MS 229 EXAM 3B Dr. Rapalje

Evaluate the limits: (6 each)

1. $\lim\limits_{x \to 0} \dfrac{\ln|\cos x|}{1-\cos x}$ 2. $\lim\limits_{x \to 0} \dfrac{e^{3x}-1}{1-\sin x}$ 3. $\lim\limits_{x \to 0} x^{x}$

Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Indicate all tests used and show all work. (10 each)

4. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n n^2 \cdot 3^n}{(n+1)!}$

7. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}(n-1)}{n^2+1}$

5. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2(n+1)}$

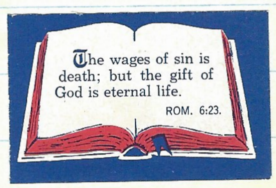
8. $\sum\limits_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)!}$

6. $\sum\limits_{n=1}^{\infty} \dfrac{n}{\ln n}$ (DO NOT USE RATIO TEST)

9. $\sum\limits_{n=1}^{\infty} e^{-n}$ (DO NOT USE RATIO TEST)

Find the values of x for which the power series converges.

(12 pts) 10. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (x-3)^n}{n \cdot 2^n}$



11. Use a Taylor series to compute the value of e
(10 pts) to 5 decimal places.

MS 329 Exam 3 Solutions.

1. $\lim_{x\to 0} \frac{\ln|\cos x|}{1-\cos x} = \frac{\ln 1}{0} = \frac{0}{0}$

$= \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{\sin x} = \lim_{x\to 0} -\frac{1}{\cos x} = \underline{\underline{-1}}$

2. $\lim_{x\to 0} \frac{e^{3x}-1}{1-\sin x} = \frac{0}{1} = \underline{\underline{0}}$

3. $\lim_{x\to 0} x^x$ Let $y = x^x$

$\ln y = x \ln x$

$\lim_{x\to 0} \frac{\ln x}{x^{-1}}$

$= \frac{1/x}{-1/x^2} = -x$

$\lim_{x\to 0} -x = 0$

$\ln y = 0$

$y = 1$

$\lim_{x\to 0} x^x = \underline{\underline{1}}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 \cdot 3^n}{(n+1)!}$

Ratio test: $\left| \frac{(n+1)^2\, 3^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{n^2 \cdot 3^n} \right|$

$\lim_{n\to\infty} \left| \frac{3}{n+2} \left(\frac{n+1}{n}\right)^2 \right|$

$= \lim_{n\to\infty} \left| \frac{3}{n+2} \cdot 1 \right| = 0$

<u>Absolutely convergent.</u>

5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2(n+1)} \le \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

<u>Absolutely convergent</u> by p series.

6. $\sum_{n=1}^{\infty} \frac{n}{\ln n}$. $\lim_{n\to\infty} \frac{n}{\ln n} = \frac{1}{1/n} = n$

$= \infty$

Diverges since $u_n \neq 0$.

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)}{n^2+1} \le \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)}{n^2-1}$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+1}$

Convergent by alternating series theorem.

$\sum_{n=1}^{\infty} \frac{n-1}{n^2+1} = \sum_{n=1}^{\infty} \frac{n}{n^2+1} - \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2} \ln(x^2+1)\Big|_1^{\infty} = \infty$

Diverges. Hence <u>conditionally convergent.</u>

8. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{(2n)!}$

Ratio test: $\left| \frac{\cancel{1\cdot 3\cdot 5 \cdots (2n-1)}(2n+1)}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{1\cdot 3 \cdots (2n-1)}} \right|$

$= \left| \frac{1}{2n+2} \right| = 0$ <u>Absolutely Convergent.</u>

9. $\sum_{n=1}^{\infty} e^{-n}$. Compare $\int_1^{\infty} e^{-x}\,dx$

$= -e^{-x}\Big|_1^{\infty} = -\left[e^{-\infty} - e^{-1}\right]$

$= -\left(\frac{1}{e^{\infty}} - \frac{1}{e}\right) = \frac{1}{e}$

<u>Absolutely convergent.</u>

10. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n\,(2^n)}$

Ratio: $\left| \frac{(x-3)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n\,2^n}{(x-3)^n} \right| < 1$

$\lim_{n\to\infty} \left| \frac{x-3}{2} \cdot \frac{n}{n+1} \right| < 1$

$|x-3| < 2$

$-2 < x-3 < 2$

$1 < x < 5$

Check $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{Diverges.}}$

$x = 5$: $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n\,2^n} = \sum \frac{(-1)^n}{n}$ Converges

$\boxed{1 < x \le 5}$

$f(x) = e^x$ let $b = 1$, $a = 0$.

$$f(b) = f(1) = e = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f'''(a)}{3!}(b-a)^3 + \ldots$$

$f(a) = 1$	$f(a) =$		1.00000	00
$f'(a) = 1$	$f'(a)(b-a) =$		1.00000	00
$f''(a) = 1$	$\frac{1}{2}(1)(1)^2$	$=$	.50000	00
$f'''(a) = 1$	$\frac{1}{6}(1)(1)^3$	$=$	.16666	67
$f^{IV}(a) = 1$	$\frac{1}{24}(1)(1)^4$	$=$	.04166	67
$f^{V}(a) = 1$	$\frac{1}{120}(1)(1)^5$	$=$	.00833	33
$f^{VI}(a) = 1$	$\frac{1}{720}(1)(1^6)$	$=$	.00138	89
$f^{VII}(a) = 1$	$\frac{1}{5040}(1)\,1^7$	$=$	.00019	84
$f^{VIII}(a) = 1$	$\frac{1}{40320}(1)\,1^8$	$=$	.00002	48
	$\frac{1}{362880}(1)\,1^9$	$=$	.00000	28

E.C.

$a = 1$

$f(x) = 3 + 2x - x^2 + 4x^3 - 2x^4$ $\quad f(a) = 6$

$f'(x) = 2 - 2x + 12x^2 - 8x^3$ $\quad f'(a) = 4$

$f''(x) = -2 + 24x - 24x^2$ $\quad f''(a) = -2$

$f'''(x) = 24 - 48x$ $\quad f'''(a) = -24 \cdot \frac{1}{3 \cdot 2}$

$f^{IV}(x) = -48$ $\quad f^{IV}(a) = -48 \cdot \frac{1}{4 \cdot 3 \cdot 2}$

$f(x) = f(a) + f'(a)(x-1) + \frac{f''(a)}{2!}(x-1)^2 + \ldots$

$$\boxed{f(x) = 6 + 4(x-1) - 1(x-1)^2 - 4(x-1)^3 - 2(x-1)^4}$$

1.0000	00
1.0000	00
.5000	00
.1666	67
.0416	67
.0083	33
.0013	89
.0001	98
.0000	24
2.7182	78

2.7183

MS 229 EXAM 3 (Again!)

(All problems 9 each.)

Evaluate the limits:

1. $\lim_{x \to 0} \dfrac{e^{3x} - 3x}{1 - \cos x}$ 2. $\lim_{x \to 0} (\cot x)^x$

Determine convergence for the following series:

Note: For positive series, "convergent" or "divergent" will suffice.

(OMIT ONE OF YOUR CHOOSING!) For alternating series, you must test to determine if "absolutely convergent," "conditionally convergent," or "divergent."

Be sure to <u>name each test used</u>.

3. $\sum_{n=1}^{\infty} \dfrac{n+1}{n\sqrt{n}}$ 4. $\sum_{n=1}^{\infty} \dfrac{(2n)!}{n^{100}}$ 5. $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{\ln n}{3n+2}$

6. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(n+1)^2}$ 7. $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$ 8. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} e^n}{n^3}$

9. $\sum \dfrac{(-1)^{n+1} 3^{n+1}}{2^{4n}}$

> Things to come: The kingdoms of this world are become the kingdoms of our Lord, and of his Christ; and he shall reign for ever and ever. Rev 11:15

For what values of x are the following convergent:

10. $\dfrac{x+1}{\sqrt{1}} + \dfrac{(x+1)^2}{\sqrt{2}} + \dfrac{(x+1)^3}{\sqrt{3}} + \dfrac{(x+1)^4}{\sqrt{4}} + \dots \dfrac{(x+1)^n}{n!}$

11. Find the Taylor series expansion of $f(x) = \ln x$ about $a = 2$. Give the n^{th} term. E.C. Find interval of convergence.

12. Use a Taylor series to compute the value of $\frac{1}{e}$ accurate to 4 decimal places. (Compute with 6, round to 4.)

1. $\lim_{x\to 0} \frac{e^{3x}-3x}{1-\cos x} = \frac{1-0}{1-1} = \frac{1}{0} = \boxed{\infty}$

2. $\lim_{x\to 0} (\cot x)^x$. Let $y = (\cot x)^x$

$\ln y = x \ln \cot x$

$= \lim_{x\to 0} \frac{\ln \cot x}{1/x} = \infty$

$= \frac{\frac{1}{\cot x}(-\csc^2 x)}{-1/x^2}$

$= \left[\frac{\sin x}{\cos x}\cdot\frac{1}{\sin^2 x}\right]\left(\frac{x^2}{1}\right)$

$= \lim_{x\to 0} \frac{x^2}{\cos x \sin x}$

$= \lim_{x\to 0} \frac{x}{\cos x}\cdot\frac{x}{\sin x}$

$= \lim_{x\to 0} \frac{x}{\cos x}\cdot \lim_{x\to 0}\frac{x}{\sin x}$

$= 0 \cdot \frac{1}{\cos x} = 0 \cdot 1 = 0$

$\ln y = 0$

$\therefore \boxed{y = 1}$

3. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{n}{n\sqrt{n}} + \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\boxed{\text{Divergent}}$ By p series.

4. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{100}}$

Ratio test:

$\frac{u_{n+1}}{u_n} = \left|\frac{[2(n+1)]!}{(n+1)^{100}}\cdot\frac{n^{100}}{(2n)!}\right|$

$= \left|\frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)^{100}}\cdot\frac{n^{100}}{\cancel{(2n)!}}\right|$

$\lim_{n\to\infty} \left|(2n+2)(2n+1)\left(\frac{n}{n+1}\right)^{100}\right| = \infty$

$\boxed{\text{Divergent}}$

5. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{3n+2}$

Alternating series test:

$\lim_{n\to\infty} u_n = \frac{1/n}{3} = 0$

Conditionally convergent.

Test for absolute convergence:

$\sum \frac{\ln n}{3n+2} \geq \sum \frac{1}{3n+2}$ Comparison test.

$\int_1^{\infty} \frac{1}{3x+2}\,dx = \frac{1}{3}\ln|3x+2|\Big]_1^{\infty}$

= Divergent

Hence $\boxed{\text{conditionally convergent}}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^2} \leq \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ (Comparison test.)

$\boxed{\text{Absolutely convergent}}$ by p series.

7. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Ratio test:

$\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$

$= \left|\frac{2}{n+1}\right| = 0$

$\boxed{\text{Absolutely convergent.}}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^n}{n^3}$: $\lim_{n\to\infty}|u_n| = \frac{e^n}{n^3}$

$= \lim_{n\to\infty} \frac{e^n}{3n^2} = \lim_{n\to\infty}\frac{e^n}{6n}$

$= \lim_{n\to\infty} \frac{e^n}{6} = \infty$

$\boxed{\text{Diverges.}}$ $u_n \neq 0$.

9. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n+1}}{2^{4n}}$ Ratio test:

$\frac{u_{n+1}}{u_n} = \left|\frac{3^{n+2}}{2^{4n+4}}\cdot\frac{2^{4n}}{3^{n+1}}\right|$

$= \left|\frac{3}{2^4}\right| = \frac{3}{16} < 1$

$\boxed{\text{Absolutely convergent.}}$

10. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}}$ Ratio test:

$\frac{u_{n+1}}{u_n} = \left|\frac{(x+1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x+1)^n}\right| < 1$

$\left(\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 1\right)$

$|x+1| < 1$

$-1 < x+1 < 1$

$-2 < x < 0$

Test $x=0$: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ Divergent by p series.

$x=-2$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ Convergent by Alt. series Theorem.

Convergent for $\boxed{-2 \leq x < 0}$

11. $f(x) = \ln x$ $\quad f(a) = \ln 2$

$f'(x) = 1/x$ $\quad f'(a) = 1/2$

$f''(x) = -1/x^2$ $\quad f''(a) = -1/2^2$

$f'''(x) = +2/x^3$ $\quad f'''(a) = +2/2^3$

$f^{IV}(x) = -\frac{3!}{x^4}$ $\quad f^{IV}(a) = -3!/2^4$

$f^{N}(x) = 4!/x^5$ $\quad f^{n}(a) = \frac{(n-1)!(-1)^{n-1}}{2^n}$

Taylor: $\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n-1)!}{2^n}\cdot\frac{(x-2)^n}{n!}$

$= \boxed{\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-2)^n}{n\,2^n}}$

11. con't. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n\,2^n}$

Ratio test: $\left| \frac{(x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(x-2)^n} \right| < 1$

$= \left| \frac{(x-2)}{2} \right| < 1$

$= |x-2| < 2$

$-2 < x-2 < 2$

$0 < x < 4$

End points: $x=0$, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-2)^n}{n(2^n)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$ Does not alternate. Diverges by p series.

$x=4$, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ = Alternating series. $\lim u_n = 0$, converges, by alt. series thm.

$\boxed{0 < x \leq 4}$

12. $f(x) = e^{-x}$ let $b=1$, $a=0$

$f(b) = f(1) = e^{-1} = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f^n(a)}{n!}(b-a)^n, \ldots$

$f(x) = e^{-x}$	$f(a) = 1$	1.0000 00	1
$f'(x) = -e^{-x}$	$f'(a) = -1$	−1.0000 00	1
$f''(x) = e^{-x}$	$f''(a) = 1$	.5000 00	1/2
$f'''(x) = -e^{-x}$	$f'''(a) = -1$	−.1666 67	1/6
$f^n(x) = (-1)^n e^{-x}$	$\vdots$	.0416 67	1/24
$(b-a)^n = 1$		−.0083 33	1/120
		.0013 89	1/720
		−.0001 98	1/720
		.0000 24	

.5430 80

−.1751 98

.3678 82

$\boxed{= .3679}$