens if the 4 doesn't divide out? Answer: Multiply numerator and denominator of the first fraction by ¼.

7. $4X^2 + 9Y^2 = 9$

$$\frac{4X^2}{9} + \frac{9Y^2}{9} = \frac{9}{9}$$

$$\frac{\frac{1}{4}\cdot 4X^2}{\frac{1}{4}\cdot 9} + \frac{9Y^2}{9} = 1$$

$$\frac{X^2}{(\quad)^2} + \frac{Y^2}{(\quad)^2} = (\quad)$$

r_x=____; r_y=____

8. $9X^2 + 25Y^2 = 100$

9. $25X^2 + 4Y^2 = 25$

10. $\frac{X^2}{9} + \frac{Y^2}{9} = 1$

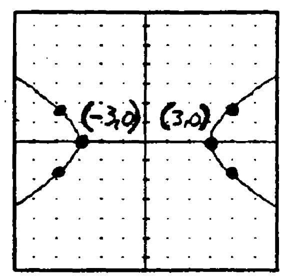
What happens if $r_x = r_y$?

Next what would happen if the $\mathbf{X^2}$ and $\mathbf{Y^2}$ terms are of opposite sign? Consider the equation: **#11. $4X^2 - 9Y^2 = 36$.** As with the ellipse, it seems convenient to graph the **X** and **Y-intercepts:**

<u>**X-Intercepts**</u>	<u>**Y-Intercepts**</u>
If Y=0, then $4X^2=36$	If X=0, then $-9Y^2=36$
$X^2= 9$	$Y^2= -4$
$X = \pm 3$	Y = No Real Values

As you can see, there are no Y-intercepts. In fact, there are no values of X allowed between -3 and 3. That is, **$X \geq 3$ or $X \leq -3$.** To obtain a few more values, for credibility, let $X=\pm 4$ or $X=\pm 5$. Substituting $X=\pm 4$, you obtain $64-Y^2=36$, so $Y^2=24$ and $Y \approx \pm 5$. By graphing these points and realizing that there is a <u>gap</u> between X=3 and X=-3, you probably see the need to connect the points with two separate smooth curves. With some imagination and a few additional points if necessary, you should obtain the following graph.

X	**Y**
±3	**0**
±4	**≈±1.7**



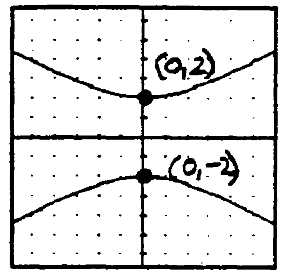
12. In a similar manner, graph the equation: $9Y^2 - 4X^2 = 36$.

<u>**Y-Intercepts**</u> — If X=0, then ________ ; Y^2= _____ ; Y = _____

<u>**X-Intercepts**</u> — If Y=0, then ________ ; X^2= _____ ; X = _____ No Real Sol.

In the equation $9Y^2 - 4X^2 = 36$, substituting $X=\pm 3$, you obtain $Y^2=8$, so $Y \approx \pm 3$. There will be no values of Y between Y=2 and Y=-2.

X	Y



These graphs, in which the coefficients of X^2 and Y^2 are of opposite sign, are called **hyperbolas**. As with the ellipses, there is an easier way to graph the hyperbola by writing the equation in a form that is equal to 1. In equations **#11. $4X^2 - 9Y^2 = 36$** and **#12. $9Y^2 - 4X^2 = 36$**, divide both sides by 36, to get the equations in a form equal to 1:

#11.
$$\frac{4X^2}{36} - \frac{9Y^2}{36} = \frac{36}{36}$$
$$\frac{X^2}{9} - \frac{Y^2}{4} = 1$$

#12.
$$\frac{9Y^2}{36} - \frac{4X^2}{36} = \frac{36}{36}$$
$$\frac{Y^2}{4} - \frac{X^2}{9} = 1$$

