# 2.10 *Inverse Functions*

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Since the beginning of your first algebra course, you have been using the idea of an **inverse.** Recall from the **number properties**, the **inverse property for addition** ( or the **additive inverse property**): $a + (-a) = 0$, and also the **inverse property for**

**multiplication** (or the **multiplicative inverse property**): $a \cdot \frac{1}{a} = 1$ .

When solving an equation, these properties are used to "undo" the equation in order to solve for x.

Remember that in the equation $3X + 5 = 35$, the X has been multiplied by 3 and then 5 was added which equals 35. In order to solve for X, you must "undo" what was done in the reverse order. You add -5 (the **additive inverse** of 5) to each side of the equation to obtain $3X = 30$. Next you divide each side of this equation by 3 (or rather you could say, "multiply each side by ⅓, the **multiplicative inverse** of 3). The final result is that the X has been isolated, and $X = 10$.

Now, in this section, the idea of **inverses** is extended to **functions.** Given a function **f(X)**, an **inverse function** of f(X) would be some function, say **g(X)**, that will "undo" the **f(X)** and leave just **X**, isolated. Likewise, the **f(X)** would "undo" the **g(X)** and leave just **X**. Therefore, **f(X)** and **g(X)** are inverse functions of one another if: $\boxed{f[g(X)] = X, \text{ for every X in the domain of } g(X), \text{ and } g[f(X)] = X, \text{ for every X in the domain of } f(X).}$

Suppose for example, $\mathbf{f(X) = 3X + 2}$. It turns out that $\mathbf{g(X) = \frac{x-2}{3}}$ is the inverse function for **f(X)**, since

$$f[g(x)] = \cancel{3}\left[\frac{x-2}{\cancel{3}}\right] + 2 \quad \text{and} \quad g[f(x)] = \frac{[3x+2]-2}{3}$$

$$= (x-2) + 2 \qquad\qquad = \frac{3x}{3}$$

$$= x \qquad\qquad = x$$

Usually the notation "$f^{-1}(x)$" is used to denote the inverse function for $f(x)$. This does <u>not</u> mean $\frac{1}{f(x)}$. In the first exercises, you will be given $f(x)$ and $f^{-1}(x)$. Show that these are indeed inverse functions of one another by showing that $f[f^{-1}(x)] = x$ and $f^{-1}[f(x)] = x$.

EXERCISES: Show that $f(x)$ and $f^{-1}(x)$ are inverse functions of one another.

1. $f(x) = 3x - 5$ $\qquad$ $f^{-1}(x) = \frac{x+5}{3}$

$$f[f^{-1}(x)] = 3[\quad] - 5 \qquad\qquad f^{-1}[f(x)] = \frac{[\quad] + 5}{3}$$

$$= \qquad\qquad\qquad\qquad =$$

$$= x \qquad\qquad\qquad\qquad = x$$

2. $f(x) = 2x + 6$ $\qquad$ $f^{-1}(x) = \frac{x-6}{2}$

$$f[f^{-1}(x)] = \qquad\qquad f^{-1}[f(x)] =$$

3. $f(x) = -2x + 6$ $\qquad$ $f^{-1}(x) = -\frac{1}{2}x + 3$

4. $f(x) = -3x - 6$ $\qquad$ $f^{-1}(x) = -\frac{1}{3}x - 2$

5. $f(x) = \dfrac{x+4}{x}$ $\qquad$ $f^{-1}(x) = \dfrac{4}{x-1}$

$f[f^{-1}(x)] = \dfrac{[\quad] + 4}{[\quad]}$ $\qquad$ $f^{-1}[f(x)] = \dfrac{4}{[\quad] - 1}$

$=$ $\qquad$ $=$

$=$ $\qquad$ $=$

$=$ $\qquad$ $=$

6. $f(x) = \dfrac{x}{x-4}$ $\qquad f^{-1}(x) = \dfrac{4x}{x-1}$

7. $f(x) = \dfrac{x^3 - 8}{4}$ $\qquad f^{-1}(x) = \sqrt[3]{4x+8}$

8. $f(x) = \sqrt[3]{2x-27}$ $\qquad f^{-1}(x) = \dfrac{x^3 + 27}{2}$

Now, suppose you were given some function $f(x)$. How would you go about finding the corresponding $f^{-1}(x)$? Begin by letting $y = f(x)$. Now: ① Interchange the $x$ and $y$; ② Solve for $y$.

Example: Given $f(x) = 3x + 2$, find $f^{-1}(x)$.

Let $y = 3x + 2$.

① Interchange: $x = 3y + 2$

② Solve for $y$: $x - 2 = 3y$

$y = \frac{x-2}{3}$, so $f^{-1}(x) = \frac{x-2}{3}$

EXERCISES: Find $f^{-1}(x)$.

9. $f(x) = 5x - 3$

Let $y = 5x - 3$

① Interchange: $(\quad) = 5(\quad) - 3$

② Solve for $y$:

10. $f(x) = -3x + 5$

11. $f(x) = \frac{5x+3}{2}$

12. $f(x) = \frac{3-x}{5}$

13. $y = \dfrac{x+1}{x}$

14. $y = \dfrac{3x+4}{5x}$

15. $y = \dfrac{1}{x-1}$

16. $y = \dfrac{4}{5x-3}$

17. $y = \dfrac{x^3+8}{3}$

18. $y = \dfrac{x^3-8}{5}$

19. $y = \sqrt[3]{3x-8}$

20. $y = \sqrt[3]{5x+8}$

21. $y = \frac{9}{5}x + 32$

22. $y = \frac{5}{9}(x - 32)$

[Where have you seen #21 and #22 before?]

23. $y = \frac{1}{x}$

24. $y = -x$

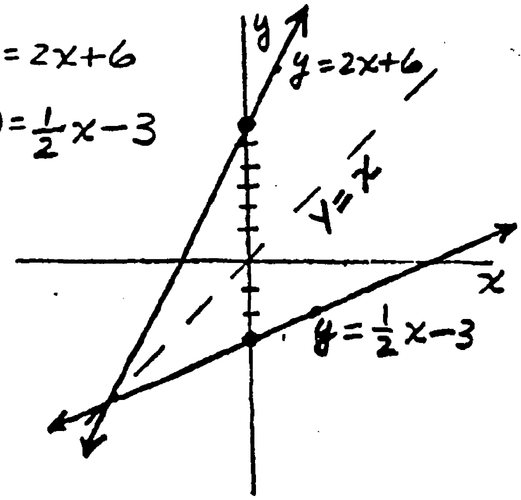
25. $y = \sqrt{4 - x^2}$
[Assume $x \geq 0$ and $y \geq 0$]

26. $y = (4 - \sqrt{x})^2$
[Assume $x \geq 0$ and $y \geq 0$]

An interesting characteristic of a function $f(x)$ and its inverse $f^{-1}(x)$ is the reflection of these graphs. The graphs will always be symmetric about the line $y = x$, as the following graphs illustrate.

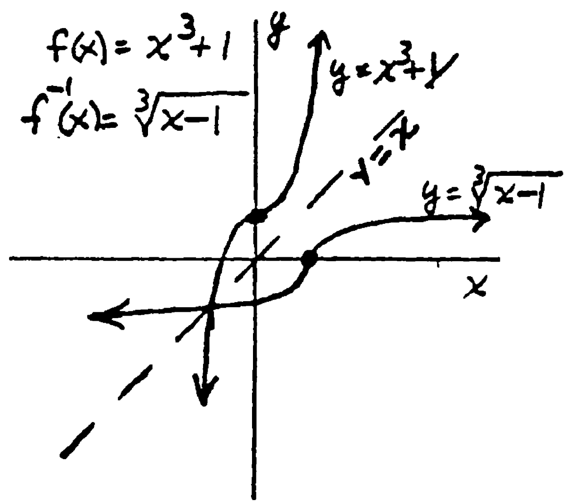
$f(x) = 2x + 6$

$f^{-1}(x) = \frac{1}{2}x - 3$




$f(x) = x^3 + 1$

$f^{-1}(x) = \sqrt[3]{x-1}$




QUESTION: Will the inverse of a function $f(x)$ always be a function?

ANSWER: Consider $f(x) = x^2$, (that is $y = x^2$).

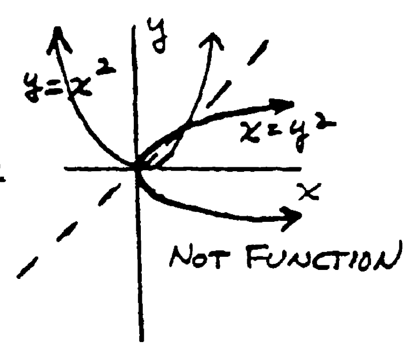
① Interchange variables: $x = y^2$

② Solve for $y$: $y^2 = x$

$y = \pm\sqrt{x}$

NOT A FUNCTION!

Moreover, the graph:




CONCLUSION: The inverse of a <u>continuous</u> function $f(x)$ will itself be a function if and only if the function is always increasing or always decreasing.

CONCLUSION: The inverse of a function will itself be a function if and only if the function $f(x)$ is a one-to-one function.

CONCLUSION :

The domain of $f(x) \Rightarrow$ range of $f^{-1}(x)$

The range of $f(x) \Rightarrow$ domain of $f^{-1}(x)$.

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9. $f^{-1}(x) = \frac{x+3}{5}$ 10. $f^{-1}(x) = \frac{5-x}{3}$ 11. $f^{-1}(x) = \frac{2x-3}{5}$

12. $f^{-1}(x) = -5x+3$ 13. $f^{-1}(x) = \frac{1}{x-1}$ 14. $f^{-1}(x) = \frac{4}{5x-3}$

15. $f^{-1}(x) = \frac{x+1}{x}$ (See # 13) 16. $f^{-1}(x) = \frac{3x+4}{5x}$ (See # 14)

17. $f^{-1}(x) = \sqrt[3]{3x-8}$ 18. $f^{-1}(x) = \sqrt[3]{5x+8}$

19. $f^{-1}(x) = \frac{x^3+8}{3}$ (See # 17) 20. $f^{-1}(x) = \frac{x^3-8}{5}$ (See # 18)

21. $f^{-1}(x) = \frac{5}{9}(x-32)$ or $f^{-1}(x) = \frac{5x-160}{9}$

22. $f^{-1}(x) = \frac{9}{5}x+32$ or $\frac{9x+160}{5}$ (See # 21)

23. $f^{-1}(x) = \frac{1}{x}$ 24. $f^{-1}(x) = -x$ 25. $f^{-1}(x) = \sqrt{4-x^2}$

26. $f^{-1}(x) = (4-\sqrt{x})^2$

[NOTE: In 23-26, each $f^{-1}(x) = f(x)$.]

p.333-339: Practice Tests. See detailed solutions p.336, 340.

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE