Math in Living C O L O R !!

1.10 Equations in Quadratic Form

College Algebra: One Step at a Time. Page 148 - 154 #2, 9, 12, 16, 19, 21, 25

Dr. Robert J. Rapalje, Retired
Central Florida, USA

See Section 1.10, with explanations, examples, and exercises, coming soon!

P. 149. # 2. $x^4 + 5x^2 - 36 = 0$

Solution: Notice that this is really just a TRINOMIAL, and as such, it can be factored into the product of two binomials. The FIRST times FIRST must be x^4 which is obviously $x^2 \bullet x^2$. Next, the LAST times LAST must give you -36, so try $9 \bullet 4$, where the numbers are of opposite sign. In order for the numbers to subtract and give you a $+5x^2$ for the middle term, it must be $9 \bullet (-4)$

$$x^4 + 5x^2 - 36 = 0$$

$$\left(x^2 + 9\right)\left(x^2 - 4\right) = 0$$

Now, you have TWO equations to solve.

$$\left(x^2 + 9\right) = 0, \quad or \quad \left(x^2 - 4\right) = 0$$

$$x^2 = -9, \quad or \quad x^2 = 4$$

$$x = \pm\sqrt{-9}, \quad or \quad x = \pm\sqrt{4}$$

Final answer: $x = \pm 3i \quad or \quad x = \pm 2$

P. 150: #9. $(x^2-4x)^2+8(x^2-4x)+15=0$

Solution: Notice that this equation seems to be built in terms of the quantity (x^2-4x). While many people may recommend solving this one by the Substitution Method, I prefer to do this by what I call the "Straight Out Factoring Method". My idea is to recognize that since this is in three parts as it is, it is really just a TRINOMIAL, and as such, it can be factored into the product of two binomials. Here are three similar warm-up problems first:

Factor: $y^2+8y+15=0$ $\quad \pi^2+8\pi+15=0$ $\quad (Junk)^2+8(Junk)+15=0$

$(y+5)(y+3)=0$ $\quad (\pi+5)(\pi+3)=0$ $\quad [(Junk)+5][(Junk)+3)]=0$

Now, notice that the next one also factors into the product of two binomials:

$$(x^2-4x)^2+8(x^2-4x)+15=0$$

$$[\qquad\qquad][\qquad\qquad]=0$$

$$[(x^2-4x)\qquad][(x^2-4x)\qquad]=0$$

Notice that the FIRST times FIRST gives you $(x^2-4x)^2$.

Next, the LAST times LAST must give you 15, so try $5\bullet 3$.

$$[(x^2-4x)+5][(x^2-4x)+3]=0$$

Next, drop the parentheses within the brackets, and change the brackets to parentheses to make the equation look simpler.

$$(x^2-4x+5)(x^2-4x+3)=0$$

Now, you have TWO equations to solve.

$$(x^2-4x+5)=0, \quad or \quad (x^2-4x+3)=0$$

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P. 150: #9 continued.

$$\left(x^2-4x+5\right)=0, \quad or \quad \left(x^2-4x+3\right)=0$$

FIRST EQUATION.

The first equation does NOT factor, so you will have to solve this one by either the quadratic formula or by completing the square. I recommend the completing the square method, unless you prefer the quadratic formula.

$$\left(x^2-4x+5\right)=0$$

Begin by adding -5 to each side of the equation.

$$x^2-4x=-5$$

Next, you need to build a perfect square trinomial on the left side by adding a number to each side of the equation.

$$x^2-4x+\_\_\_=-5+\_\_\_$$

You can find this number by taking HALF of the -4 and squaring it. Half of -4 is -2, and $(-2)^2$ is 4, so add $+4$ to each side of the equation.

$$x^2-4x+4=-5+4$$

$$(x-2)^2=-1$$

Take the square root of each side. Don't forget the $\pm$ sign!

$$\sqrt{(x-2)^2}=\pm\sqrt{-1}$$

$$x-2=\pm i$$

Finally, add $+2$ to each side:

$$x=2\pm i$$

SECOND EQUATION.

The second equation DOES factor, so this one should be easy.

$$\left(x^2-4x+3\right)=0$$

$$(x-3)(x-1)=0$$

$$x=3, \quad x=1$$

FINAL ANSWER. There are four answers: $x=2\pm i$ or $x=3, \quad x=1$

This problem can also be solved by the Method of Substitution!

