3.04 *Factoring by Synthetic Division*
Rational Root Theorem, Descartes Rule

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

At the beginning of this chapter (section 3.01), you performed polynomial division and synthetic division, and you were told that synthetic division can be used as means of factoring polynomial expressions. Consider the following example.

Suppose you knew that X-1 is a factor of $X^3 - 2X^2 - 5X + 6$. If you divide $X^3 - 2X^2 - 5X + 6$ by X-1, you will get a remainder of 0 (of course), and the other factor which is quadratic, as illustrated below. The easiest way to divide is by synthetic division.

1. Factor $X^3 - 2X^2 - 5X + 6$, given that X-1 is a factor.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & \downarrow & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

The remainder is zero (of course!)

The quotient is $X^2 - 1X - 6$, which can be factored:

(   )(   ).

Therefore, the answer is **(   )(   )(   ).**

In 2-10, factor each polynomial expression.

2. $X^3 + 4X^2 + X - 6$, given that X-1 is a factor.

3. $X^3 - 4X^2 + X + 6$, given that X+1 is a factor.

4. $X^3 - 3X^2 - 22X + 24$, given that $X+4$ is a factor.

5. $X^3 - 13X + 12$, given that $X-1$ is a factor.
[Hint: Don't forget the zero coefficent of X^2.]

6. $X^3 - 52X + 96$, given that $X-2$ is a factor.

7. $2X^3 - 7X^2 + 7X - 2$, given that $X-2$ is a factor.

8. $3X^3 - X^2 - 3X + 1$, given that $X-1$ is a factor.

9. Factor $3X^3 - X^2 - 3X + 1$ by the method of grouping.

10. $X^3 + 19X^2 + 114X + 216$ given that $X+4$ is a factor.

This method can also be applied to solving polynomial equations in which one of the roots is known. For example:

11. Solve the equation $X^3 - 3X^2 - 22X + 24 = 0$, given that $X = -4$ is a root of the equation.

Solution: **Since $X = -4$ is a root of $X^3 - 3X^2 - 22X + 24 = 0$, this means that $(X+4)$ is a factor of $X^3 - 3X^2 - 22X + 24$. Now proceed as in the previous exercises.** ***The difference is that before the problem was to factor an expression, and now it is to solve an equation.*** **The procedure is the same, but the form of the answer is different.**

$$\begin{array}{r|rrrr} -4 & 1 & -3 & -22 & 24 \\ & \downarrow & -4 & 28 & -24 \\ \hline & 1 & -7 & 6 & 0 \end{array}$$

As anticipated the remainder is zero, and the quadratic equation that remains (called the reduced or the depressed equation) is $X^2 - 7X + 6 = 0$.

Solving: ( )( ) = 0

X = ____; X = ____

Therefore, the total answer is X = ____, ____, or ____.

In 12- 32, solve the polynomial equations:

12. $X^3 + 2X^2 - 5X - 6 = 0$, given that $X = -1$ is a root.

13. $X^3 - 8X^2 + 19X - 12 = 0$, given that $X = 3$ is a root.

14. $X^3 + 2X^2 - 13X + 10 = 0$, **given that** $X = 2$ **is a root.**

15. $2X^3 + 3X^2 - 3X - 2 = 0$, **given that** $X = 1$ **is a root.**

16. $3X^3 - 2X^2 - 3X + 2 = 0$, **given that** $X = 1$ **is a root.**

[After thought: Could this have been factored by grouping?]

It is frequently necessary to apply synthetic division more than once in order to solve certain polynomial equations. Consider the next exercise.

17. $X^4 + 2X^3 - 7X^2 - 8X + 12 = 0$, given that $X=1$ and $X=-2$ are roots.

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & -7 & -8 & 12 \\ & \downarrow & 1 & 3 & -4 & -12 \\ \hline & 1 & 3 & -4 & -12 & 0 \end{array}$$

The remainder is zero.

Reduced equation: $x^3 + 3x^2 - 4x - 12 = 0$

Now:

$$\begin{array}{r|rrrr} -2 & 1 & 3 & -4 & -12 \\ & \downarrow & -2 & -2 & 12 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

(Again $R=0$!)

