6 CALCULUS III   EXAM 3A   chpter 14 (L+H)

Show all work on separate paper.   Calculators allowed.

1. Find $\frac{\partial^2 f}{\partial x^2}$ for $f(x,y) = e^{x^2 y}$

2. Find the differential dz for $z = \ln(x^2+y^2)$.
Use this differential to approximate the change in z as (x,y) moves from $(0,1)$ to $(.01, .98)$.

3. Let $w = \sqrt{x^2+y^2}$, $x = s+t$, $y = 2t - s$.
Find $\frac{\partial w}{\partial s}$ when $s = t = 1$.

4. Find the directional derivative for $f(x,y,z) = z^2 e^{xy}$.
Find the maximum value of the directional derivative at the point $(-1,0,3)$.

5. Find the equation of the tangent plane to the surface $f(x,y) = 16 - 2x^2 - y^2$ at the point $(1,-2,10)$.

6. Use Lagrange multipliers to find the maximum temperature $T(x,y,z) = x+y+z$ on the surface $x^2+y^2+z^2 = 9$.

1. $f(x,y) = e^{x^2y}$

$\frac{\partial f}{\partial x} = 2xy\,e^{x^2y}$

$\frac{\partial^2 f}{\partial x^2} = 2xy\,e^{x^2y}(2xy) + e^{x^2y}\,2y$

$= 2y\,e^{x^2y}(2x^2y+1)$

2. $z = \ln(x^2+y^2)$ $(0,1)$ to $(.01, .98)$

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$= \frac{2x}{x^2+y^2}dx + \frac{2y}{x^2+y^2}dy$

$\Delta z \approx dz$, Let $x=0,\ y=1$

$dx = .01 \quad dy = -.02$

$dz = 0 + \left(\frac{2(1)}{0+1}\right)(-.02) = -.04$

3. $w = \sqrt{x^2+y^2} \quad x = s+t \quad y = 2t-s$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$

$= \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}x)\cdot 1 + \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}y)\cdot(-1)$

$= \frac{x-y}{\sqrt{x^2+y^2}}$

When $s=1,\ t=1,\ x=2,\ y=1$

$\frac{\partial w}{\partial s} = \frac{2-1}{\sqrt{4+1}} = \frac{1}{\sqrt{5}}$

4. $f(x,y,z) = z^2e^{xy}$

$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\vec{\nabla} f = yz^2e^{xy}\hat{i} + xz^2e^{xy}\hat{j} + 2ze^{xy}\hat{k}$

Gradient f

At $(-1,0,3)$, $\vec{\nabla} f = 0\hat{i} - 9e^0\hat{j} + 6e^0\hat{k}$

$\vec{\nabla} f = -9\hat{j} + 6\hat{k}$

Max $\vec{\nabla} f = \sqrt{(-9)^2+6^2} = 3\sqrt{3^2+2^2}$

$= 3\sqrt{13}$

Let $\vec{u}$ = unit vector

$= \frac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

$D_{\vec{u}}f = \frac{e^{xy}}{\sqrt{a^2+b^2+c^2}}\left(ayz^2 + bxz^2 + 2cz\right)$

Also, $D_{\vec{u}}f = e^{xy}\left(yz^2\cos\alpha + xz^2\cos\beta + 2z\cos\gamma\right)$

5. $f(x,y) = 16 - 2x^2 - y^2 \quad (1,-2,10)$

$\frac{\partial f}{\partial x} = -4x = -4 \quad \frac{\partial f}{\partial y} = -2y = 4$

$\langle -4, 4, -1\rangle \sim \langle 4, -4, 1\rangle$

→ vector ⊥ tangent plane

$4(x-1) - 4(y+2) + 1(z-10) = 0$

$4x - 4 - 4y - 8 + z - 10 = 0$

$4x - 4y + z = 22$

6. $T(x,y,z) = x+y+z$

constraint: $x^2+y^2+z^2-9=0$

$F(x,y,z,\lambda) = x+y+z-\lambda(x^2+y^2+z^2-9)$

$$\left.\begin{aligned} F_x &= 1-2x\lambda = 0 & x &= \tfrac{1}{2\lambda} \\ F_y &= 1-2y\lambda = 0 & y &= \tfrac{1}{2\lambda} \\ F_z &= 1-2z\lambda = 0 & z &= \tfrac{1}{2\lambda} \end{aligned}\right\}\ x = y = z$$

$F_\lambda = x^2+y^2+z^2-9 = 0 \longrightarrow 3x^2 - 9 = 0$

$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - 9 = 0.$

$x = \pm\sqrt{3}$

$T(x,y,z) = x+y+z \le 3\sqrt{3}$