Math in Living C O L O R !!

2.03 The Parabola

College Algebra: One Step at a Time, **Page 219 - 228: #37, 43, 46, 47, 68, 71, 72, 75**

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See Section 2.03, with explanations, examples, and exercises, coming soon!

General Explanation

There are quite a number of different explanations and presentations that can be made for graphing parabolas. In the analytic geometry explanation, usually found in a Precalculus or Calculus course, the parabola is defined to be the set of all points that are equally distant from a given point called the focal point and a given line, which is called the directrix of the parabola.

By contrast, and for simplicity in lower math courses, a parabola can be explained in terms of simple graphs of equations in the form of $y=x^2$, $y=-x^2$, $x=y^2$, or $x=-y^2$. In the more general case, equations in the form $y=ax^2+bx+c$ and $x=ay^2+by+c$ can be shown to represent graphs of parabolas.

For $y=ax^2+bx+c$, the parabola will open up if a is positive
down if a is negative.

For $x=ay^2+by+c$, the parabola will open right if a is positive
left if a is negative.

There are several different ways to explain and graph parabolas in this form. You can just make a table of values and plot points. I don't recommend this—it's too tedious and mechanical. You can also graph by finding an intercept (the easiest and a very important point to graph!) and the vertex of the parabola by completing the square. For the past 20 years or so, in my book *College Algebra: One Step at a Time*, I explained parabolas by this method. However, I think it might be easier find the vertex of a parabola by using a simple formula that is easy to derive and remember.

For a parabola in the form $y = ax^2 + bx + c$, the vertex will have as its x coordinate $x = \frac{-b}{2a}$. This is not a new formula. It actually comes from the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where the radical is zero. After you find the x-coordinate of the vertex, then you just substitute this value of x into the original equation, and solve for *y* to find the *y*-coordinate of the vertex. Remember, the vertex consists of TWO coordinates, both *x* and *y*. You must find BOTH of them.

It is also a good idea to find the *y*-intercept. This is both an easy point and an important point to graph, since it indicates where the graph crosses the *y*-axis. It is also very easy to find, since you just let *x*=0, and solve for *y* in the original equation.

For an equation of a parabola in the form $x = ay^2 + by + c$, the vertex will have as its *y* coordinate $y = \frac{-b}{2a}$. Then substitute this value of *y* back into the original equation to find the x coordinate. Don't forget, you must find BOTH coordinates, and when you give ordered pair notation (*x*, *y*), you must always put the x coordinate first

It is also a good idea to find the *x*-intercept. This is both an easy point and an important point to graph, since it indicates where the graph crosses the *x*-axis. It is also very easy to find, since you just let *y*=0, and solve for x in the original equation.

Parabola Summary

$y = ax^2 + bx + c$ -- Opens UP if a > 0

-- Opens DOWN if a < 0

Vertex is at $x = \frac{-b}{2a}$

$x = ay^2 + by + c$ -- Opens RIGHT if a > 0

-- Opens LEFT if a < 0

Vertex is at $y = \frac{-b}{2a}$

P. 222. # 37. $y = 3x^2 - 18x + 24$

Solution: Before you ever start the problem, you know from the fact that the equation is in the form $y = x^2$, that it is a **PARABOLA**, and it opens **UP**! It should be very easy to graph, if you can find the VERTEX and the *y*-intercept.

Of course, the easiest point to find is the *y*-intercept, which is where $x = 0$, and therefore $y = 24$. This is the point $(0, 24)$.

There are many ways to find the vertex, but perhaps the easiest way is to know that the vertex will be at $x = \frac{-b}{2a}$, where $a = 3$ and $b = -18$. Therefore

$$x = \frac{-b}{2a} = \frac{-(-18)}{2 \bullet 3} = \frac{18}{6} = 3$$

To find the *y*-coordinate, substitute $x = 3$ back into the original equation for *y*.

$$y = 3x^2 - 18x + 24$$

$$y = 3(3)^2 - 18(3) + 24$$

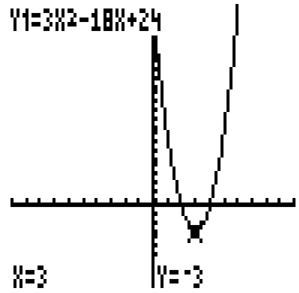
$$y = 27 - 54 + 24$$

$$y = -3$$

Therefore the vertex is $(3, -3)$.

Sketch the graph by locating these two points, $(3, -3)$ and $(0, 24)$, and draw a parabola from the vertex at $(3, -3)$, opening UP, and passing through $(0, 24)$.

$y = 3x^2 - 18x + 24$



P. 228. # 43. $y=-3x^2-18x+24$

Solution: From the fact that the equation is in the form $y=-x^2$, you know that it is a PARABOLA, and it opens DOWN! Knowing this, you can sketch the graph, if you can find the VERTEX and the *y*-intercept.

Of course, the easiest point to find is the y-intercept, which is where $x=0$, and therefore $y=24$. This is the point $(0,24)$

There are many ways to find the vertex, but perhaps the easiest way is to know that the vertex will be at $x=\frac{-b}{2a}$, where $a=-3$ and $b=-18$. Therefore

$$x=\frac{-b}{2a}=\frac{-(-18)}{2(-3)}=\frac{18}{-6}=-3$$

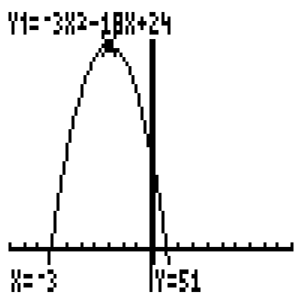
To find the *y*-coordinate, substitute $x=-3$ back into the original equation for *y*.

$$y=-3x^2-18x+24$$
$$y=-3(-3)^2-18(-3)+24$$
$$y=-27+54+24$$
$$y=51$$

Therefore the vertex is $(-3,51)$.

Sketch the graph by locating these two points, $(-3,51)$ and $(0,24)$, and draw a parabola from the vertex $(-3,51)$, opening DOWN, and passing through $(0,24)$.

$y=-3x^2-18x+24$



P. 228. # 46. $y = 4x^2 + 4x + 12$

Solution: You know from the fact that the equation in the form $y = x^2$, that this is a PARABOLA, and it opens UP! Knowing this, it should be very easy to graph, if you can find the VERTEX and the *y*-intercept.

Of course, the easiest point to find is the *y*-intercept, which is where $x = 0$, and therefore $y = 12$. This is the point $(0,12)$

There are many ways to find the vertex, but perhaps the easiest way is to know that the vertex will be at $x = \frac{-b}{2a}$, where $a = 4$ and $b = 4$. Therefore

$$x = \frac{-b}{2a} = \frac{-(4)}{2(4)} = \frac{-4}{8} = -\frac{1}{2}$$

To find the *y*-coordinate, substitute $x = -\frac{1}{2}$ back into the original equation for *y*.

$$y = 4x^2 + 4x + 12$$

$$y = 4\left(-\frac{1}{2}\right)^2 + 4 \bullet \left(-\frac{1}{2}\right) + 12$$

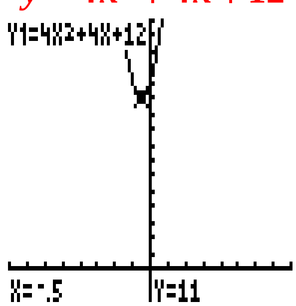
$$y = 4 \bullet \frac{1}{4} + 4 \bullet \left(-\frac{1}{2}\right) + 12$$

$$y = 11$$

Therefore the vertex is $(-\frac{1}{2}, 11)$.

Sketch the graph by locating these two points, $(-\frac{1}{2}, 11)$ and $(0,12)$, and draw a parabola from the vertex at $(-\frac{1}{2}, 11)$, opening UP, and passing through $(0,12)$.

$y = 4x^2 + 4x + 12$



P. 228. # 47. $y = 2x^2 - 2x - 8$

Solution: Since the equation in the form of $y = x^2$, you know it is a PARABOLA, and it opens UP! Knowing this, it should be very easy to graph, if you can find the VERTEX and the *y*-intercept.

Of course, the easiest point to find is the *y*-intercept, which is where $x = 0$, and therefore $y = -8$. This is the point $(0, -8)$

There are many ways to find the vertex, but perhaps the easiest way is to know that the vertex will be at $x = \frac{-b}{2a}$, where $a = 2$ and $b = -2$. Therefore

$$x = \frac{-b}{2a} = \frac{-(-2)}{2(2)} = \frac{2}{4} = \frac{1}{2}$$

To find the *y*-coordinate, substitute $x = \frac{1}{2}$ back into the original equation for *y*.

$$y = 2x^2 - 2x - 8$$

$$y = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) - 8$$

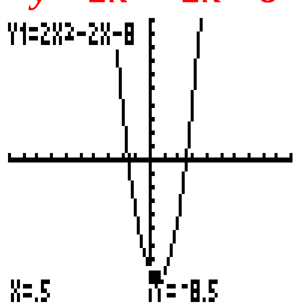
$$y = 2 \bullet \frac{1}{4} - 1 - 8$$

$$y = \frac{1}{2} - 9 = -8.5$$

Therefore the vertex is $(0.5, -8.5)$.

Sketch the graph by locating these two points, $(0.5, -8.5)$ and $(0, -8)$, and draw a parabola from the vertex at $(0.5, -8.5)$, opening UP, and passing through $(0, -8)$.

$y = 2x^2 - 2x - 8$



P. 228. # 68. $x = 2y^2 - 3y + 1$

Solution: Since this is an equation in the form of $x = y^2$, you know that it is a PARABOLA, and it opens to the RIGHT! Knowing this, you can sketch the graph, if you can find the VERTEX and the x-intercept.

Of course, the easiest point to find is the x-intercept, which is where $y = 0$, and therefore $x = 1$. This is the point $(1,0)$

There are many ways to find the vertex, but perhaps the easiest way is to realize that the vertex will be at $y = \frac{-b}{2a}$, where $a = 2$ and $b = -3$. Therefore

$$y = \frac{-b}{2a} = \frac{-(-3)}{2(2)} = \frac{3}{4}$$

To find the x-coordinate, substitute $y = \frac{3}{4}$ back into the original equation for x.

$$x = 2y^2 - 3y + 1$$

$$x = 2\left(\frac{3}{4}\right)^2 - 3 \bullet \frac{3}{4} + 1$$

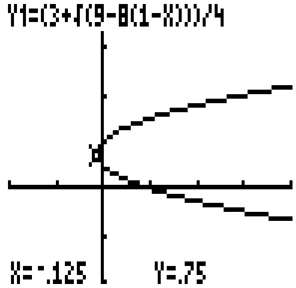
$$x = 2 \bullet \frac{9}{16} - \frac{9}{4} + 1$$

$$x = \frac{9}{8} - \frac{18}{8} + \frac{8}{8} = -\frac{1}{8}$$

Therefore the vertex is $(-\frac{1}{8}, \frac{3}{4})$.

Sketch the graph by locating these two points, $(-\frac{1}{8}, \frac{3}{4})$ and $(1,0)$, and draw a parabola from the vertex at $(-\frac{1}{8}, \frac{3}{4})$, opening RIGHT, and passing through $(1,0)$.

$$x = 2y^2 - 3y + 1$$



ALTERNATE SOLUTION--Completing the Square Method

P. 228. # 68. $x = 2y^2 - 3y + 1$

Solution: As before, you know that since the equation in the form of $x = y^2$, it is a PARABOLA, and it opens to the RIGHT! Knowing this, if you can find the VERTEX and the x-intercept, you can sketch the graph.

Of course, the easiest point to find is the x-intercept, which is where $y = 0$, and therefore $x = 1$. This is the point $(1,0)$

In order to find the vertex by the completing the square method, you must have the coefficient of y^2 to be 1 by factoring out the 2 -- even if fractions result:

$$x\_\_\_ = 2(y^2 - \frac{3}{2}y + \_\_\_) + 1$$

You must complete the square to fill in the blank space within the parentheses by taking half of the $-\frac{3}{2}$, which is $-\frac{3}{4}$, and squaring in order to obtain $\frac{9}{16}$. Then you placed a $\frac{9}{16}$ in the parentheses on the right side of the equation, but you that $\frac{9}{16}$ was actually multiplied times the 2 that was outside the parentheses. Therefore, you must add $2 \bullet \frac{9}{16}$, or $\frac{9}{8}$ to the left side of the equation. It should look like this:

$$x\_\_\_ = 2(y^2 - \frac{3}{2}y + \_\_\_) + 1$$

$$x + 2 \bullet \frac{9}{16} = 2(y^2 - \frac{3}{2}y + \frac{9}{16}) + 1$$

$$x + \frac{9}{8} = 2(y - \frac{3}{4})^2 + 1$$

To be continued next page!!

P. 228: #68 continued.

$$x+\frac{9}{8}=2(y-\frac{3}{4})^2+1$$

Next subtract 1 from each side:

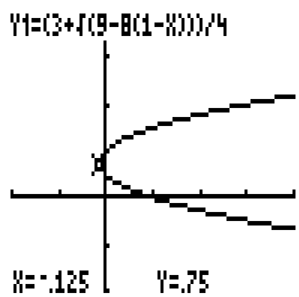
$$x+\frac{9}{8}-1=2(y-\frac{3}{4})^2+1-1$$

$$x+\frac{1}{8}=2(y-\frac{3}{4})^2$$

From this form of the equation, you can see that the vertex (i.e., what zeros out the x term and what zeros out the y term) is $x=-\frac{1}{8}$ and $y=\frac{3}{4}$. The vertex is therefore the point $(-\frac{1}{8},\frac{3}{4})$.

As before, sketch the graph by locating these two points, $(-\frac{1}{8},\frac{3}{4})$ and $(1,0)$, and draw a parabola from the vertex at $(-\frac{1}{8},\frac{3}{4})$, opening LEFT, and passing through $(1,0)$.

$$x=2y^2-3y+1$$



P. 228. # 71. $x = -y^2 - 6y - 5$

Solution: You know that, since the equation in the form of $x = -y^2$, this is a PARABOLA, and it opens to the LEFT! You can graph it, if you find the VERTEX and the x-intercept.

Of course, the easiest point to find is the x-intercept, which is where $y = 0$, and therefore $x = -5$. This is the point $(-5, 0)$

There are many ways to find the vertex, but perhaps the easiest way is to know that the vertex will be at $y = \frac{-b}{2a}$, where $a = -1$ and $b = -6$. Therefore

$$y = \frac{-b}{2a} = \frac{-(-6)}{2(-1)} = -3$$

To find the x-coordinate, substitute $y = -3$ back into the original equation for x.

$$x = -y^2 - 6y - 5$$

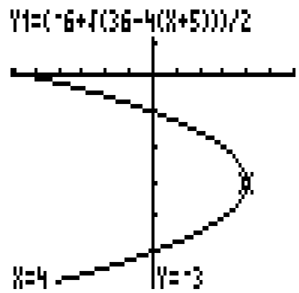
$$x = -(-3)^2 - 6(-3) - 5$$

$$x = -9 + 18 - 5 = 4$$

Therefore the vertex is $(4, -3)$.

Sketch the graph by locating these two points, $(4, -3)$ and $(-5, 0)$, and draw a parabola from the vertex at $(4, -3)$, opening LEFT, and passing through $(-5, 0)$. This is how it looks on the graphing calculator:

$$x = -y^2 - 6y - 5$$



P. 228. # 72. $x = y^2 + 8y + 12$

Solution: Since this equation in the form of $x = y^2$, it is a PARABOLA that opens to the RIGHT! Find the VERTEX and the x-intercept to sketch the graph.

Of course, the easiest point to find is the x-intercept, which is where $y = 0$, and therefore $x = 12$. This is the point $(12, 0)$

There are many ways to find the vertex, but perhaps the easiest way is to know that the vertex will be at $y = \frac{-b}{2a}$, where $a = 1$ and $b = 8$. Therefore

$$y = \frac{-b}{2a} = \frac{-(8)}{2(1)} = -4$$

To find the x-coordinate, substitute $y = -4$ back into the original equation for x.

$$x = y^2 + 8y + 12$$

$$x = (-4)^2 + 8(-4) + 12$$

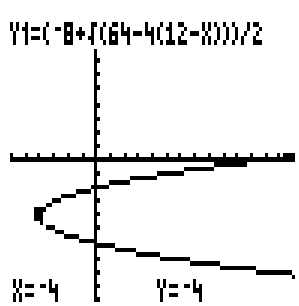
$$x = 16 + (-32) + 12 = -4$$

Therefore the vertex is $(-4, -4)$.

Sketch the graph by locating these two points, $(-4, -4)$ and $(12, 0)$, and draw a parabola from the vertex at $(-4, -4)$, opening RIGHT, and passing through $(12, 0)$.

This is how it looks on the graphing calculator:

$$x = y^2 + 8y + 12$$



P. 228. # 75. $x = 4y^2 + 6y + 3$

Solution: Since this equation in the form $x = y^2$, it is a PARABOLA that opens to the RIGHT! Knowing this, you can sketch the graph using the VERTEX and the x-intercept.

Of course, the easiest point to find is the x-intercept, which is where $y = 0$, and therefore $x = 3$. This is the point $(3,0)$

To find the vertex, use the formula $y = \frac{-b}{2a}$, where $a = 4$ and $b = 6$. Therefore,

$$y = \frac{-b}{2a} = \frac{-(6)}{2(4)} = -\frac{6}{8} = -\frac{3}{4}$$

To find the x-coordinate, substitute $y = -\frac{3}{4}$ back into the original equation for x.

$$x = 4y^2 + 6y + 3$$

$$x = 4\left(\frac{-3}{4}\right)^2 + 6\left(\frac{-3}{4}\right) + 3$$

$$x = 4 \bullet \frac{9}{16} - \frac{18}{4} + 3$$

$$x = \frac{9}{4} - \frac{18}{4} + \frac{12}{4} = \frac{3}{4}.$$

Therefore the vertex is at $\left(\frac{3}{4}, \frac{-3}{4}\right)$.

Sketch the graph by locating these two points, $\left(\frac{3}{4}, \frac{-3}{4}\right)$ and $(3,0)$, and draw a parabola from the vertex at $\left(\frac{3}{4}, \frac{-3}{4}\right)$, opening RIGHT, and passing through $(3,0)$.

$$x = 4y^2 + 6y + 3$$