Solve: $x^2 + x - 6 = 0$

$(x \quad)(x \quad) = 0$

$x =$ ___ $x =$ ___

TOTAL ANSWER IS $x =$ ________________

18. $X^4 - X^3 - 16X^2 + 4X + 48 = 0$, given that $X=2$ and $X=4$ are roots.

19. $X^4 - 4X^3 - 15X^2 + 58X - 40 = 0$, given that $X=1$ and $X=-4$ are roots.

20. $X^4 + 2X^3 - 16X^2 - 2X + 15 = 0$, given that $X=1$ and $X=3$ are roots.

21. $X^3 + 5X^2 + 9X + 5 = 0$, given that $X=-1$ is a root.
[NOTE: Frequently the quadratic formula or completing the square is necessary--sometimes even complex numbers!]

22. $X^3 + 3X^2 + 4X - 8 = 0$, given that $X=1$ is a root.

23. $X^4 - 3X^3 + 3X^2 - 3X + 2 = 0$, given that $X=1$ and $X=2$ are roots.

24. $X^4 + X^3 + 2X^2 + 4X - 8 = 0$, given that $X=1$ and $X=-2$ are roots.

25. $X^4 + 6X^3 + 10X^2 - 2X - 15 = 0$, given that $X=1$ and $X=-3$ are roots.

26. $X^4 - X^3 - 2X^2 + X + 1 = 0$, given that $X=1$ and $X=-1$ are roots.

27. $X^3 + 2X^2 - 7X + 4 = 0$, given that $X=1$ is a root.
[NOTE: Frequently there may be multiple roots. Be sure to give multiplicity of roots.]

28. $X^3 - 2X^2 - 7X - 4 = 0$, given that $X=4$ is a root.

29. $X^4 - 8X^3 + 23X^2 - 28X + 12 = 0$, given that there is a double root at $X=2$.

30. $X^4 - 6X^3 + 13X^2 - 12X + 4 = 0$, given that there is a double root at $X=2$.

31. $X^4 - 4X^3 + 6X^2 - 4X + 1 = 0$, **given that there is a multiple root at** $X=1$.

32. $X^4 + 8X^3 + 24X^2 + 32X + 16 = 0$, **given that there is a multiple root at** $X=-2$.

Perhaps by now you have noticed that, counting complex roots and multiplicity of roots, X^2 equations have 2 solutions, X^3 equations have 3 solutions, X^4 equations have 4 solutions, and in general X^n equations have "n" solutions.

Also, in the equations given so far, you were always given one or more roots so you would be able to do sythetic division to find the other roots. Perhaps you wondered if there is a way of knowing what the roots are. There is a way of determining the roots, but it is a trial and error process. It may help to work a problem "backwards." Consider the equation (X-1)(X-2)(X+6)=0, which has roots at X=1, X=2, and X=-6. If this equation were multiplied out, you would get the expanded equation $X^3+3X^2-16X+12=0$. Notice that in the process of multiplying (X-1)(X-2)(X+6)=0

to obtain $\mathbf{X^3}+3X^2-16X\mathbf{+12}=0$,

the $\mathbf{X}^3$ came from **X·X·X** and **"+12"** came from **(-1)·(-2)·(+6)**. Working the problem backwards beginning with

$$X^3 + 3X^2 - 16X + \mathbf{12} = 0$$

to obtain a factored form: ( )( )( ) = 0

you can see that the numbers you need for the factored form must be "±" factors of **"+12."** It is the constant term, then, that determines the possible roots of the equation, provided the leading coefficient (coefficient of the highest power term) is **1**.

In general. the equation $\mathbf{1 \cdot X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 = 0}$ has possible rational roots only at **X = ±(factors of a_0)**. It then is a process of trial and error to decide which ones work!

