Math in Living C O L O R !!

3.04 Factoring by Synthetic Division
Rational Root Theorem, Descartes Rule

***College Algebra: One Step at a Time*, Pages 378 - 405: #9, 15, 21, 25, 27, 30, 31, 42, 43, 53, Extra Problem**

Special Graphing Calculator Solution for TI 84 (or TI 83+)!! See Page 397: #42, 43, 53

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See Section 3.04, with explanations, examples, and exercises, coming soon!

p. 380. #9. Factor $3x^3 - x^2 - 3x + 1$ by grouping.

Solution: Group the first two terms, and factor out the x^2. Group the last two terms, and factor out a -1 .

$$3x^3 - x^2 - 3x + 1$$

$$x^2(3x-1) - 1(3x-1)$$

Now factor out the common factor $(3x-1)$:

$$(3x-1)(x^2-1)$$

Factor the difference of squares:

$$(3x-1)(x-1)(x+1).$$

p. 384. # 15. $2x^3 + 3x^2 - 3x - 2 = 0$, given that $x = 1$ is a root.

Solution: If $x = 1$ is a root, then $x - 1 = 0$, so $x - 1$ is a factor of the polynomial. Therefore you must do synthetic division by the number 1 in order to find the other factor, which gives you the reduced equation. First write down the coefficients of the polynomial, which are $2 \quad 3 \quad -3 \quad -2$ and prepare to do synthetic division with 1.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -3 & -2 \\ & \downarrow & 2 & 5 & 2 \\ \hline & 2 & 5 & 2 & 0 \end{array}$$

Of the resulting numbers $2 \quad 5 \quad 2 \quad 0$, the last number 0 is the remainder. (Of course it is ZERO! If $x - 1$ is a factor, then the remainder MUST be 0!!) Also, the first three numbers $2 \quad 5 \quad 2$ are the coefficients of the quotient. The quotient always begins with an exponent that is one less than the highest power of the polynomial. The quotient will therefore begin with $2x^2$. The quotient is $2x^2 + 5x + 2$, and the reduced equation that must now be solved is $2x^2 + 5x + 2 = 0$. Fortunately, this factors:

$$2x^2 + 5x + 2 = 0$$

$$(2x + 1)(x + 2) = 0$$

$$2x = -1, \quad x = -2$$

$$x = -\frac{1}{2}, \quad x = -2$$

Final Answer: There are three roots: $x = 1, \quad x = -\frac{1}{2}, \quad x = -2$

p. 384. # 21. $x^3+5x^2+9x+5=0$, given that $x=-1$ is a root.

Solution: If $x=-1$ is a root, then $x+1=0$, so $x+1$ is a factor of the polynomial. Therefore you must do synthetic division by the number -1 in order to find the other factor, which gives you the reduced equation. First write down the coefficients of the polynomial, which are $1 \quad 5 \quad 9 \quad 5$ and prepare to do synthetic division with -1.

$$\begin{array}{r|rrrr} -1 & 1 & 5 & 9 & 5 \\ & \downarrow & -1 & -4 & -5 \\ \hline & 1 & 4 & 5 & 0 \end{array}$$

Of the resulting numbers $1 \quad 4 \quad 5 \quad 0$, the last number 0 is the remainder. (Of course it is ZERO! If $x+1$ is a factor, then the remainder MUST be 0!!) Also, the first three numbers $1 \quad 4 \quad 5$ are the coefficients of the quotient. The quotient always begins with an exponent that is one less than the highest power of the polynomial. The quotient will therefore begin with $1x^2$. The quotient is $1x^2+4x+5$, and the reduced equation that must now be solved is $x^2+4x+5=0$. Unfortunately, this does NOT factor, so you will have to solve it by quadratic formula or completing the square. Completing the square in this case is actually easier for most.

$$x^2+4x+5=0$$

$$x^2+4x+\_\_\_=-5+\_\_\_$$

$$x^2+4x+\underline{\ 4\ }=-5+\underline{\ 4\ }$$

$$(x+2)^2 = -1$$

$$x+2 = \pm\sqrt{-1}$$

$$x=-2 \pm i$$

Final Answer: There are three roots: $x=-1, \quad x=-2 \pm i$

p. 386. # 25. $x^4 + 6x^3 + 10x^2 - 2x - 15 = 0$, given that $x = 1$ and $x = -3$ are roots.

Solution: If $x = 1$ is a root, then you must do synthetic division by the number 1 in order to find the other factor, which gives you a reduced equation. As always, first write down the coefficients of the polynomial, which are $1 \quad 6 \quad 10 \quad -2 \quad -15$ and prepare to do synthetic division with 1.

$$
\begin{array}{r|rrrrr}
1 & 1 & 6 & 10 & -2 & -15 \\
 & \downarrow & 1 & 7 & 17 & 15 \\
\hline
 & 1 & 7 & 17 & 15 & 0
\end{array}
$$

Of the resulting numbers $1 \quad 7 \quad 17 \quad 15 \quad 0$, of course the last number 0 is the remainder, and the first four numbers $1 \quad 7 \quad 17 \quad 15$ are the coefficients of the quotient. Since the quotient always begins with an exponent that is one less than the highest power of the polynomial, the reduced equation begins with x^3. The reduced equation that must now be solved is $x^3 + 7x^2 + 17x + 15 = 0$. This requires synthetic division this time b by -3.

$$
\begin{array}{r|rrrr}
-3 & 1 & 7 & 17 & 15 \\
 & \downarrow & -3 & -12 & -15 \\
\hline
 & 1 & 4 & 5 & 0
\end{array}
$$

Now the reduced equation is

$$x^2 + 4x + 5 = 0$$

which does NOT factor! It must be solved by quadratic formula, completing the square, or graphing calculator methods. Completing the square is a good way in this situation.

$$x^2 + 4x + \_\_\_ = -5 + \_\_\_$$

$$x^2 + 4x + 4 = -5 + 4$$

$$(x+2)^2 = -1$$

Take the square root of each side:

$$x + 2 = \pm\sqrt{-1}$$

$$x = -2 \pm i$$

$x = -2 \pm i$ (also $x = 1$ and $x = -3$ from the synthetic division!)

Final Answer: There are four roots: $x = -2 \pm i,\ 1,\ -3$.

p. 386. # 27. $x^3+2x^2-7x+4=0$, given that $x=1$ is a root.

Solution: If $x=1$ is a root, then you must do synthetic division by the number 1 in order to find the other factor, which gives you the reduced equation. First write down the coefficients of the polynomial, which are $1 \quad 2 \quad -7 \quad 4$ and prepare to do synthetic division with 1.

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -7 & 4 \\ & \downarrow & 1 & 3 & -4 \\ \hline & 1 & 3 & -4 & 0 \end{array}$$

Of the resulting numbers $1 \quad 3 \quad -4 \quad 0$, the last number 0 is the remainder. (Of course it is ZERO! It MUST be 0!!) Also, the first three numbers $1 \quad 3 \quad -4$ are the coefficients of the quotient. The quotient always begins with an exponent that is one less than the highest power of the polynomial. The quotient is x^2+3x-4 , and the reduced equation that must now be solved is $x^2+3x-4=0$, which can be solved by factoring:

$$x^2+3x-4=0$$

$$(x+4)(x-1)=0$$

$x=-4, \quad x=1$ (also $x=1$ that was given!)

Final Answer: There are three roots: $x=-4, \quad x=1$ (mult 2)

p. 388. # 30. $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$, given that there is a double root at $x = 2$.

Solution: If $x = 2$ is a root, then you must do synthetic division by the number 2 in order to find the other factor, which gives you a reduced equation. The fact that it is a double root means that you will be able to use synthetic division with $x = 2$ twice. As always, first write down the coefficients of the polynomial, which are $1 \quad -6 \quad 13 \quad -12 \quad 4$ and prepare to do synthetic division (repeatedly) with 2.

$$\begin{array}{r|rrrrr} 2 & 1 & -6 & 13 & -12 & 4 \\ & \downarrow & 2 & -8 & 10 & -4 \\ \hline & 1 & -4 & 5 & -2 & 0 \end{array}$$

Of the resulting numbers $1 \quad -4 \quad 5 \quad -2 \quad 0$, of course the last number 0 is the remainder, and the first four numbers $1 \quad -4 \quad 5 \quad -2$ are the coefficients of the quotient. Since the quotient always begins with an exponent that is one less than the highest power of the polynomial, the reduced equation begins with x^3. The reduced equation that must now be solved is $x^3 - 4x^2 + 5x - 2 = 0$. This requires synthetic division again by 2.

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 5 & -2 \\ & \downarrow & 2 & -4 & 2 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Now the reduced equation is

$$x^2 - 2x + 1 = 0$$

which factors into

$$(x-1)(x-1) = 0$$

$x = 1, \quad x = 1$ (also $x = 2$ and $x = 2$ from the synthetic division!)

Final Answer: There are four roots: $x = 1 \text{ (mult 2)}, \quad 2 \text{ (mult 2)}$.

p. 388. # 31. $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0$, given that $x = 1$ is a multiple root.

Solution: If $x = 1$ is a root, then you must do synthetic division by the number 1 in order to find the other factor, which gives you a reduced equation. The fact that it is a multiple root means that you will be able to use synthetic division with $x = 1$ more than once. As always, first write down the coefficients of the polynomial, which are $1 \quad -4 \quad 6 \quad -4 \quad 1$ and prepare to do synthetic division (repeatedly) with 1.

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 6 & -4 & 1 \\ & \downarrow & 1 & -3 & 3 & -1 \\ \hline & 1 & -3 & 3 & -1 & 0 \end{array}$$

Of the resulting numbers $1 \quad -3 \quad 3 \quad -1 \quad 0$, of course the last number 0 is the remainder, and the first four numbers $1 \quad -3 \quad 3 \quad -1$ are the coefficients of the quotient. Since the quotient always begins with an exponent that is one less than the highest power of the polynomial, the reduced equation begins with x^3. The reduced equation that must now be solved is $x^3 - 3x^2 + 3x - 1 = 0$. This requires synthetic division again by 1.

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & \downarrow & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Now the reduced equation is

$$x^2 - 2x + 1 = 0$$

which factors into

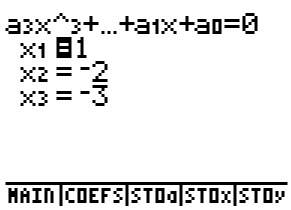
$$(x-1)(x-1) = 0$$

$x = 1, \quad x = 1$ (also $x = 1$ and $x = 1$ from the synthetic division!)

Final Answer: There are four roots: $x = 1$ **(mult 4)**

p. 393. # 42. Find all roots of $x^3 + 4x^2 + x - 6 = 0$.

Solution: In Descartes' day (or in my day for that matter!), it was necessary to find roots of a polynomial equation like this by trial and error process., using factors of 6 and synthetic division . Today, we can find roots of a polynomial equation with a graphing calculator by either graphing techniques or a program in the TI 83+ or TI 84 called POLYSMLT. (In the TI 85/86 look for [2nd] [POLY]. It works the same way!). In the TI 83+ or TI 84, begin with the [APPS] button, and in this menu, look for the program POLYSMLT, and press [ENTER]. The calculator asks for the Degree of the Polynomial (for a TI 85/86, the word "Order" is used instead of "Degree"—it means the same thing!), which is the highest power of x in the equation, which in this case is 3. Type the number 3, and press [ENTER]. Next, enter each coefficient followed by [ENTER] beginning with the highest power. Don't forget, that if a term is missing, you must enter the coefficient as 0. When you have entered all the coefficients, press the [F5] key, which represents [SOLVE]. The TI 84 comes back with this screen:



The roots (or zeros!) are obviously $x = 1, -2,\ and\ -3$. You can now do synthetic division with any of these three numbers, which will reduce the polynomial equation to a quadratic equation that can be solved by ordinary methods.

As always, first write down the coefficients of the polynomial, which are $1 \quad 4 \quad 1 \quad -6$ and prepare to do synthetic division (choose any of the three numbers!!) with 1.

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 1 & -6 \\ & \downarrow & 1 & 5 & 6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

Of the resulting numbers $1 \quad 5 \quad 6 \quad 0$, of course the last number 0 is the remainder, and the first three numbers $1 \quad 5 \quad 6$ are the coefficients of the quotient. Since the quotient always begins with an exponent that is one less than the highest power of the polynomial, the reduced equation begins with x^2.

The reduced equation is

$x^2 + 5x + 6 = 0$, which can be solved by factoring!

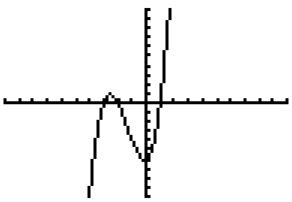
$(x+3)(x+2) = 0$

$x = -3, \;\; x = -2$

Final Answer: There are three roots: $x = 1, \; -3, \; -2$

ALTERNATE METHOD (especially for those who do NOT have POLYSMLT!!):

These roots can also be obtained by graphing $y1 = x^3 + 4x^2 + x - 6$, which gives this graph in the standard window.

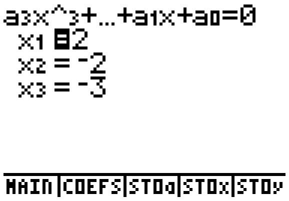


From this graph, you can see that the graph crosses the x axis at x = 1, -2, and -3. With this information, you can use synthetic division with any of these roots, which reduces the equation to a quadratic equation to find the other two roots.

p. 393. # 43. Find all roots of $x^3 + 3x^2 - 4x - 12 = 0$.

Solution: In the past, it was necessary to find roots of a polynomial equation like this by trial and error process., using factors of 12 and synthetic division . Today, we can find roots of a polynomial equation with a graphing calculator by either graphing techniques or a program in the TI 83+ or TI 84 called POLYSMLT. (In the TI 85/86 look for [2nd] [POLY]. It works the same way!). In the TI 83+ or TI 84, begin with the [APPS] button, and in this menu, look for the program POLYSMLT, and press [ENTER]. The calculator asks for the Degree (or Order!) of the Polynomial, which is the highest power of x in the equation, which in this case is 3. Type the number 3, and press [ENTER]. Next, enter each coefficient followed by [ENTER] beginning with the highest power. Don't forget, that if a term is missing, you must enter the coefficient as 0. When you have entered all the coefficients, press the [F5] key, which represents [SOLVE].

The TI 84 comes back with this screen:



The roots (or zeros!) are obviously $x = 2, -2, \textit{ and } -3$. You can now do synthetic division with any of these three numbers, which will reduce the polynomial equation to a quadratic equation that can be solved by ordinary methods.

As always, first write down the coefficients of the polynomial, which are $1 \quad 3 \quad -4 \quad -12$ and prepare to do synthetic division (choose any of the three numbers!!) with 1.

2	1	3	-4	-12
	$\downarrow$	2	10	12
	1	5	6	0

Of the resulting numbers $1 \quad 5 \quad 6 \quad 0$, of course the last number 0 is the remainder, and the first three numbers $1 \quad 5 \quad 6$ are the coefficients of the quotient. Since the quotient always begins with an exponent that is one less than the highest power of the polynomial, the reduced equation begins with x^2. The reduced equation is

$$x^2 + 5x + 6 = 0,$$ which can be solved by factoring!

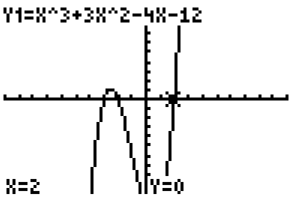
$$(x+3)(x+2) = 0$$

$$x = -3, \quad x = -2$$

Final Answer: There are three roots: $x = 2, \ -3, \ -2$

ALTERNATE METHOD (especially for those who do NOT have POLYSMLT!!):

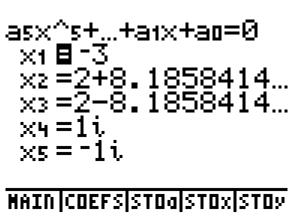
These roots can also be obtained by graphing $y1 = x^3 + 3x^2 - 4x - 12$, which gives this graph in the standard window. The graph below, obtained by using [TRACE] with x=2, illustrates the root or zero at x=2 . Of course the zeros in this case can be easily seen from just looking at the graph, or from [2nd] [F5 (TABLE)].



From this graph, you can see that the graph crosses the x axis at x = 2, -2, and -3. With this information, you can use synthetic division with any of these roots, which reduces the equation to a quadratic equation to find the other two roots.

p. 397. # 53. Find all roots of $x^5 - x^4 - 7x^3 + 11x^2 - 8x + 12 = 0$.

Solution: **In Descartes' day (or in my day for that matter!), it was necessary to find roots of a polynomial equation like this by trial and error process., using factors of 12 and synthetic division . Today, we can find roots of a polynomial equation with a graphing calculator by either graphing techniques or a program in the TI 83+ or TI 84 called POLYSMLT. (In the TI 85/86 look for [2nd] [POLY]. It works the same way!). In the TI 83+ or TI 84, begin with the [APPS] button, and in this menu, look for the program POLYSMLT, and press [ENTER]. The calculator asks for the Degree of the Polynomial, which is the highest power of x in the equation, which in this case is 5. Type the number 5, and press [ENTER]. Next, enter each coefficient followed by [ENTER] beginning with the highest power. Don't forget, that if a term is missing, you must enter the coefficient as 0. When you have entered all the coefficients, press the [F5] key, which represents [SOLVE]. It may take a few seconds to solve, but the TI 84 comes back with this screen:**



It looks as if the calculator is giving you roots at $x = -3,\ 2 \pm 8.1858414\ldots,\ \pm i$. The middle numbers, however, are misleading. If you scroll down to one of these numbers and press the right arrow, you will see that these are actually $x = 2 \pm 8.1858414\,35\text{E}\,{-7}\,i\ldots$ The E -7 is the scientific notation for a number divided by 10 million, which is essentially zero! This means that these two middle numbers actually represent a double root at $x = 2$. You can now do synthetic division consecutively with the three numbers $x = -3,\ 2,\ 2$ which will reduce the polynomial equation down to a quadratic equation that can be solved by ordinary methods.

As always, first write down the coefficients of the polynomial, which are $1\ \ -1\ \ -7\ \ 11\ \ -8\ \ 12$ and prepare to do synthetic division (repeatedly) with -3.

$$\begin{array}{r|rrrrrr} -3 & 1 & -1 & -7 & 11 & -8 & 12 \\ & \downarrow & -3 & 12 & -15 & 12 & -12 \\ \hline & 1 & -4 & 5 & -4 & 4 & 0 \end{array}$$

Of the resulting numbers $1\ \ -4\ \ 5\ \ -4\ \ 4\ \ 0$, of course the last number 0 is the remainder, and the first five numbers $1\ \ -4\ \ 5\ \ -4\ \ 4$ are the coefficients of the quotient. Since the quotient always begins with an exponent that is one less than the highest power of the polynomial, the reduced equation begins with x^4. The reduced equation that must now be solved is $1x^4 - 4x^3 + 5x^2 - 4x + 4 = 0$. Next, perform synthetic division by 2.

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 5 & -4 & 4 \\ & \downarrow & 2 & -4 & 2 & -4 \\ \hline & 1 & -2 & 1 & -2 & 0 \end{array}$$

This time the reduced equation is $x^3 - 2x^2 + x - 2 = 0$

Perform synthetic division by 2 again, since it is a double root (it occurs twice!).

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 1 & -2 \\ & \downarrow & 2 & 0 & -2 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

Now the reduced equation is

$$x^2 + 0x + 1 = 0 \text{ or } x^2 + 1 = 0.$$

This can be solved by adding -1 to each side of the equation:

$$x^2 = -1$$

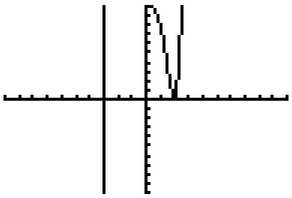
and then take the square root of each side:

$x = \pm i$ (also $x = -3$ and $x = 2$ (mult 2) from the synthetic division!)

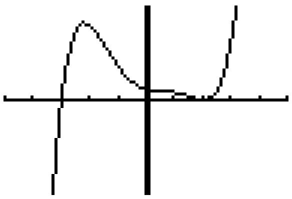
Final Answer: There are five roots: $x = -3,\ 2 \text{ (mult 2)},\ \pm i$

ALTERNATE METHOD (especially for those who do NOT have POLYSMLT!!):

These roots can also be obtained by graphing $y1 = x^5 - x^4 - 7x^3 + 11x^2 - 8x + 12$, which gives this graph in the standard window.



Adjusting the window may make it easier to see what the graph really does. I suggest using a window such as x=[-5,5] and *y*=[-100,100]:



From either of these graphs, even without knowing what the graph actually looks like, you can see that the graph crosses the x axis at x = -3, and it bounces at x = 2. This means that there is an even multiplicity (i.e., a double root!) at x = 2. With this information, you can proceed to use synthetic division to find the other two roots which happen to be imaginary roots.

Additional Problem. (See #43 above!!)

If $x = -3$ is a root of $x^3 + 3x^2 - ax - 12 = 0$, what is the value of a?

Solution:

As always, first write down the coefficients of the polynomial, which are $1 \quad 3 \quad -a \quad -12$ and prepare to do synthetic division using with $x = -3$.

$$\begin{array}{r|rrrr} -3 & 1 & 3 & -a & -12 \\ & \downarrow & -3 & 0 & 3a \\ \hline & 1 & 0 & -a & 3a-12 \end{array}$$

In order for $x = -3$ to be a root, the remainder must be zero. This means that $3a - 12 = 0$

$$3a = 12$$

$$a = 4$$