P. 151. #12. $\left(x+\frac{12}{x}\right)^2-15\left(x+\frac{12}{x}\right)+56=0$

Solution: Notice that this equation seems to be built in terms of the quantity $\left(x+\frac{12}{x}\right)$. It is therefore appropriate to name a new variable (you can use any letter that you like, so let's use the letter u), and write $u=x+\frac{12}{x}$. This is called the Substitution Method, because you will be substituting this u back into the original problem and write this:

$$\left(x+\frac{12}{x}\right)^2-15\left(x+\frac{12}{x}\right)+56=0$$

$$(u)^2 \quad - \quad 15(u) \quad + \quad 56 \quad =0$$

Now, you can see that this equation is in the shape of a quadratic equation. That is why we call this section "quadratic form." Not only is this in the form of a quadratic equation, it factors.

$$u^2 - 15u + 56 = 0$$
$$(u-7)(u-8) = 0$$
$$u = 7, \quad u = 8$$

Now, you must substitute the formula for u, and solve for the original variable which is x. This gives you the following two equations to solve:

$$u = 7 \qquad\qquad u = 8$$

$$x+\frac{12}{x} = 7 \qquad\qquad x+\frac{12}{x} = 8$$

Multiply both sides of each equation by the common denominator which is x.

$$x\bullet x+x\bullet\frac{12}{x} = x\bullet 7 \qquad\qquad x\bullet x+x\bullet\frac{12}{x} = x\bullet 8$$

$$x^2+12 = 7x \qquad\qquad x^2+12 = 8x$$

$$x^2-7x+12 = 0 \qquad\qquad x^2-8x+12 = 0$$

Interestingly enough, both of these trinomials factor!

$$(x-4)(x-3) = 0 \qquad\qquad (x-6)(x-2) = 0$$

Final answer: $x=4, \quad x=3$ $\qquad\qquad$ $x=6, \quad x=2$

P. 152: #16. $$\frac{x+7}{x}+\frac{16x}{x+7}=10$$

Solution: In this equation, the expression $\frac{x}{x+7}$ is the reciprocal of $\frac{x+7}{x}$. Let $u=\frac{x+7}{x}$. This means that the reciprocal $\frac{1}{u}=\frac{x}{x+7}$. Make these substitutions:

$$\frac{x+7}{x}+\frac{16x}{x+7}=10$$

$$u + 16\bullet\frac{1}{u} = 10$$

Next, multiply both sides of the equation times u in order to clear the fraction.

$$u\bullet u + u\bullet 16\bullet\frac{1}{u} = u\bullet 10$$

The denominator divides out leaving:

$$u\bullet u + \cancel{u}\bullet 16\bullet\frac{1}{\cancel{u}} = u\bullet 10$$

$$u^2 + 16 = 10u$$

$$u^2 - 10u + 16 = 0$$

This factors into: $$(u-8)(u-2) = 0$$

$$u=8, \quad u=2$$

Now, you must substitute the formula for u, and solve for the original variable which is x. This gives you the following two equations to solve:

$$u=8 \qquad\qquad u=2$$

$$\frac{x+7}{x}=8 \qquad\qquad \frac{x+7}{x}=2$$

Multiply both sides of each equation by the common denominator which is x.

$$x\bullet\frac{x+7}{x}=x\bullet 8 \qquad\qquad x\bullet\frac{x+7}{x}=x\bullet 2$$

$$\cancel{x}\bullet\frac{x+7}{\cancel{x}}=x\bullet 8 \qquad\qquad \cancel{x}\bullet\frac{x+7}{\cancel{x}}=x\bullet 2$$

$$x+7=8x \qquad\qquad x+7=2x$$

$$-x \quad -x \qquad\qquad -x \quad -x$$

$$7=7x \qquad\qquad 7=x$$

$$x=1$$

Final answer: $x = 1$ or $x = 7$

These answers are guaranteed, since you did NOT square both sides. You did multiply both sides by the variable x, but since $u \neq 0$ and $x \neq 0$, checking the answers is not required. There will be NO extraneous roots!!

P. 153: #19. $$\frac{x^2+12}{x}-\frac{32x}{x^2+12}=4$$

Solution: In this equation, the expression $\frac{x}{x^2+12}$ is the reciprocal of $\frac{x^2+12}{x}$. It is therefore appropriate to let $u=\frac{x^2+12}{x}$. This means that the reciprocal $\frac{1}{u}=\frac{x}{x^2+12}$. When these two substitutions are made into the original equation, it looks like this:

$$\frac{x^2+12}{x}-\frac{32x}{x^2+12}=4$$

$$u - 32\bullet\frac{1}{u} = 4$$

Next, multiply both sides of the equation times u in order to clear the fraction.

$$u\bullet u - u\bullet 32\bullet\frac{1}{u} = u\bullet 4$$

The denominator divides out leaving:

$$u\bullet u - \cancel{u}\bullet 32\bullet\frac{1}{\cancel{u}} = u\bullet 4$$

$$u^2 - 32 = 4u$$

$$u^2 - 4u - 32 = 0$$

This factors into: $$(u-8)(u+4) = 0$$

$$u=8,\quad u=-4$$

Now, you must substitute the formula for u, and solve for the original variable which is x. This gives you the following two equations to solve:

$$u=8 \qquad\qquad u=-4$$

$$\frac{x^2+12}{x}=8 \qquad\qquad \frac{x^2+12}{x}=-4$$

Multiply both sides of each equation by the common denominator which is x.

$$x\bullet\frac{x^2+12}{x}=x\bullet 8 \qquad\qquad x\bullet\frac{x^2+12}{x}=x\bullet -4$$

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P. 153: #19 continued.

$$x \bullet \frac{x^2+12}{x} = x \bullet 8 \qquad x \bullet \frac{x^2+12}{x} = x \bullet -4$$

$$\not{x} \bullet \frac{x^2+12}{\not{x}} = x \bullet 8 \qquad \not{x} \bullet \frac{x^2+12}{\not{x}} = x \bullet -4$$

$$x^2+12 = 8x \qquad x^2+12 = -4x$$

$$x^2-8x+12=0 \qquad x^2+4x+12=0$$

The first problem above can be factored, but the second problem does NOT factor, so it must be solved by completing the square or by quadratic formula. In this case, I think completing the square works best. First, the factoring problem:

$$x^2-8x+12=0$$

$$(x-2)(x-6)=0$$

$$x=2 \text{ or } x=6$$

For the second problem use the completing the square method:

$$x^2+4x+12=0$$

$$x^2+4x+\underline{\quad\quad} = -12+\underline{\quad\quad}$$

Take half of the 4 and square which is 4:

$$x^2+4x+\underline{4} = -12+\underline{4}$$

$$(x+2)^2=-8$$

Take the square root of each side:

$$\sqrt{(x+2)^2} = \pm\sqrt{-8}$$

$$x+2=\pm\sqrt{-4}\sqrt{2}$$

$$x+2=\pm 2i\sqrt{2}$$

$$x=-2\pm 2i\sqrt{2}$$

Final answer: $x=2$, $x=6$, or $x=-2\pm 2i\sqrt{2}$

These answers are guaranteed, since you did NOT square both sides. You did multiply both sides by the variable x, but since $u \neq 0$ and $x \neq 0$, checking the answers is not required. There will be NO extraneous roots!!

P. 153: #21. $(x^2-6x)^{\frac{2}{3}}-(x^2-6x)^{\frac{1}{3}}-6=0$

Solution: It appears that the expression (x^2-6x) is the building block of this equation. It is a good idea to let $u=(x^2-6x)^{\frac{1}{3}}$. Squaring both sides gives you $u^2=(x^2-6x)^{\frac{2}{3}}$. Make these substitutions into the original equation.

$$(x^2-6x)^{\frac{2}{3}}-(x^2-6x)^{\frac{1}{3}}-6=0$$

$$u^2 - u - 6 = 0$$

Notice that this factors: $(u-3)(u+2) = 0$

$$u=3, \quad u=-2$$

Now, substitute the formula for u, and solve for the original variable which is x. This gives you the following two equations to solve:

$$u=3 \qquad\qquad u=-2$$

$$(x^2-6x)^{\frac{1}{3}}=3 \qquad\qquad (x^2-6x)^{\frac{1}{3}}=-2$$

Since the $\frac{1}{3}$ power means "cube root", you must cube both sides of the equations to "undo" the $\frac{1}{3}$ power and solve for x.

$$\left((x^2-6x)^{\frac{1}{3}}\right)^3=(3)^3 \qquad\qquad \left((x^2-6x)^{\frac{1}{3}}\right)^3=(-2)^3$$

$$x^2-6x = 27 \qquad\qquad x^2-6x = -8$$

Solve these quadratic equations. Notice that when set equal to zero, they both factor!! Hmmm!

$$x^2-6x - 27=0 \qquad\qquad x^2-6x + 8=0$$

$$(x-9)(x+3)=0 \qquad\qquad (x-4)(x-2)=0$$

Final answer: $x=9, \quad x=-3$ $\qquad\qquad$ $x=4, \quad x=2$

These answers are guaranteed, since you did NOT square both sides. (Cubing both sides does NOT generate extraneous real solutions!) There will be NO extraneous roots!!

P. 154: #25. $x^2 + 8x + 12 = 7\sqrt{x^2 + 8x}$

Solution: While this looks at first like a radical equation, in which you might square both sides in order to eliminate the radical, if you do this, you will have to square a trinomial, which is a very LARGE equation to solve. Now, if you happen to have a graphing calculator that can solve large polynomial equations, maybe this is not such a bad thing after all! Nevertheless, there is a building block of $(x^2 + 8x)$ within this equation. It might be a good idea to let $u = \sqrt{x^2 + 8x}$. Squaring both sides gives you $u^2 = x^2 + 8x$. When these two substitutions are made into the original equation, it looks like this:

$$(x^2 + 8x) + 12 = 7(\sqrt{x^2 + 8x})$$

$$u^2 \quad + 12 = 7u$$

Set this equal to zero, and notice that it factors:

$$u^2 - 7u + 12 = 0$$
$$(u - 4)(u - 3) = 0$$
$$u = 4, \quad u = 3$$

Now, you must substitute these values into the formula for u, and solve for the original variable which is x. This gives the following two equations to solve:

$$u = 4 \qquad\qquad u = 3$$

$$\sqrt{x^2 + 8x} = 4 \qquad\qquad \sqrt{x^2 + 8x} = 3$$

Square both sides of the equations to "undo" the square roots, and solve for x.

$$\left(\sqrt{x^2 + 8x}\right)^2 = (4)^2 \qquad\qquad \left(\sqrt{x^2 + 8x}\right)^2 = (3)^2$$

$$x^2 + 8x = 16 \qquad\qquad x^2 + 8x = 9$$

Solve these quadratic equations. Notice that when set equal to zero, one of them factors, but the other one does not!! Too bad!! Sometimes life comes out even, sometimes it does not!

$$x^2 + 8x - 16 = 0$$

(Does not factor! Solve by completing the square. See next page for solution.

$$x^2 + 8x - 9 = 0$$
$$(x + 9)(x - 1) = 0$$
$$x = -9, \quad x = 1$$

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P. 154: #25 continued.

For the first problem use the completing the square method:

$$x^2 + 8x - 16 = 0$$

$$x^2 + 8x + \_\_\_\_ = 16 + \_\_\_\_$$

Take half of the 8 and square which is 16:

$$x^2 + 8x + \underline{16} = 16 + \underline{16}$$

$$(x+4)^2 = 32$$

Take the square root of each side:

$$\sqrt{(x+4)^2} = \pm\sqrt{32}$$

$$x + 4 = \pm\sqrt{16}\sqrt{2}$$

$$x + 4 = \pm 4\sqrt{2}$$

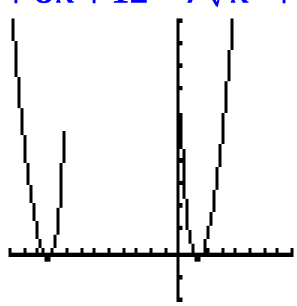
$$x = -4 \pm 4\sqrt{2}$$

Final answer: $x = -9$, $x = 1$, or $x = -4 \pm 4\sqrt{2}$ (approximately 1.65 and -9.65)

The check on this is tricky, but since you squared both sides of the equations, it is necessary! Substituting each value of x back into the original equation, especially the radical answers obtained by completing the square, is way too tedious and hard. There must be an easier way—and there is!

Check: You can set this equation $x^2 + 8x + 12 = 7\sqrt{x^2 + 8x}$ equal to zero, giving you $x^2 + 8x + 12 - 7\sqrt{x^2 + 8x} = 0$. With a graphing calculator, you can sketch the graph of $y1 = x^2 + 8x + 12 - 7\sqrt{x^2 + 8x}$ and determine the zeros (or roots) of this graph. That is, you can find the values at which the graph crosses the x-axis. These values will be the solutions to the equation. This graph clearly has 4 points of intersection, each of which corresponds to the solution values that were obtained.

$$y1 = x^2 + 8x + 12 - 7\sqrt{x^2 + 8x}$$



Notice also that there are NO values of this graph between x=-8 and x=0, because these x values result in a negative in the radical, which is not allowed in the real number system. More on that when we get to Domain and Range.