In each respective case, the more general forms would be:

$$\frac{X^2}{r_x^2} - \frac{Y^2}{r_y^2} = 1 \qquad\qquad \frac{Y^2}{r_y^2} - \frac{X^2}{r_x^2} = 1 .$$

Notice that in either case, the **center** is at **(0,0)**, r_x**=3**, and r_y**=2**. In **#11. $4X^2 - 9Y^2 = 36$**, the r_x=3 represents the X-intercepts and since there are no Y-intercepts, it appears that r_y=2 is of no value. In **#12. $9Y^2 - 4X^2 = 36$**, the r_y=2 represents the Y-intercepts and since there are no X-intercepts, it appears that r_x=3 is of no value. Nevertheless, on each graph mark off both r_x and r_y on the X and Y-axes respectively and construct a rectangle with vertical and horizontal **dotted** lines, as illustrated on the next page. This is called the **central rectangle** of the hyperbola. The dotted **diagonals** of this rectangle form the **asymptotes** of each hyperbola that serve as guidelines for graphing. Finally, notice that equations of the form $\frac{X^2}{r_x^2} - \frac{Y^2}{r_y^2} = 1$ always **open right and left (sideways)**, and equations of the form $\frac{Y^2}{r_y^2} - \frac{X^2}{r_x^2} = 1$ always open **up and down.**

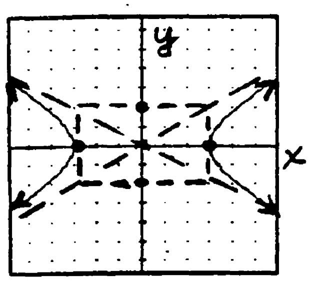
Graph #11 and #12 by sketching the central rectangle and diagonals:

11. $\frac{X^2}{9} - \frac{Y^2}{4} = 1$

r_x = ____

r_y = ____

Opens: ____



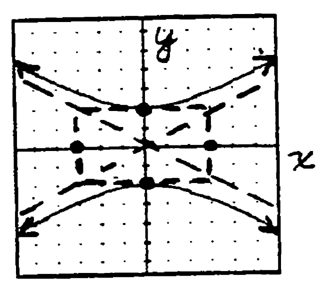
$\frac{X^2}{9} - \frac{Y^2}{4} = 1$

12. $\frac{Y^2}{4} - \frac{X^2}{9} = 1$

r_x = ____

r_y = ____

Opens: ____



$\frac{Y^2}{4} - \frac{X^2}{9} = 1$

In 13-20, express the equations of the hyperbolas in standard form equal to 1, graph the central rectangles with diagonals, decide whether the graph opens "right/left" or "up/down", and sketch the graph.

13. $9X^2 - 4Y^2 = 36$

$$\frac{9X^2}{36} - \frac{4Y^2}{36} = \frac{36}{36}$$

$$\frac{X^2}{(\quad)} - \frac{Y^2}{(\quad)} = (\quad)$$

$$\frac{X^2}{(\quad)^2} - \frac{Y^2}{(\quad)^2} = (\quad)$$

r_x=_____; r_y=_____

14. $4Y^2 - 9X^2 = 36$

15. $4Y^2 - 25X^2 = 100$

16. $25X^2 - 4Y^2 = 100$

17. $25X^2 - Y^2 = 25$

18. $4X^2 - 9Y^2 = -9$

$$\frac{4X^2}{-9} - \frac{9Y^2}{-9} = \frac{-9}{-9}$$

$$\frac{Y^2}{1} - \frac{4X^2}{9} = 1$$

$$\frac{Y^2}{1} - \frac{\frac{1}{4}\cdot 4X^2}{\frac{1}{4}\cdot 9} = 1$$

$$\frac{Y^2}{(\quad)^2} - \frac{X^2}{(\quad)^2} = (\quad)$$

r_x=_____; r_y=_____

19. $9X^2 - 25Y^2 = -100$

20. $25Y^2 - 4X^2 = -25$

In 21-30, identify the type of graph and sketch its graph.

