3.07 *Systems of Equations (2X2) and Inequalities*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

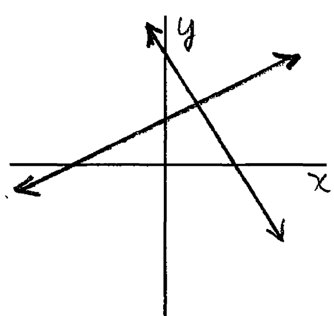
When solving systems of linear equations (that is, two equations and two unknowns), **three types of solutions** are possible. These types are described and illustrated below.

I. The two equations may represent lines that intersect in a **single point**. There is a **unique solution**. This is called an **independent system.**

II. The two equations may represent lines that are **parallel**. In this case, there is **no solution** or **∅**. This is called an **inconsistent system.**

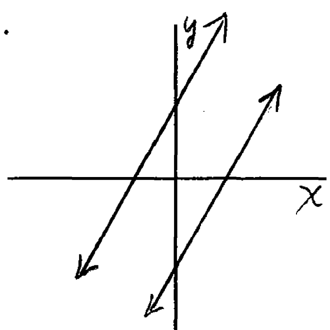
III. The two equations may represent the **same line**. In this case, there are **infinitely many solutions** to the system of equations. This is called a **dependent system.**

I.



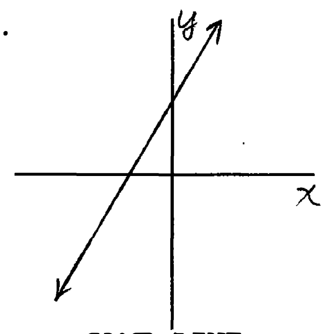
UNIQUE SOLUTION

II.



PARALLEL LINES

III.



SAME LINE

The most common methods of solving such systems include the **elimination method** (also known as the "addition" method), the **substitution method**, and the **graphical method.** If both equations are given in **standard form** (i.e., **AX + BY = C**), then usually elimination is the easiest method. If one or more of the equations is given in the **slope-intercept form** (i.e., **Y = mX + b**) or in the form of **X =** ________, then substitution is usually the easiest method. Yet another method involving **determinants** known as **Cramer's Rule** will be introduced in the next section.

EXAMPLE 1: Solve by the Elimination Method.

$$\begin{aligned} \mathbf{5X + 3Y} &= \mathbf{14} \quad \text{The common multiple of X coef. is 45} \\ \mathbf{9X + 4Y} &= \mathbf{7} \quad \text{The common multiple of Y coef. is 12} \end{aligned}$$

It is easier to eliminate the Y terms. Multiply both sides of the first equation by **4** and multiply both sides of the second equation by **-3** to eliminate the Y term.

$$\begin{aligned} 4(5X + 3Y) &= 4(14) \quad \text{or} \quad & 20X + 12Y &= 56 \\ -3(9X + 4Y) &= -3(7) \quad \text{or} \quad & -27X - 12Y &= -21 \\ & & -7X &= 35 \\ & \text{so} & X &= -5 \end{aligned}$$

Now substitute into the first equation:

$$\begin{aligned} 5X + 3Y &= 14 \\ 5(-5) + 3Y &= 14 \\ -25 + 3Y &= 14 \\ 3Y &= 39 \\ Y &= 13 \end{aligned}$$

Check:

$$\begin{aligned} 9X + 4Y &= 7 \\ 9(-5) + 4(13) &= 7 \\ -45 + 52 &= 7 \quad \checkmark \end{aligned}$$

The solution is the point (-5,13).

EXAMPLE 2: Solve by the Elimination Method.

$$\begin{aligned} \mathbf{3X + 5Y} &= \mathbf{2} \quad \text{To eliminate Y, mult. first eq. by } \mathbf{-2}. \\ \mathbf{6X + 10Y} &= \mathbf{-2} \end{aligned}$$

$$\begin{aligned} -2(3X + 5Y) &= -2(2) \quad \text{or} \quad & -6X - 10Y &= -4 \\ 6X + 10Y &= -2 \quad \text{or} \quad & 6X + 10Y &= -2 \\ & & 0 &= -6 \end{aligned}$$

Whenever eliminating one variable "by chance" results in the elimination of <u>both</u> variables, and an impossible statement such as **0 = -6**, or **0 = any non-zero number,** there is **No Solution** possible. This is the case of the **two parallel lines.**

Whenever eliminating one variable "by chance" results in the elimination of <u>both</u> variables <u>and</u> the constants (number terms) as well, then the statement **0 = 0** results. This statement is an **identity**, which means there are **many solutions.** In fact, this is the case in which the two equations represent the **same line**. The solution is the entire line.

