Math in Living C O L O R !!

1.04 Factoring

College Algebra: One Step at a Time

Pages 36-44: 19, 37, 49, 51, 55, 67, 72, 75, 77, 78, 81, 89, 99, 100, 108, 110, 115, 117

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See Section 1.04, with explanations, examples, and exercises, coming soon!

Guidelines to Factoring

1. Common Factor
2. Trinomials
3. Difference of Squares,
   Difference and Sum of Cubes
4. Grouping

P. 33. # 19. $x^5 - 5x^3 - 36x$

Solution: Notice that there are three separate terms in this polynomial. This makes it look like it might be a trinomial. But first, notice that there is a common factor that should be taken out. Begin by factoring out the x factor.

$$x\left(x^4 - 5x^2 - 36\right)$$

Next, notice that within the parentheses is a trinomial that factors. The FIRST times FIRST must be x^4 , the LAST times LAST must be −36, and the OUTER times OUTER and INNER times INNER must add up to $-5x^2$.

Let's start off with the FIRST times FIRST being $x^2 \bullet x^2$.

$$x\left(x^2 \quad\right)\left(x^2 \quad\right)$$

Next, find two numbers, LAST times LAST, whose product is −36 and whose sum (MIDDLE TERM) is −5. The correct combination is −9 and 4:

$$x\left(x^2 - 9\right)\left(x^2 + 4\right)$$

So the trinomial factored into a product of a difference of two squares times a sum of two squares. Of course, the difference of two squares factors again. The sum of squares does NOT re-factor!

Final answer: $x\left(x-3\right)\left(x+3\right)\left(x^2 + 4\right)$

P. 35. # 37. $(2x-3y)^2-14(2x-3y)+49$

Solution: The first step is to recognize that this is a trinomial. Can you see that it is in three parts? Just in case you don't see the "trinomial" in this exercise, let's do an easier problem first. Suppose the problem had been this:

$$x^2-14x+49$$

Surely you can see that this is a very simple trinomial that factors into

$$(x-7)(x-7) \text{ or } (x-7)^2$$

In the same way, you can surely see that

$$y^2-14y+49$$

factors into $(y-7)(y-7)$ or $(y-7)^2$,

and that $(Junk)^2-14(Junk)+49$

factors into $(Junk-7)(Junk-7)$ or $(Junk-7)^2$.

In the same way,

$$(2x-3y)^2-14(2x-3y)+49$$

factors into $\left[(2x-3y)\quad\right]\left[(2x-3y)\quad\right]$

$$\left[(2x-3y)\ -7\right]\left[(2x-3y)\ -7\right]$$

Cleaned up a bit it looks like this:

Final answer: $(2x-3y-7)(2x-3y-7)$ or $(2x-3y-7)^2$

P. 36. # 49. $(x^2-7x)^2+16(x^2-7x)+60$

Solution: The first step is to recognize that this is a trinomial. Can you see that it is in three parts?

$$(x^2-7x)^2+16(x^2-7x)+60$$

The FIRST times FIRST must be $(x^2-7x)^2$, the LAST times LAST must be 60, and the OUTER times OUTER and INNER times INNER must add up to $16(x^2-7x)$. $(x^2-7x)^2+16(x^2-7x)+60$. You must find two numbers whose product is 60 and whose sum is 16.

The FIRST times FIRST must be $(x^2-7x)^2$

$$\left[(x^2-7x) \qquad\qquad\right]\left[(x^2-7x) \qquad\qquad\right]$$

Next, find two numbers whose product is 60 and whose sum is 16. That would be 10 and 6:

$$\left[(x^2-7x) \quad + \quad 10\right]\left[(x^2-7x) \quad + \quad 6\right]$$

This can be "cleaned up" to make it look like the product of two ordinary trinomials,

$$\left((x^2-7x) \quad + \quad 10\right)\left((x^2-7x) \quad + \quad 6\right)$$

These may be "ordinary" trinomials, but do you think maybe they will re-factor? As "chance" would have it, they DO!! Always FACTOR COMPLETELY!!

Final answer: $(x-5)\ (x-2)\ \ (x-6)\ (x-1)$

P. 37. # 51. $(x^2 - 5x)^2 - 36$

Solution: The first step is to recognize that this is a difference of two squares! The FIRST times FIRST must be $(x^2 - 5x)^2$, the LAST times LAST is the perfect square 36 which is 6 times 6, and MIDDLE TERM must subtract out!

The FIRST times FIRST must be $(x^2 - 5x)^2$

$$\left[(x^2 - 5x) \qquad\qquad \right]\left[(x^2 - 5x) \qquad\qquad \right]$$

$$\left[(x^2 - 5x) - 6\right]\left[(x^2 - 5x) + 6\right]$$

This can be "cleaned up" to make it look like the product of two regular trinomials,

$$(x^2 - 5x - 6)(x^2 - 5x + 6)$$

and as "chance" would have it, these trinomials each can be factored!!

$$(x - 6)\ (x + 1)\ \ (x - 2)\ (x - 3)$$

P. 37. # 55. $(2x + 3y)^5 - 9(2x + 3y)^3$

Solution: Notice that there is a common factor which is $(2x + 3y)^3$! So take out the common factor, which leaves

$$(2x + 3y)^3\left[(2x + 3y)^2 - 9\right]$$

What is left in the brackets above is the difference of two squares:

$$(2x + 3y)^3\left[(2x + 3y) \qquad \right]\bullet\left[(2x + 3y) \qquad \right]$$

$$(2x + 3y)^3\left[(2x + 3y) - 3\right]\bullet\left[(2x + 3y) + 3\right]$$

If you clean it up, it looks like this:

$$(2x + 3y)^3(2x + 3y - 3)(2x + 3y + 3)$$

P. 39. # 67. $x^2 - y^2 + 4y - 4$

Solution: This is most likely a grouping problem, but the trick is to get the right grouping. You may have tried to group the first two together, which is a difference of two squares, and the last two, in which there is a common factor of 4. It would look like this:

$$x^2 - y^2 + 4y - 4$$

$$(x - y)(x + y) + 4(y - 1)$$

Now what?? There is no common factor, so it turns out that this is a dead end!! When you reach a dead end, you have to turn around and go back to the beginning!!

Now, did you do the exercises on the previous page? In particular, did you do problems 63 and 65? Exercises 63 – 65 included a hint to group the last three terms together. It turns out that this is the method that works in this exercise. So, group the last three terms together, factoring out a "negative" or a "-1" from the last three terms:

$$x^2 - y^2 + 4y - 4$$

$$x^2 - 1(y^2 - 4y + 4)$$

Notice that the last three terms form a perfect square trinomial, so the result is actually the difference of two squares!

$$x^2 - 1(y - 2)^2$$

It starts out like this:

$$[x \qquad]\,[x \qquad]$$

Then this:

$$[x - 1(y - 2)]\,[x + 1(y - 2)]$$

When you remove the blue parentheses inside by distributive property, you can change the black brackets to parentheses and "clean it up" a bit:

Final answer: $(x - y + 2)(x + y - 2)$

P. 39. # 72. $x^2 - 4xy + 4y^2 + 3x - 6y + 2$

Solution: First notice that, because of the number of terms involved here, this must be a grouping problem. Did you notice that the first three terms look good together? It turns out that these first three terms form a perfect square trinomial. Then try grouping the next two terms together from which you can factor out a common factor of 3. The last term stays by itself. Putting this into grouping by color may help you see it better:

$$x^2 - 4xy + 4y^2 + 3x - 6y + 2$$

Rewrite it in this form

$$(x - 2y)^2 + 3(x - 2y) + 2$$

and recognize that this is a trinomial. Can you see that it is in three parts?

$$(x - 2y)^2 + 3(x - 2y) + 2$$

The FIRST times FIRST must be $(x - 2y)^2$, the LAST times LAST must be 2, and the OUTER times OUTER and INNER times INNER must add up to $3(x - 2y)$. To factor $(x - 2y)^2 + 3(x - 2y) + 2$, you must find two numbers whose product is 2 and whose sum is 3.

The FIRST times FIRST must be $(x - 2y)^2$

$$\left[(x - 2y) \qquad \right]\left[(x - 2y) \qquad \right]$$

Next, find two numbers whose product is 2 and whose sum is 3. Try 2 times 1.

$$\left[(x - 2y) \;\; + \;\; 2\right]\left[(x - 2y) \;\; + \;\; 1\right]$$

This can be "cleaned up" to make it look like the product of two regular trinomials,

$$\left(x - 2y \;\; + \;\; 2\right)\left(x - 2y \;\; + \;\; 1\right)$$

P. 39. # 75. $x^2+2xy+y^2+9x+9y-10$

Solution: First notice that, because of the number of terms involved here, this must be a grouping problem. Did you notice that the first three terms look good together? It turns out that these first three terms form a perfect square trinomial. Then try grouping the next two terms together from which you can factor out a common factor of 3. The last term stays by itself. Putting this into grouping by color may help you see it better:

$$x^2+2xy+y^2+9x+9y-10$$

Rewrite it in this form

$$(x+y)^2+9(x+y)-10$$

and recognize that this is a trinomial. Can you see that it is in three parts?

$$(x+y)^2+9(x+y)-10$$

The FIRST times FIRST must be $(x+y)^2$, the LAST times LAST must be -10, and the OUTER times OUTER and INNER times INNER must subtract to give $9(x+y)$. To factor $(x+y)^2+9(x+y)-10$, you must find two numbers whose product is -10 and whose difference is +9.

The FIRST times FIRST must be $(x+y)^2$

$$\left[(x+y)\qquad\right]\left[(x+y)\qquad\right]$$

Next, find two numbers whose product is -10 and whose difference is +9. That would be +10 and -1:

$$\left[(x+y)\ +\ 10\right]\left[(x+y)\ -\ 1\right]$$

This can be "cleaned up" to make it look like the product of two regular trinomials,

$$\left(x+y\ +\ 10\right)\left(x+y\ -\ 1\right)$$

P. 40. # 77. $4x^2 - 12xy + 9y^2 - 12x + 18y + 9$

Solution: First notice that, because of the number of terms involved here, this must be a grouping problem. Did you notice that the first three terms look good together? It turns out that these first three terms form a perfect square trinomial. Then try grouping the next two terms together from which you can factor out a common factor of 3. The last term stays by itself. Putting this into grouping by color may help you see it better:

$$4x^2 - 12xy + 9y^2 - 12x + 18y + 9$$

Rewrite it in this form

$$(2x - 3y)^2 - 6(2x - 3y) + 9$$

and recognize that this is a trinomial. Can you see that it is in three parts?

$$(2x - 3y)^2 - 6(2x - 3y) + 9$$

The FIRST times FIRST must be $(2x - 3y)^2$, the LAST times LAST must be 9, and the OUTER times OUTER and INNER times INNER must add up to $-6(2x - 3y)$. To factor $(2x - 3y)^2 - 6(2x - 3y) + 9$, you must find two numbers whose product is 9 and whose sum is −6.

The FIRST times FIRST must be $(2x - 3y)^2$

$$\left[(2x - 3y) \qquad \right]\left[(2x - 3y) \qquad \right]$$

Next, find two numbers whose product is 9 and whose sum is −6. That would be −3 times −3:

$$\left[(2x - 3y) - 3\right]\left[(2x - 3y) - 3\right]$$

This can be "cleaned up" to make it look like the product of two regular trinomials,

Final answer: $(2x - 3y - 3)(2x - 3y - 3)$ or $(2x - 3y - 3)^2$

P. 40. # 78. $4x^2 - 12xy + 9y^2 - 9$

Solution: First notice that this is probably a grouping problem. Did you notice that the first three terms look good together? It turns out that these first three terms form a perfect square trinomial. The last term stays by itself. Putting this into grouping by color may help you see it better:

$$4x^2 - 12xy + 9y^2 - 9$$

Rewrite it in this form:

$$(2x - 3y)^2 - 9$$

and recognize that this is a difference of two squares.

The FIRST times FIRST must be $(2x - 3y)^2$, and the LAST times LAST must be -9. The middle term must subtract out.

The FIRST times FIRST must be $(2x - 3y)^2$

$$\left[(2x - 3y) \qquad \right]\left[(2x - 3y) \qquad \right]$$

Next, find two numbers whose product is −9.. That would be −3 times +3:

$$\left[(2x - 3y) \quad - \quad 3\right]\left[(2x - 3y) \quad + \quad 3\right]$$

This can be "cleaned up" to make it look like the product of two regular trinomials,

$$\left(2x - 3y \ - \ 3\right)\left(2x - 3y \ + \ 3\right) .$$

P. 40. # 81. $x^6 - 9x^3 + 8$

Solution: The first step is to recognize that this is a trinomial. Can you see that it is in three parts?

The FIRST times FIRST must be x^6 , the LAST times LAST must be 8, and the OUTER times OUTER and INNER times INNER must add up to $-9x^3$. You must find two numbers whose product is 8 and whose sum is −9.

The FIRST times FIRST must be x^6. Try $x^3 \bullet x^3$

$$\left(x^3 \qquad \right)\left(x^3 \qquad \right)$$

Next, find two numbers whose product is 8 and whose sum is −9. That would be −8 and −1:

$$\left(x^3 - 8\right)\left(x^3 - 1\right)$$

Each of these factors represent the difference of cubes, which can be factored using the formula: $\left(x^3 - y^3\right) = (x - y)\left(x^2 + xy + y^2\right)$

$$\left(x^3 - 8\right)\left(x^3 - 1\right)$$

$$(x-2)\left(x^2 + 2x + 2^2\right)(x-1)\left(x^2 + 1x + 1^2\right)$$

$$(x-2)\left(x^2 + 2x + 4\right)(x-1)\left(x^2 + x + 1\right)$$

These trinomials CANNOT be factored, so this is your final answer!!

P. 41. # 89. $x^6 - 64$

Solution: The first step is to recognize that this is a difference of two squares! (You can also think of this as the difference of two cubes, but this method is NOT recommended! The difference of squares is a MUCH easier method!)

The FIRST times FIRST must be x^6 , the LAST times LAST is the perfect square 64 which is 8 times 8, and MIDDLE TERM must subtract out!

The FIRST times FIRST must be x^6, which would be $x^3 \bullet x^3$

$$\left(x^3 \qquad\right)\left(x^3 \qquad\right)$$

$$\left(x^3 - 8\right)\left(x^3 + 8\right)$$

Each of these factors represent the difference or sum of cubes, which can be factored using the formulas: $\left(x^3 - y^3\right) = \left(x - y\right)\left(x^2 + xy + y^2\right)$

and $\left(x^3 + y^3\right) = \left(x + y\right)\left(x^2 - xy + y^2\right)$

$$\left(x^3 - 8\right)\left(x^3 + 8\right)$$

$$\left(x-2\right)\left(x^2+2x+2^2\right)\left(x+2\right)\left(x^2-2x+2^2\right)$$

$$\left(x-2\right)\left(x^2+2x+4\right)\left(x+2\right)\left(x^2-2x+4\right)$$

These trinomials CANNOT be factored, so this is your **final answer**!!

P. 42. # 99. $x - 64x^{-2}$

Solution: First, recognize that there is a common factor of x. When factoring the common factor, take the *lowest power of the factor*, which is x^{-2}.

$$x - 64x^{-2} = x^{-2}(\qquad)$$

Remember, when you factor out a common factor, you put down the common factor, which in this case is x^{-2}, and then you *subtract* exponents:

$$x - 64x^{-2} = x^{-2}\left(x^{1-(-2)} - 64\right)$$
$$= x^{-2}\left(x^{3} - 64\right)$$

Remember that to eliminate the negative exponent, you convert to a fraction, and $\left(x^{3} - 64\right)$ is a difference of two cubes:

$$x^{-2}\left(x^{3} - 64\right) = \frac{1}{x^2}(x-4)\left(x^2+4x+16\right) \text{ or } \frac{(x-4)\left(x^2+4x+16\right)}{x^2}$$

P. 42. # 100. $x - 81x^{-3}$

Solution: First, recognize that there is a common factor of x. When factoring the common factor, take the *lowest power of the factor*, which is x^{-3}.

$$x - 81x^{-3} = x^{-3}(\qquad)$$

Remember, when you factor out a common factor, you put down the common factor, which in this case is x^{-2}, and then you *subtract* exponents:

$$x - 81x^{-3} = x^{-3}\left(x^{1-(-3)} - 81\right)$$
$$= x^{-3}\left(x^{4} - 81\right)$$

Remember that to eliminate the negative exponent, you convert to a fraction. Also, $\left(x^{4} - 81\right)$ is the difference of two squares:

$$x^{-3}\left(x^{4} - 81\right) = \frac{1}{x^3}\left(x^2-9\right)\left(x^2+9\right)$$
$$= \frac{1}{x^3}(x-3)(x+3)\left(x^2+9\right) \text{ or } \frac{(x-3)(x+3)\left(x^2+9\right)}{x^3}$$

P. 43. # 108. $x^{\frac{5}{2}} + 8x^{-\frac{1}{2}}$

Solution: First, recognize that there is a common factor of x. When factoring the common factor, *take out the lowest power of the factor*, which is $x^{-\frac{1}{2}}$.

$$x^{\frac{5}{2}} + 8x^{-\frac{1}{2}} = x^{-\frac{1}{2}}\left(\qquad \right)$$

Remember, when you factor out a common factor, you put down the common factor, which in this case is $x^{-\frac{1}{2}}$, and then you *subtract* exponents:

$$x^{\frac{5}{2}} + 8x^{-\frac{1}{2}} = x^{-\frac{1}{2}}\left(x^{\frac{5}{2}-\left(-\frac{1}{2}\right)} + 8 \right)$$

$$= x^{-\frac{1}{2}}\left(x^{3} + 8 \right)$$

Remember that to remove the negative exponent, you can convert to a fraction. Does the sum of cubes $\left(x^3 + 8\right)$ look familiar??

$$x^{-\frac{1}{2}}\left(x^{3} + 8 \right) = \frac{1}{x^{\frac{1}{2}}}(x+2)\left(x^2 - 2x + 4\right) \text{ or } \frac{(x+2)\left(x^2-2x+4\right)}{x^{\frac{1}{2}}}$$

P. 43. # 110. $x^{\frac{10}{3}} - 81x^{-\frac{2}{3}}$

Solution: First, recognize that there is a common factor of x. When factoring the common factor, take the *lowest power of the factor*, which is $x^{-\frac{2}{3}}$

$$x^{\frac{10}{3}} - 81x^{-\frac{2}{3}} = x^{-\frac{2}{3}}(\qquad\qquad)$$

Remember, when you factor out a common factor, you put down the common factor, which in this case is $x^{-\frac{2}{3}}$, and then you *subtract* exponents:

$$x^{\frac{10}{3}} - 81x^{-\frac{2}{3}} = x^{-\frac{2}{3}}\left(x^{\frac{10}{3}-\left(-\frac{2}{3}\right)} - 81\right)$$

$$= x^{-\frac{2}{3}}\left(x^{\frac{10}{3}+\frac{2}{3}} - 81\right)$$

$$= x^{-\frac{2}{3}}\left(x^4 - 81\right)$$

Remember that you can remove a negative exponent by converting to a fraction. Also, the binomial $\left(x^4 - 81\right)$ can be factored again, as the difference of two squares:

$$x^{-\frac{2}{3}}\left(x^4 - 81\right) = \frac{1}{x^{\frac{2}{3}}}\left(x^2 - 9\right)\left(x^2 + 9\right)$$

$$= \frac{1}{x^{\frac{2}{3}}}\left(x - 3\right)\left(x + 3\right)\left(x^2 + 9\right)$$

Final answer: $= \dfrac{(x-3)(x+3)\left(x^2+9\right)}{x^{\frac{2}{3}}}$

P. 44. # 115. $(x^2+4)^{\frac{1}{2}} + x^2(x^2+4)^{-\frac{1}{2}}$

Solution:

First, recognize that there is a common factor of (x^2+4) . When factoring the common factor, take the *lowest power of the factor*, which is $(x^2+4)^{-\frac{1}{2}}$.

Remember, when you factor out a common factor, you put down the common factor, which in this case is $(x^2+4)^{-\frac{1}{2}}$, and then you *subtract* exponents:

$$(x^2+4)^{-\frac{1}{2}}\left[(x^2+4)^{\frac{1}{2}-\left(-\frac{1}{2}\right)} + x^2\right]$$

Cleaning it up a bit, it looks like this:

$$(x^2+4)^{-\frac{1}{2}}\left[(x^2+4)^1 + x^2\right]$$

$$(x^2+4)^{-\frac{1}{2}}\left[x^2+4+ x^2\right]$$

$$(x^2+4)^{-\frac{1}{2}}\left(2x^2+4\right)$$

which re-factors, by taking out the common factor of **2**. Also, remember that to remove a negative exponent, you can convert to fractional notation.

$$\frac{1}{(x^2+4)^{\frac{1}{2}}}2\left(x^2+2\right)$$

Final answer: or $$\frac{2\left(x^2+2\right)}{(x^2+4)^{\frac{1}{2}}}$$

P. 44. # 117. $(4-x^3)^{\frac{1}{3}} - x^3(4-x^3)^{-\frac{2}{3}}$

Solution:

First, recognize that there is a common factor of $(4-x^3)$. When factoring the common factor, take the *lowest power of the factor*, which is $(4-x^3)^{-\frac{2}{3}}$.

Remember, when you factor out a common factor, you put down the common factor, which in this case is $(4-x^3)^{-\frac{2}{3}}$, and then you *subtract* exponents:

$$(4-x^3)^{-\frac{2}{3}}\left[(4-x^3)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x^3\right]$$

$$(4-x^3)^{-\frac{2}{3}}\left[(4-x^3)^1 - x^3\right]$$

Cleaning it up a bit, it looks like this:

$$(4-x^3)^{-\frac{2}{3}}\left[4-x^3 - x^3\right]$$

$$(4-x^3)^{-\frac{2}{3}}\left[4-2x^3\right]$$

This re-factors, by taking out the common factor of **2**. Also, remember that to remove a negative exponent, you convert to fractional notation.

Final answer: $\frac{1}{(4-x^3)^{\frac{2}{3}}}\left[2(2-x^3)\right]$ or $\frac{2(2-x^3)}{(4-x^3)^{\frac{2}{3}}}$