33. **Find all <u>possible</u> rational roots for $X^3+3X^2-16X+12 = 0$.**
Factors of 12: ± ____, ±____, ±____, ±____, ±____, ±____

34. **Find all <u>possible</u> rational roots for $X^4-4X^3-15X^2+58X-40 = 0$.**

Factors of 40: ±

35. **Find all <u>possible</u> rational roots for $X^4+13X^3+59X^2+107X+60 = 0$.**

Factors of _____:

36. **Find all <u>possible</u> rational roots for $X^4+3X^3-8X^2-12X+16 = 0$.**

Factors of _____:

If the leading coefficient is other than 1, then the process becomes a bit more complicated. Again, it may help to work a problem backwards. Consider the equation:

$$(3X - 5)(2X - 1)(2X - 3) = 0,$$ which in expanded form becomes $12X^3 - 44X^2 + 49X - 15 = 0$.

Notice as before that when

$$(3X - 5)(2X - 1)(2X - 3) = 0$$ is multiplied

out to become $$12X^3 - 44X^2 + 49X - 15 = 0$$ it is the

$(3X)\cdot(2X)\cdot(2X)$ that becomes $12X^3$

and $(-5)\cdot(-1)\cdot(-3)$ that becomes -15.

In reverse, this means that to go from the

expanded form $12X^3 - 44X^2 + 49X - 15 = 0$

to factored form $(3X - 5)(2X - 1)(2X - 3) = 0$

$5\cdot1\cdot3 = 15$

with roots $X = \frac{5}{3},\ X = \frac{1}{2},\ X = \frac{3}{2},$

$3\cdot2\cdot2 = 12$

it is the **constant coefficient** **(-15)** that determines the factors in the numerators of the roots, and it is the **leading coefficient** **(12)** that determines the denominator factors of the roots.

RULE: The equation $a_n\cdot X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 = 0$ has possible roots at $X = \pm\, p/q$, where p is a factor of a_0 and q is a factor of a_n.

37. Find all possible rational roots for $6X^3+19X^2-19X+14 = 0$.

Numerator factors of 14: ±1, ±2, ±7, ±14

Denominator factors of 6: 1, 2, 3, 6 (Do you need ± ?)

Possible roots: ± 1, ± 2, ± 7, ± 14;

± 1/2, ± 7/2 ;

± 1/3, ± 2/3, ± 7/3, ± 14/3;

± 1/6, ± 7/6.

[Notice that all fractions that reduced were already listed!]

38. Find all possible rational roots for $3X^3-2X^2-7X-2 = 0$.

Numerator factors of 2: ±1, ±2

Denominator factors of 3: 1, 3

Possible roots:

39. Find all possible rational roots for $12X^3-44X^2+49X-15 = 0$.

Numerator factors of _____:

Denominator factors of _____:

Possible roots:

40. Find all possible rational roots for $6X^3+19X^2-19X+4 = 0$.

Numerator factors of 4: ±1, ±2, ±4

Denominator factors of 6: 1, 2, 3, 6 (Do you need ± ?)

Possible roots:

In 41-65, solve the polynomial equations. Use a trial and error approach to determine some possible roots. Use synthetic division to see if the remainder is zero. This process also gives you the depressed equation which can then be used to find the rest of the roots.

41. $X^3 - 2X^2 - 5X + 6 = 0$ [Hint: try factors of ______]
Remember--trial and error! Keep trying until remainder = 0.

42. $X^3 + 4X^2 + X - 6 = 0$

43. $X^3 + 3X^2 - 4X - 12 = 0$

44. $X^4 + 2X^3 - 7X^2 - 8X + 12 = 0$
[Hint: You will need to use two successive divisions.]

45. $X^3 - 5X^2 + 7X - 3 = 0$
[A value that gives a remainder of 0 can be tried again!]

46. $X^3 + X^2 - 8X - 12 = 0$

47. $X^4 + X^3 - 7X^2 - 13X - 6 = 0$

48. $X^4 - 5X^3 + X^2 + 21X - 18 = 0$

49. $X^5 - 6X^4 + 6X^3 + 20X^2 - 39X + 18 = 0$

50. $X^5 - 6X^4 + 5X^3 + 16X^2 - 12X - 16 = 0$

51. $X^4 + 2X^3 - 2X^2 - 3X + 2 = 0$

52. $X^4 + 4X^3 + 4X^2 - 4X - 5 = 0$

53. $X^5 - X^4 - 7X^3 + 11X^2 - 8X + 12 = 0$

54. $X^5 + 4X^4 - 2X^3 - 14X^2 - 3X - 18 = 0$

55. $3X^3 - 2X^2 - 7X - 2 = 0$

56. $3X^3 - 10X^2 + X + 6 = 0$

57. $12X^3 + 8X^2 - 29X - 15 = 0$
[Hint: No integral roots!]

58. $12X^3 - 44X^2 + 49X - 15 = 0$
[Hint: No integral roots!]

When a polynomial equation has rational coefficients**, as in this section they do, all complex or irrational roots must occur in conjugate pairs. This means that if $X = 3 + 4i$ is a root of an equation with rational coefficients, then $X = 3 - 4i$ is also a root of this equation. If $X = 2 - \sqrt{3}$ is a root of such an equation, then $X = 2 + \sqrt{3}$ is also a root of this equation. This also implies that if a polynomial equation has complex or irrational roots, there must be an even number of such roots. To find complex roots when there is more than one pair of irrational or complex roots, ti is necessary that you be given one of the roots. There are two methods.

** In the rest of this section, all polynomial equations will be assumed to have rational coefficients.

59. Find the roots of $X^4 - 4X^3 + 5X^2 - 2X - 2 = 0$ given that $X = 1 - i$ is a root. [Note: other root = $1 + i$]

METHOD I: (Synthetic Division Method)

$$
\begin{array}{r|rrrrr}
1-i & 1 & -4 & 5 & -2 & -2 \\
 & \downarrow & 1-i & -4+2i & 3+i & 2 \\
\hline
 & 1 & -3-i & 1+2i & 1+i & 0
\end{array}
$$

$$\underbrace{1 \quad -3-i \quad 1+2i \quad 1+i}_{\text{Reduced Equation}} \qquad \underset{\text{Remainder}}{0}$$

SCRATCH PAPER

$(1-i)(-3-i)$
$= -3+2i+i^2$
$= \underline{\underline{-4+2i}}$

$(1-i)(1+2i)$
$= 1+i-2i^2$
$= \underline{\underline{3+i}}$

$(1-i)(1+i)$
$= 1-i^2 = \underline{\underline{2}}$

$$
\begin{array}{r|rrrr}
1+i & 1 & -3-i & 1+2i & 1+i \\
 & \downarrow & 1+i & -2-2i & -1-i \\
\hline
 & 1 & -2 & -1 & 0
\end{array}
$$

$$x^2-2x-1=0$$

Solve by completing square or quadratic formula.

You finish:

59. **Find the roots of $X^4 - 4X^3 + 5X^2 - 2X - 2 = 0$ given that $X = 1 - i$ is a root.**

METHOD II: (Method of Quadratic Factors)

$$x = 1 - i \qquad x = 1 + i$$

$$(x-1+i) = 0 \qquad (x-1-i) = 0$$

$$(x-1+i)(x-1-i) = 0$$

$$\left[(x-1)+i\right]\left[(x-1)-i\right] = 0$$

$$(x-1)^2 - i\cancel{(x-1)} + i\cancel{(x-1)} - i^2 = 0$$

$$x^2 - 2x + 1 \quad + 1 = 0$$

$x^2 - 2x + 2$ is one quadratic factor.

Use long division to find the other factor:

$$x^2 - 2x - 1 = \text{other quadratic factor}$$

$$\begin{array}{r|l}
x^2-2x+2 & x^4 - 4x^3 + 5x^2 - 2x - 2 \\
 & \underline{-x^4 + 2x^3 - 2x^2} \\
 & \quad -2x^3 + 3x^2 - 2x - 2 \\
 & \quad \underline{+2x^3 - 4x^2 + 4x} \\
 & \qquad\quad -x^2 + 2x - 2 \\
 & \qquad\quad \underline{+x^2 - 2x + 2} \\
\end{array}$$

Now solve $x^2 - 2x - 1 = 0$ as before

In 60-64, solve by the preferred method:

60. $X^4 + 2X^3 - X^2 - 2X + 10 = 0$ **given** $X = -2 + i$

other root = ________

61. $X^4 + 4X^3 + 23X^2 + 10X + 250 = 0$ **given** $X = 1 - 3i$

62. $X^4 + 2X^3 - 4X^2 + 4X + 4 = 0$ given $-2 - \sqrt{2}$

63. $X^4 + 4X^3 + 40X + 32 = 0$ given $1 - i\sqrt{7}$

64. $X^4 + 4X^3 - 7X^2 + 14X + 6 = 0$ given $-3 + \sqrt{7}$

 # 65 EXTRA CHALLENGE

65. $X^7 + 2X^6 - 10X^5 - 12X^4 + 37X^3 + 10X^2 - 52X + 24 = 0$

In 66 - 76, use synthetic division to find zeros of the polynomial functions and sketch their graphs.

66. $Y = X^3 + 4X^2 + X - 6$

67. $Y = X^3 - 4X^2 + X + 6$

68. $Y = X^3 - 13X + 12$

69. $Y = X^4 - X^3 - 16X^2 + 4X + 48$

70. $Y = X^4 + 2X^3 - 16X^2 - 2X + 15$

71. $Y = X^3 + 2X^2 - 7X + 4$

72. $Y = X^3 - 5X^2 + 7X - 3$

73. $Y = X^5 - X^4 - 2X^3 + 2X^2 + X - 1$

74. $Y = X^5 + X^4 - 2X^3 - 2X^2 + X + 1$

EXTRA CHALLENGE

75. $Y = X^6 - 7X^5 + 12X^4 + 14X^3 - 59X^2 + 57X - 18$

76. $Y = X^6 - 2X^5 - 4X^4 + 6X^3 + 7X^2 - 4X - 4$

DESCARTES' RULE of SIGNS

It is frequently helpful to know the number of positive or negative real roots of a polynomial equation. This can be determined by <u>Descartes' Rule of Signs</u>:

Given $P(x) = +x^5 - 7x^4 + 3x^3 + x^2 - 6 = 0$,

count the number of sign changes from left to right (ignore any zero terms). The number of positive real roots will equal the number of sign changes or will be less than this number by some even number. In this case there will be either 3 or 1 positive real roots.

To find the number of negative real roots, substitute $(-x)$ for x and determine the number of sign changes:

$$P(-x) = (-x)^5 - 7(-x)^4 + 3(-x)^3 + (-x)^2 - 6 = 0$$
$$= -x^5 - 7x^4 - 3x^3 + x^2 - 6$$

Therefore there will be either 2 or 0 negative real roots.

The possibility of decreasing the number of sign changes "by some even number" above is necessary to allow for the possibility of complex roots, which, if they exist, are always in conjugate pairs.

UPPER and LOWER BOUNDS

A particularly useful application of Descartes' Rule of Signs is the case in which all terms of the polynomial are positive:

$$P(x) = x^7 + 3x^5 + 2x^4 + 5x^3 + 2x + 12$$

Since all of the signs are positive, there are no sign changes and therefore there are no positive real roots. In the search for rational roots, half of the possibilities are eliminated.

It is also helpful in search for rational roots to establish an <u>upper</u> and <u>lower bound</u>. (This is to further restrict the possible roots that must be tested.)

If synthetic division is performed with a positive number resulting in all positive or zero terms in the quotient line of the division, then this <u>positive</u> number is an <u>upper bound</u>.

EXAMPLE:

$$\begin{array}{r|rrrrr} 4 & 1 & -3 & 3 & -2 & -5 \\ & \downarrow & 4 & 4 & 28 & 104 \\ \hline & 1 & 1 & 7 & 26 & 99 \end{array}$$

Since all terms here are positive or zero, $x=4$ is an upper bound of $x^4-3x^3+3x^2-2x-5$. There will be no roots greater than $x=4$.

To establish a lower bound, you must perform synthetic division with a negative number resulting in terms in the quotient line alternating in sign (you may disregard zero terms!)

EXAMPLE:

$$\begin{array}{r|rrrrr} -2 & 1 & -3 & 3 & -2 & -5 \\ & \downarrow & -2 & 10 & -26 & 56 \\ \hline & 1 & -5 & 13 & -28 & 51 \end{array}$$

Since these terms alternate in sign and $x=-2$ is negative, $x=-2$ is a lower bound. There will be no roots less than -2.

Therefore, since $x^4-3x^3+3x^2-2x-5=0$ has lower bound of -2 and upper bound of 4, any real or rational roots, (if there are any!), must be between -2 and 4.

Finally, it is possible to "close in" on a real root by comparing two successive synthetic divisions (or substitutions).

EXAMPLE: $x^4+2x^3-16x^2-2x+15=0$

$$\begin{array}{r|rrrrr} 2 & 1 & 2 & -16 & -2 & 15 \\ & \downarrow & 2 & 8 & -16 & -36 \\ \hline & 1 & 4 & -8 & -18 & -21 \end{array} \qquad \begin{array}{r|rrrrr} 4 & 1 & 2 & -16 & -2 & 15 \\ & \downarrow & 4 & 24 & 32 & 120 \\ \hline & 1 & 6 & 8 & 30 & 135 \end{array}$$

Because $x=2$ gives a negative remainder and $x=4$ gives a positive remainder, there is a root <u>between</u> $x=2$ and $x=4$.

Theorem: Given $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 = 0$.
If $a < b$, and $P(a)$ and $P(b)$ are of opposite sign, then there is at least one real root between a and b.

This theorem really opens the door to using technology (computers and calculators) to find roots of polynomial equations. The computer can easily be programmed to find an interval $[a, b]$ where there is a sign change between $f(a)$ and $f(b)$. Then this interval is split in half, and the half-interval with the sign change is selected. Then this half-interval is split in half, ... Eventually the interval becomes so small that the root is determined to a prescribed accuracy.

There are more sophisticated and faster ways to find roots in higher mathematics, but this is probably the simplist and obvious method.

3.04 p. 378-408:

1. $(x-1)(x-3)(x+2)$ 2. $(x-1)(x+2)(x+3)$ 3. $(x+1)(x-2)(x-3)$

4. $(x+4)(x-6)(x-1)$ 5. $(x-1)(x-3)(x+4)$ 6. $(x-2)(x+8)(x-6)$

7. $(x-2)(2x-1)(x-1)$ 8. $(x-1)(x+1)(3x-1)$ 9. $(x-1)(x+1)(3x-1)$

10. $(x+4)(x+6)(x+9)$ 11. $-4, 6, 1$ 12. $-1, 2, -3$

13. $3, 1, 4$ 14. $2, 1, -5$ 15. $1, -2, -\frac{1}{2}$ 16. $1, -1, \frac{2}{3}$

17. $1, -2, 2, -3$ 18. $2, 4, -2, -3$ 19. $1, -4, 2, 5$ 20. $1, 3, -1, -5$

21. $-1, -2 \pm i$ 22. $1, -2 \pm 2i$ 23. $1, 2, \pm i$ 24. $1, -2, \pm 2i$

25. $1, -3, -2 \pm i$ 26. $1, -1, \frac{1 \pm \sqrt{5}}{2}$ 27. 1 (mult 2), -4

28. -1 (mult 2), 4 29. 2 (mult 2), $1, 3$ 30. 2 (mult 2), 1 (mult 2)

31. 1 (mult 4) 32. -2 (mult 4) 33. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

34. $\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 20, \pm 40$

35. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 12, \pm 15, \pm 20, \pm 30, \pm 60$

36. $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$ 38. $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

39. $\pm 1, \pm 3, \pm 5, \pm 15; \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}; \pm \frac{1}{3}, \pm \frac{5}{3}; \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{5}{4}, \pm \frac{15}{4};$
$\pm \frac{1}{6}, \pm \frac{5}{6}; \pm \frac{1}{12}, \pm \frac{5}{12};$ 40. $\pm 1, \pm 2, \pm 4; \pm \frac{1}{2}; \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}; \pm \frac{1}{6}$

41. $1, 3, -2$ 42. $1, -2, -3$ 43. $2, -2, -3$ 44. $1, 2, -2, -3$

45. 1 (mult 2), 3 46. -2 (mult 2), 3 47. -1 (mult 2), $3, -2$

48. 3 (mult 2), $1, -2$ 49. 3 (mult 2), 1 (mult 2), -2

50. 2 (mult 2), -1 (mult 2), 4 51. $1, -2, \frac{-1 \pm \sqrt{5}}{2}$

52. $1, -1, -2 \pm i$ 53. 2 (mult 2), $-3, \pm i$

54. -3 (mult 2), $2, \pm i$ 55. $2, -1, -\frac{1}{3}$ 56. $1, 3, -\frac{2}{3}$

57. $\frac{3}{2}, -\frac{1}{2}, -\frac{5}{3}$ 58. $\frac{1}{2}, \frac{3}{2}, \frac{5}{3}$ 59. $1 \pm i, 1 \pm \sqrt{2}$

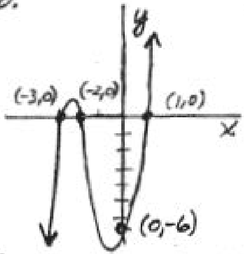
60. $-2 \pm i, 1 \pm i$ 61. $1 \pm 3i, -3 \pm 4i$ 62. $-2 \pm \sqrt{2}, 1 \pm i$

63. $1 \pm \sqrt{7}i, -3 \pm \sqrt{5}$ 64. $-3 \pm \sqrt{7}, 1 \pm i\sqrt{2}$

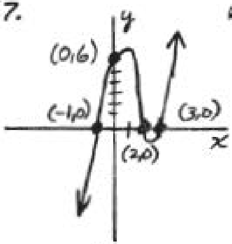
65. 1 (mult 3), -2 (mult 2), $2, -3$

3.04 p.378-408:

66.



67.



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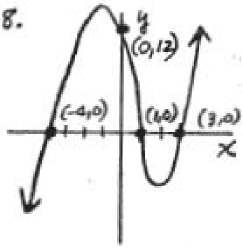

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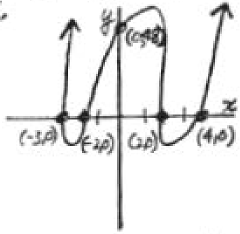

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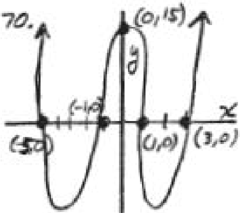

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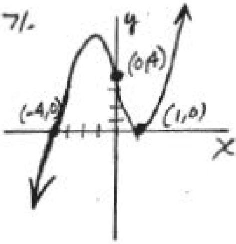

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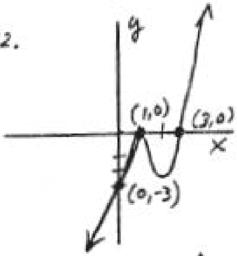

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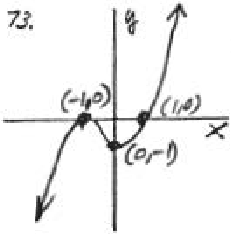

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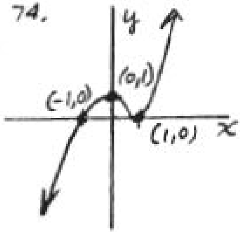

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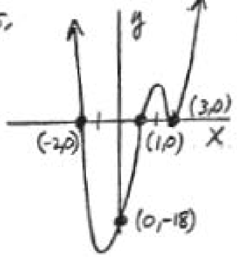

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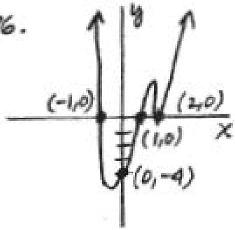

Dr. Robert J. Rapalje

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE