4.05 *Applications--Growth and Decay*

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Of all the times the questions are asked, **"What good is math? How can this possibly be used in real life?"**, logarithms provides one of the best opportunities to answer! It can be proven in higher mathematics (differential equations) that **if growth rate of a population** y **is proportional to the population at any time** t, **where the initial population is** y_0 **and the constant of proportionality is** k, **then the population** y **may be expressed as a function of time** t **as follows:**

$$Y = Y_0 e^{kt}$$

This is an **exponential** growth equation, valid for population **growth** problems, where **k** is **positive**, and also for **decay** problems, such as radioactive decay, carbon decay, or population declines, where the value of **k** is **negative**. Moreover, the assumption that the growth rate be proportional to the population at any time **t** is, under normal circumstances, a reasonable assumption. In a population of people, dogs, cats, bacteria, or whatever, the more you have in the population, the more you have reproducing. In other words, the more you have, the more you get! More formally stated, **population growth rate is proportional to the population at any time** t.

There are many other applications of this principle. In business, where **P** is the **principle** invested, **r** is the **rate** of interest per year, **t** is the **time** (number of years of investment), **I** is the **interest**, **A** is the **compound amount**, and **A = P + I**, the **compound amount** may be regarded in the same way as the population growth in the previous examples. In particular, interest may be compounded annually, semi-annually, quarterly, monthly, daily, or even continuously. If interest is compounded continuously, then $A = P \cdot e^{rt}$.

APPLICATION EXERCISES

In #1-10, the number of rabbits on a farm is given by $\mathbf{Y = Y_0 \cdot e^{0.05t}}$, where Y_0 is the initial population and t is the time in years.

1. If the initial population is 100, how many rabbits will there be in one year?

   SOLUTION: $Y = 100\ e^{(0.05)(1)}$

   $= 100\ e^{.05}$

   $=$ __________.

2. If the initial population is 100, how many rabbits will there be at the end of ten years?

3. If the initial population is 100, how many rabbits will there be at the end of 50 years?

4. If the initial population is 200, how many rabbits will there be at the end of 75 years?

5. If the initial population is 100, how long will it take the population of rabbits to reach 1000?

SOLUTION: $Y = Y_0 \cdot e^{0.05t}$

$1000 = 100 \cdot e^{0.05t}$

$10 = e^{0.05t}$ Take ln of both sides.

$\ln 10 = \ln e^{0.05t}$

$\ln 10 = 0.05t$ Since $\ln e^X = X$

$\frac{\ln 10}{0.05} = t$ Use calculator to evaluate.

t = ________ years

6. If the initial population is 100, how long will it take the population to reach 10,000?

7. If the initial population is 10, how long will it take the population to reach 500?

8. If the initial population is 1000, how long will it take the population to reach 1,000,000?

9. How long will it take this population of rabbits to double? [Hint: the population doubles when $Y = 2 \cdot Y_0$.]

SOLUTION: $Y = Y_0 \cdot e^{0.05t}$ and $Y = 2 \cdot Y_0$

$$2 \cdot Y_0 = Y_0 \cdot e^{0.05t}$$

$$2 = e^{0.05t}$$

$$\ln 2 = \ln e^{0.05t}$$

$$\ln 2 = 0.05t$$

$$\frac{\ln 2}{0.05} = t$$ Use calculator to evaluate.

t = ________ years

10. How long will it take this population of rabbits to triple?

In 11-16, the amount of radioactive substance present at any time is given by $Y = Y_0 e^{-0.0004t}$, where Y_0 is the initial amount in grams and t is the time in years.

11. If there are initially 1000 grams of the substance, how many grams are left after 10 years?

12. If there are initially 1000 grams of the substance, how many grams are left after 1000 years?

13. If after 1000 years, there remain 500 grams of substance, how much was there originally?

SOLUTION: $Y = Y_0 e^{-0.0004t}$

$500 = Y_0 e^{(-0.0004)(1000)}$

$500 = Y_0 e^{-0.4}$ or $\frac{500}{e^{-0.4}} = Y_0$

$500 = Y_0 \div e^{0.4}$ $\_\_\_\_\_\_\_ = Y_0$

$Y_0 = 500(e^{0.4})$

$Y_0 = \_\_\_\_\_\_\_\_\_\_$ grams

14. If after 2000 years, there remain 500 grams of substance, how much was there originally?

15. If there are 25 grams of the substance left after 2000 years, how much was there originally?

16. If there are 25 grams of the substance left after 5000 years, how much was there originally?

In most problems, the exponential equation is not given. What is usually given is some statement to the effect that "the rate of growth is proportional to the population at any time **t**." This, of course, means that $\mathbf{Y = Y_0 \cdot e^{kt}}$, but the value of **k**, the constant of proportionality, is not usually given. Instead, you may be given the **initial population Y_0**, and you must also be given the population at some other given time **t**. There are two ways to solve the problem as illustrated below. In **Method I**, use the given information and logarithms to solve for **k**. In **Method II**, instead of using logarithms to solve for k, solve for $\mathbf{e^k}$. In method II, logarithms are not required.

17. **The population of a certain city in 2000 was 100,000. In 2002 the population of the city was 120,000. What will be the population in the year 2010, assuming that the growth rate is proportional to the population at any given time?**

METHOD I: Solve for k.

$$y = y_0 e^{kt}$$

$$y = 100{,}000\, e^{kt}$$

At $t=2$, $y = 120{,}000$

$$120{,}000 = 100{,}000\, e^{k \cdot 2}$$ Divide by 100,000

$$1.2 = e^{2k}$$ Take ln both sides

$$\ln 1.2 = \ln e^{2k}$$

$$\ln 1.2 = 2k$$

$$k = \frac{\ln 1.2}{2} \approx .09116$$ [NOTE: In future calculations, use the calculator value! Do NOT use rounded value]

Let $t = 10$:

$$y = 100000\, e^{(.09116)(10)}$$

$$y = 100{,}000\, e^{.9116\ldots}$$ use calculator value!

$y =$ ________

17. The population of a certain city in 2000 was 100,000. In 2002 the population of the city was 120,000. What will be the population in the year 2010, assuming that the growth rate is proportional to the population at any given time?

METHOD II: Solve for e^k.

$$y = y_0 e^{kt} = y_0 (e^k)^t$$

$120{,}000 = 100{,}000\,(e^k)^2$, since $y = 120{,}000$ when $t = 2$.

$$1.2 = (e^k)^2$$

$e^k = \sqrt{1.2}$ Take square root both sides.

$$y = 100{,}000\,(e^k)^t$$

$$y = 100{,}000\,(\sqrt{1.2})^{10}$$

$$= 100{,}000\,(1.2)^5 = \_\_\_\_\_\_\_\_$$

18. Use both methods to determine how long will it take the population in the previous exercise to reach 300,000?

METHOD I

$$300{,}000 = 100{,}000\, e^{kt}$$

$$3 = e^{(.09116\ldots)t}$$

$$\ln 3 = \ln e^{(.09116\ldots)t}$$

$$\ln 3 = (.09116\ldots)t$$

$$t =$$

METHOD II

$$300{,}000 = 100{,}000\, e^{kt}$$

$$3 = (e^k)^t$$

$$3 = (\sqrt{1.2})^t$$

$$\ln 3 = \ln (\sqrt{1.2})^t$$

$$\ln 3 = t \ln \sqrt{1.2}$$

$$t =$$

19. Use both methods to determine how long it will take the population to reach 1,000,000?

METHOD I

METHOD II

In 20-25, the population of a city was 40,000 in 1990. In 1995, the population of the city was 50,000.

20. Find the value of k or e^k (method I or method II, your choice).

$y = 40{,}000\, e^{kt}$ at $t = 5$, $y = 50{,}000$

$(\quad\quad) = 40{,}000\, e^{k(\ )}$

21. Use the value obtained in #20 to find the expected population in 2005.

22. Find the expected population in 2008.

23. In what year will the population double (that is, reach 80,000)?

24. In what year will the population triple?

25. In what year will the population reach 200,000?

26. The population of a city was 85,000 in 2000 and 88,000 in 2002. At this rate of growth, what population should be expected in 2005?

27. At this rate of growth, how long will it take the population in the city in the previous exercise to double?

NOTE: Please forgive and skip over these OLD and OUTDATED problems!! Someone younger can revise them!

28. In 1980, the national debt was $914 billion. (In case you don't realize how much this is, in silver dollars it would cover an area of one square mile 45.4 inches deep!) In 1990, the national debt was $3.8 trillion. Assuming exponential growth in the national debt, what would be the debt in 1995? How deep was the debt (1 sq. mi. in silver dollars) in 1990?

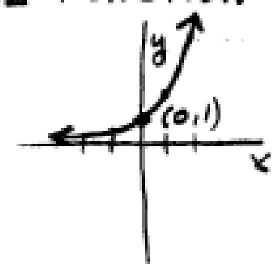
29. Use the information in #28 to estimate the national debt in the year 2000? (How far can they stretch a rubber band??)

30. According to data from the Center for Disease Control, up to August, 1988, there had been 71,171 cases of AIDS reported worldwide. By December, 1991, (40 months later) there had been 206,392 cases reported. Assuming exponential growth (the spread of the virus being proportional to the number of cases), how many cases would you expect by March, 1992 (43 months from August, 1988)? [The actual number of reported cases was 218,301, lower than expected. Can you speculate why? See editorial on next page.]

LOGARITHMS SUMMARY

EXPONENTIAL FUNCTION

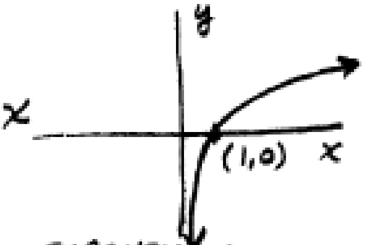
$y = 2^x$



INVERSE of EXPONENTIAL FUNCTION

$x = 2^y$

or $y = \log_2 x$



DEFINITION of LOGARITHM: <u>LOG FORM</u> $y = \log_b x \Leftrightarrow$ <u>EXPONENTIAL FORM</u> $b^y = x$, $x > 0$, $b > 0$, $b \neq 1$

NATURAL LOGS: $e \approx \left(1 + \frac{1}{n}\right)^n$ for large n, ≈ 2.718281828

$$\ln x = \log_e x$$

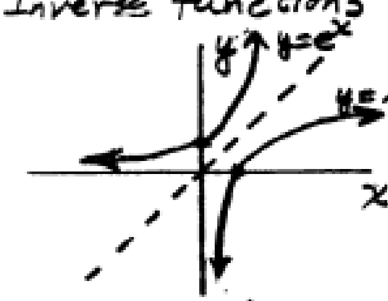
RELATIONSHIP of $y = e^x$ and $y = \ln x$: Inverse functions

$f(x) = e^x \qquad f^{-1}(x) = \ln x$

$f[f^{-1}(x)] = e^{\ln x} = x$

$f^{-1}[f(x)] = \ln e^x = x$

$\ln e = 1$



LAWS of LOGARITHMS: for any $b > 0$, $b \neq 1$.

I. $\log_b MN = \log_b M + \log_b N$

II. $\log_b \frac{M}{N} = \log_b M - \log_b N$

III. $\log_b M^N = N \log_b M$

also, $\log_b x = \frac{\ln x}{\ln b}$

APPLICATIONS of LOGS:

I. By taking ln or $\log_b$ of both sides of an equation, an exponential equation can be reduced to an algebraic equation.

II. When rate of growth (or decline) of a population is in proportion to the population y at any time t, the population equation is $y = y_0 e^{kt}$ where y_0 is the initial population and k is the constant of proportionality.

Editorial Page: COMING SOON!!

(maybe)

ANSWERS -- 4.05

4.05 p.532-540:

1. 105.1 2. 164.9 3. 1218.2 4. 8504.2 5. 46.05 yrs.
6. 92.10 yrs. 7. 78.24 yrs. 8. 138.16 yrs. 9. 13.86 10. 21.97 yrs.
11. 996 g. 12. 670.3 g. 13. 745.9 g. 14. 1112.8 g. 15. 55.6 g.
16. 184.7 g. 17. 248,832 18. 12.05 yrs. 19. 25.26 yrs. 20. $e^k = \sqrt[5]{1.25}$
$k = 0.04462871026$ 21. 78125 22. 89317 23. 2006 (15.5 yrs) 24. 2015 (24.6 yrs)
25. 36.1 yrs, 2026 26. 92700 27. 40 yrs. 28. \$7.748 Trillion; 15.74 ft. 29. \$15.799 Trillion 30. 223,548.

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