3.06 *Polynomial and Fractional Inequalities*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

The topic of inequalities (linear, quadratic, and absolute value) was previously introduced in section 1.11. Now, after the study of polynomial equations and functions, it is appropriate to continue with a more advanced section on inequalities, which includes polynomial and fractional expressions.

The strategy, as before, is to:

1. **find all endpoints**
2. **set the inequality to zero, and graph "Y1=_____."**
3. **determine which intervals between the endpoints are above or below the X-axis.**
4. **determine if the endpoints are included or not included**
5. **give the final solution in interval notation.**

For **polynomial inequalities**, the endpoints are obtained by simply solving the corresponding **polynomial equation.** For **fractional equations**, the there will be "asymptotes" so the graphs and the process in general is somewhat more complicated. More on that later . . .

When graphing Y1 > 0, shade ABOVE the X-axis.

When graphing Y1 ≥ 0, shade ON or ABOVE the X-axis.

When graphing Y1 < 0, shade BELOW the X-axis.

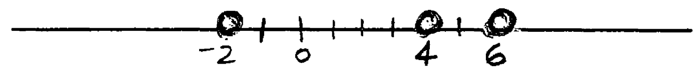
When graphing Y1 ≤ 0, shade ON or BELOW the X-axis.

EXAMPLE I. $(X - 4)(X + 2)(X - 6) < 0$ BELOW X-axis!

Change to equation: $(X - 4)(X + 2)(X - 6) = 0$

Find endpoints: $X = 4;\ X = -2;\ X = 6$

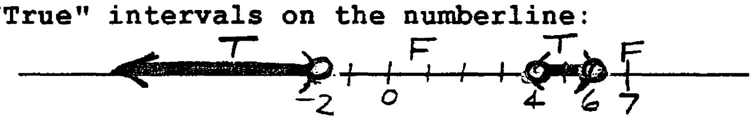
On numberline:



Test one point from each interval in the original inequality:

X = 7:	$(7 - 4)(7 + 2)(7 - 6) < 0$		
	$+ \quad + \quad + \quad < 0$	**False**	
X = 5:	$(5 - 4)(5 + 2)(5 - 6) < 0$		
	$+ \quad + \quad - \quad < 0$	**True**	
X = 3:	$(3 - 4)(3 + 2)(3 - 6) < 0$		
	$- \quad + \quad - \quad < 0$	**False**	
X = -3:	$(-3 - 4)(-3 + 2)(-3 - 6) < 0$		
	$- \quad - \quad - \quad < 0$	**True**	

Indicate the "True" intervals on the numberline:



Notice that endpoints are not included (because of "<").

Give answer in interval notation: $(-\infty, -2) \cup (4,6)$

Did you notice that in the above example, the intervals alternated "False," "True," "False," "True." This alternating occurred because each endpoint was a **root of odd multiplicity**. Thus, **it was only necessary to test one interval, and then alternate "False," "True," "False," etc.**

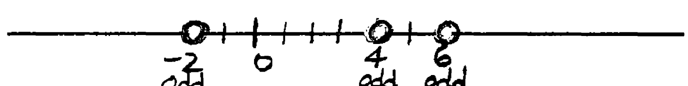
Easier way to solve Example I:

Change to equation: $(X - 4)(X + 2)(X - 6) = 0$

Find endpoints: $X = 4;\ X = -2;\ X = 6$

Multiplicity: odd odd odd

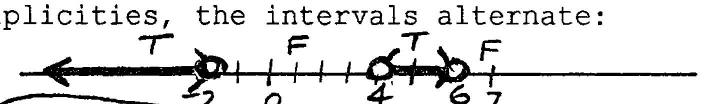
On numberline:



Test X = 7: $(7 - 4)(7 + 2)(7 - 6) < 0$ **False**

Because of odd multiplicities, the intervals alternate:

On numberline:



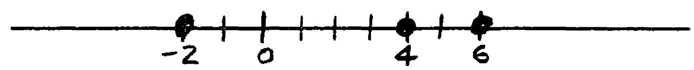
Interval notation: $(-\infty, -2) \cup (4,6)$

EXAMPLE II. $(X - 4)^2(X + 2)(X - 6) \le 0$ On or Below x-axis!

Change to equation: $(X - 4)^2(X + 2)(X - 6) = 0$

Find endpoints: $X = 4;\ X = -2;\ X = 6$

On numberline:



Test one point from each interval in the original inequality:

$X = 7$: $(7 - 4)^2(7 + 2)(7 - 6) \le 0$

$+ \quad + \quad + \quad \le 0$ **False**

$X = 5$: $(5 - 4)^2(5 + 2)(5 - 6) \le 0$

$+ \quad + \quad - \quad \le 0$ **True**

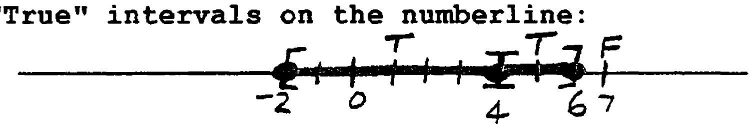
$X = 3$: $(3 - 4)^2(3 + 2)(3 - 6) \le 0$

$+ \quad + \quad - \quad \le 0$ **True**

$X = -3$: $(-3 - 4)^2(-3 + 2)(-3 - 6) \le 0$

$+ \quad - \quad - \quad \le 0$ **False**

Indicate the "True" intervals on the numberline:



Notice that endpoints <u>are</u> included (because of "≤").

Give answer in interval notation: [-2,6]

Did you notice that in this ~~example~~, the intervals did **<u>not</u>** alternate? The intervals did **not** alternate because of the **even multiplicity of the root at X = 4.** Again, this provides a shortcut, and again, **it is <u>not</u> necessary to test a point in each interval.** Rather, just **alternate "True/False" at endpoints of odd multiplicity,** and **keep the truth value the same (True/True or False/False) at endpoints of even multiplicity,** as follows:

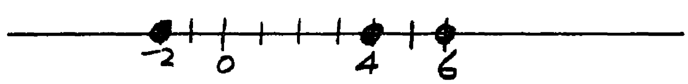
Easier way to solve Example II:

Change to equation: $(X - 4)^2(X + 2)(X - 6) = 0$

Find endpoints: $X = 4;\ X = -2;\ X = 6$

Multiplicity: even odd odd

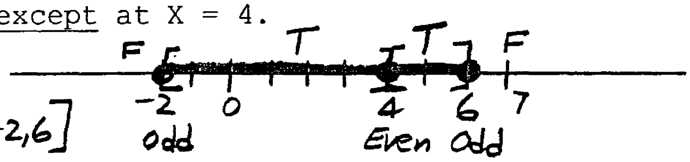
On numberline:



Test X = 7: $(7 - 4)^2(7 + 2)(7 - 6) \le 0$ False

Intervals alternate <u>except</u> at X = 4.

On numberline:



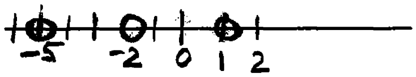
Interval notation: [-2,6]

EXERCISES:

1. $(X + 2)(X - 1)(X + 5) > 0$ — *y1 above x-axis!*

Endpts:
Mult:

Include endpts?

N'line: 

Test X = 2:

Interval notation:

2. $(X - 2)(X + 3)(X - 4) < 0$ — *y1 below x-axis!*

Test X = 5:

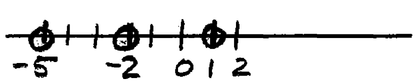
3. $(X - 5)(X + 2)(X - 1) \le 0$ — *y1 on or below x-axis!*

4. $(X + 2)(X + 4)(X - 3) \ge 0$ — *y1 on or above x-axis!*

5. $(X + 2)(X - 1)(X + 5)^2 > 0$

Endpts:
Mult:

Include endpts?

N'line: 

Test X = 2:

Interval notation:

6. $(X - 2)^2(X + 3)(X - 4) < 0$

Test X = 5:

7. $(X - 5)^3(X + 2)(X - 1)^2 \le 0$

8. $(X + 2)^3(X + 4)^2(X - 3) \ge 0$

9. $(X + 2)^2(X - 1)^4(X + 5) \ge 0$

10. $(X - 2)(X + 3)^4(X - 4)^2 < 0$

11. $X^3 - 4X^2 - 4X + 16 \le 0$

12. $X^3 + 2X^2 - 9X - 18 \ge 0$

13. $X^3 + 4X^2 - 16X - 64 < 0$

14. $X^3 - 3X^2 - 9X + 27 \geq 0$

15. $X^4 - 10X^2 + 9 \geq 0$

16. $X^4 - 10X^2 + 9 < 0$

For **fractional inequalities**, there are **two sources of endpoints**:

1. **solve corresponding equations** (as before with polynomial inequalities), and

2. **set denominators not equal to zero.**

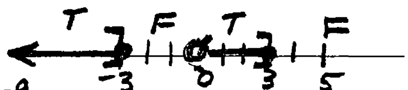
Exercises and examples to illustrate these concepts continue on the next page.

17. $\frac{X^2 - 9}{X} \le 0$

Denom endpt: (Never included) $x \neq 0$ On or Below x axis!

Other endpts: (Included!) $X^2 - 9 = 0$

N'line:



Test X=5: $\frac{25-9}{5} \le 0$ False

Intervals alternate! Yes

Interval notation: (,] ∪ (,]

18. $\frac{X}{X^2 - 9} \ge 0$ On or above x-axis!

19. $\frac{X^2 - 9}{X^2 - 4X} < 0$

20. $\frac{X^2 - 4X}{X^2 - 9} \ge 0$

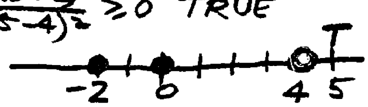
21. $\frac{X(X + 2)}{(X - 4)^2} \ge 0$ On or Above x-axis!

Denom endpts: $x \neq 4$
Mult: Even
Included? No

Other endpts: $x=0$ $x=-2$
Mult: odd odd
Included? Yes Yes

Test $x=5$: $\frac{5(5+2)}{(5-4)^2} \ge 0$ TRUE

N'line:



Int. notation:

22. $\frac{(X - 4)^2}{X(X + 2)} \ge 0$

23. $\frac{X^3(X-3)^4}{X^2-4} \geq 0$

24. $\frac{X^3(X-3)^4}{X^2-4} < 0$

Consider the innocent-looking problem $\frac{4}{X} < 2$. The obvious approach (and **wrong!!**) is to multiply both sides of the equation by the variable **X**. Remember that if you multiply both sides of an inequality by a **negative** number, then you must **reverse the direction of the inequality sign.** However, if you multiply both sides of an inequality by a **positive** number, then you do **not** reverse the direction of the inequality sign! So, what if you multiply both sides of an inequality by a **variable, which could be either positive or negative?** You just don't know!! Therefore, you should **never multiply both sides of an inequality by a variable**, unless you know that the variable is always positive (or negative)!

PRINCIPLE
Never multiply both sides of an inequality by a variable, unless the variable is known to be always positive (or always negative)!

Example III on the next page illustrates a correct method of solving the problem $\frac{4}{X} < 2$ following the general summary for inequalities below.

SOLVING INEQUALITIES SUMMARY

I. **Find all endpoints.**

   A. **Set inequal to 0, and simplify. Set numerators equal to 0.**

   B. **Set denominators not equal to 0.**

II. **Determine if endpoints are included or not included.**

   A. **Denominator endpoints are NEVER included.**

   B. **Other endpoints are included for ≤ or ≥; not included for < or >.**

III. **Determine which intervals between endpoints are "True" and which are "False."**

   A. **You may either test one point in each interval, OR**

   B. **Use "multiplicity of roots" to determine "sign changes."**

IV. **Give answer in interval notation.**

EXAMPLE III. $\frac{4}{X} < 2$

I. Find all endpoints:

A. Set inequality to 0: $\frac{4}{X} - 2 < 0$ *Below x-axis!*

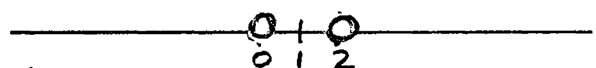
$$\frac{4 - 2X}{X} < 0$$

Set numerator = 0: $4 - 2X = 0$

$X = 2$ (Endpt <u>not</u> included!)

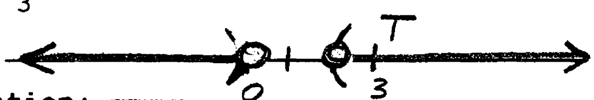
B. Set denominator ≠ 0: $X \neq 0$ (Denom endpts <u>never</u> included!)
Summary of endpoints: $X = 2$; $X = 0$.
multiplicity: odd odd

Numberline:



Test X = 3: $\frac{4}{3} < 2$, *True* (Intervals alternate)

Numberline:



Interval notation: $(-\infty, 0) \cup (2, \infty)$

EXAMPLE IV.

I. Find all endpoints: $\frac{9}{X} \leq X$

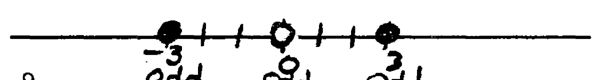
A. Set inequality to 0: $\frac{9}{X} - X \leq 0$ *On or below x-axis!*

$$\frac{9 - X^2}{X} \leq 0$$

Set numerator = 0: $X = \pm 3$ (Endpts included!)

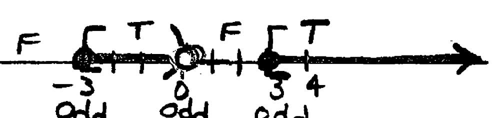
B. Set denominator ≠ 0: $X \neq 0$ (Denom endpts <u>never</u> included!)
Summary of endpoints: $X = 3$; $X = -3$; $X \neq 0$.
multiplicity: odd odd odd

Numberline:



Test X = 4: $\frac{9}{4} \leq 4$, *True* (Intervals alternate)

Numberline:



Interval notation: $[-3, 0) \cup [3, \infty)$

25. $\frac{4}{X} \geq 2$

26. $\frac{12}{X} < 3$

Find endpts: (On or Above!)

A. Set to 0: $\frac{4}{x} - 2 \geq 0$

$\frac{4-2x}{x} \geq 0$

B. Denom ≠ 0:

27. $\frac{12}{X} < 3X$

28. $\frac{12}{X} \geq 3X$

29. $\frac{12}{X^2 - 4X} > 1$

30. $\frac{12}{X^2 + 4X} < 1$

31. $\frac{12}{X^2 + 4X} \le 1$

32. $\frac{12}{X^2 - 4X} \ge 1$

33. $\frac{6}{X^2 + 4X} \ge \frac{1}{X}$

34. $\frac{6}{X^2 - 4X} \ge \frac{1}{X}$

ANSWERS 3.06

p.416-424:

1. $(-5,-2)\cup(1,\infty)$; **2.** $(-\infty,-3)\cup(2,4)$; **3.** $(-\infty,-2]\cup[1,5]$;

4. $[-4,-2]\cup[3,\infty)$; **5.** $(-\infty,-5)\cup(-5,-2)\cup(1,\infty)$;

6. $(-3,2)\cup(2,4)$; **7.** $[-2,5]$; **8.** $(-\infty,-2]\cup[3,\infty)$; **9.** $[-5,\infty)$

10. $(-\infty,-3)\cup(-3,2)$; **11.** $(-\infty,-2]\cup[2,4]$; **12.** $[-3,-2]\cup[3,\infty)$;

13. $(-\infty,-4)\cup(-4,4)$; **14.** $[-3,\infty)$; **15.** $(-\infty,-3]\cup[-1,1]\cup[3,\infty)$;

16. $(-3,-1)\cup(1,3)$; **17.** $(-\infty,-3]\cup(0,3]$; **18.** $(-3,0]\cup(3,\infty]$;

19. $(-3,0)\cup(3,4)$; **20.** $(-\infty,-3)\cup[0,3)\cup[4,\infty)$;

21. $(-\infty,-2]\cup[0,4)\cup(4,\infty)$; **22.** $(-\infty,-2)\cup(0,\infty)$;

23. $(-2,0]\cup(2,\infty)$; **24.** $(-\infty,-2)\cup(0,2)$; **25.** $(0,2]$;

26. $(-\infty,0)\cup(4,\infty)$; **27.** $(-2,0)\cup(2,\infty)$; **28.** $(-\infty,-2]\cup(0,2]$;

29. $(-2,0)\cup(4,6)$; **30.** $(-\infty,-6)\cup(-4,0)\cup(2,\infty)$;

31. $(-\infty,-6]\cup(-4,0)\cup[2,\infty)$; **32.** $[-2,0)\cup(4,6]$;

33. $(-\infty,-4)\cup(0,2]$; **34.** $(-\infty,0)\cup(4,10]$.

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