EXAMPLE 3: Solve by the Substitution Method.

Since the second equation is in the form **X =** _____, the substitution method is appropriate for this problem.

$$5Y - 3X = 34$$
$$X = 7 - 2Y$$

Rewrite the first equation: $5Y - 3(\quad) = 34$

and substitute 7-2Y for X: $5Y - 3(7-2Y) = 34$

$$5Y - 21 + 6Y = 34$$
$$11Y - 21 = 34$$
$$11Y = 55$$
$$Y = 5$$

The best place to find X is to substitute **Y=5** into **X = 7 - 2Y**:

$$X = 7 - 2Y$$
$$X = 7 - 2(5)$$
$$X = 7 - 10$$
$$X = -3$$

Check: $5Y - 3X = 34$ (You must use the other equation!)

$$5(5) - 3(-3) = 34$$
$$25 + 9 = 34$$

EXERCISES: In #1-12, solve the systems of equations by the "appropriate" method. Indicate if the equations represent parallel lines or the same line.

1. $3X + 7Y = 6$
   $2X + 3Y = -1$

2. $-3X + 7Y = 4$
   $2X - 3Y = -6$

3. $9X - 4Y = 2$
$2X + 5Y = -29$

4. $50X - 9Y = 1$
$7X - 2Y = -8$

5. $2X - 6Y = 12$
$-X + 3Y = -6$

6. $X = 3Y + 18$
$6Y - 2X = 36$

7. $5X - 4Y = 22$
$Y = -4X + 5$

8. $-8X + 6Y = 32$
$X = 2Y + 6$

9. $17X + 8Y = 4$
$32X + 18Y = -16$

10. $4X - 2Y = -8$
$2X - Y = -4$

11. $4X - 2Y = 8$
$Y = 2X + 4$

12. $12Y + 5X = 41$
$X = 4 - 3Y$

When graphing a line whose equation is in the form **$Y = mX + b$,** it is usually easiest to use the **Y-intercept** and **slope** to draw the line. If the equation is in **standard form**, **$AX + BY = C$**, then it is usually easiest to find the **X** and **Y-intercepts.**

When graphing a **linear inequality** there are essentially three steps:

I. Change the inequality to an equation and graph the line.

A. If **Y = mX + b**, then use **Y-intercept/slope** method.
B. If **AX + BY = C** form, then use **intercepts** method.

II. Decide whether the line is included or not included.

A. If "<" or ">", then use a **dotted line**.
B. If "≤" or "≥", then use a **solid line**.

III. Decide whether to **shade above or below** the line.

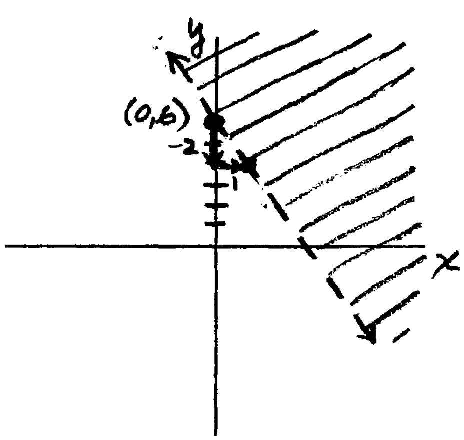
A. If the equation has a **positive Y-coefficient** and "<" or "≤", then **shade** <u>**below**</u> **the graph** of the line.
B. If the equation has a **positive Y-coefficient** and ">" or "≥", then **shade** <u>**above**</u> **the graph** of the line.
C. If the equation has a **negative Y-coefficient**, then multiply both sides of the inequality by -1, which reverses the direction of the inequality sign. Then shade above or below as indicated.

EXAMPLE 4: Graph Y > -2X + 6

Solution: I. Graph Y = -2X + 6
Easiest to use slope-intercept method
Y-int = (0,6); m = -2
(See below!)

II. Use a dotted line

III. Since +Y > , you must shade above the line.

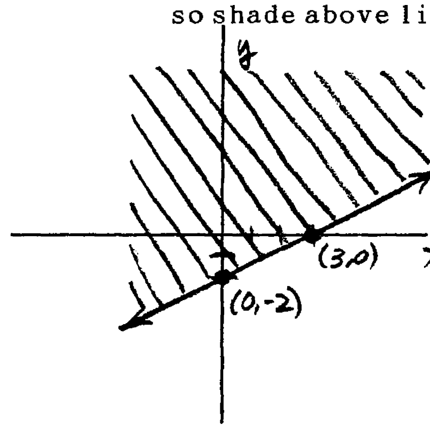


EXAMPLE 5: Graph 2X - 3Y ≤ 6

Solution: I. Graph 2X - 3Y = 6
Easiest to use two intercepts method
X-int = (3, 0)
Y-int = (0,-2)

II. Use a solid line

III. Since -Y ≤, divide both sides by -1. This means +Y ≥ , so shade above line.



EXERCISES. Graph each of the following inequalities. Shade the appropriate areas.

1. $Y < 3X + 2$

2. $Y > -2X + 4$

3. $Y \geq -X - 4$

4. $Y \leq 2X - 4$

5. $-X + 3Y \leq 6$

6. $3X + 2Y < -12$

7. $3X - Y < -6$

8. $-2X - Y \leq 8$

9. $3X - 4Y > 12$ 10. $3X - 2Y \geq -12$

It is frequently necessary to graph a **system of inequalities**--that is, two or more inequalities with two unknowns. In such problems, there will be two or more regions to graph, with instructions to shade the **union** or the **intersection** of the regions. As before, remember that the **union** includes all shaded regions, while the **intersection** of the regions includes only the regions common to both (or all) of the shadings. It will be very helpful to use colored pencils, using a different color for each equation/shaded region.

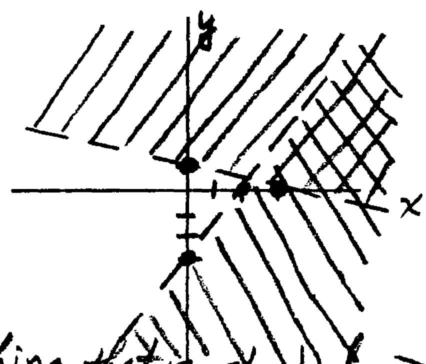
EXAMPLE 6a) Graph the region represented by the union of

$$3X - 2Y > 6$$
$$X + 3Y > 3$$

$3x - 2y = 6$

x	y
0	-3
2	0

Dotted line. Shade below the line.



$x + 3y = 3$

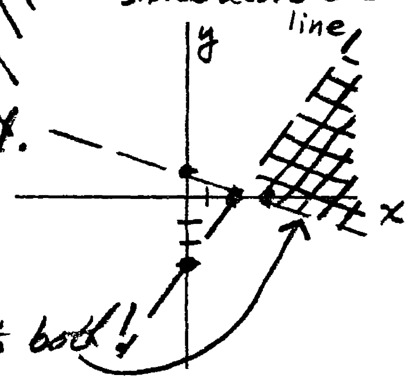
x	y
0	1
3	0

Dotted line. Shade above the line.

Union is everything that is shaded.

EXAMPLE 6b) Graph the intersection:

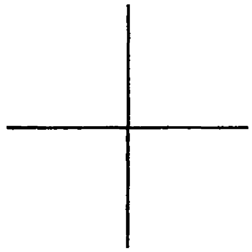
$$3X - 2Y > 6$$
$$X + 3Y > 3$$



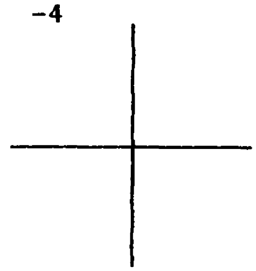
Only what is common to both!

In 11 - 14, find the region represented by the unions of the following inequalities.

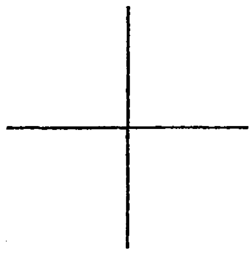
11. $Y > X + 2$
$Y < -X$



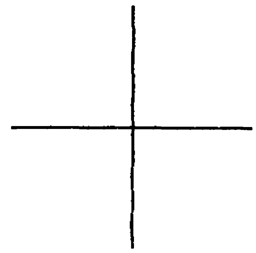
12. $4X + Y \le -8$
$X - 2Y \le -4$



13. $2X - 3Y \ge -12$
$-Y \le 4X - 8$

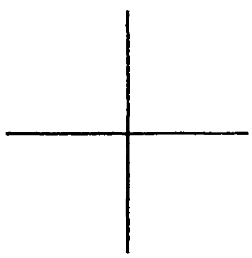


14. $Y < 4X - 4$
$Y > -2X + 4$

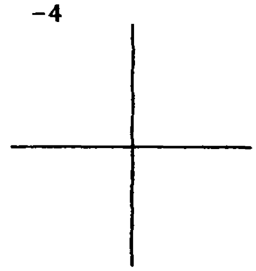


In 15 - 18, find the region represented by the intersections of each of the following.

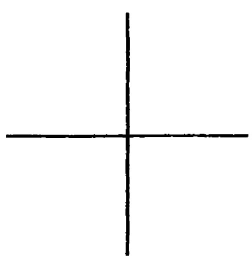
15. $Y \le -X + 2$
$Y \ge X$



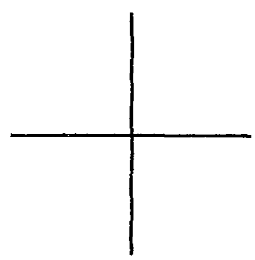
16. $4X - Y > -8$
$X + 2Y < -4$



17. $2X - 3Y < -12$
$-Y > 4X - 8$



18. $Y \ge -4X + 4$
$Y \le 2X - 4$



EXAMPLE 7: Graph the region represented by the intersection of

$$3X + 2Y < 12$$
$$X \geq 0$$
$$Y \geq -3$$

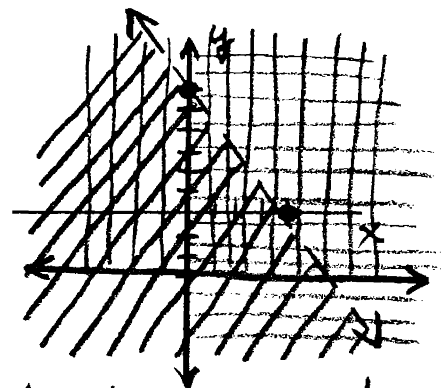
$3x + 2y = 12$

x	y
0	6
4	0

Dotted line
Shade below.

$x = 0$
y axis
Solid
Shade to right.

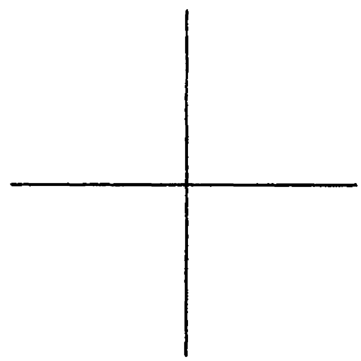
$y = -3$
Horizontal line
Solid
Shade above



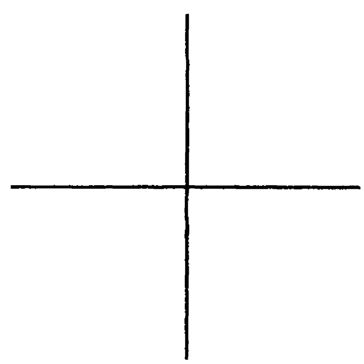
(Solution is triangular area.)

In 19 - 20, graph the region represented by the intersections of each of the following inequalities.

19. $3X - 2Y \leq 12$
$X \geq 0$
$Y \leq 0$



20. $X + 3Y > -6$
$X < 2$
$Y < 3$



EXAMPLE 8: Graph the region represented by the intersection of

$$\left.\begin{array}{r} 3X - 2Y < 6 \\ X + 2Y < 2 \\ X + Y > -3 \end{array}\right\}$$

[Hint: Use 3 colors!]

$3x - 2y = 6$

x	y
0	-3
2	0

Dotted
Shade above

$x + 2y = 2$

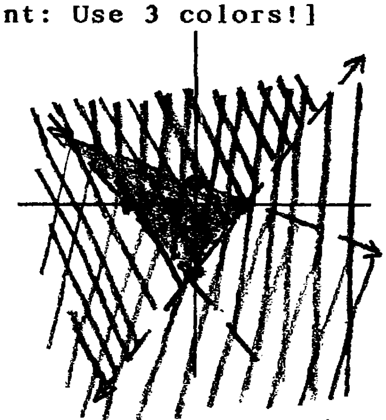
x	y
0	1
2	0

Dotted
Shade below

$x + y = -3$

x	y
0	-3
-3	0

Dotted
Shade above



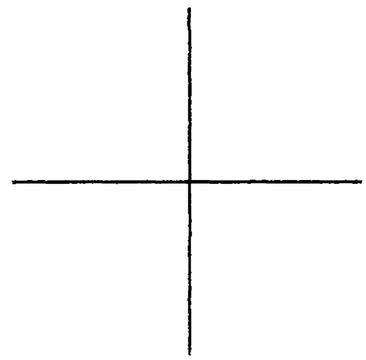
(Solution is triangular area!)

In 21 - 28, graph the region represented by the intersections of each of the following inequalities.

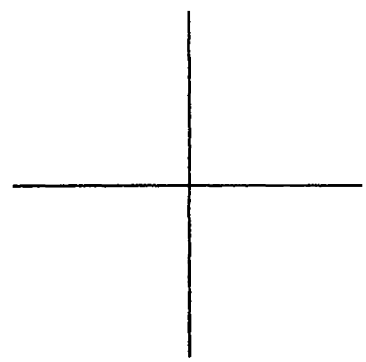
21. $3X - 2Y \le 6$
$X + 2Y \le 2$
$X \ge 0$

22. $X + 2Y \ge -4$
$X - 2Y \ge -4$
$Y \le -3X + 2$

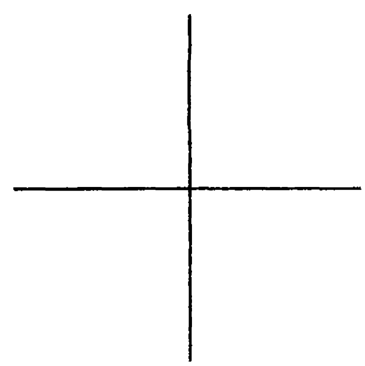
23. $Y < 3X + 2$
$Y < -3X + 2$
$4X - Y < 8$



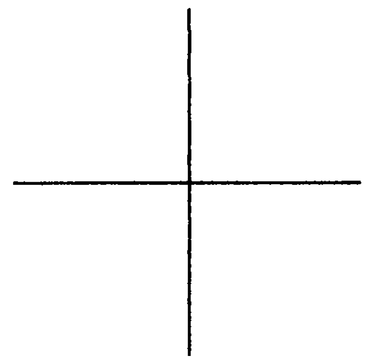
24. $2X - 3Y < 6$
$X + Y < 3$
$Y < 2X - 2$



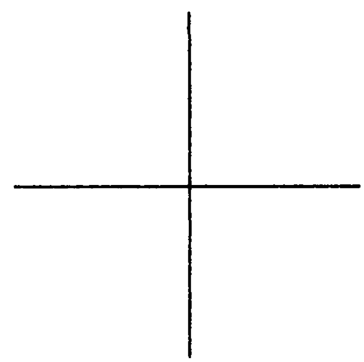
25. $X + 2Y < -4$
$X - 2Y > -4$
$Y < -3X + 2$



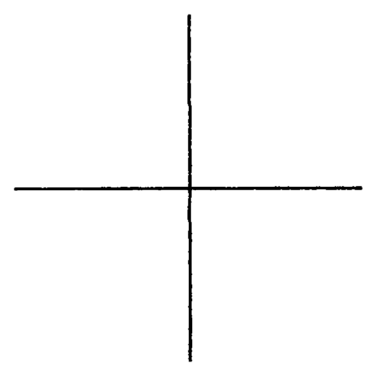
26. $2X - 3Y \leq 6$
$X + Y \geq 3$
$Y \leq 2X - 2$



27. $X - Y \leq -2$
$2X + 3Y \geq 12$
$X - 3Y \geq 6$



28. $X - Y \geq -6$
$X + Y \leq 6$
$X \geq -2$
$Y \geq 0$



APPLICATIONS OF TWO EQUATIONS AND TWO UNKNOWNS

Many of the word problems from Section I can also be solved using two equations and two unknowns. In Section I, the procedure was to state, "Let x = _____" and then to express everything in terms of x. In this section, the procedure is to identify two variables, "Let x = ______, and let y = ______". In order to solve for the two variables, it will be necessary to have <u>two</u> equations.

Not all problems from Section I can be conveniently solved by the methods of this section. For example, if a problem contained three unknowns, then three equations would be required to solve it (beyond the scope of this book). While there are no new problems in this section, it does give depth in problem solving techniques.

Solve the following exercises using two equations and two unknowns.

1. The sum of two numbers is 25. Their difference is 11. Find the numbers.

   SOLUTION: Let x = First number
   y = Second number

   Write two equations from the problem:

   $$\begin{aligned} x + y &= 25 \\ \underline{x - y} &\underline{= 11} \end{aligned}$$

   Solve: $2x$ = ______ (Addition method, eliminates y)

   x = ______

   Solve for y:

   $$\begin{aligned} x + y &= 25 \\ 18 + y &= 25 \\ y &= \_\_\_\_\_ \end{aligned}$$

   Check original word problem: Sum of x and y = __________.

   Difference of x and y = __________.

2. The sum of two numbers is 18. Three times the first number plus twice the second number is 32.

SOLUTION: Let x = First number
y = Second number

Write two equations from the problem:

$$x + y = 18$$
$$3x + 2y = 32$$

Solve:

Check:

3. The sum of two numbers is 120, and their difference is 24. Find the numbers.

4. The sum of two numbers is 48. The second number is four less than three times the first. Find the numbers.

5. The perimeter of a rectangle is 96. The length is three more than twice the width. Find the dimensions of the rectangle.

SOLUTION: Let x = width
y = length

$2x + 2y =$ ________
$y = 2x + 3$

.Solve for x and y:

6. The perimeter of a rectangle is 100. The length is two more than three times the width. Find the dimensions of the rectangle.

7. A box contains 24 coins, some quarters and the rest dimes. The value of the coins is $3.90. How many of each coin are there?

SOLUTION: Let x = number of quarters
y = ____________________

Two Equations:	$x + y = 24$	Number of coins
	$25x + 10y = 390$	Value of coins (in cents)

Solve:

8. A box containing 30 coins in nickels and quarters has a value of \$2.70. How many of each coin are there?

9. A sum of money was invested at 8% simple interest, and three times as much at 10%. The total interest earned for the year was \$190. How much was invested at each rate?

SOLUTION: Let x = investment at 8%
y = investment at 10%

Equations:

$$y = 3x$$
$$.08x + .10y = 190.00$$

10. A sum of money was invested at 12% simple interest, and \$1000 more than twice this amount at 10%. The total interest earned for the year was \$260. How much was invested at each rate?

11. A total of $10,000 was invested, some at 12% and the rest at 10% simple interest. The total interest earned for the year was $1060. How much was invested at each rate?

SOLUTION: Let x = ____________________

y = ____________________

Equations: $x + y =$ ______________

$.12x + .10y =$ ______________

Solve:

12. A total of $2500 was invested, part at 12% simple interest, and the rest at 10%. The total interest earned for the year was $260. How much was invested at each rate?

SOLUTION: Let x = ____________________

y = ____________________

Equations:

Solve:

13 A merchant mixes some candy worth $3.50 per pound with cheap stuff worth $1 per pound. The total value of the mixture is $28, and there are 10 more pounds of the cheap stuff than the more expensive candy. How many pounds of each are there?

SOLUTION: Let x = ____________________

y = ____________________

14. Twenty liters of 16% alcohol solution is to be formed by mixing some 10% solution with 30% solution. How much of each must be used?

SOLUTION: Let x = Number of liters of 10% solution

y = Number of liters of 30% solution

Equations: $x + y =$ ______________

$.10x + .30y = .16(20)$

15. How many liters of 10% alcohol solution must be mixed with some 5% alcohol solution to make 25 liters of 7% alcohol solution?

SOLUTION: Let x = Number of liters of 10% solution

y = Number of liters of 5% solution

Equations:

16. A certain farmer has chickens and pigs. There are a total of 60 heads and 200 feet. How many chickens and how many pigs does the farmer have?

SOLUTION: Let x = ______________

y = ______________

Equations:

ANSWERS 3.07

p.427-429:

1. (-5,3); **2.** (-6,-2); **3.** (-2,-5); **4.** (2,11);

5. Same line, sometimes written {(X,Y)| 2X-6Y=12};

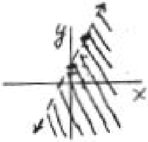
6. e, parallel lines; **7.** (2,-3); **8.** (-10,-8); **9.** (4,-8);

10. Same line, may be written {(X,Y)| 2X-Y=-4};

11. e, parallel lines; **12.** (25,-7).

p. 431-437:

1.



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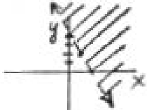

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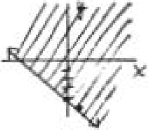

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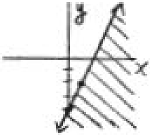

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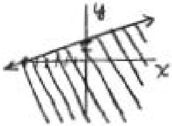
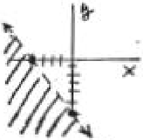


6.



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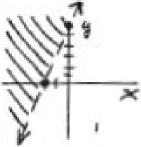

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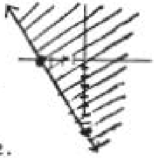

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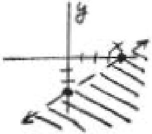

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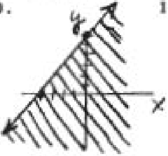

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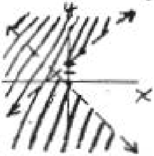

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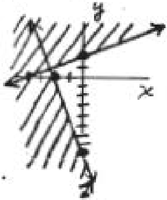

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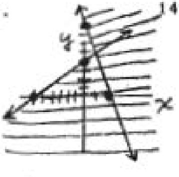

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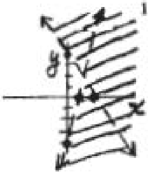

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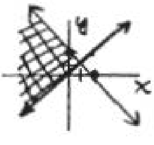

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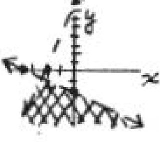

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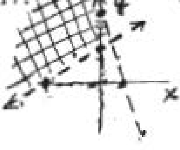

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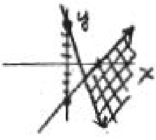

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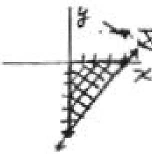

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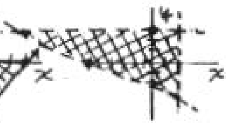
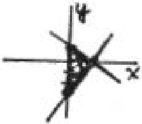


p. 431-437:

21.



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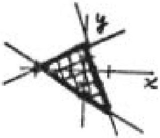
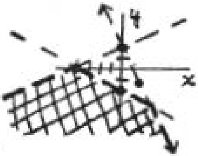


23.

24.

25.



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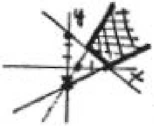

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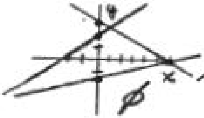

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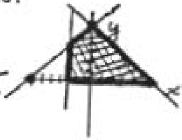

p. 438-443:

1. 18, 7; 2. -4, 22; 3. 72, 48; 4. 13, 35; 5. w = 15, L = 33; 6. W = 12, L = 38; 7. a = 10, d = 14; 8. n = 24, a = 6; 9. \$500 at 8%, \$1500 at 10%; 10. \$500 at 12%, \$2000 at 10%; 11. \$3000 at 12%, \$7000 at 10%; 12. \$500 at 12%, \$2000 at 10%; 13. 4 lb at \$3.50, 14 lb. cheap stuff;
14. 14 l. of 10%, 6 l. of 30% 15. 10 l. of 10%, 15 l. of 5%
16. 40 pigs, 20 chickens.

Dr. Robert J. Rapalje

ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE