Math in Living C O L O R !!

2.08 Algebra of Functions, Composite Functions Piecewise Functions

College Algebra: One Step at a Time, Pages 296-302: 7a), b), c), 8a, b), c), Extra, 12
Pages 303-305: 1, 4, 8

Dr. Robert J. Rapalje, Retired
Central Florida, USA

See Section 2.08 with explanations, examples, and exercises, coming soon!

NOTE: In this section, there are TWO very important concepts:

1. Substitution **2. Simplification**

The first step is usually not hard, and the second step sometimes is easy. But then again, the second step can be VERY difficult. Don't be intimidated!!

Page 298: 7a) Find $f[g(x)]$, given $f(x)=\dfrac{2x+3}{4x}$ and $g(x)=x^2-3x+6$

Step 1: Substitution: $f(x)=\dfrac{2(x)+3}{4(x)}$

$$f[g(x)]=\frac{2\left(x^2-3x+6\right)+3}{4\left(x^2-3x+6\right)}$$

Step 2: Simplification: $f[g(x)]=\dfrac{2\left(x^2-3x+6\right)+3}{4\left(x^2-3x+6\right)}$

$$f[g(x)]=\frac{2x^2-6x+12+3}{4\left(x^2-3x+6\right)}$$

7a) Final answer: $f[g(x)]=\dfrac{2x^2-6x+15}{4\left(x^2-3x+6\right)}$

Page 298: 7b) Find $g[f(x)]$ where $f(x)=\dfrac{2x+3}{4x}$ and $g(x)=x^2-3x+6$

Step 1: Substitution: $g[x]=(x)^2-3(x)+6$

$$g[f(x)]=\left(\frac{2x+3}{4x}\right)^2-3\left(\frac{2x+3}{4x}\right)+6$$

Step 2: Simplification: $g[f(x)]=\left(\dfrac{2x+3}{4x}\right)^2-3\left(\dfrac{2x+3}{4x}\right)+6$

$$g[f(x)]=\left(\frac{4x^2+12x+9}{16x^2}\right)-\frac{3}{1}\left(\frac{2x+3}{4x}\right)+\frac{6}{1}$$

Before going any further, the LCD is $16x^2$, so multiply numerator and denominator of the second fraction by $4x$, and the third "fraction" by $16x^2$. Hang on! This one might get ugly!

$$g[f(x)]=\left(\frac{4x^2+12x+9}{16x^2}\right)-\frac{3}{1}\bullet\frac{4x}{4x}\bullet\left(\frac{2x+3}{4x}\right)+\frac{6}{1}\bullet\frac{16x^2}{16x^2}$$

$$g[f(x)]=\frac{(4x^2+12x+9)-12x(2x+3)+6\bullet 16x^2}{16x^2}$$

$$g[f(x)]=\frac{4x^2+12x+9-24x^2-36x+96x^2}{16x^2}$$

7b) Final answer: $g[f(x)]=\dfrac{76x^2-24x+9}{16x^2}$

Solution: 7c) Find $f[f(x)]$, where $f(x)=\dfrac{2x+3}{4x}$

Step 1: Substitution: $f(x)=\dfrac{2(x)+3}{4(x)}$

$$f[f(x)]=\frac{2f(x)+3}{4f(x)}, \text{ where } f(x)=\frac{2x+3}{4x}$$

$$f[f(x)]=\frac{2\left(\frac{2x+3}{4x}\right)+3}{4\left(\frac{2x+3}{4x}\right)}$$

Page 298: 7c) continued

Step 2: Simplification: $f[f(x)]=\dfrac{2\left(\dfrac{2x+3}{4x}\right)+3}{4\left(\dfrac{2x+3}{4x}\right)}$

Of course, this is a complex fraction, which can be tricky. Method I or Method II? Let's try Method II. The LCD for the entire fraction is $4x$, so multiply numerator and denominator of the complex fraction by $4x$.

$$f[f(x)]=\frac{\dfrac{4x}{1}\bullet\left[2\left(\dfrac{2x+3}{4x}\right)+3\right]}{\dfrac{4x}{1}\bullet\left[4\left(\dfrac{2x+3}{4x}\right)\right]}$$

$$f[f(x)]=\frac{\left[\dfrac{4x}{1}\bullet 2\left(\dfrac{2x+3}{4x}\right)+\dfrac{4x}{1}\bullet 3\right]}{\dfrac{4x}{1}\bullet\left[4\left(\dfrac{2x+3}{4x}\right)\right]}$$

Now THAT'S UGLY!! But it cleans up nicely! ALL these ugly fractions divide out, leaving this:

$$f[f(x)]=\frac{\left[\dfrac{\cancel{4x}}{1}\bullet 2\left(\dfrac{2x+3}{\cancel{4x}}\right)+\dfrac{4x}{1}\bullet 3\right]}{\dfrac{\cancel{4x}}{1}\bullet\left[4\left(\dfrac{2x+3}{\cancel{4x}}\right)\right]}$$

$$f[f(x)]=\frac{2(2x+3)+12x}{4(2x+3)}$$

$$f[f(x)]=\frac{4x+6+12x}{4(2x+3)}$$

$$f[f(x)]=\frac{16x+6}{4(2x+3)}$$

$$f[f(x)]=\frac{2(8x+3)}{4(2x+3)}$$

7c) Final answer: $f[f(x)]=\dfrac{8x+3}{2(2x+3)}$

Page 298: 8a) Find $f[g(x)]$ where $f(x)=\dfrac{3x-2}{4x}$ and $g(x)=5x^2-2x$

Step 1: Substitution: $$f(x)=\frac{3(x)-2}{4(x)}$$

$$f[g(x)]=\frac{3(5x^2-2x)-2}{4(5x^2-2x)}$$

Step 2: Simplification: $$f[g(x)]=\frac{3(5x^2-2x)-2}{4(5x^2-2x)}$$

8a) Final answer: $$f[g(x)]=\frac{15x^2-6x-2}{4x(5x-2)}$$

Page 298: 8b) Given $f(x)=\dfrac{3x-2}{4x}$ and $g(x)=5x^2-2x$, find $g[f(x)]$.

Step 1: Substitution: $$g[x]=5(x)^2-2(x)$$

$$g[f(x)]=5\left(\frac{3x-2}{4x}\right)^2-2\left(\frac{3x-2}{4x}\right)$$

Step 2: Simplification: $$g[f(x)]=\frac{5}{1}\left(\frac{3x-2}{4x}\right)^2-\frac{2}{1}\left(\frac{3x-2}{4x}\right)$$

$$g[f(x)]=\frac{5}{1}\left(\frac{9x^2-12x+4}{16x^2}\right)-\frac{2}{1}\left(\frac{3x-2}{4x}\right)$$

Before going any further, the LCD is $16x^2$, so multiply numerator and denominator of the last fraction by $4x$. Hang on! This one might get ugly!

$$g[f(x)]=\frac{5}{1}\left(\frac{9x^2-12x+4}{16x^2}\right)-\frac{2}{1}\bullet\frac{4x}{4x}\bullet\left(\frac{3x-2}{4x}\right)$$

$$g[f(x)]=\frac{5\bullet(9x^2-12x+4)-8x(3x-2)}{16x^2}$$

$$g[f(x)]=\frac{45x^2-60x+20-24x^2+16x}{16x^2}$$

8b) Final answer: $$g[f(x)]=\frac{21x^2-44x+20}{16x^2}$$

Page 298: 8c) Find $f[f(x)]$, where $f(x)=\dfrac{3x-2}{4x}$ and $g(x)=5x^2-2x$

Step 1: Substitution:

$$f(x)=\frac{3(x)-2}{4(x)}$$

$$f[f(x)]=\frac{3f(x)-2}{4f(x)}, \text{ where } f(x)=\frac{3x-2}{4x}$$

$$f[f(x)]=\frac{3\left(\frac{3x-2}{4x}\right)-2}{4\left(\frac{3x-2}{4x}\right)}$$

Step 2: Simplification:

Of course, this is a complex fraction, which can be tricky. Method I or Method II? Let's try Method II. The LCD for the entire fraction is $4x$, so multiply numerator and denominator of the complex fraction by $4x$.

$$f[f(x)]=\frac{\frac{4x}{1}\bullet\left[3\left(\frac{3x-2}{4x}\right)-2\right]}{\frac{4x}{1}\bullet\left[4\left(\frac{3x-2}{4x}\right)\right]}$$

$$f[f(x)]=\frac{\left[\frac{4x}{1}\bullet 3\left(\frac{3x-2}{4x}\right)-\frac{4x}{1}\bullet 2\right]}{\frac{4x}{1}\bullet\left[4\left(\frac{3x-2}{4x}\right)\right]}$$

Now THAT'S UGLY!! But it cleans up nicely! ALL these ugly fractions divide out, leaving this:

$$f[f(x)]=\frac{\left[\frac{\cancel{4x}}{1}\bullet 3\left(\frac{3x-2}{\cancel{4x}}\right)-\frac{4x}{1}\bullet 2\right]}{\frac{\cancel{4x}}{1}\bullet\left[4\left(\frac{3x-2}{\cancel{4x}}\right)\right]}$$

$$f[f(x)]=\frac{3(3x-2)-8x}{4(3x-2)}$$

8c) Final answer: $$f[f(x)]=\frac{9x-6-8x}{4(3x-2)}=\frac{x-6}{4(3x-2)}$$

Composite Functions

(also known as Composition of Functions)

Using the notation of higher math, the notation is sometimes written

$$f[g(x)] = (f \circ g)(x)$$
$$g[f(x)] = (g \circ f)(x)$$

Extra Application (from Soledad at Brevard Community College).

The function to convert Fahrenheit to Celsius is given by:

C(f) = 5/9 (f - 32)

The function to convert Celsius to Kelvin is given by:

K(C) = C + 273.15

a) Find a composite function that represents temperature on the Kelvin scale in terms of degrees Fahrenheit. That is, find K o C(f)

$K(C) = C + 273.15$ and $C(f) = \frac{5}{9}(f - 32)$

$$K \circ (C(f)) = K[C(f)] = \frac{5}{9}(f - 32) + 273.15$$

b) Convert 0 degrees Fahrenheit to Kelvin using your composite function. If $f = 0$, then

$$K \circ (C(0)) = \frac{5}{9}(0 - 32) + 273.15 = \frac{-160}{9} + 273.15 \approx 255.37$$

c) Convert 100 degrees F to Kelvin using your composite function.
Solution: If $f = 100$, then

$$K \circ (C(100)) = \frac{5}{9}(100 - 32) + 273.15$$

$$K \circ (C(100)) = \frac{5}{9}(68) + 273.15 = \frac{340}{9} + 273.15 \approx 310.93$$

Page 301: 12. $f(x)=\dfrac{3x-4}{x}$ and $g(x)=x^2+4x-8$

Solution:

First find: $f(-4)=$ ______ and $g(-4)=$ ______

$f(-4)=\dfrac{3(-4)-4}{(-4)}$ $\qquad$ $g(-4)=(-4)^2+4(-4)-8$

$f(-4)=\dfrac{-16}{-4}$ $\qquad$ $g(-4)=16-16-8$

$f(-4)=4$ $\qquad$ $g(-4)=-8$

12a) $(f+g)(-4)$

$f(-4)+g(-4)$

$4 + (-8)$

-4

b) $(f-g)(-4)$

$f(-4)-g(-4)$

$4 - (-8)$

12

c) $(f \bullet g)(-4)$

$f(-4)\bullet g(-4)$

$4 \bullet (-8)$

-32

d) $(f/g)(-4)$

$\dfrac{f(-4)}{g(-4)}$

$\dfrac{4}{(-8)}$

$-\dfrac{1}{2}$

e) $(f\circ g)(-4)$

$f[g(-4)]$ where $g(-4)=-8$

$f[-8]=\dfrac{3(-8)-4}{(-8)}$

$f[-8]=\dfrac{-28}{-8}$

$(f\circ g)(-4)=f[-8]=\dfrac{7}{2}$

f) $(g\circ f)(-4)$

$g[f(-4)]$ where $f(-4)=4$

$g[4]=4^2+4\bullet 4-8$

$g[4]=16+16-8$

$(g\circ f)(-4)=g[4]=24$

Page 301: 12 continued) $f(x)=\frac{3x-4}{x}$ and $g(x)=x^2+4x-8$

g) $(g\circ g)(-4)$

$g[g(-4)]$ where $g(-4)=-8$

$g[-8]=(-8)^2+4\bullet(-8)-8$

$g[-8]=64-32-8$

$(g\circ g)(-4)=g[-8]=24$

h) $(f\circ f)(-4)$

$f[f(-4)]$ where $f(-4)=4$

$f[4]=\frac{3(4)-4}{(4)}$

$f[4]=\frac{8}{4}$

$(f\circ f)(-4)=f[4]=2$

Page 303: #1. $f(x)=\begin{cases} x^2 & \text{if } x\geq 0 \\ x-3 & \text{if } x<0 \end{cases}$ **(call this category 1)** / **(call this category 2)**

Solution:

a) $f(2)=$ ______ This means that $x=2$, which means that x is in **category 1**, since $x\geq 0$.

$f(2)=2^2$

$f(2)=4$

b) $f(-2)=$ ______ This means that $x=-2$, which means that x is in **category 2**, since $x\leq 0$.

$f(-2)=-2-3$

$f(-2)=-5$

c) $f(-8)=$ ______ This means that $x=-8$, which means that x is in **category 2**, since $x\leq 0$.

$f(-8)=-8-3$

$f(-8)=-11$

d) $f(8)=$ ______ This means that $x=8$, which means that x is in **category 1**, since $x\geq 0$.

$f(8)=8^2$

$f(8)=64$

e) $f(0)=$ ______ This means that $x=0$, which means that x is in **category 1**, since $x\geq 0$.

$f(0)=0^2$

$f(0)=0$

f) $f(-25)=$ ______ This means that $x=-25$, which means that x is in **category 2**, since $x\leq 0$.

$f(-25)=-25-3$

$f(-25)=-28$

Page 304: #4. $f(x) = \begin{cases} x^2 & \text{if } x \le -2 \quad \text{(call this category 1)} \\ 3 & \text{if } -2 < x \le 0 \quad \text{(call this category 2)} \\ -2x & \text{if } x > 0 \quad \text{(call this category 3)} \end{cases}$

Solution:

a) $f(2) =$ _____ This means that $x=2$, which means that x is in **category 3**, since $x > 0$.
$f(2) = -2 \bullet 2$
$f(2) = -4$

b) $f(-2) =$ _____ This means that $x=-2$, which means that x is in **category 1**, since $x \le -2$.
$f(-2) = (-2)^2$
$f(-2) = 4$

c) $f(-8) =$ _____ This means that $x=-8$, which means that x is in **category 1**, since $x \le -2$.
$f(-8) = (-8)^2$
$f(-8) = 64$

d) $f(8) =$ _____ This means that $x=8$, which means that x is in **category 3**, since $x > 0$.
$f(8) = -2 \bullet (8)$
$f(8) = -16$

e) $f(0) =$ _____ This means that $x=0$, which means that x is in **category 2**, since $-2 < x \le 0$.
$f(0) = 3$

f) $f(-13) =$ ____ This means that $x=-13$, which means that x is in **category 1**, since $x \le -2$.
$f(-13) = (-13)^2$
$f(-13) = 169$

Page 305: #8. $f(x)=\begin{cases}-x^2+5 & \text{if } x<-3 & \textbf{(call this category 1)}\\ 6-5x & \text{if } -3\le x<2 & \textbf{(call this category 2)}\\ -3-2x & \text{if } x\ge 2 & \textbf{(call this category 3)}\end{cases}$

Solution:

a) $f(2)=$ ______ This means that $x=2$, which means that x is in **category 3**, since $x\ge 2$.

$f(2)=-3-2(2)$

$f(2)=-3-4$

$f(2)=-7$

b) $f(-3)=$ _____ This means that $x=-3$, which means that x is in **category 2**, since $-3\le x<2$.

$f(-3)=6-5(-3)$

$f(-3)=6+15$

$f(-3)=21$

c) $f(-5)=$ ______ This means that $x=-5$, which means that x is in **category 1**, since $x<-3$.

$f(-5)=-(-5)^2+5$

$f(-5)=-25+5$

$f(-5)=-20$

d) $f(8)=$ ______ This means that $x=8$, which means that x is in **category 3**, since $x\ge 2$.

$f(8)=-3-2(8)$

$f(8)=-3-16$

$f(8)=-19$

e) $f(0)=$ ______ This means that $x=0$, which means that x is in **category 2**, since $-3\le x<2$.

$f(0)=6-5(0)$

$f(0)=6$

f) $f(-8)=$ _____ This means that $x=-8$, which means that x is in **category 1**, since $x<-3$.

$f(-8)=-(-8)^2+5$

$f(-8)=-64+5$

$f(-8)=-59$