Math in Living C O L O R !!

4.04 Part I

Exponential Equations with Logarithms

College Algebra: One Step at a Time.

Page 523: #3, 5, 16, 17, 19, Extra Prob

Dr. Robert J. Rapalje, Retired
Central Florida, USA

See Section 4.04, with explanations, examples, and exercises, coming soon!

See Math in Living C O L O R, Section 4.04 Part II, coming soon!

P. 521 # 3. Three Solutions for $27^{x+2} = 9^{2x-1}$

First Solution: Did you happen to notice that this is a special problem in which both sides of the equation can be written as a power of the same number 3? In mathematics we call this a "contrived problem," one that we made up especially because of this special property, and therefore the method is of little or no use to real life problems, where things usually don't come out even.

$$27^{x+2} = 9^{2x-1}$$

Notice that $27 = 3^3$ and $9 = 3^2$, and substitute these into the original problem:

$$(3^3)^{x+2} = (3^2)^{2x-1}$$

Remember that when you raise a power to a power, you multiply exponents:

$$3^{3x+6} = 3^{4x-2}$$

Since the base number on both sides of the equation is the same, you can use what I call it the "This Equals That Theorem":

$$\textit{If } 3^{this} = 3^{that}, \textit{ then } \text{this} = \text{that}!$$

Likewise, If $3^{3x+6} = 3^{4x-2}$, then $3x+6 = 4x-2$, which is a linear equation!

$$-x = -8$$
$$x = 8$$

Final answer: $x = 8$

Second solution: Using Logarithms.

How would you solve an exponential equation if it does NOT have the same base number on both sides of the equation?

$$27^{x+2} = 9^{2x-1}$$

Take the ln of both sides of the equation:

$$\ln 27^{(x+2)} = \ln 9^{(2x-1)}$$

Using the second law of logarithms:

$$(x+2)\ \ln\ 27 = (2x-1)\ln\ 9$$

Using the distributive property:

$$x\ \ln\ 27 + 2\ln 27 = 2x\ln\ 9 - 1\ln\ 9$$

Get all x terms on the left side by subtracting $2x\ \ln\ 9$ and $2\ln 27$ from both sides of the equation:

$$x\ \ln\ 27 + \cancel{2\ln 27} = \cancel{2x\ln}\ 9 - 1\ln\ 9$$

$$-2x\ \ln\ 9\ \ - \cancel{2\ln 27}\ \ \ - \cancel{2x\ln}\ 9 - 2\ln 27$$

$$x\ \ln\ 27 - 2x\ \ln\ 9 = -1\ln\ 9 - 2\ln 27$$

Get the x variable in one place by factoring out the common factor of x:

$$x\ (\ln\ 27 - 2\ln\ 9) = -1\ln\ 9 - 2\ln 27$$

Solve for x by dividing each side of the equation by $(\ln\ 27\ - 2\ln\ 9)$

$$\frac{x\ (\ln\ 27 - 2\ln\ 9)}{(\ln\ 27 - 2\ln\ 9)} = \frac{(-1\ln\ 9 - 2\ln 27)}{(\ln\ 27 - 2\ln\ 9)}$$

In the final step, parentheses are included that would be suitable for calculating the answer with a TI 83 or TI 84. Notice that for ALL calculators, you must have parentheses around the numerator and the denominator. Notice also that at the beginning of the numerator, it is a NEGATIVE (not minus!) of the ln 9, but the other two signs in the problem are MINUS (not negative!) signs. Also, for the TI 83/84 calculator, be reminded that there must be a DOUBLE closed parentheses at the end of the numerator!

$$x = \frac{(\text{-}\ln\ (9) - 2\ln\ (27))}{(\ln\ (27)\ - 2\ln\ (9))}$$

Calculate with your calculator:

x = 8

Third solution: Using a Graphing Calculator—TI84 (or TI83+)

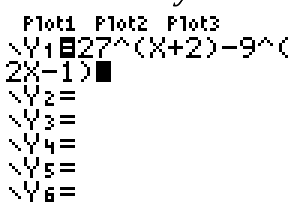
Given the exponential equation $27^{x+2} = 9^{2x-1}$

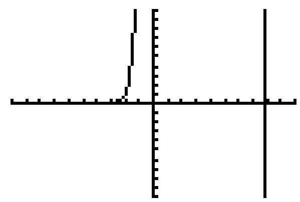
First set the equation equal to zero:

$$27^{x+2} - 9^{2x-1} = 0$$

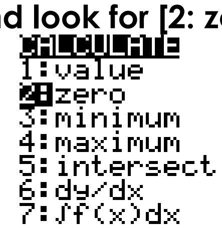
Since the exponents are binomials, be sure to put parentheses around each of the exponents, and write this equation in the form y1= for graphing purposes. Graph the equation as given below with the graphing calculator. The standard window will be appropriate for this one.

$$y1 = 27^{(x+2)} - 9^{(2x-1)}$$

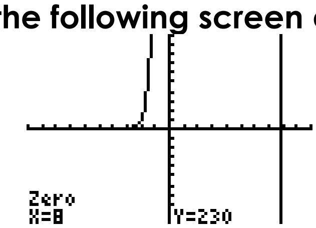




You can probably see that the answer to this problem is the root (zero) that occurs at x = 8. However, you can't be sure from looking at it that the answer is exactly 8. Also, most of the time, the answer will NOT come out even for these types of problems, so it may be helpful to find the zero. With a TI 83/84, use the keystrokes [2nd] [CALC F4] and look for [2: zero] as illustrated in the next picture.



You will be asked for a "left bound", so give the calculator a value to the left of what you think the zero is. You can enter any value less than 8, and press [ENTER]. Then the calculator will ask for a "right bound", so enter a value larger than 8, say a 9 or a 10 and press [ENTER]. Then the calculator says "Guess?", so press [ENTER] again, and the following screen appears:



This means that the zero or root is at x = 8. Don't worry about the fact that it says y = 230. You are working with an exponential graph, which is UNBELIEVABLY steep. On this calculator one pixel (that is one DOT!) of movement to the right moves about .21 units to the right or left, but it moves a vertical distance of about 4.3 x10^{13} up or down. This is close as the calculator can come to y = 0.

Final Answer: x= 8 .

P. 521 # 5. Two Solutions for $8^{x-3} = 32^{x+1}$

First solution: Laws of exponents--a "contrived" problem.

Begin by noticing that this is a very "special" problem in which both sides of the equation can be written as a power of the same number 2. In mathematics we call this a "contrived problem," one that we made up especially because of this special property, and therefore the method is of little or no use to real life problems, where things usually don't come out even.

$$8^{x-3} = 32^{x+1}$$

Notice that $8 = 2^3$ and $32 = 2^5$, and substitute these into the original problem:

$$\left(2^3\right)^{x-3} = \left(2^5\right)^{x+1}$$

Remember that when you raise a power to a power, you multiply exponents:

$$2^{3x-9} = 2^{5x+5}$$

Notice that the base number on both sides of the equation is the same. You can use what I call the "This Equals That Theorem":

$$\textit{If } 3^{this} = 3^{that}, \textit{ then } \text{this} = \text{that!}$$

If $2^{3x-9} = 2^{5x+5}$, then $3x-9 = 5x+5$, and solve for x.

$$\begin{aligned} 3x-9 &= 5x+5 \\ \underline{-5x+9} &\underline{= -5x+9} \\ -2x &= 14 \\ x &= -7 \end{aligned}$$

Final answer: x = −7.

Second solution: Using Logarithms.

How would you solve an exponential equation if it does NOT have the same base number on both sides of the equation?

$$8^{x-3} = 32^{x+1}$$

Take the ln of both sides of the equation:

$$\ln 8^{x-3} = \ln 32^{x+1}$$

Using the second law of logarithms:

$$(x-3)\ln 8 = (x+1)\ln 32$$

Using the distributive property:

$$x\ln 8 - 3\ln 8 = x\ln 32 + 1\ln 32$$

Get all x terms on the left side by subtracting $x\ln 32$ **from each side:**

$$\begin{array}{rcl} x\ln 8 - 3\ln 8 & = & x\ln 32 + 1\ln 32 \\ -x\ln 32 & & -x\ln 32 \\ \hline x\ln 8 - x\ln 32 - 3\ln 8 & = & 1\ln 32 \end{array}$$

Next, get all the non-x terms on the right side by adding $+3\ln 8$ **to each side:**

$$\begin{array}{rcl} x\ln 8 - x\ln 32 - 3\ln 8 & = & 1\ln 32 \\ +3\ln 8 & & +3\ln 8 \\ \hline x\ln 8 - x\ln 32 & = & 1\ln 32 + 3\ln 8 \end{array}$$

Get the x variable in one place by factoring out the common factor of x:

$$x(\ln 8 - \ln 32) = 1\ln 32 + 3\ln 8$$

Solve for x by dividing each side of the equation by $(\ln 8 - \ln 32)$

$$\frac{x(\ln 8 - \ln 32)}{(\ln 8 - \ln 32)} = \frac{(\ln 32 + 3\ln 8)}{(\ln 8 - \ln 32)}$$

$$x = \frac{(\ln 32 + 3\ln 8)}{(\ln 8 - \ln 32)}$$

In the final step, don't forget that if/when your calculator OPENS parentheses for you, you must remember to CLOSE the parentheses. Notice that for ALL calculators, you must have parentheses around the numerator and the denominator. Also, for TI83/84 calculators, be reminded that there must be a DOUBLE closed parentheses at the end of the numerator!

Calculate with your calculator: x = -7 Final Answer!

P. 523 # 16. $10^x = 5^{(x-2)}$

Solution:

Take the ln of both sides of the equation:

$$\ln 10^x = \ln 5^{(x-2)}$$

Using the second law of logarithms:

$$x \ln 10 = (x - 2) \ln 5$$

Using the distributive property:

$$x \ln 10 = x \ln 5 - 2 \ln 5$$

Get all x terms on the left side by subtracting $x \ln 5$ from each side:

$$x \ln 10 - x \ln 5 = -2 \ln 5$$

Get the x variable in one place by factoring out the common factor of x:

$$x (\ln 10 - \ln 5) = -2 \ln 5$$

Solve for x by dividing each side of the equation by $(\ln 10 - \ln 5)$

$$\frac{x (\ln 10 - \ln 5)}{(\ln 10 - \ln 5)} = \frac{-2 \ln 5}{(\ln 10 - \ln 5)}$$

In the final step, parentheses are included that would be suitable for calculating the answer with a TI 83 or TI 84. To calculate with a TI 85 or TI 86, use parentheses as indicated in the previous step.

$$x = \frac{-2 \ln (5)}{(\ln (10) - \ln (5))}$$

Calculate with your calculator:

Final answer: x ≈ -4.64385618978 . . . or -4.64

You may also want to use the graphing calculator to obtain the approximate solution to this exercise by setting the equation $10^x = 5^{(x-2)}$ equal to zero, and graphing $y1 = 10^x - 5^{(x-2)}$. Never mind what the graph looks like, just use the "zeros" function [2nd] [F4 (Calc)] [2 (zeros)] of the TI 83 or TI 84. Amazingly enough, it will be exactly the same (at least to calculator accuracy) as the answer given above!

P. 523 # 17.

$$4^{(x-1)} = 27^{(x+1)}$$

Solution:

Take the ln of both sides of the equation:

$$\ln 4^{(x-1)} = \ln 27^{(x+1)}$$

Using the second law of logarithms:

$$(x-1) \ln 4 = (x+1) \ln 27$$

Using the distributive property:

$$x \ln 4 - 1 \ln 4 = x \ln 27 + 1 \ln 27$$

Get all x terms on the left side by subtracting $x \ln 27$ from each side of the equation, and get the non-x terms on the right side by adding $+1 \ln 4$:

$$\begin{array}{rcl} x \ln 4 - \cancel{1 \ln 4} & = & \cancel{x \ln 27} + 1 \ln 27 \\ -x \ln 27 + \cancel{1 \ln 4} & & -\cancel{x \ln 27} + 1 \ln 4 \\ \hline x \ln 4 - x \ln 27 & = & 1 \ln 27 + 1 \ln 4 \end{array}$$

Get the x variable in one place by factoring out the common factor of x:

$$x(\ln 4 - \ln 27) = 1 \ln 27 + 1 \ln 4$$

Solve for x by dividing each side of the equation by $(\ln 4 - \ln 27)$

$$\frac{x\cancel{(\ln 4 - \ln 27)}}{\cancel{(\ln 4 - \ln 27)}} = \frac{(1 \ln 27 + 1 \ln 4)}{(\ln 4 - \ln 27)}$$

In the final step, parentheses are included that would be suitable for calculating the answer with a TI 83 or TI 84.

$$x = \frac{(\ln 27 + \ln 4)}{(\ln 4 - \ln 27)}$$

Calculate with your calculator:

Final answer: $x \approx -2.4519649\ldots$ or $x \approx -2.45$

P. 523 # 19. $9^{(x-1)} = 27^{(x+1)}$

Solution:

Take the ln of both sides of the equation:

$$\ln 9^{(x-1)} = \ln 27^{(x+1)}$$

Using the second law of logarithms:

$$(x-1)\ln 9 = (x+1)\ln 27$$

Using the distributive property:

$$x\ln 9 - 1\ln 9 = x\ln 27 + 1\ln 27$$

Get all x terms on the left side by subtracting $x \ln 27$ from each side of the equation and get the non-x terms on the right side by adding $+1 \ln 9$:

$$x\ln 9 - \cancel{1\ln 9} = \cancel{x\ln 27} + 1\ln 27$$

$$-x\ln 27 + \cancel{1\ln 9} \qquad \cancel{-x\ln 27} + 1\ln 9$$

$$x\ln 9 - x\ln 27 = 1\ln 27 + 1\ln 9$$

Get the x variable in one place by factoring out the common factor of x:

$$x(\ln 9 - \ln 27) = 1\ln 27 + 1\ln 9$$

Solve for x by dividing each side of the equation by $(\ln 9 - \ln 27)$

$$\frac{x\cancel{(\ln 9 - \ln 27)}}{\cancel{(\ln 9 - \ln 27)}} = \frac{(1\ln 27 + 1\ln 9)}{(\ln 9 - \ln 27)}$$

In the final step, parentheses are included that would be suitable for calculating the answer with a TI 83 or TI 84. To calculate with a TI 85 or TI 86, use parentheses as indicated in the previous step.

$$x = \frac{(\ln 27 + \ln 9)}{(\ln 9 - \ln 27)}$$

Calculate with your calculator:

Final answer: $x = -5$

Note: It came out even!!! Could this be a "contrived" problem?? Could it be solved as in the first few problems in this section using the laws of exponents?? Hmmm . . .

Extra Problem $12^{(3x-2)} = 5^x$

Find the exact value, and also calculate the answer to calculator accuracy.

Solution:

Take the ln of both sides of the equation:

$$\ln 12^{(3x-2)} = \ln 5^x$$

Using the second law of logarithms:

$$(3x-2) \ln 12 = x \ln 5$$

Using the distributive property:

$$3x \ln 12 - 2 \ln 12 = x \ln 5$$

Get all x terms on the left side by subtracting $x \ln 5$ from both sides:

$$\begin{array}{r} 3x \ln 12 - 2 \ln 12 = x \ln 5 \\ -x \ln 5 \qquad\qquad -x \ln 5 \\ \hline 3x \ln 12 - x \ln 5 - 2 \ln 12 = 0 \end{array}$$

Get the non-x term on the right side by adding $+2 \ln 12$ to each side:

$$3x \ln 12 - x \ln 5 \cancel{-2 \ln 12} \cancel{+2 \ln 12} = 2 \ln 12$$

$$3x \ln 12 - x \ln 5 = 2 \ln 12$$

Get the x variable in one place by factoring out the common factor of x:

$$x(3 \ln 12 - \ln 5) = 2 \ln 12$$

Solve for x by dividing each side of the equation by $(3 \ln 12 - \ln 5)$

$$\frac{x \cancel{(3 \ln 12 - \ln 5)}}{\cancel{(3 \ln 12 - \ln 5)}} = \frac{2 \ln 12}{(3 \ln 12 - \ln 5)}$$

$$x = \frac{2 \ln 12}{(3 \ln 12 - \ln 5)}$$ **This is the exact value!!**

In the final step, parentheses are included that would be suitable for calculating the answer with a TI 83 or TI 84. Notice that for ALL calculators, you must have parentheses around the denominator. Parentheses around the numerator are not needed here since the numerator is a monomial. Remember that these are only approximations!!

Final answer:

$$x \approx 0.8502264336$$

$$x \approx 0.85$$