CALCULUS II EXAM 3

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Term II, 1996

Show all work on separate paper.

1. $\int x \arccos x \, dx$

2. $\int x^2 e^{3x} \, dx$

3. $\int \frac{\cos^3 x}{\sqrt{\sin x}} \, dx$

4. $\int \tan^3 t \sec^3 t \, dt$

5. $\int \frac{1}{x^2\sqrt{25-x^2}} \, dx$

6. $\int \frac{\sqrt{x^2-4}}{x} \, dx$

7. $\int \frac{x+2}{x^2-4x} \, dx$

8. $\int \frac{6x^2-3x+14}{x^3-2x^2+4x-8} \, dx$

9a) $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$

b) $\lim_{x\to\infty} \frac{3x^2-2x+1}{2x^2+3}$

10. $\lim_{x\to 0^+} (1+x)^{1/x}$

11a) $\int \frac{e^x}{1+e^x} \, dx$ b) $\int \frac{e^x}{1+e^{2x}} \, dx$

In 12–14, determine convergence or divergence of the improper integral. Evaluate the integral if it converges. (See # 11)

12. $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

13. $\int_{-1}^{2} \frac{dx}{x^3}$

14a) $\int_0^\infty \frac{e^x}{1+e^x} \, dx$

b) $\int_0^\infty \frac{e^x}{1+e^{2x}} \, dx$

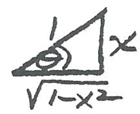
CALCULUS II Exam 3 Solutions

Dr. Rapalje

1. $\int x \arccos x \, dx$ let $u = \arccos x$ $dV = x\,dx$

$du = -\frac{1}{\sqrt{1-x^2}}dx$ $V = \frac{x^2}{2}$

$= \frac{x^2}{2}\arccos x + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$



$= \frac{x^2}{2}\arccos x + \frac{1}{2}\int \frac{\sin^2\theta \cancel{\cos\theta} d\theta}{\cancel{\cos\theta}}$ $x = \sin\theta$ $\sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\, d\theta$

$= \frac{x^2}{2}\arccos x + \frac{1}{2}\int \frac{1-\cos 2\theta}{2}d\theta$

$= \frac{x^2}{2}\arccos x + \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]$

$= \frac{x^2}{2}\arccos x + \frac{1}{4}\left(\arcsin x - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + c$

$= \frac{x^2}{2}\arccos x + \frac{1}{4}\left(\arcsin x - x\sqrt{1-x^2}\right) + C$

2. $\int x^2 e^{3x} dx$ let $u = x^2$ $dV = e^{3x}dx$

$du = 2x\,dx$ $V = \frac{1}{3}e^{3x}$

$= \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3} x e^{3x} dx$ let $u = x$ $dV = e^{3x}dx$

$du = dx$ $V = \frac{e^{3x}}{3}$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[\frac{x e^{3x}}{3} - \int \frac{e^{3x}dx}{3}\right]$

$= \frac{1}{3}x^2 e^{3x} - \frac{2x}{9}e^{3x} + \frac{2}{27}e^{3x} + c$

$= e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) + c$

3. $\int \frac{\cos^3 x\, dx}{\sqrt{\sin x}}$ let $u = \sin x$

$du = \cos x\, dx$

$= \int \frac{\cos^2 x \cos x\, dx}{\sqrt{\sin x}} = \int \frac{(1-\sin^2 x)\cos x\, dx}{\sqrt{\sin x}}$

$= \int \frac{(1-u^2)}{u^{1/2}}du = \int \left[u^{-1/2} - u^{3/2}\right] du$

$= \frac{2u^{1/2}}{1} - \frac{2u^{5/2}}{5} + c$

$= 2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{5/2} + c$

or $2\sqrt{\sin x}\left[1 - \frac{1}{5}\sin^2 x\right] + c$

$\frac{2\sqrt{\sin x}}{5}\left(5 - \sin^2 x\right) + c$

(tan to odd power)

4. $\int \tan^3 t \sec^3 t\, dt$

$u = \sec t$

$du = \sec t \tan t\, dt$

$= \int \tan^2 t \sec^2 t\, (\sec t \tan t\, dt)$

$= \int (\sec^2 t - 1)\sec^2 t\, (du)$

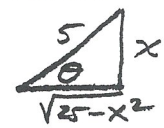
$= \int (u^2 - 1)u^2 du$

$= \int u^4 - u^2\, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + c$

$= \frac{1}{5}\sec^5 t - \frac{1}{3}\sec^3 t + c$

5. $\int \frac{1}{x^2\sqrt{25-x^2}}dx$



$\int \frac{1}{25\sin^2\theta\, 5\cos\theta} 5\cos\theta\, d\theta$ $\sin\theta = \frac{x}{5}$

$x = 5\sin\theta$

$= \frac{1}{25}\int \csc^2\theta\, d\theta$ $dx = 5\cos\theta\, d\theta$

$= \frac{1}{25}(-\cot\theta) + c$ $\sqrt{25-x^2} = 5\cos\theta$

$= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + c$

6. $\int \frac{\sqrt{x^2-4}}{x}dx$

$\sec\theta = \frac{x}{2}$

$\int \frac{2\tan\theta}{\cancel{2\sec\theta}}\, 2\cancel{\sec\theta}\tan\theta\, d\theta$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$= 2\int \tan^2\theta\, d\theta$ $\sqrt{x^2-4} = 2\tan\theta$

$= 2\int (\sec^2\theta - 1)\, d\theta$

$= 2(\tan\theta - \theta) + c$

$= 2\frac{\sqrt{x^2-4}}{2} - 2\,\text{arcsec}\,\frac{x}{2} + c$

$= \sqrt{x^2-4} - 2\,\text{arcsec}\,\frac{x}{2} + c$

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7. $\frac{x+2}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$

$x+2 = A(x-4) + Bx$

Let $x=0$: $2 = -4A$ $\quad A = -\frac{1}{2}$

$x=4$: $6 = 4B$ $\quad B = \frac{3}{2}$

$\int \frac{x+2}{x(x-4)}dx = \int\left[\frac{-\frac{1}{2}}{x} + \frac{\frac{3}{2}}{x-4}\right]dx$

$= -\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x-4| + C$

or $= \ln x^{-1/2} + \ln(x-4)^{3/2} + C$

$= \ln \frac{(x-4)^{3/2}}{x^{1/2}} + C$

9a) $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \frac{0}{0}$ L'Hopital

$= \lim_{x\to 0} \frac{2\cos 2x}{3\cos 3x} = \frac{2}{3}$

b) $\lim_{x\to\infty} \frac{3x^2-2x+1}{2x^2+3} = \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{6x-2}{4x} = \frac{\infty}{\infty}$

$= \frac{6}{4} = \frac{3}{2}$

11a) $\int \frac{e^x}{1+e^x}dx$

Let $u = 1+e^x$

$du = e^x dx$

$\int \frac{du}{u} + c$

$= \ln u + C$

$= \ln(1+e^x) + C$

b) $\int \frac{e^x}{1+e^{2x}}dx = \int \frac{du}{1+u^2}$

Let $u = e^x$

$du = e^x dx$

$= \arctan u = \arctan e^x + C$

12. $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

Let $u = \sqrt{x}$ $\quad u^2 = x$

$2u\,du = dx$

$\int \frac{2u\,du}{u(u^2+1)} = \int \frac{2\,du}{u^2+1}$

$= 2\arctan u$

$= 2\arctan\sqrt{x}\Big|_0^\infty$

$= 2\left(\frac{\pi}{2} - 0\right) = \pi$

CONVERGES

8. $\frac{6x^2-3x+14}{x^3-2x^2+4x-8} = \frac{}{x^2(x-2)+4(x-2)}$

$= \frac{6x^2-3x+14}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$

$6x^2-3x+14 = A(x^2+4) + (Bx+C)(x-2)$

$x=2$: $24-6+14 = A\cdot 8$

$32 = 8A$ $\quad A=4$

$x=0$: $14 = 4A + C\cdot(-2)$

$14 = 16 - 2C$

$-2 = -2C$ $\quad C=1$

$x=1$: $6-3+14 = 5A + (B+C)(-1)$

$17 = 20 - B - 1$ $\quad B=2$

$\int\left(\frac{4}{x-2} + \frac{2x+1}{x^2+4}\right)dx$

$= \int \frac{4}{x-2}dx + \int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}dx$

$= 4\ln|x-2| + \ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C$

10a) $\lim_{x\to 0^+} (1+x)^{1/x} = 0^\infty$

Let $y = (1+x)^{1/x}$

$\ln y = \ln(1+x)^{1/x}$

$\lim_{x\to 0^+}$ $= \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$

$= \frac{0}{0}$

$\lim_{x\to 0^+}$ $= \frac{\frac{1}{1+x}}{1} = 1$

$\ln y = 1$, then $y = e$

13. $\int_{-1}^{0} \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}$

$-x^{-2}\Big|_{-1}^{0}$

DIVERGES

14a) $\ln(1+e^x)\Big|_0^\infty$ = DIVERGES

b) $\arctan e^x\Big|_0^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

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6. Alt. method: $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{x^2-4}\;x\,dx}{x^2}$ Let $u=\sqrt{x^2-4}$

$u^2 = x^2-4$

$2u\,du = 2x\,dx$

$u\,du = x\,dx$

$x^2 = u^2+4$

$$= \int \frac{u\;u\,du}{u^2+4}$$

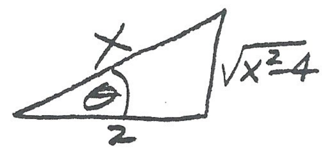
$$= \int \frac{u^2\,du}{u^2+4}$$

$$= \int \frac{(u^2+4-4)\,du}{u^2+4}$$

$$= \int \left(1 - \frac{4}{u^2+4}\right)du$$

$$= u - 4\cdot\frac{1}{2}\arctan\frac{u}{2} + C$$

$$= \sqrt{x^2-4} - 2\arctan\frac{\sqrt{x^2-4}}{2} + C$$



or $= \sqrt{x^2-4} - 2\,\mathrm{arcsec}\,\dfrac{x}{2} + C$

12. What if $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$ Let $u=\sqrt{x}+1$

$du = \frac{1}{2}x^{-1/2}\,dx$

$2\,du = \dfrac{dx}{\sqrt{x}}$

$$= \int \frac{2\,du}{u}$$

$$= 2\ln u$$

$$= 2\ln\left|\sqrt{x}+1\right| \Big|_0^\infty$$

$= \ln\infty$ DIVERGES