CALCULUS COMPREHENSIVE PRACTICE TEST

Evaluate the limits:

1. $\lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

2. $\lim\limits_{x\to 2} \dfrac{4-x^2}{3-\sqrt{x^2+5}}$

3. $\lim\limits_{x\to \infty} \dfrac{3x^2-4x+12}{9x^2-12}$

4. $\lim\limits_{x\to 0^-} x\sqrt{4+\dfrac{9}{x^2}}$

5. $\lim\limits_{x\to +\infty} \left(2x-\sqrt{4x^2+12x}\right)$

Find the first derivatives and simplify:

6. $f(x) = (x^2+4x+5)\sqrt{x^2+4x+5}$.

7. $f(x) = (2x+3)^4(3x-2)^{7/2}$.

8. $f(t) = \dfrac{\sqrt[3]{t^2+2t}}{t^2}$.

9. Find $\frac{dy}{dx}$ for $x^2+2xy+2y^2+3x-y=9$.

10. Find y'' for $x^2+4xy-y^2=8$.

11. Find the equations of the normal and tangent lines to $y = 2x - \dfrac{4}{\sqrt{x}}$ at $x=4$.

12. The sum of one number and three times a second number is 60. Find the two numbers that have the maximum product.

13. A cylindrical can is to have volume of 27 in^3. If top and bottom are to be cut from squares and the residue wasted, find the dimensions of the can so that the ~~total~~ metal used, including waste, is a minimum.

14. A trough is 12 m. long and its ends are isosceles triangles having altitude of 6 m. and a base of 4 m. Water is flowing into the trough at 2 m^3/min. How fast is the water level rising when the water is 3 m. deep? (Assume vertex is

15. Determine relative maximums and minimums for $f(x) = x^4 + \frac{4}{3}x^3 - 12x^2$. Give intervals in which $f(x)$ is increasing and decreasing.

16. If $f(x) = 5x^{2/3} - x^{5/3}$, find $f'(x)$ and $f''(x)$. Find all relative extrema, points of inflection, and discuss concavity.

17. Given the table: $\lim\limits_{x \to -\infty} f(x) = -\infty$

$\lim\limits_{x \to +\infty} f(x) = +\infty$

x		-2		-1		0		1		3	
f		2		0		2		∞		0	
f'	$+$	0	$-$	∞	$+$	0	$+$	∞	$+$	$+$	$+$
f''	$-$	$-$	$-$	∞	$-$	0	$+$	∞	$-$	0	$+$

Identify relative extrema, points of inflection, asymptotes, vertical tangents, intervals of increasing and decreasing, concave up, concave down. Sketch a graph of $f(x)$.

Evaluate the integrals:

18. $\displaystyle\int_1^2 x^2\sqrt{x^3+8}\,dx$

19. $\displaystyle\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$

20. $\displaystyle\int_1^4 \frac{2+\sqrt{x}}{x^2}\,dx$

Find the area between the curves:

21. $y = x^2 - 4x$ and $2x + y = 8$

22. $y^2 = 2x - 2$ and $y = x - 5$

CALCULUS PRACTICE TEST Solutions

1. $\lim_{h\to 0} \dfrac{x^2+2xh+h^2-x^2}{h}$

$= \lim_{h\to 0} \dfrac{\cancel{h}(2x+h)}{\cancel{h}}$

$= 2x$

2. $\lim_{x\to 2} \dfrac{4-x^2}{3-\sqrt{x^2+5}} = \dfrac{0}{0}$

$= \lim_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})}$

$= \lim_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{9-x^2-5}$

$= \lim_{x\to 2} (3+\sqrt{x^2+5}) = 6$

3. $\lim_{x\to\infty} \dfrac{\frac{1}{x^2}(3x^2-4x+12)}{\frac{1}{x^2}(9x^2-12)}$

$= \lim_{x\to\infty} \dfrac{3-\frac{4}{x}+\frac{12}{x^2}}{9-\frac{12}{x^2}}$

$= \dfrac{3}{9} = \dfrac{1}{3}$

4. $\lim_{x\to 0^-} x\sqrt{4+\dfrac{9}{x^2}}$

$= \lim_{x\to 0^-} x\sqrt{\dfrac{4x^2+9}{x^2}}$

$= \lim_{x\to 0^-} \dfrac{x}{|x|}\sqrt{4x^2+9}$

$= -1\cdot\sqrt{9} = -3$

5. $\lim_{x\to+\infty} (2x-\sqrt{4x^2+12x})$

$= \infty - \infty$

$= \lim_{x\to+\infty} \dfrac{(2x-\sqrt{4x^2+12x})(2x+\sqrt{4x^2+12x})}{2x+\sqrt{4x^2+12x}}$

$= \lim_{x\to+\infty} \dfrac{4x^2-4x^2-12x}{2x+\sqrt{4x^2(1+\frac{3}{x})}}$

$= \lim_{x\to+\infty} \dfrac{-12x}{2x(1+\sqrt{1+3/x})} = \dfrac{-6}{2} = -3$

6. $f(x) = (x^2+4x+5)^{3/2}$

$f'(x) = \frac{3}{2}(x^2+4x+5)^{1/2}(2x+4)$

$= 3(x+2)\sqrt{x^2+4x+5}$

7. $f(x) = (2x+3)^4(3x-2)^{7/2}$

$f'(x) = (2x+3)^4\,\frac{7}{2}(3x-2)^{5/2}\cdot 3 + (3x-2)^{7/2}\,4(2x+3)^3\cdot 2$

$= (2x+3)^3(3x-2)^{5/2}\left[\frac{21}{2}(2x+3)+8(3x-2)\right]$

$= \frac{1}{2}(2x+3)^3(3x-2)^{5/2}(90x+31)$

8. $f(t) = \dfrac{\sqrt[3]{t^2+2t}}{t^2}$

$f'(t) = \dfrac{t^2\,\frac{1}{3}(t^2+2t)^{-2/3}(2t+2) - \sqrt[3]{t^2+2t}\cdot 2t}{t^4}$

$= \dfrac{2t(t^2+2t)^{-2/3}\left[\frac{1}{3}t(t+1)-(t^2+2t)\right]}{t^4}$

$= \dfrac{2(t^2+t-3t^2-6t)}{3t^3(t^2+2t)^{2/3}} = \dfrac{-2(2t+5)}{3t^2(t^2+2t)^{2/3}}$

9. $x^2+2xy+2y^2+3x-y=9$

$2x+2x\frac{dy}{dx}+y\cdot 2+4y\frac{dy}{dx}+3-\frac{dy}{dx}=0$

$\frac{dy}{dx}(2x+4y-1) = -2x-2y-3$

$\dfrac{dy}{dx} = \dfrac{2x+2y+3}{-2x-4y+1}$

10. $x^2+4xy-y^2=8$

$2x+4x\frac{dy}{dx}+y\cdot 4-2y\frac{dy}{dx}=0$

$2x+4y = (2y-4x)\frac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{x+2y}{y-2x}$

$\dfrac{d^2y}{dx^2} = \dfrac{(y-2x)\cdot(1+2\frac{dy}{dx}) - (x+2y)(\frac{dy}{dx}-2)}{(y-2x)^2}$

$= \dfrac{5(y^2-4xy-x^2)}{(y-2x)^3} = \dfrac{-40}{(y-2x)^3}$

or $\dfrac{40}{(2x-y)^3}$

11. $y = 2x - 4x^{-1/2}$ $y(4) = 8-2 = 6$

$y' = 2+2x^{-3/2}$ at $x=4$

$y'(4) = 2+2\cdot\frac{1}{8} = \frac{9}{4}$

TANGENT LINE

$y-6 = \frac{9}{4}(x-4)$

$y-6 = \frac{9}{4}x-9$

$y = \frac{9}{4}x-3$

NORMAL LINE

$y-6 = -\frac{4}{9}(x-4)$

$y-6 = -\frac{4}{9}x+\frac{16}{9}$

$y = -\frac{4}{9}x+\frac{70}{9}$

12. $x + 3y = 60$

Max $P = xy$

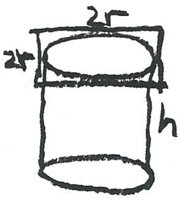
$= (60-3y)y$

$= 60y - 3y^2$

$P' = 60 - 6y = 0$

$y = 10, \quad x = 30$

13. $V = \pi r^2 h \quad h = \frac{V}{\pi r^2}$



Material $= 2\pi r h + 2(2r)^2$

$= 2\pi r \frac{V}{\pi r^2} + 8r^2$

$= 2Vr^{-1} + 8r^2$

$M' = -2Vr^{-2} + 16r = 0$

$16r = \frac{2V}{r^2}$

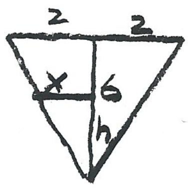
$r^3 = \frac{V}{8}$

Now if $V = 27$

$r = \sqrt[3]{\frac{27}{8}} = \frac{3}{2}$

$h = \frac{27}{\pi \frac{9}{4}} = \frac{12}{\pi}$ in.

14.



Let h = depth of water

$2x$ = dist across top of water.

$\frac{x}{h} = \frac{2}{6}$ so $x = \frac{h}{3}$, $\frac{dV}{dt} = 2$ m³/min.

Volume $= 12 \cdot \frac{1}{2}(2xh)$

$= 12xh$

$= 12 \frac{h}{3} \cdot h = 4h^2$

$\frac{dV}{dt} = 8h \frac{dh}{dt}$

$2 = 8 \cdot 3 \frac{dh}{dt} \quad \frac{dh}{dt} = \frac{1}{12}$ m/min.

15. $f(x) = x^4 + \frac{4}{3}x^3 - 12x^2$

$f'(x) = 4x^3 + 4x^2 - 24x$

$4x(x^2 + x - 6) = 0$

$4x(x+3)(x-2) = 0$

$x = 0 \quad x = -3 \quad x = 2$

$f'(x) < 0$, Decreasing $(-\infty, -3) \cup (0, 2)$

$f'(x) > 0$, Increasing $(-3, 0) \cup (2, \infty)$

16. $f(x) = 5x^{2/3} - x^{5/3}$

$f'(x) = \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{2/3}$

$= \frac{5}{3}x^{-1/3}(2 - x) = 0$

$f''(x) = -\frac{10}{9}x^{-4/3} - \frac{10}{9}x^{-1/3}$

$= -\frac{10}{9}x^{-4/3}(x+1)$

Local Min $(0,0)$

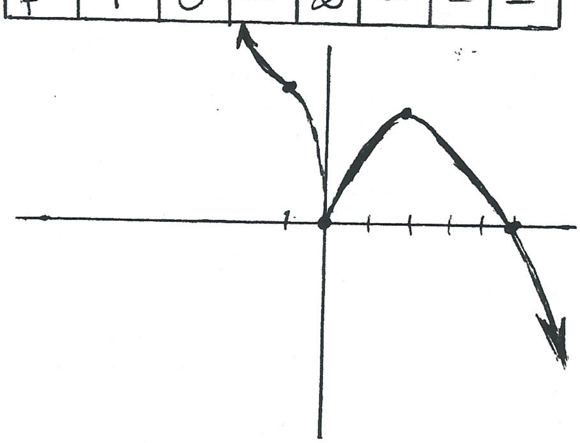
Local Max $(2, 3\sqrt[3]{4})$

Pt Inflection $(-1, 6)$

Concave Up $(-\infty, -1)$

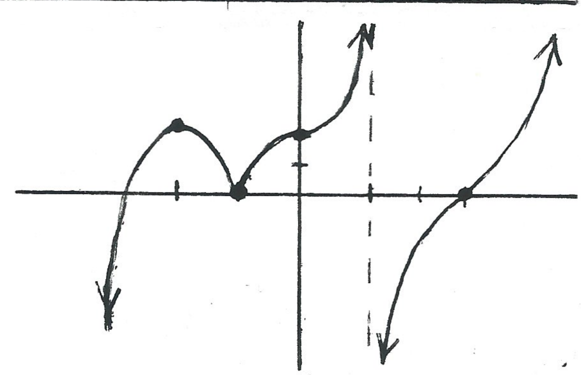
Down $(-1, 0) \cup (0, \infty)$

x		-1		0		2	
f	▨	6	▨	0	▨	$3\sqrt[3]{4}$	▨
f'	−	−	−	∞	+	0	−
f''	+	0	−	∞	−	−	−



17.

x		-2		-1		0		1		3	
f	▨	2	▨	0	▨	2	▨	∞	▨	0	▨
f'	+	0	−	∞	+	0	+	∞	+	+	+
f''	−	−	−	∞	−	0	+	∞	−	0	+



Relative Max: $(-2, 1)$

Min: $(-1, 0)$

Point of Infl: $(0, 2)$ $(3, 0)$

Asymptote: $x = 1$

Vert. Tangent: $(-1, 0)$

Increasing: $(-\infty, -2) \cup (-1, 1) \cup (1, \infty)$

Decreasing: $(-2, -1)$

Concave Up: $(0, 1) \cup (3, \infty)$

18. $\int_1^2 x^2\sqrt{x^3+8}\,dx$ Let $u = x^3+8$, $du = 3x^2dx$, $\frac{du}{3} = x^2dx$

$= \int u^{1/2}\frac{du}{3}$

$= \frac{2}{3}\frac{u^{3/2}}{3}$

$= \frac{2}{9}(x^3+8)^{3/2}\Big|_1^2$

$= \frac{2}{9}\left[16^{3/2} - 9^{3/2}\right]$

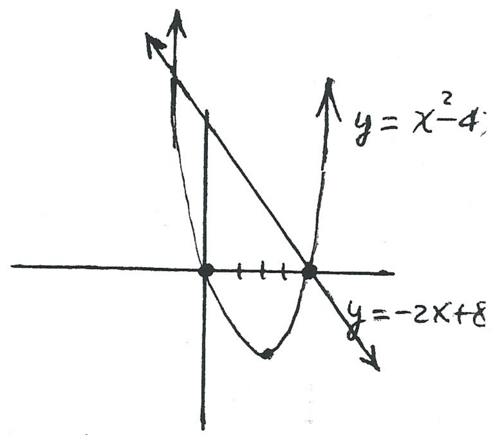
$= \frac{2}{9}(64-27) = \frac{74}{9}$

19. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$ Let $u = 1+\sqrt{x}$, $du = \frac{1}{2}x^{-1/2}dx$, $2du = \frac{dx}{\sqrt{x}}$

$= \int \frac{2du}{u^2}$

$= 2\int u^{-2}du = 2\frac{u^{-1}}{-1} + c$

$= -2\frac{1}{1+\sqrt{x}} + c = \frac{-2}{1+\sqrt{x}} + c$



20. $\int_1^4 \frac{2+\sqrt{x}}{x^2}dx$

$= \int_1^4 (2x^{-2} + x^{-3/2})dx$

$= \frac{2x^{-1}}{-1} + \frac{-2}{1}x^{-1/2}\Big|_1^4$

$= -2\left(\frac{1}{x} + \frac{1}{\sqrt{x}}\right)\Big|_1^4$

$= -2\left(\frac{1}{4}+\frac{1}{2}\right) + 2(1+1)$

$= -2\cdot\frac{3}{4} + 4 = \frac{5}{2}$

21. $y = x^2-4x$
Parabola – Up.
Vertex $x = \frac{4}{2} = 2$
$y = 4-8 = -4$

$2x+y = 8$
$2x + (x^2-4x) = 8$
$x^2-2x-8 = 0$
$(x-4)(x+2) = 0$
$x=4$, $y=0$ $\quad$ $x=-2$, $y=12$
Points of Intersection.

$\int_{-2}^4 (-2x+8) - (x^2-4x)\,dx$

$= \int_{-2}^4 (-x^2+2x+8)\,dx$

$= -\frac{x^3}{3} + x^2 + 8x\Big|_{-2}^4$

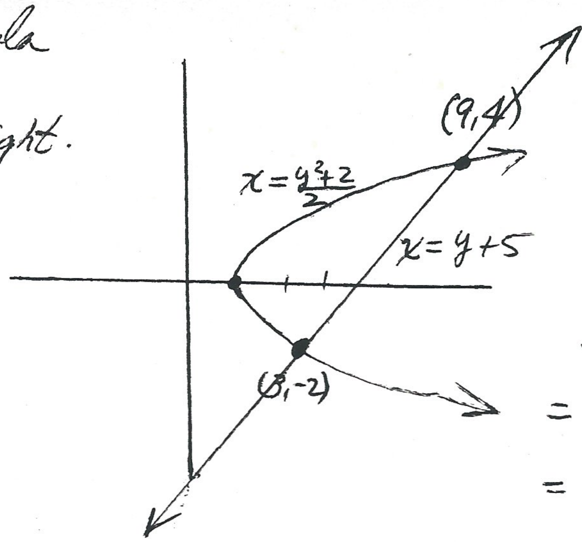
$= \left(-\frac{64}{3} + 16 + 32\right) - \left(\frac{8}{3} + 4 - 1\right.$

$= -\frac{72}{3} + 48 + 12$

$= -24 + 60 = 36$

22. $y^2 = 2x-2$ Parabola
$y^2 = 2(x-1)$
V(1,0) Opens Right.

$y = x-5$
$(x-5)^2 = 2x-2$
$x^2-10x+25 = 2x-2$
$x^2-12x+27 = 0$
$(x-9)(x-3) = 0$
$x=9 \quad x=3$
$y=4 \quad y=-2$



$\int_{-2}^4 (y+5) - \left(\frac{y^2+2}{2}\right)dy$

$= \int_{-2}^4 \left(-\frac{y^2}{2} + y + 4\right)dy$

$= -\frac{y^3}{6} + \frac{y^2}{2} + 4y\Big|_{-2}^4$

$= -\frac{64}{6} + \frac{16}{2} + 16 - \left(\frac{8}{6} + \frac{4}{2} - 8\right)$

$= -\frac{72}{6} + 24 + 6 =$

$= -12 + 30 = 18$