3.09 *Systems of Equations (3X3)*

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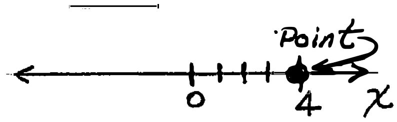
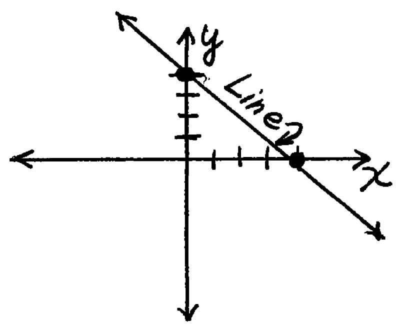
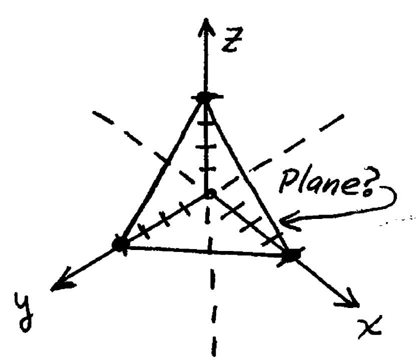
ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

In the previous section you solved 2 equations and 2 unknowns by the method of elimination, the method of substitution, and by Cramer's Rule. Now, in this section, you will be solving systems of equations involving 3 unknowns (variables). Usually, when there are 3 variables, there will be 3 equations, but not always. Geometrically, it will require 3 dimensions to describe an equation involving 3 variables. As always, it is helpful to recall what you know to be true, and then to speculate beyond this:

The EQUATION	represents	GRAPH
X = 4	**a point on a line (1 dimension)**	
X + Y = 4	**a line in XY-plane (2 dimensions)**	
X + Y + Z = 4	**a ______ in XYZ-space (3 dimensions)**	

From this illustration it should be no quantum leap of faith to see that the linear equation in 3 variables **AX + BY + CZ = D** is indeed a **plane**. Can you see that two planes, if they intersect, will intersect in a **line**? Then, if there is a third plane that intersects the **line** of intersection of the first two planes, the result is usually (but not always!) a **point** of intersection. Therefore, 3 planes usually intersect in a **point.**

The problem at hand, then, is to solve systems of 3 equations and 3 unknowns, by the elimination method. **Or, if you would rather, try the TI 85/86 using [2nd] [simult], using "Number" equal to [3]!**

EXAMPLE. Solve the system:

$$\begin{aligned} \text{I:}\quad & 3X + Y + Z = 8 \\ \text{II:}\quad & 2X + 2Y - Z = 10 \\ \text{III:}\quad & \underline{X - 3Y + 2Z = -4} \end{aligned}$$

SOLUTION: If you just add the first two equations together, you get **5X + 3Y + 18.** The problem is that there are still two variables, **X** and **Y.** What it takes to solve equations with two **variables** is two **equations.** In other words, you need another equation with the same variable **Z** eliminated. Therefore, to eliminate the **Z-variable** again, you should multiply equation II by 2, and add this to equation III. It will be helpful to organize the work as follows:

$$\begin{aligned} \text{I:}\quad & 3X + Y + Z = 8 \\ \text{II:}\quad & 2X + 2Y - Z = 10 \\ \text{III:}\quad & X - 3Y + 2Z = -4 \end{aligned}$$

$$\begin{array}{rl} \text{I:} & 3X + Y + Z = 8 \\ \text{II:} & \underline{2X + 2Y - Z = 10} \\ & 5X + 3Y \quad = 18 \end{array} \qquad \begin{array}{rl} \text{2 II:} & 4X + 4Y - 2Z = 20 \\ \text{III:} & \underline{X - 3Y + 2Z = -4} \\ & 5X + Y \quad = 16 \end{array}$$

Now, you have "2 equations and 2 unknowns" to solve:

$$\begin{array}{rl} & 5X + 3Y = 18 \\ -1 & \underline{(5X + Y = 16)} \end{array}$$

$$\begin{array}{r} 5X + 3Y = 18 \\ \underline{-5X - Y = -16} \\ 2Y = 2 \\ Y = 1 \end{array}$$

Now, use Y to solve for X:

$$\begin{aligned} 5X + 3Y &= 18 \\ 5X + 3(1) &= 18 \\ 5X &= 15 \\ X &= 3 \end{aligned}$$

Finally, go back to one of the original equations and, using X = 3 and Y = 1, solve for Z:

$$\begin{aligned} 3X + Y + Z &= 8 \\ 3(3) + (1) + Z &= 8 \\ 10 + Z &= 8 \\ Z &= -2 \end{aligned}$$

You may check the by substituting the values of X, Y, and Z into equation **III** (or equation **II**):

$$\begin{aligned} X - 3Y + 2Z &= -4 \\ (3) - 3(1) + 2(-2) &= -4 \\ -4 &= -4 \end{aligned}$$

SUMMARY: **When solving 3 equations with 3 unknowns, you must eliminate one variable, that is the SAME variable, twice. This gives 2 equations and 2 unknowns which then must be solved.**

EXERCISES: **Solve the systems for X, Y, and Z.**

1. (It looks like Y will be the easiest to eliminate.)

$$
\begin{array}{rl}
\text{I:} & 2X + 3Y - 3Z = 9 \\
\text{II:} & 5X - 2Y - 8Z = 6 \\
\text{III:} & 4X - Y - 5Z = -1
\end{array}
$$

Mult 3rd Eq. by 3 | Mult 3rd Eq. by -2

$$
\begin{array}{rrcl}
\text{I:} & 2X + 3Y - 3Z & = & 9 \\
3(\text{III}): & 12X - 3Y - 15Z & = & -3 \\
\hline
 & 14X \qquad - 18Z & = & 6
\end{array}
\qquad
\begin{array}{rrcl}
\text{II:} & 5X - 2Y - 8Z & = & 6 \\
-2(\text{III}): & -8X + 2Y + 10Z & = & 2 \\
\hline
 & -3X \qquad + 2Z & = & 8
\end{array}
$$

Now, solve the two equations and two unknowns:

$$
\begin{array}{rl}
 & 14X - 18Z = 6 \\
9\,(& -3X + 2Z = 8\,) \\
\hline
\end{array}
$$

Solve for X and Z!

Now, substitute the values of X and Z into the first equation:

$$
\begin{array}{rl}
\text{I:} & 2X \quad + 3Y - \quad 3Z \quad = 9 \\
 & 2(\quad) + 3Y - 3(\quad) = 9
\end{array}
$$
and solve for Y:

Now, check by substituting X, Y, and Z into third equation:

$$
\begin{array}{rl}
\text{III:} & 4X \quad - \quad Y \quad - \quad 5Z \quad = -1 \\
 & 4(\quad) - (\quad) - 5(\quad) = -1
\end{array}
$$

Does it check??

2.

$$\begin{aligned} 3X + Y + Z &= 8 \\ 2X + 2Y - Z &= 10 \\ X - 3Y + 2Z &= -4 \end{aligned}$$

3.

$$\begin{aligned} X - 5Y + Z &= 14 \\ -2X + Y + 2Z &= -6 \\ 4X + 4Y - Z &= 3 \end{aligned}$$

4.

$$
\begin{aligned}
3X + 2Y + Z &= 23 \\
2X + Y + Z &= 11 \\
-X + 3Y + Z &= -10
\end{aligned}
$$

5.

$$
\begin{aligned}
4X - Y + 3Z &= 0 \\
2X + 3Y - Z &= 4 \\
X + 2Y + 6Z &= -18
\end{aligned}
$$

6.

$$
\begin{aligned}
X + 2Y + Z &= -1 \\
2X + 3Y + 2Z &= 5 \\
-X + Y + 5Z &= 4
\end{aligned}
$$

7.

$$
\begin{aligned}
3X - Y - Z &= -5 \\
X + 5Y + 2Z &= 8 \\
4X + 2Y - Z &= 5
\end{aligned}
$$

8.

$$\begin{aligned} 3X + 4Y + Z &= 2 \\ 7X + 2Y + 4Z &= 5 \\ -2X + Y - 2Z &= -6 \end{aligned}$$

9.

$$\begin{aligned} 4X + 5Y - 3Z &= -5 \\ 2X - 3Y - 2Z &= 1 \\ 7X + 4Y - 4Z &= 1 \end{aligned}$$

10.

$$\begin{aligned} 3X + 5Y - 2Z &= 4 \\ 5X + 2Y - 6Z &= 2 \\ 4X + 3Y - 5Z &= 5 \end{aligned}$$

11.

$$\begin{aligned} 3X + 2Y - 4Z &= 5 \\ 2X - 2Y + 4Z &= 10 \\ -6X + 3Y + 2Z &= 0 \end{aligned}$$

12.

$$\begin{aligned} X + 2Y - Z &= 2 \\ X - 2Y + Z &= 6 \\ 3X + 5Y + 3Z &= -4 \end{aligned}$$

In the next problem, since the first equation is in X and Y only (the Z-term is missing!), use the first equation as is, and eliminate the Z-term using equations II and III.

13.

$$\begin{aligned} 3X - 5Y \quad &= 1 \\ 2(4X \quad + 3Z &= 0) \\ -3(\quad 3Y + 2Z &= 2) \end{aligned}$$

I: $3X - 5Y = 1$

$$\begin{aligned} 2(II):&\quad 8X \quad + 6Z = 0 \\ -3(III):&\quad - 9Y - 6Z = -6 \\ &\quad \overline{8X - 9Y \quad = -6} \end{aligned}$$

You finish:

14.

$$\begin{aligned} 3X + 2Y \phantom{{}-3Z} &= -2 \\ 2Y - 3Z &= 1 \\ X - 2Y + 2Z &= 4 \end{aligned}$$

15.

$$\begin{aligned} 8X + 3Y + 2Z &= 3 \\ 4X + 5Y \phantom{{}+2Z} &= 7 \\ 2Y - 3Z &= -9 \end{aligned}$$

16.

$$\begin{aligned} 3X + 2Y + 5Z &= 2 \\ 2X - 3Y + 2Z &= -4 \\ X + 4Y + 2Z &= 2 \end{aligned}$$

17.

$$\begin{aligned} -2X + 5Y + 4Z &= 4 \\ 5X - 2Y - Z &= 2 \\ 3X + 3Y + 2Z &= 0 \end{aligned}$$

18.

$$\begin{aligned} 4X + 2Y + 3Z &= 6 \\ X - Y + Z &= 2 \\ -X + 4Z &= 8 \end{aligned}$$

19. If you solve a system of 3 equations and 3 unknowns by eliminating the same variable twice, reducing the problem to 2 equations and 2 unknowns, then how would you solve a system of 4 equations and 4 unknowns? Can you see how Cramer's Rule might apply? Is Cramer's Rule a good idea in the next exercise?

20. Solve the system:

$$\begin{aligned}
\text{I:}\quad & W + 2X + Y + 3Z = 0 \\
\text{II:}\quad & -W + X + Y + Z = -4 \\
\text{III:}\quad & 2W - 2X - 3Y + 2Z = 1 \\
\text{IV:}\quad & 2W + X - Y + 2Z = 5
\end{aligned}$$

ANSWERS 3.09

p.462-471:

1. (-6,2,-5); 2. (3,1,-2); 3. (3,-2,1);
4. (9,3,-10); 5. (2,-1,-3); 6. (9,-7,4);
7. (-2,4,-5); 8. (-5,2,9); 9. (3,-1,4);
10. (-4,2,-3); 11. (3,4,3); 12. (4,-2,-2);
13. (-3,-2,4); 14. (2,-4,-3); 15. (-2,3,5);
16. (-6,0,4); 17. (0,-4,6); 18. (0,0,2);
19. Eliminate the <u>SAME</u> unknown 3 times in order to reduce the problem to 3 equations and 3 unknowns. Be sure to eliminate the <u>SAME</u> variable . . .

20. w, x, y, z
(3, 2, -1, -2).

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