21. $4X^2 + 25Y^2 = 100$
Type: ___________

22. $4X^2 - 25Y^2 = 100$
Type: ___________

23. $4X^2 - 25Y^2 = -100$
Type: ___________

24. $25X^2 + 4Y^2 = 100$
Type: ___________

25. $4X^2 + Y^2 = 16$
Type: __________

26. $4Y^2 - X^2 = -16$
Type: __________

27. $4X^2 - 4Y^2 = 16$
Type: __________

28. $X^2 + 4Y^2 = 16$
Type: __________

29. $4X^2 + 4Y^2 = 16$
Type: __________

30. $-X^2 + 4Y^2 = -16$
Type: __________

All of the **ellipses** and **hyperbolas** encountered so far were centered at **(0,0)**. As you probably have guessed, this is not always the case! Remember how the equation of a **circle** with center at **(0,0)** compares to the equation of a circle with center **(h,k)**:

	Center at (0,0)	**Center at (h,k)**
CIRCLE:	$X^2 + Y^2 = r^2$	$(X - h)^2 + (Y - k)^2 = r^2$

In the same way, the equations of ellipses and hyperbolas with center at **(h,k)** compare as follows:

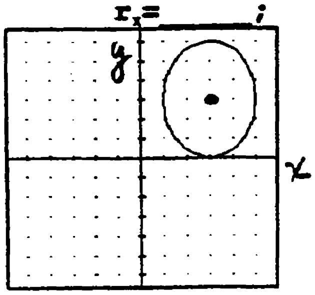
ELLIPSE:	$\dfrac{X^2}{r_x^2} + \dfrac{Y^2}{r_y^2} = 1$	$\dfrac{(X - h)^2}{r_x^2} + \dfrac{(Y - k)^2}{r_y^2} = 1$
HYPERBOLA:	$\dfrac{X^2}{r_x^2} - \dfrac{Y^2}{r_y^2} = 1$	$\dfrac{(X - h)^2}{r_x^2} - \dfrac{(Y - k)^2}{r_y^2} = 1$
	$\dfrac{Y^2}{r_y^2} - \dfrac{X^2}{r_x^2} = 1$	$\dfrac{(Y - k)^2}{r_y^2} - \dfrac{(X - h)^2}{r_x^2} = 1$

[BE CAREFUL: in the last case the center of the hyperbola is still at (h,k)!!]

In 31-40, identify the type of graph, the center, r_x, r_y, and sketch the graph. For hyperbolas, locate the center, sketch the central rectangle and the asymptotes.

31. $\dfrac{(X-3)^2}{4} + \dfrac{(Y-3)^2}{9} = 1$

Type__________; **Center**______
r_x=______; **r_y=**______



32. $\dfrac{(X+1)^2}{25} + \dfrac{(Y-2)^2}{9} = 1$

Type__________; **Center**______
r_x=______; **r_y=**______

33. $\dfrac{(X+2)^2}{1} + \dfrac{(Y-3)^2}{16} = 1$

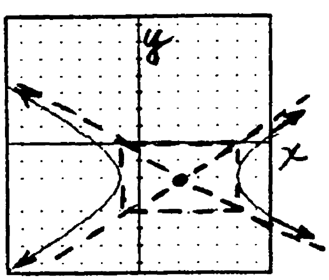
Type__________; **Center**______
r_x=______; **r_y=**______

