4.03 *Properties of Logarithms*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

In the previous section, some "patterns" of logarithmic numbers emerged. These are the **properties** or **laws of logarithms**, and they can be proven by using the laws of exponents and the definition of the logarithm:

DEFINITION OF LOGARITHM

$Y = \log_b X \quad \Leftrightarrow \quad b^Y = X$

assuming that $b>0$, $b\neq 1$, *and* $X>0$.

LAWS of LOGARITHMS

I. $\log_b M \cdot N = \log_b M + \log_b N$

II. $\log_b \frac{M}{N} = \log_b M - \log_b N$

III. $\log_b M^N = N \log_b M$

PROOF: I. $\log_b MN = \log_b M + \log_b N$

Let $X = \log_b M$, $Y = \log_b N$, $Z = \log_b MN$

By definition: $b^X = M$, $b^Y = N$, $b^Z = MN$

$$M \cdot N = MN$$
$$b^X \cdot b^Y = b^Z$$
$$b^{X+Y} = b^Z$$
$$X + Y = Z$$
$$\therefore \log_b M + \log_b N = \log_b MN$$

The first law says that when you take the **log of a product**, $\log_b MN$, you **add** logarithms, $\log_b M + \log_b N$. This is like the law of exponents which says "when you **multiply** with the same base number, you **add** the exponents."

Law II is similar to **Law I.** It should be clear that, if, when you take the log of a **product** you **add** logarithms, then, if you take the log of a **quotient**, you must **subtract** logarithms.

That is, $\log_b \frac{M}{N} = \log_b M - \log_b N$. This is like the law of exponents that says, "When you **divide** with the same base number, you **subtract** the exponents."

Law III is an extension of **Law I.**

Given: $\log_b M \cdot N = \log_b M + \log_b N$

$\Rightarrow$ $\log_b M \cdot M = \log_b M + \log_b M$

$= 2 \log_b M$

$\Rightarrow$ $\log_b M \cdot M \cdot M = \log_b M + \log_b M + \log_b M$

$= 3 \log_b M$

In general, $\log_b M^N = = N \log_b M$

This third law has several specific applications:

Given: $\log_b M^N = = N \log_b M$

1. $\log_b \sqrt{X} = \log_b X^{\frac{1}{2}} = \frac{1}{2} \log_b X$

2. $\log_b \sqrt[3]{X} = \log_b X^{\frac{1}{3}} = \frac{1}{3} \log_b X$

3. $\log_b \sqrt[n]{X} = \log_b X^{\frac{1}{n}} = \frac{1}{n} \log_b X$

4. $\log_b \frac{1}{X} = \log_b X^{-1} = -1 \cdot \log_b X$ *or* $- \log_b X$

It is very important to **know** the basic **definition of logarithms** (Section 4.01) and the **laws of logarithms** and to **practice** using them. It is important to recognize the laws of logarithms "backwards" and "forward." First, the "forward" practice.

In each of the following, expand according to laws of logarithms to express as a sum/difference of logs:

1. $\log_b X^2 Y^3 = \log_b X^2 + \log_b Y^3$

   $= \_\_\_ \log_b X + \_\_\_ \log_b Y$

2. $\log_b X^4 Z^9 =$

3. $\log_b \frac{X^3}{Y^5} =$ $\log_b \_\_\_\_\_ - \log_b \_\_\_\_\_$

   $= \_\_\_ \log_b \_\_\_ - \_\_\_ \log_b \_\_\_$

4. $\log_b \frac{Y^2}{X^9} =$

5. $\log_b \frac{X^4}{\sqrt{Y}} =$

6. $\log_b \frac{\sqrt[3]{X}}{Y^2} =$

7. $\log_b \frac{1}{X^3} =$

8. $\log_b \frac{1}{Z^2} =$

9. $\log_b \frac{X^3Y^2}{Z} =$

10. $\log_b \frac{XY^3}{Z^2} =$

11. $\log_b \frac{X^4}{Y^2Z^3} = \log_b X^4 - [\log_b (Y^2Z^3)]$

$= \log_b X^4 - [\log_b \_\_\_\_\_ + \log_b \_\_\_\_\_)]$

$=$

12. $\log_b \frac{Y^3}{X^4Z^2} =$

13. $\log_b \frac{\sqrt{a}}{X^3\sqrt{Z}} =$

14. $\log_b \frac{X^3\sqrt{a}}{Y^2\sqrt[3]{c}} =$

15. $\log_b \frac{1}{X^2Y^3} =$

16. $\log_b \frac{1}{X^3\sqrt{Y}} =$

17. $\log_b \frac{1}{X\sqrt[3]{Y}} =$

18. $\log_b \frac{1}{X^2\sqrt[3]{Y^2}} =$

It is also frequently necessary to recognize the laws of logarithms in "reverse." That is, given a sum or difference of logarithms, you must express this as a single log.

19. $2 \log_b X + 3 \log_b Y = \log_b X^2 + \log_b Y^3$
$= \log_b$ _____

20. $4 \log_b Y + 2 \log_b X =$

$=$

21. $2 \log_b X + \frac{1}{2} \log_b Y - 7 \log_b Z$

22. $\frac{1}{3} \log_b Y + 2 \log_b Z - 4 \log_b X$

23. $2 \log_b X - 4 \log_b Y + \log_b Z$

24. $2 \log_b X - \log_b Y + 4 \log_b Z$

25. $3 \log_b X - \frac{1}{2} \log_b Y - 2 \log_b Z$

26. $\log_b X - \log_b Y - \log_b Z$

27. $\log_b X - 2 [\log_b Y + \log_b Z]$
[Hint: Distribute the 2!]

28. $\log_b X - 2 [\log_b Y + 3 \log_b Z]$

29. $2 \log_b X - 3 [\log_b Z - 2 \log_b Y]$

30. $\frac{1}{2} \log_b X - 5 [\log_b Y - \log_b Z]$

31. $\frac{1}{2} \log_b (5X+3) - 2 [\log_b (X+2) + \log_b (3X+1)]$

32. $4 \log_b (X-1) - 3 [\log_b (X+1) - \log_b (X^2+1)]$

Use the definition of logarithms (i.e., convert to exponential form) to simplify the following. You will probably notice a major concept--the relationship between "log base b" and "b raised to a power"--that will allow you to simplify in one step.

33. $\log_b b =$ _____

Let $\log_b b = X$ and find X.

$b^X = b$

$X =$ _____

34. $\log_b b^3 =$ _____

Let $\log_b b^3 = X$

$b^X = b^3$

$X =$ _____

35. $\log_b \sqrt{b} =$ _____

36. $\log_b \sqrt[3]{b} =$ _____

37. $\log_b \frac{1}{b^3} =$ _____

38. $\log_b \frac{1}{b^5} =$ _____

39. $\log_b \frac{1}{\sqrt{b}} =$ _____

40. $\log_b 1 =$ _____

41. $\log_5 25 =$ _____

42. $\log_5 125 =$ _____

43. $\log_5 \frac{1}{5} =$ _____

44. $\log_5 \frac{1}{25} =$ _____

45. $\log_5 0.04 =$ _____

46. $\log_5 0.2 =$ _____

47. $\log_5 1 =$ _____

48. $\log_{10} 1 =$ _____

49. $\ln e^5 =$ _____

50. $\ln e^4 =$ _____

51. $\ln \frac{1}{e^2} =$ _____

52. $\ln e =$ _____

53. $\ln e^0 =$ _____

54. $\ln 1 =$ _____

Use the laws of logarithms to expand as a sum or difference of logarithms. Simplify where possible. [Hint: Remember, there are no laws of logarithms dealing with **log (X+Y)**, **log (X-Y)**, or **(ln X)^n**.]

55. $\log_b \frac{a^3 c^4}{b^3} = \log_b a^3 + \log_b c^4 - \log_b b^3$ [Hint: See #34.]

$= \_\_\_\_\_ + \_\_\_\_\_ - \_\_\_\_\_$

56. $\log_b \frac{a^3 b^2}{c^4} =$

57. $\log_5 \frac{2\sqrt{3}}{25} =$

58. $\log_5 \frac{5\sqrt{6}}{7} =$

59. $\log_2 \frac{5\sqrt{7}}{8} =$

60. $\log_3 \frac{27\sqrt{2}}{5} =$

61. $\log_3 \frac{9\sqrt{3}}{2}$ =

62. $\log_2 \frac{32\sqrt{2}}{5}$ =

63. $\log_7 7\sqrt{7}$ =

64. $\log_5 125\sqrt{5}$ =

65. $\log_3 \frac{5\sqrt{3}}{3}$ =

66. $\log_5 \frac{7\sqrt[3]{5}}{5}$ =

ANSWERS -- 4.03

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1. $2\log_b x + 3\log_b y$ 2. $4\log_b x + 9\log_b z$ 3. $3\log_b x - 5\log_b y$

4. $2\log_b y - 9\log_b x$ 5. $4\log_b x - \frac{1}{2}\log_b y$ 6. $\frac{1}{3}\log_b x - 2\log_b y$

7. $-3\log_b x$ (Remember, $\log_b 1 = 0$) 8. $-2\log_b z$

9. $3\log_b x + 2\log_b y - \log_b z$ 10. $\log_b x + 3\log_b y - 2\log_b z$

11. $4\log_b x - 2\log_b y - 3\log_b z$ 12. $3\log_b y - 4\log_b x - 2\log_b z$

13. $\frac{1}{2}\log_b a - 3\log_b x - \frac{1}{2}\log_b z$ 14. $3\log_b x + \frac{1}{2}\log_b a - 2\log_b y - \frac{1}{3}\log_b c$

15. $-2\log_b x - 3\log_b y$ 16. $-3\log_b x - \frac{1}{2}\log_b y$

17. $-\log_b x - \frac{1}{3}\log_b y$ 18. $-2\log_b x - \frac{2}{3}\log_b y$

19. $\log_b x^2y^3$ 20. $\log_b x^2y^4$ 21. $\log_b \frac{x^2\sqrt{y}}{z^7}$ 22. $\log_b \frac{z^2\sqrt[3]{y}}{x^4}$

23. $\log_b \frac{x^2z}{y^4}$ 24. $\log_b \frac{x^2z^4}{y}$ 25. $\log_b \frac{x^3}{z^2\sqrt[3]{y}}$ 26. $\log_b \frac{x}{y\sqrt{z}}$

27. $\log_b \frac{x}{y\sqrt{z}}$ 28. $\log_b \frac{x}{y\sqrt[3]{z^2}}$ 29. $\log_b \frac{x^2y}{z^3}$ 30. $\log_b \frac{\sqrt{x}\,z^5}{y^5}$

31. $\log_b \frac{\sqrt{5x+3}}{(x+2)^2(3x+1)^2}$ 32. $\log_b \frac{(x-1)^2(x^3+1)^3}{(x+1)^3}$ 33. 1 34. 3

35. $\frac{1}{2}$ 36. $\frac{1}{3}$ 37. -3 38. -5 39. $-\frac{1}{2}$ 40. 0 41. 2

42. 3 43. -1 44. -2 45. -2 46. -1 47. 0 48. 0

49. 3 50. 4 51. -2 52. 1 53. 0 54. 0

55. $3\log_b a + 4\log_b c - 3$ 56. $3\log_b a + 2 - 4\log_b c$

57. $\log_5 2 + \frac{1}{2}\log_5 3 - 2$ 58. $1 + \frac{1}{2}\log_5 6 - \log_5 7$

59. $\log_2 5 + \frac{1}{2}\log_2 7 - 3$ 60. $3 + \frac{1}{2}\log_3 2 - \log_3 5$

61. $\frac{5}{2} - \log_3 2$ 62. $\frac{11}{2} - \log_2 5$ 63. $\frac{3}{2}$ 64. $\frac{7}{2}$

65. $\log_3 5 - \frac{1}{2}$ 66. $\log_5 7 - \frac{2}{3}$