3.01 *Polynomial Division, Synthetic Division*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

You probably remember from your previous algebra background that there are two types of division problems: **division by a monomial** and **division by a polynomial.** When dividing by a **monomial**, such as $\frac{a+b+c}{d}$, simply break it into separate fractions: $\frac{a+b+c}{d} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d}$, and, if possible, reduce each fraction. When dividing by a polynomial (two or more terms), use the method known as **long division** or perhaps the shortcut **synthetic division.** Consider the example: $\frac{X^4 + 5X^3 + 6X - 2}{X + 4}$.

Remember first to **arrange in descending** (or ascending) **powers of the variable** and **if there are terms missing, insert place-holder zeros or spaces for the missing terms.** Remember, after setting it up in this way, there are four steps in the process:

LONG DIVISION
1. Divide first into first.
2. Multiply.
3. Change the signs and add.
4. Bring down the next terms.
Repeat the process.

$$
\begin{array}{r|l}
 & x^3 + x^2 - 4x + 22 - \frac{90}{x+4} \\
\hline
X + 4 & X^4 + 5X^3 + 0X^2 + 6X - 2 \\
 & -x^4 - 4x^3 \\
\hline
 & \quad x^3 + 0x^2 + 6x - 2 \\
 & \quad -x^3 - 4x^2 \\
\hline
 & \qquad -4x^2 + 6x - 2 \\
 & \qquad +4x^2 + 16x \\
\hline
 & \qquad\qquad 22x - 2 \\
 & \qquad\qquad -22x - 88 \\
\hline
 & \qquad\qquad\quad -90
\end{array}
$$

The shortcut of **synthetic division** may be used if the **divisor** is of the form **X - a** (of course this includes **X + a**). In the method of synthetic division, as with regular long division, be sure the terms are in descending order, with zeros as placeholders for missing terms. Then write down the coefficients only of the dividend. In the current example, write down the coefficients **1 5 0 6 -2** . To write the divisor X + 4 in the form X - a, let **a** = -4. In other words, if the divisor is X + 4, you perform synthetic division using -4. If the divisor is X - 4, you perform synthetic division using +4. After writing the coefficients as illustrated below, bring down the first coefficient, then repeatedly multiply, add, multiply, add, multiply, add, etc. until you run out of numbers.

Bring down "1"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & & & & \\ \hline & 1 & & & & \end{array}$$

Mult "1" times "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & & & \\ \hline & 1 & & & & \end{array}$$

Add "5" and "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & & & \\ \hline & 1 & 1 & & & \end{array}$$

Mult "1" times "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & & \\ \hline & 1 & 1 & & & \end{array}$$

Add "0" and "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & & \\ \hline & 1 & 1 & -4 & & \end{array}$$

Mult "-4" times "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & \\ \hline & 1 & 1 & -4 & & \end{array}$$

Add "6" and "16"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & \\ \hline & 1 & 1 & -4 & 22 & \end{array}$$

Mult "22" times "-4"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & -88 \\ \hline & 1 & 1 & -4 & 22 & \end{array}$$

Add "-2" and "-88"

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & -88 \\ \hline & 1 & 1 & -4 & 22 & -90 \end{array}$$

The numbers obtained on the last line above are the coefficients of the quotient, with the last number being the remainder. The exponent of the first term will always be one less than the degree of the dividend. Therefore, the answer is $1X^3+1X^2-4X+22$ r. -90.

In the illustration below, notice the similarity (and amount!) of the work in the long division process illustrated on the left and the synthetic division process on the right:

LONG DIVISION

$$\begin{array}{rl} & x^3 + x^2 - 4x + 22 - \frac{90}{x+4} \\ X + 4 \;\big)\!\! & \cancel{X^4} + 5X^3 + 0X^2 + 6X - 2 \\ & -\cancel{x^4} - 4x^3 \\ \hline & \cancel{x^3} + 0x^2 + 6x - 2 \\ & -\cancel{x^3} - 4x^2 \\ \hline & -\cancel{4x^2} + 6x - 2 \\ & +4x^2 + 16x \\ \hline & 22\cancel{x} - 2 \\ & -\cancel{22x} - 88 \\ \hline & -90 \end{array}$$

SYNTHETIC DIVISION

$$\begin{array}{r|rrrrr} -4 & 1 & 5 & 0 & 6 & -2 \\ & \downarrow & -4 & -4 & 16 & -88 \\ \hline & 1 & 1 & -4 & 22 & -90 \end{array}$$

$$1x^3 + 1x^2 - 4x + 22 - \frac{90}{x+4}$$

Perform the divisions, using synthetic division if appropriate.

1. $\dfrac{X^3 - 4X^2 + 5X - 19}{X - 4}$

2. $\dfrac{X^3 - 5X^2 - 6X - 2}{X - 3}$

3. $\dfrac{2X^3 + 5X^2 + 6X - 2}{X + 3}$

4. $\dfrac{2X^3 + 5X^2 - 6X + 2}{X + 4}$

5. $\dfrac{X^4 + 5X^2 - 6X - 2}{X - 2}$

6. $\dfrac{X^4 - 5X^2 + 7X + 2}{X + 1}$

7. $\dfrac{X^5 - 1}{X + 1}$

8. $\dfrac{X^5 - 1}{X - 1}$

9. $\dfrac{12X^4 - 2X^3 - 7X - 2}{3X + 4}$

10. $\dfrac{12X^4 - X^3 - 29X^2 - 4}{3X - 4}$

11. $\dfrac{X^4 - 4X^3 + 5X^2 - 2X - 2}{X^2 - 2X + 2}$

12. $\dfrac{X^4 - 4X^3 + 5X^2 - 2X - 2}{X^2 - 2X - 1}$

THE REMAINDER THEOREM, FACTOR THEOREM
(Division and Substitution)