34. $\dfrac{(X-3)^2}{1} + \dfrac{(Y+1)^2}{4} = 1$

Type__________; **Center**______
r_x=______; **r_y=**______

35. $\dfrac{(X-2)^2}{9} - \dfrac{(Y+2)^2}{4} = 1$

Type__________; **Center**______
r_x=______; **r_y=**______



36. $\dfrac{(X+3)^2}{4} - \dfrac{(Y-1)^2}{1} = 1$

Type__________; **Center**______
r_x=______; **r_y=**______

37. $\frac{(Y+2)^2}{4} - \frac{(X-3)^2}{1} = 1$

Type__________; **Center**_____
r_x=_____; r_y=_____

38. $\frac{(Y-2)^2}{16} - \frac{(X+1)^2}{9} = 1$

Type__________; **Center**_____
r_x=_____; r_y=_____

39. $\frac{Y^2}{4} - \frac{(X-3)^2}{9} = -1$

Type__________; **Center**_____
r_x=_____; r_y=_____

40. $\frac{X^2}{4} - \frac{(Y+3)^2}{25} = -1$

Type__________; **Center**_____
r_x=_____; r_y=_____

Frequently, the equations of ellipses and hyperbolas are given in an **expanded form, $AX^2 + CY^2 + DX + EY + F = 0$**, rather than the **standard form** you have seen so far. When the equation is given in the expanded form, it is necessary to complete the square in order to identify the center, r_x, and r_y. The process is very similar to the completing the square required for circles, only a bit more tricky, because of the coefficients of X^2 and Y^2. The following exercise illustrates the process.

41. $9X^2 + 25Y^2 - 36X + 100Y - 89 = 0$ — Collect X terms, Y terms, and add +89 to each side. Leave spaces to complete the square.

$9X^2 - 36X + \underline{\quad\quad} + 25Y^2 + 100Y + \underline{\quad\quad} = 89$ — Next **always** factor coef of X^2 and Y^2.

$9(X^2 - 4X + \underline{\quad\quad}) + 25(X^2 + 4Y + \underline{\quad\quad}) = 89$ — Fill in the blanks by completing the square. Add the **appropriate number** to each side. However, if you put **+4** in each blank, remember you are <u>actually</u> adding **9·4=36** and **25·4=100** to each side.

$9(X^2 - 4X + \underline{\ 4\ }) + 25(X^2 + 4Y + \underline{\ 4\ }) = 89 + \underline{36} + \underline{100}$

$9(X-2)^2 \quad + \quad 25(x+2)^2 \quad = 225$ — Now, divide both sides by **225**, reduce fractions.

$$\frac{9(X-2)^2}{225} + \frac{25(Y+2)^2}{225} = \frac{225}{225}$$

$$\frac{(X-2)^2}{25} + \frac{(Y+2)^2}{9} = 1$$

Center $(\underline{\quad\quad}, \underline{\quad\quad})$;

$r_x = \underline{\quad\quad}$; $r_y = \underline{\quad\quad}$

42. $X^2 + 4Y^2 - 2X - 24Y + 33 = 0$

43. $4X^2 + 25Y^2 + 24X - 50Y - 39 = 0$

44. $9X^2 + 16Y^2 + 36X + 160Y + 292 = 0$

45. $16X^2 - 25Y^2 - 32X + 150Y - 609 = 0$ Collect X and Y terms

$16X^2-32X+ \quad -25Y^2+150Y \quad = 609$ Factor 16 and -25

$16(X^2-2X+\_\_\_\_) -25(Y^2-6Y+\_\_\_\_) = 609$ **BE CAREFUL**--you are adding a **+16·1 and -25·9**

$16(X^2-2X+\underline{\ 1\ }) - 25(Y^2-6Y+\underline{\ 9\ }) = 609 + \underline{\ 16\ } - \underline{\ 225\ }$

$16(X-1)^2 - 25(Y-3)^2 = 400$

46. $9X^2 - 25Y^2 + 36X - 100Y - 289 = 0$

47. $X^2 - 4Y^2 - 6X - 16Y + 9 = 0$

48. $9X^2 - Y^2 + 36X + 4Y + 41 = 0$

49. $25X^2 - 4Y^2 - 100X = 0$

50. $16X^2 + 9Y^2 + 96X - 90Y + 225 = 0$

51. $X^2 + 4Y^2 - 6X - 16Y + 21 = 0$

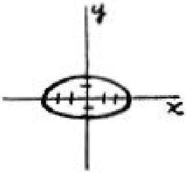
52. $X^2 - 4Y^2 - 2X + 24Y - 39 = 0$

53. $25X^2 - 9Y^2 - 100X - 54Y + 244 = 0$

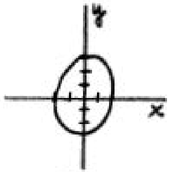
54. $25X^2 + 9Y^2 - 100X + 54Y - 44 = 0$

