Math in Living C O L O R !!

4.05 Population Growth and Decay

College Algebra: One Step at a Time. Page 539: #15, 20, 21, 23, 26 – 32, 2 Extras

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See Section 4.05, with explanations, examples, and exercises, coming soon!

P. 535 # 15.

The amount of radioactive substance present at any time is given by $y = y_0 e^{-0.0004t}$, where y_0 is the initial amount and t is the time in years. If there are 25 grams of the substance left after 2000 years, how much was there originally?

Solution:

$y = 25$ when t = 2000, and y_0 is the unknown!

$y = y_0 e^{-0.0004t}$ Solve for y_0.

$25 = y_0 e^{(-0.0004 \bullet 2000)}$

$25 = y_0 e^{-0.8}$

Find y_0, by dividing both sides of the equation by $e^{-0.8}$:

$$\frac{25}{e^{-0.8}} = \frac{y_0 e^{-0.8}}{e^{-0.8}}$$

$$\frac{25}{e^{-0.8}} = \frac{y_0 \cancel{e^{-0.8}}}{\cancel{e^{-0.8}}}$$

$y_0 \approx 55.63852321$ or approximately 55.6 grams.

P. 538 # 20.

The population of a city was 40,000 in the year 1990. In 1995, the population of the city was 50,000. Find the value of k in the formula $y = y_0 e^{kt}$.

Solution:

$y_0 = 40{,}000$ and $y = 50{,}000$ when $t = 5$

$$y = y_0 e^{kt}$$

$$50{,}000 = 40{,}000e^{(k \bullet 5)}$$

First find k, by dividing both sides of the equation by 40,000:

$$\frac{50{,}000}{40{,}000} = \frac{40{,}000e^{(k \bullet 2)}}{40{,}000}$$

When calculating this, I strongly recommend that you leave the result in fractional form. Do NOT calculate the decimal value. If you write down a decimal value, you *may* be rounding off the answer, which could lead to an incredible error in the final answer. It's okay to round off the final answer, but NEVER round off a number and then use that number in subsequent calculations!

$$\frac{5}{4} = e^{(k \bullet 5)}$$

[NOTE: In this case, if you calculate the decimal value, the result is 1.25, which is EXACT! No rounding takes place, so there is NO round off error. In this case, using 1.25 is equally correct! However, don't do this in the rest of the exercises!]

Now, take the ln of each side of the equation to "undo" the "e^ ":

$$\ln\left(\frac{5}{4}\right) = \ln e^{(k \bullet 5)}$$

$$\ln\left(\frac{5}{4}\right) = k \bullet 5$$

Divide both sides by 5: $k = \dfrac{\ln\left(\frac{5}{4}\right)}{5}$

The calculator needs parentheses: $k = \dfrac{\ln\left(\frac{5}{4}\right)}{5} \approx 0.0446287\ldots$

You may want to store this number in your calculator using the [STO→] button. For TI 83/84 press [STO→] [ALPHA] [*k*].

P. 538 # 21.

21. Use the value of k obtained in #20 to find the expected population in 2005.

Solution:

Notice that since y_0 means that t=0 in 1990, in 2005, t=15.

$$y = y_0 e^{(k \bullet t)}$$

$$y = 40,000e^{(k \bullet 15)} \quad \text{or} \quad y = 40,000e^{(15k)}$$

To use your calculator do this, having already entered the value of k in the calculator, press [40000] [2nd] [ln] [your calculator may or may not automatically open parentheses] [15] [ALPHA] [k] [closed parentheses].

$y \approx 78{,}125$.

P. 538 # 23.

23. Use the value of k obtained in #20 to determine how long it will take the population to double.

Solution:

If the population doubles, where y_0 was the initial population, the new population will be $2y_0$, and you can substitute into the equation as follows:

$$y = y_0 e^{kt}$$

$$2y_0 = y_0 e^{kt}$$

Divide both sides by y_0:

$$\frac{2\cancel{y_0}}{\cancel{y_0}} = \frac{\cancel{y_0}e^{kt}}{\cancel{y_0}}$$

$$2 = e^{kt}$$

Take the ln of each side:

$$\ln(2) = \ln(e^{kt})$$

$$\ln(2) = kt$$

To solve for t, divide both sides by k.

$$\frac{\ln(2)}{k} = \frac{kt}{k}$$

$$t = \frac{\ln(2)}{k}$$

If you have the value of k in the calculator already from the previous exercises, just divide $\ln(2)$ by [alpha] [k] .

t =15.5314186, or t= 15.5 years

P. 539 # 26 – 27.

26. The population of a city was 85,000 in the year 2000, and 88,000 in 2002. At this rate of growth, what population should be expected in 2005?

Solution: $y_0 = 85,000$ and $y = 88,000$ when $t = 2$

$$y = y_0 e^{(k \bullet t)}$$

$$88,000 = 85,000 e^{(k \bullet 2)}$$

First find k, by dividing both sides of the equation by 85,000:

$$\frac{88,000}{85,000} = \frac{85,000 e^{(k \bullet 2)}}{85,000}$$

When calculating this, I strongly recommend that you leave the result in fractional form. Do NOT calculate the decimal value. If you write down a decimal value, you are rounding off the answer, which could lead to an incredible error in the final answer. It's okay to round off the final answer, but NEVER round off a number and then use that number in subsequent calculations!

$$\frac{88}{85} = e^{(k \bullet 2)}$$

Now, take the ln of each side of the equation to "undo" the "e^ ":

$$\ln\left(\frac{88}{85}\right) = \ln e^{(k \bullet 2)}$$

$$\ln\left(\frac{88}{85}\right) = k \bullet 2$$

Divide both sides by 2: $k = \dfrac{\ln\left(\frac{88}{85}\right)}{2}$

The calculator needs parentheses: $k = \dfrac{\ln\left(\frac{(88}{85)}\right)}{2} \approx 0.017342779\ldots$

You may want to store this number in your calculator using the [STO→] button. For TI 83/84 press [STO→] [ALPHA] [*k*].

Now, use this value of *k* to answer the question: "Find *y* when *t*=5."

$$y = y_0 e^{(k \bullet t)}$$

$$y = 85,000 e^{(k \bullet 5)} \quad \text{or} \quad y = 85,000 e^{(5k)}$$

To use your calculator do this, having already entered the value of *k* in the calculator, press [85000] [2nd] [ln] [your calculator may or may not automatically open parentheses] [5] [ALPHA] [*k*] [closed parentheses].

$y \approx 92699.69$, which rounds off to 92,700.

P. 539 # 27.

27. The population of a city was 85,000 in the year 2000, and 88,000 in 2002. At this rate of growth, how long will the population to double?

Solution:

When the population doubles, whatever y_0 is, $y = 2y_0$.

$y = y_0 e^{(k \bullet t)}$, where k is the value in the calculator from #26!

$$2y_0 = y_0\, e^{(k \bullet t)}$$

First find k, by dividing both sides of the equation by y_0:

$$\frac{2y_0}{y_0} = \frac{y_0\, e^{(k \bullet t)}}{y_0}$$

$$2 = e^{(k \bullet t)}$$

Take the ln of each side of the equation to "undo" the "e^ ":

$$\ln(2) = \ln\, e^{(k \bullet t)}$$

$$\ln(2) = k \bullet t$$

Now, since you are solving for *t*, you must divide both sides by *k*:

$t = \dfrac{\ln(2)}{k}$, where *k* is the value in your calculator from #26!

t ≈ 39.867 years (approximately 40 years!)

NEW EXERCISES!!!

28. The population of a city was 85,000 in the year 2000, and 88,000 in 2002. How long (from year 2000) will it take the population to reach 200,000?

Solution:

$y_0 = 85{,}000$, $y = 200{,}000$, and k is in the calculator.

$$y = y_0 e^{(k \bullet t)}$$

$$200{,}000 = 85{,}000 e^{(k \bullet t)}$$

Solve for t, by dividing both sides of the equation by $85{,}000$:

$$\frac{200{,}000}{85{,}000} = \frac{85{,}000 e^{(k \bullet t)}}{85{,}000}$$

$$\frac{200}{85} = e^{(k \bullet t)}$$

$$\ln\left(\frac{200}{85}\right) = \ln e^{(k \bullet t)}$$

$$\ln\left(\frac{200}{85}\right) = k \bullet t$$

Divide both sides by k, the value in the calculator. The answer is t.

$$t = \frac{\ln\left(\frac{200}{85}\right)}{k} \approx 4.910 \quad \approx 4.9 \text{ years.}$$ Final Answer

In 29 - 32, the population of a city was 184,000 in the year 1996, and 310,000 in the year 2003.

29. At this rate of growth, find the value of k, and estimate the population that should be expected:

a. in 2008? b. in 2010? c. in 2015?

Solution (Find the value of k):

$y_0 = 184,000$ and $y = 310,000$ when $t = 7$

$$y = y_0 e^{(k \bullet t)}$$

$$310,000 = 184,000 e^{(k \bullet 7)}$$

First find k, by dividing both sides of the equation by $184,000$:

$$\frac{310,000}{184,000} = \frac{184,000 e^{(k \bullet 7)}}{184,000}$$

Remember, it's better to leave the result in fractional form. If you calculate the decimal value, you will be tempted to round off the answer, which could lead to an incredible error in the final answer. As I said before, it's okay to round off the final answer, but NEVER round off a number and then use that number in subsequent calculations!

$$\frac{310}{184} = e^{(k \bullet 7)}$$

Now, take the ln of each side of the equation to "undo" the "e^ ":

$$\ln\left(\frac{310}{184}\right) = \ln e^{(k \bullet 7)}$$

$$\ln\left(\frac{310}{184}\right) = k \bullet 7$$

Divide both sides by 7: $k = \dfrac{\ln\left(\frac{310}{184}\right)}{7}$

The calculator needs parentheses: $k = \dfrac{\ln\left(\frac{(310}{184)}\right)}{7} \approx 0.074519057\ldots$

You may want to store this number in your calculator using the [STO→] button. For TI 83/84 press [STO→] [ALPHA] [k].

29a) Now, use that value of *k* to find *y* in 2008. In 2008, t=12."

$$y = y_0 e^{(k \bullet t)}$$

$$y = 184{,}000e^{(k \bullet 12)} \quad \text{or} \quad y = 184{,}000e^{(12k)}$$

To use your calculator do this, having already entered the value of *k* in the calculator, press [184000] [2nd] [ln] [your calculator may or may not automatically open parentheses] [12] [ALPHA] [*k*] [closed parentheses].

y ≈ 449,965.0108 , which rounds off to 450,000.

29b) Now, use that value of *k* to find *y* in 2010. In 2010, t=14."

$$y = y_0 e^{(k \bullet t)}$$

$$y = 184{,}000e^{(k \bullet 14)} \quad \text{or} \quad y = 184{,}000e^{(14k)}$$

To use your calculator do this, having already entered the value of *k* in the calculator, press [184000] [2nd] [ln] [your calculator may or may not automatically open parentheses] [14] [ALPHA] [*k*] [closed parentheses].

y ≈ 522,282.6087 , which rounds off to 522,000.

29c) Now, use that value of *k* to find *y* in 2015. In 2015, t=19."

$$y = y_0 e^{(k \bullet t)}$$

$$y = 184{,}000e^{(k \bullet 19)} \quad \text{or} \quad y = 184{,}000e^{(19k)}$$

To use your calculator do this, having already entered the value of *k* in the calculator, press [184000] [2nd] [ln] [your calculator may or may not automatically open parentheses] [19] [ALPHA] [*k*] [closed parentheses].

y ≈ 785,093.2247 , which rounds off to 785,000.

30. The population of the city in the previous problem was 184,000 in the year 1996, and 310,000 in the year 2003. How long will it take the population:

a. to double? **b.** to triple?

Solution:

30a. When the population doubles, whatever y_0 is, $y = 2y_0$.

$$y = y_0 e^{(k \bullet t)}$$, where k is the value in the calculator!

$$2y_0 = y_0 \, e^{(k \bullet t)}$$

First find k, by dividing both sides of the equation by y_0:

$$\frac{2y_0}{y_0} = \frac{y_0 \, e^{(k \bullet t)}}{y_0}$$

$$2 = e^{(k \bullet t)}$$

Take the ln of each side of the equation to "undo" the "e^ ":

$$\ln(2) = \ln \, e^{(k \bullet t)}$$

$$\ln(2) = k \bullet t$$

Now, since you are solving for *t*, you must divide both sides by *k*:

$$t = \frac{\ln(2)}{k}$$, where *k* is the value in your calculator!

$t \approx 9.30155$ years (approximately 9.3 years!)

30b. When the population triples, whatever y_0 is, $y = 3y_0$.

$$y = y_0 e^{(k \bullet t)}$$, where k is the value in the calculator!

$$3y_0 = y_0 \, e^{(k \bullet t)}$$

First find k, by dividing both sides of the equation by y_0:

$$\frac{3y_0}{y_0} = \frac{y_0 \, e^{(k \bullet t)}}{y_0}$$

$$3 = e^{(k \bullet t)}$$

Take the ln of each side of the equation to "undo" the "e^ ":

$$\ln(3) = \ln \, e^{(k \bullet t)}$$

$$\ln(3) = k \bullet t$$

Now, since you are solving for *t*, you must divide both sides by *k*:

$$t = \frac{\ln(3)}{k}$$, where *k* is the value in your calculator!

$t \approx 14.7426$ years (approximately 14.7 years!)

31. The population of the city in the previous problem was 184,000 in the year 1996, and 310,000 in the year 2003. How many years will it take this population reach 500,000?

Solution:

$y_0 = 184,000$, $y = 500,000$, and k is in the calculator.

$$y = y_0 e^{(k \bullet t)}$$

$$500,000 = 184,000 e^{(k \bullet t)}$$

Solve for t, by dividing both sides of the equation by $184,000$:

$$\frac{500,000}{184,000} = \frac{184,000 e^{(k \bullet t)}}{184,000}$$

$$\frac{500}{184} = e^{(k \bullet t)}$$

$$\ln\left(\frac{500}{184}\right) = \ln e^{(k \bullet t)}$$

$$\ln\left(\frac{500}{184}\right) = k \bullet t$$

Divide both sides by k, the value in the calculator. The answer is t.

$$t = \frac{\ln\left(\frac{500}{184}\right)}{k} \approx 13.4149 \approx 13.4 \text{ years.}$$ **Final Answer**

32. The population of the city in the previous problem was 184,000 in the year 1996, and 310,000 in the year 2003. How many years will it take this population to reach 1,000,000?

Solution:

$y_0 = 184,000$, $y = 1,000,000$, and k is in the calculator.

$$y = y_0 e^{(k \bullet t)}$$

$$1,000,000 = 184,000 e^{(k \bullet t)}$$

Solve for t, by dividing both sides of the equation by $184,000$:

$$\frac{1,000,000}{184,000} = \frac{184,000 e^{(k \bullet t)}}{184,000}$$

$$\frac{1000}{184} = e^{(k \bullet t)}$$

$$\ln\left(\frac{1000}{184}\right) = \ln e^{(k \bullet t)}$$

$$\ln\left(\frac{1000}{184}\right) = k \bullet t$$

Divide both sides by k, the value in the calculator. The answer is t.

$$t = \frac{\ln\left(\frac{1000}{184}\right)}{k} \approx 22.71646 \quad \approx 22.7 \text{ years}.$$ Final Answer

NOTE: As a check, subtract the answers from the last two problems (#31 and #32) and compare to #30a) where the population doubles!

(Notice that 22.7 − 13.4 = 9.3 years)

Extra Problem #1

A pharmaceutical company makes a vaccine that grows at a rate of 2.5% per hour. How many units of this organism must they have initially to have 1000 units after 5 days.

Solution: In this problem, rate of growth is 2.5%, so $k = 0.025$.

Also, $y = 1,000$ when $t = 5 \bullet 24 = 120$ hours.

You must solve for y_0 using this formula

$$y = y_0 e^{kt}$$

$$1,000 = y_0 e^{(0.025 \bullet 120)}$$

$$1,000 = y_0 e^{(3)}$$

$$y_0 = \frac{1,000}{e^{(3)}}$$

$y_0 \approx 49.79$ or 50 units .

Extra Problem #2

Some substances, such as carbon 14, have very long half lives and are used by archaeologists to date fossils, plants and animals fairly accurately. The half life of carbon 14 is about 5600 years. At an archaeological site, a thigh bone was found and contained approximately one third of the carbon 14 normally found in a living thigh bone. Calculate the age of the bone to the nearest year.

Solution:

$$y = y_0 e^{kt}$$

When $t = 5600$ **years, half of the carbon remains, so**

$$y = \frac{1}{2} y_0$$

$$\frac{1}{2} y_0 = y_0 e^{k \bullet 5600}$$

Divide both sides by y_0

$$\frac{1}{2} = e^{k \bullet 5600}$$

Take the ln of each side:

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{k \bullet 5600}\right)$$

$$\ln 1 - \ln 2 = k \bullet 5600$$

$$-\ln 2 = k \bullet 5600$$

Divide both sides by 5600

$$k = -\frac{\ln 2}{5600} \approx -1.23776 \times 10^{-4}$$

Extra Problem #2 Continued.

Now, the question is this: If one-third of the carbon remains (i.e., if $y = \frac{1}{3} y_0$), how old is the bone (i.e., solve for t)??

$$y = y_0 e^{kt}$$

$$\frac{1}{3} y_0 = y_0 e^{kt}$$

Divide both sides by y_0.

$$\frac{1}{3} = e^{kt}$$

Take the ln of each side:

$$\ln\left(\frac{1}{3}\right) = \ln\left(e^{kt}\right)$$

$$\ln 1 - \ln 3 = kt$$

$$-\ln 3 = kt$$

Divide both sides by k

$$t = -\frac{\ln 3}{k} \approx -\frac{\ln 3}{-1.23776 \times 10^{-4}} \approx 8876 \text{ years}$$

Notes about the accuracy of carbon dating and age estimation

1. The formula $y = y_0 e^{kt}$ is mathematically derived in Calculus, assuming that the rate of growth is proportional to the population at any time t.

2. In this formula, the value of k is the growth "constant." This formula ASSUMES that the conditions that affect growth are constant. It is a leap of faith to assume that the conditions that affect growth are constant for a short period of time, let alone for eons of time! Can you imagine ANYTHING being constant for millions of years???

3. The growth curve is exponential! When it comes to exponential growth, very small errors in one variable can result in HUGE calculation errors in another variable.