EXAMPLE 1: **Find the remainder if $P(X) = X^5 - 1$ is divided by $X + 1$. Then, find $P(-1)$.**

Solution:

$$\begin{array}{r|rrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & -1 \\ & \downarrow & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -2 = R \end{array}$$

??

$$P(-1) = (-1)^5 - 1 = -2$$

EXAMPLE 2: **Find the remainder if $P(X) = X^3 - 4X^2 + 5X - 19$ is divided by $X - 2$. Then, find $P(2)$.**

Solution:

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 5 & -19 \\ & \downarrow & 2 & -4 & 2 \\ \hline & 1 & -2 & 1 & -17 = R \end{array}$$

??

$$P(2) = 2^3 - 4(2)^2 + 5(2) - 19$$
$$= 8 - 16 + 10 - 19 = -17$$

SPECULATION: **Let $P(X)$ be divided by $X - r$, yielding a remainder of R. Then, $P(r) = R$.**

APPLICATION: **The method of substitution can be used instead of synthetic division (to find a remainder) and vice versa.**

REMAINDER THEOREM

If P(X) is divided by X-r, resulting in a quotient Q(X) and a remainder R, then P(r) = R .

PROOF:

$$\frac{P(X)}{X-r} = Q(X) + \frac{R}{X-r}$$

$$P(X) = Q(X)\cdot(X-r) + R$$

$$P(r) = Q(r)\cdot(r-r) + R$$

$$P(r) = Q(r)\cdot 0 + R$$

$$P(r) = R$$

In 1 - 4, use substitution as a short-cut for synthetic division to find the remainder. Check by synthetic division. (Which way was easier??)

1. Find the remainder if $X^5 - 3X + 1$ is divided by $X - 1$.

$P(X) = x^5 - 3x + 1$

$P(1) =$

$=$ $= R$

Check by syn. division:

$1 \rfloor \; 1 \;\; 0 \;\; 0 \;\; 0 \;\; -3 \;\; 1$

$\downarrow$

$\bigcirc = R$

2. **Find the remainder if $X^5 - 3X + 1$ is divided by $X - 2$.**

$P(x) = x^5 - 3x + 1$

$P(\quad) =$

3. **Find the remainder if $X^4 + 6X + 2$ is divided by $X + 2$.**

4. **Find the remainder if $X^4 + 6X + 2$ is divided by $X - 3$.**

FACTOR THEOREM

If P(X) is divided by X-r, resulting in a remainder of 0, then X-r is a factor of P(X). Furthermore, IF P(r)= 0, THEN X-r IS A FACTOR OF P(X).

In 5 - 9, use the method of substitution to:

5. determine if $X-1$ is a factor of $X^5 - 1$.

$P(x) = x^5 - 1$

$P(1) =$

6. determine if $X+1$ is a factor of $X^5 - 1$.

7. determine if $X+1$ is a factor of $X^5 - 2X + 1$.

8. determine if $X-1$ is a factor of $X^5 - 2X + 1$.

9. determine if $X+2$ is a factor of $X^5 - 3X^3 + 8$.

In 10 - 11, use <u>synthetic division</u> as a short-cut for substitution to evaluate the polynomial. Check by substitution.

10. If $P(X) = X^5 + 6X^4 - 30X^3 - 20X^2 - 70X - 40$, use synthetic division to find $P(4)$.

$$\begin{array}{r|rrrrrr} 4 & 1 & 6 & -30 & -20 & -70 & -40 \\ & \downarrow & & & & & \\ \hline & 1 & & & & & \bigcirc = P(4) \end{array}$$

$P(4) = 4^5 + 6 \cdot 4^4 - 30 \cdot 4^3 - 20 \cdot 4^2 - 70 \cdot 4 - 40$

$=$

11. If $P(X) = X^5 - 5X^4 + 7X^3 + 19X^2 - 16X + 3$, use synthetic division to find $P(3)$.

In 12-16, use <u>synthetic division</u> as a shortcut:

12. If $P(X) = X^4 - 10X^3 + 12X^2 + 8X + 3$, find $P(4)$.

$4\,|$

13. If $P(X) = X^4 - 10X^3 + 12X^2 - 8X + 3$, find $P(6)$.

14. If $P(X) = X^4 + 10X^3 + 12X^2 + 2X + 3$, find $P(-4)$.

15. If $P(X) = X^4 + 10X^3 + 12X^2 - 52X - 240$, find $P(-6)$.

16. If $P(X) = X^4 + 7X^3 + 15X^2 + 52X - 12$, find $P(-6)$.

3.01 p. 344-345:

1. $x^2 + 5 + \frac{1}{x-4}$; 2. $x^2 - 2x - 12 - \frac{38}{x-3}$;

3. $2x^2 - x + 9 - \frac{29}{x+3}$; 4. $2x^2 - 3x + 6 - \frac{22}{x+4}$;

5. $x^3 + 2x^2 + 9x + 12 + \frac{22}{x-2}$; 6. $x^3 - x^2 - 4x + 11 - \frac{9}{x+1}$;

7. $x^4 - x^3 + x^2 - x + 1 - \frac{2}{x+1}$; 8. $x^4 + x^3 + x^2 + x + 1$;

9. $4x^3 - 6x^2 + 8x - 13 + \frac{50}{3x+4}$; 10. $4x^3 + 5x^2 - 3x - 4 - \frac{20}{3x-4}$;

11. $x^2 - 2x - 1$; 12. $x^2 - 2x + 2$.

p. 347-351:

1. -1	2. 27	3. 6	4. 101
5. Yes	6. No	7. No	8. Yes
9. Yes	10. 0	11. 153	12. -157
13. -477	14. -197	15. -360	16. 0