2.05 p.242-261:

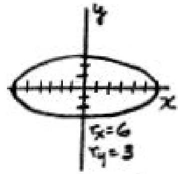
1. $4x^2+9y^2=36$



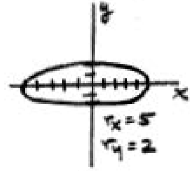
2. $9x^2+4y^2=36$



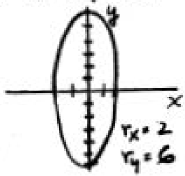
3. $x^2+4y^2=36$



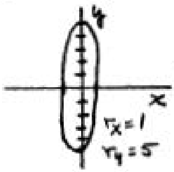
4. $4x^2+25y^2=100$



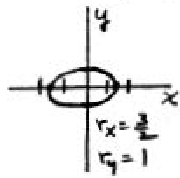
5. $9x^2+y^2=36$



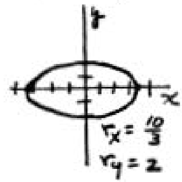
6. $25x^2+y^2=25$



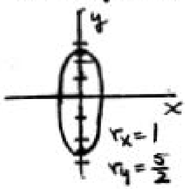
7. $4x^2+9y^2=9$



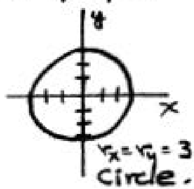
8. $9x^2+25y^2=100$



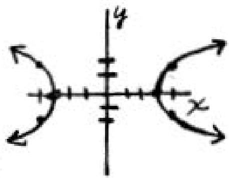
9. $25x^2+4y^2=25$



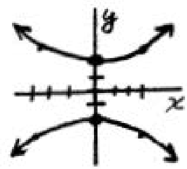
10. $\frac{x^2}{9}+\frac{y^2}{9}=1$



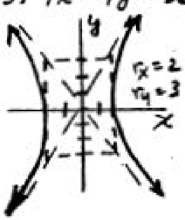
11. $4x^2-9y^2=36$



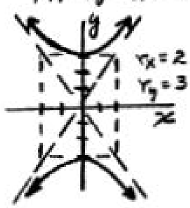
12. $9y^2-4x^2=36$



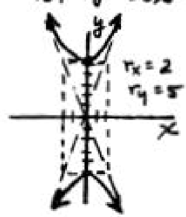
13. $9x^2-4y^2=36$



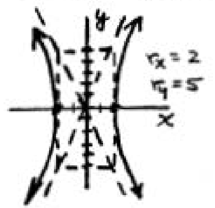
14. $4y^2-9x^2=36$



15. $4y^2-25x^2=100$



16. $25x^2-4y^2=100$



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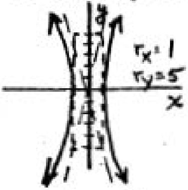
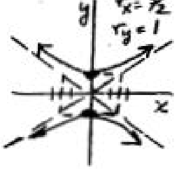
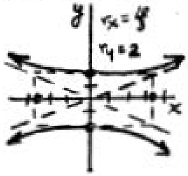
17. $25x^2 - y^2 = 25$



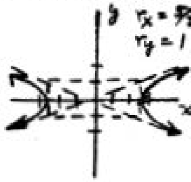
18. $4x^2 - 9y^2 = -9$



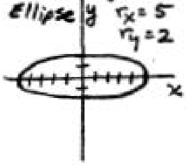
19. $9x^2 - 25y^2 = -100$



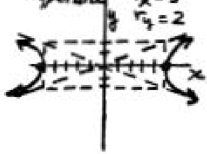
20. $25y^2 - 4x^2 = -25$



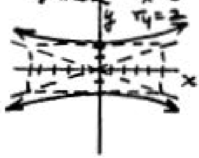
21. $4x^2 + 25y^2 = 100$



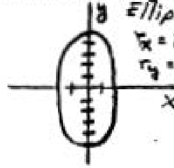
22. $4x^2 - 25y^2 = 100$



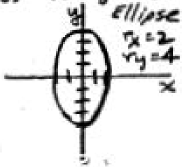
23. $4x^2 - 25y^2 = -100$



24. $25x^2 + 4y^2 = 100$

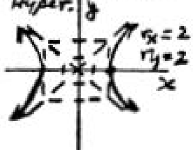


25. $4x^2 + y^2 = 16$

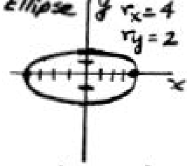


26. $4y^2 - x^2 = -16$

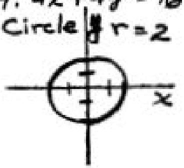
27. $4x^2 - 4y^2 = 16$



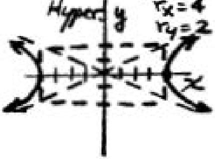
28. $x^2 + 4y^2 = 16$



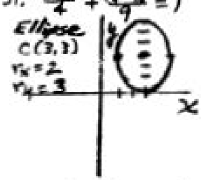
29. $4x^2 + 4y^2 = 16$



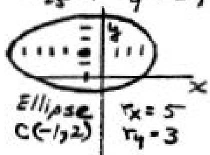
30. $-x^2 + 4y^2 = -16$



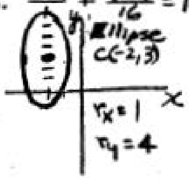
31. $\frac{(x-3)^2}{4} + \frac{(y-3)^2}{9} = 1$



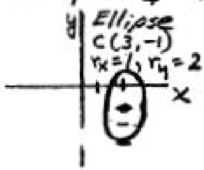
32. $\frac{(x+1)^2}{25} + \frac{(y-2)^2}{9} = 1$



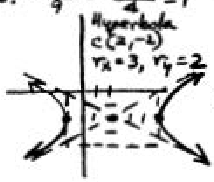
33. $\frac{(x+2)^2}{1} + \frac{(y-3)^2}{16} = 1$



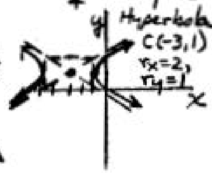
34. $\frac{(x-3)^2}{1} + \frac{(y+1)^2}{4} = 1$



35. $\frac{(x-2)^2}{9} - \frac{(y+2)^2}{4} = 1$



36. $\frac{(x+3)^2}{4} - \frac{(y-1)^2}{1} = 1$



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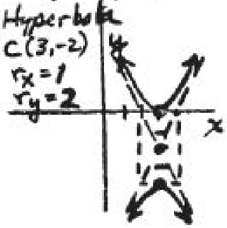
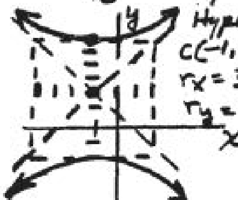
37. $\frac{(y+2)^2}{4}-\frac{(x-3)^2}{1}=1$

Hyperbola
C(3,-2)
$r_x=1$
$r_y=2$



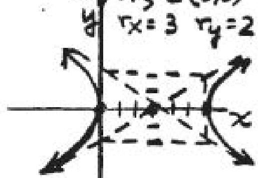
38. $\frac{(y-2)^2}{16}-\frac{(x+1)^2}{9}=1$

Hyper.
C(-1,2)
$r_x=3$
$r_y=4$



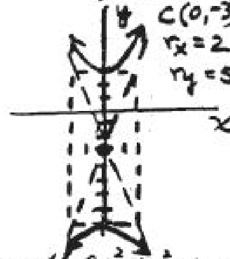
39. $\frac{y^2}{4}-\frac{(x-3)^2}{9}=-1$

Hyper; C(3,0)
$r_x=3$ $r_y=2$



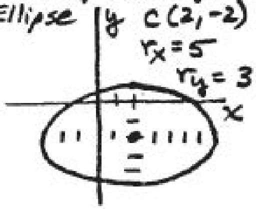
40. $\frac{x^2}{4}-\frac{(y+3)^2}{25}=-1$

Hyperbola
C(0,-3)
$r_x=2$
$r_y=5$

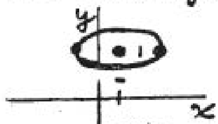


41. $9x^2+25y^2-36x+100y-89=0$

Ellipse C(2,-2)
$r_x=5$
$r_y=3$



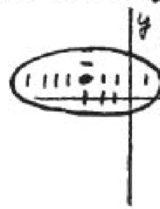
42. $x^2+4y^2-2x-24y+33=0$



Ellipse
C(1,3)
$r_x=2, r_y=1$

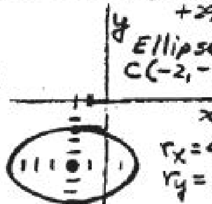
43. $4x^2+25y^2+24x-50y-39=0$

Ellipse
C(-3,1)
$r_x=5; r_y=2$



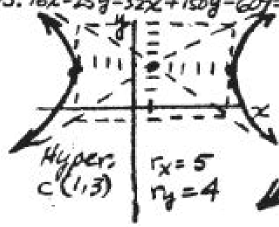
44. $9x^2+16y^2+36x+160y+292=0$

Ellipse
C(-2,-5)



$r_x=4$
$r_y=3$

45. $16x^2-25y^2-32x+150y-609=0$



Hyper.
C(1,3)
$r_x=5$
$r_y=4$

46. $9x^2-25y^2+36x-100y-289=0$

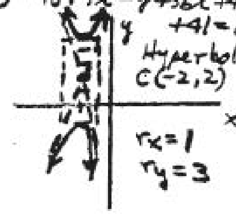
Hyper C(-2,-2)
$r_x=5; r_y=3$

47. $x^2-4y^2-6x-16y+9=0$

Hyperbola
C(3,-2)
$r_x=4; r_y=2$

48. $9x^2-y^2+36x+4y+41=0$

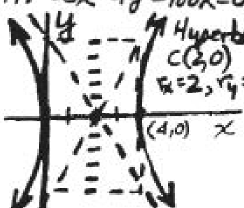
Hyperbola
C(-2,2)



$r_x=1$
$r_y=3$

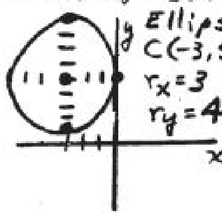
49. $25x^2-4y^2-100x=0$

Hyperbola
C(2,0)
$r_x=2, r_y=5$



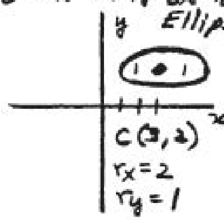
50. $16x^2+9y^2+96x-90y+225=0$

Ellipse
C(-3,5)
$r_x=3$
$r_y=4$



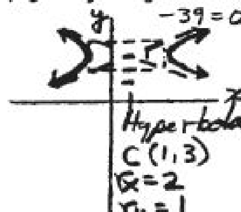
51. $x^2+4y^2-6x-16y+21=0$

Ellipse



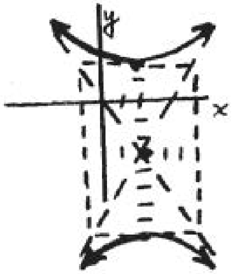
C(3,2)
$r_x=2$
$r_y=1$

52. $x^2-4y^2-2x+24y-39=0$



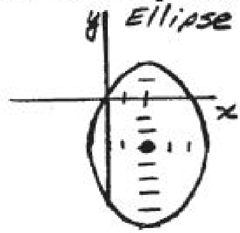
Hyperbola
C(1,3)
$r_x=2$
$r_y=1$

53. $25x^2-9y^2-100x-54y+244=0$



54. $25x^2+9y^2-100x+54y-44=0$

Ellipse
C(2,-3)
$r_x=3$ $r_y=5$



Dr. Robert J. Rapalje

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE