4.02 *Natural Logarithm, the Number "e"*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

You recall from the previous section that a logarithm can exist to any base **b > 0**, provided **b ≠ 1**. In fact, for any such number **b**, there exists an entire logarithm system. In years past (before calculators) the most *common* system of logarithms was the **base ten** system of logs, which is called **common logarithms**. Today, however, the system of **natural logs**, or **Napier logs** (named after the Scottish mathematician **John Napier** who first introduced logarithms in 1614) are most frequently used. **Natural logs** have the base **"e"**, approximately **2.71828**, are important because of their special properties and applications to "natural" phenomena in physical sciences, business, social science, and life science. Most of these "special properties" will be studied in greater depth in calculus and differential equations. When referring to natural logarithms, you usually write $\log_e x$, read "log base e of x" or **ln x**, read "ell en [for **l**og **n**atural] of x."

One of the main applications of natural logs relates to **exponential growth.** When the rate of growth of a population (bacteria, people, etc.) is proportional to the population at any time **"t"**, then the population **Y** at any time **t** is given by the equation $Y = Y_o e^{kt}$, where Y_o is the initial population, and **k** is called the constant of proportionality. If instead of growth there is decay, such as in radioactivity and negative population growth, then the equation is the same, except that the constant "k" is negative.

In business, if **P** represents the **principal** invested, **r** is the **rate** of interest per year, **t** is the **time** in years, **I** is the **interest** earned, **A** is the total **amount** accumulated or the **compound amount**, then $A = P + I$. The compound amount may be regarded in the same way as the population in the growth equation of the previous example. Interest may be compounded annually, semi-annually, quarterly, monthly, daily, or even continuously. In particular, if interest is compounded continuously, then $A = P \cdot e^{rt}$.

With your calculator, try now to find the value of **e**. To do this, you should check your calculator manual, since different calculators operate in a variety of ways. Here are a few suggestions that may help. If your calculator has a button labelled **"e"**, then enter **"1"** in the calculator and press **"e"** in order to calculate $\mathbf{e^1}$. However, many calculators do not have an **"e"** button. For these, you must use enter **"1"** and press the **"alt"** (sometimes it is the **"inv"**, **"2nd function"**, or **"shift"**) and the **"ln"** keys. [If you do not have the **"ln"** key, then get another calculator for this section.] For graphics calculators, press **"shift"**, **"ln"**, and **"1"**. Write your answer here:

My calculator value for "e" is: ____________________.

Another way to calculate the value of **e** is to use large values of **n** in the formula:

$$e \approx \left(1 + \frac{1}{n}\right)^n \quad \textbf{or} \quad e \approx \left(\frac{n+1}{n}\right)^n \quad \textbf{for large values of n.}$$

Using your calculator, calculate each of the following values, using the formula given above: [NOTE: You may use <u>either</u> formula to calculate!]

1. **n = 1,** $e \approx \left(1+\frac{1}{1}\right)^1 = 2$

2. **n = 2,** $e \approx \left(1+\frac{1}{2}\right)^2 \approx$ _____

3. **n = 3,** $e \approx \left(1+\frac{1}{3}\right)^3 \approx$ _____

4. **n = 10,** $e \approx \left(1+\frac{1}{\ }\right) \approx$ _____

5. **n = 100,** $e \approx \left(1+\frac{1}{\ }\right) \approx$

6. **n = 1000,** $e \approx$

7. **n = 10,000** $e \approx$

8. **n = 1,000,000** $e \approx$

9. **How large a value of n must you use to achieve your calculator value of "e"?**

With calculators giving answers with 8-digit accuracy (or more!), how should such answers be rounded off? First, be sure that when you are using a calculator value to obtain a subsequent answer, ***NEVER ROUND IT OFF*!** Rather, you should use the fullest accuracy of the calculator when performing operations to calculate other values. This is expecially important when multiplying by large numbers, dividing by extremely small numbers, or when raising to powers. What may seem like a small round-off error could result in a large error in the final answer.

After the final answer is obtained, instructions as to rounding off may be given. You may be instructed to round to the nearest hundredth (or two decimal places), nearest thousandth (or three decimal places), etc. You may be instructed to round to two significant figures (like 1.5, or 1500, or 0.0015) or three significant figures (like 1.56, or 1560, or 0.00156). If you are not instructed how to round, then use your own judgment based upon the information in the problem. For example, if a problem is given with only two or three digits of accuracy, then it is certainly not sensible to give an answer with more than two or three digits of accuracy.

In the next exercises, use your calculator to find the values. Remember that it is the **"ln"** key that represents the **"natural logarithms"** (or **log base "e"**). Do not confuse this with the **"log"** key which represents **"common logarithms"** or **log base 10**. Also, look for patterns of numbers to develop across the "rows" of exercises. **To facilitate seeing these patterns, express your answers rounded to the nearest tenth. These "patterns"** (or **"speculations"**) **will be proven in Section 4.03.**

1. $\ln 2 =$ ________ **2.** $\ln 4 =$ ________ **3.** $\ln 8 =$ ________

4. $\ln 3 =$ ________ **5.** $\ln 9 =$ ________ **6.** $\ln 27 =$ ________

7. $\ln 4 =$ ________ **8.** $\ln 16 =$ ________ **9.** $\ln 64 =$ ________

10. $\ln 5 =$ ________ **11.** $\ln 25 =$ ________ **12.** $\ln 125 =$ ________

13. $\ln 10 =$ ________ **14.** $\ln 100 =$ ________ **15.** $\ln 1000 =$ ________

16. How do the second and third columns above compare to the first? Do these patterns apply in log base 10? (See 17-25.)

17. $\log_{10} 2 =$ ______ **18.** $\log_{10} 4 =$ ______ **19.** $\log_{10} 8 =$ ______

20. $\log_{10} 5 =$ ______ **21.** $\log_{10} 25 =$ ______ **22.** $\log_{10} 125 =$ ______

23. $\log_{10} 10 =$ ______ **24.** $\log_{10} 100 =$ ______ **25.** $\log_{10} 1000 =$ ______

Use the scientific notation function of your calculator to find the following logarithms (base 10). Do you notice other patterns in these exercises?

26. $\log_{10} 10^{12} =$ ________ **27.** $\log_{10} 10^{-12} =$ ________

28. $\log_{10} (5.67 \times 10^{12}) =$ ________ **29.** $\log_{10} (5.67 \times 10^{27}) =$ ________

In the following exercises round to the nearest tenth. Again, compare the <u>exercises</u> in the third column to <u>exercises</u> in the first two columns. Then compare the <u>answers</u> in the third column to the <u>answers</u> in the first two columns. Do you see a pattern for the <u>exercises</u> and <u>answers</u>?

30. ln 2 = ________ 31. ln 3 = ________ 32. ln 6 = ________

33. ln 5 = ________ 34. ln 7 = ________ 35. ln 35 = ________

36. ln 10 = ________ 37. ln 47 = ________ 38. ln 470 = ________

39. $\log_{10}$ 2 = ______ 40. $\log_{10}$ 3 = ________ 41. $\log_{10}$ 6 = ________

42. $\log_{10}$ 10 = ______ 43. $\log_{10}$ 47 = ________ 44. $\log_{10}$ 470 = ______

45. How does the third column compare to the first two columns?

In the following exercises, you may round answers to two decimal places, but remember--when using a calculated answer to calculate other answers, be sure to use the full accuracy of the calculator. Again, look for patterns.

46. e = __________ 47. e^2 = __________ 48. e^{10} = ___________

49. $\ln(e)$ = _______ 50. $\ln(e^2)$ = _______ 51. $\ln(e^{10})$ = ________

52. Can you generalize about $\ln(e^X)$? ______________________________

53. ln 1 = ________ 54. ln 2 = ________ 55. ln 10 = ________

56. $e^{(\ln 1)}$ = ______ 57. $e^{(\ln 2)}$ = ______ 58. $e^{(\ln 10)}$ = _______

59. Can you generalize about $e^{(\ln X)}$?

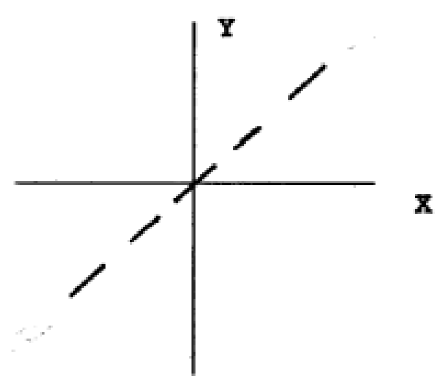
60. Use your calculator to obtain values and graph the functions $Y = e^X$ and $X = e^Y$ on the same graph grid.
[Hint: use different colors--see p. 494, #9!]

$Y = e^X$

X	Y
0	
1	
2	
-1	
-2	

$X = e^Y$

X	Y
	0
	1
	2
	-1
	-2



Since **ln X** really means **$\log_e$ X**, you could say that **Y = ln X** is equivalent to **Y = $\log_e$ X**. Then, by the definition of logarithms, you may change this from logarithmic form to exponential form: **Y = $\log_e$ X** is equivalent to **e^Y = X.**

$$Y = \ln X \quad \Leftrightarrow \quad Y = \log_e X$$

$$Y = \log_e X \quad \Leftrightarrow \quad e^Y = X$$

This means that **Y = ln X** is equivalent to **e^Y = X**, which is the inverse of **Y = e^X**. Therefore, **Y = e^X** and **Y = ln X** are ***inverse functions.***

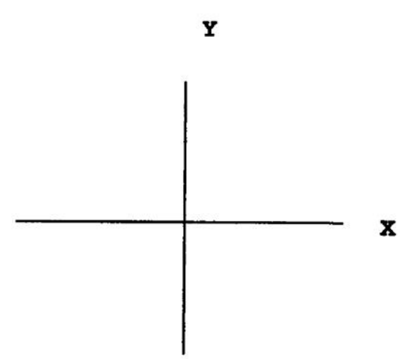
61. Use your calculator to obtain values and graph similar to #60 on the previous page. [Hint: Again, use different colors!]

Y = e^X

X	Y
0	
1	
2	
-1	
-2	

Y = ln X

X	Y
.1	
.5	
1	
2	
3	



62a) Find the largest value of X for which your calculator will compute e^X. (By trial and error using your calculator!)

b) Find the maximum e^X value for your calculator.

c) Find the largest value of X for which your calculator will compute ln (X).

d) Find the maximum ln (X) value for your calculator.

e) Is there a connection among these values? Speculate . . .

ANSWERS -- 4.02

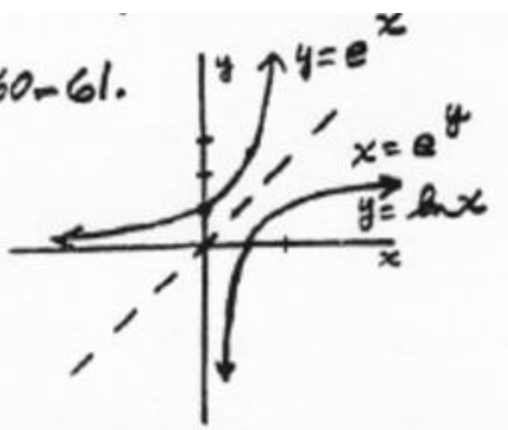
4.02 p.503: $e \approx 2.718281828$

1. 2 2. 2.25 3. 2.37037069 4. 2.59374246
5. 2.704813829 6. 2.716923932 7. 2.718145927
8. 2.718280469 9. (Casio fx-7000: $n = 10^9$ comes close.
For $n = 10^{10}$, calculator rounds $1 + \frac{1}{n}$ to 1 giving $1^n = 1.000$)

4.02 p. 505-508:

1. 0.7 2. 1.4 3. 2.1 4. 1.1 5. 2.2 6. 3.3
7. 1.4 8. 2.8 9. 4.2 10. 1.6 11. 3.2 12. 4.8
13. 2.3 14. 4.6 15. 6.9 17. 0.3 18. 0.6 19. 0.9
20. 0.7 21. 1.4 22. 2.1 23. 1 24. 2 25. 3
26. 12 27. -12 28. 12.8 29. 27.8 30. 0.7 31. 1.1 32. 1.8
33. 1.6 34. 1.9 35. 3.6 36. 2.3 37. 3.9 38. 6.2
39. 0.3 40. 0.5 41. 0.8 42. 1 43. 1.7 44. 2.7
45. Answer in column 3 is the sum of answers in first two.
46. 2.72 47. 7.39 48. 22026.47 49. 1 50. 2 51. 10
52. x 53. 0 54. 0.69 55. 2.30 56. 1 57. 2 58. 10
59. x

60-61.



62a) Graphing calculator: 230.2585
b) $9.99999999 \times 10^{99}$
c) $9.99999999 \times 10^{99}$
d) 230.2585
e) Two digit power of 10??

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