MS 229 EXAM 3C (Again!) Dr. Rapalje

(All problems 9 each.)

Evaluate the limits:

1. $\lim_{x \to 0} \dfrac{e^{3x} - 3x}{1 - \cos x}$ 2. $\lim_{x \to 0} (\cot x)^x$

Determine convergence for the following series:

(OMIT ONE OF YOUR CHOOSING!)

Note: For positive series, "convergent" or "divergent" will suffice.
For alternating series, you must test to determine if "absolutely convergent," "conditionally convergent," or "divergent."
Be sure to <u>name each test used</u>.

3. $\sum_{n=1}^{\infty} \dfrac{n+1}{n\sqrt{n}}$ 4. $\sum_{n=1}^{\infty} \dfrac{(2n)!}{n^{100}}$ 5. $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{\ln n}{3n+2}$

6. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(n+1)^2}$ 7. $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$ 8. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} e^n}{n^3}$

9. $\sum \dfrac{(-1)^{n+1} 3^{n+1}}{2^{4n}}$

> Things to come: The kingdoms of this world are become the kingdoms of our Lord, and of his Christ; and he shall reign for ever and ever. Rev 11:15

For what values of x are the following convergent:

10. $\dfrac{x+1}{\sqrt{1}} + \dfrac{(x+1)^2}{\sqrt{2}} + \dfrac{(x+1)^3}{\sqrt{3}} + \dfrac{(x+1)^4}{\sqrt{4}} + \dots \dfrac{(x+1)^n}{n!}$

11. Find the Taylor series expansion of $f(x) = \ln x$ about $a = 2$. Give the n^{th} term. E.C. Find interval of convergence.

12. Use a Taylor series to compute the value of $\frac{1}{e}$ accurate to 4 decimal places. (Compute with 6, round to 4.)

Calc III, Exam 3C Solutions Dr. Rapalje

1. $\lim_{x\to 0} \frac{e^{3x}-3x}{1-\cos x} = \frac{1-0}{1-1} = \frac{1}{0} = \boxed{\infty}$

2. $\lim_{x\to 0} (\cot x)^x$. Let $y = (\cot x)^x$

$\ln y = x \ln \cot x$

$= \lim_{x\to 0} \frac{\ln \cot x}{1/x} = \infty$

$= \frac{\frac{1}{\cot x}(-\csc^2 x)}{-1/x^2}$

$= \left[\frac{\sin x}{\cos x}\cdot\frac{1}{\sin^2 x}\right]\left(\frac{x^2}{1}\right)$

$= \lim_{x\to 0} \frac{x^2}{\cos x \sin x}$

$= \lim_{x\to 0} \frac{x}{\cos x}\cdot\frac{x}{\sin x}$

$= \lim_{x\to 0}\frac{x}{\cos x}\cdot\lim_{x\to 0}\frac{x}{\sin x}$

$= 0 \cdot \frac{1}{\cos x} = 0\cdot 1 = 0$

$\ln y = 0$

$\therefore \boxed{y = 1}$

3. $\sum_{n=1}^{\infty}\frac{n+1}{n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{n}{n\sqrt{n}} + \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$

$= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $\boxed{\text{Divergent}}$ By p series.

4. $\sum_{n=1}^{\infty}\frac{(2n)!}{n^{100}}$

Ratio test:

$\frac{u_{n+1}}{u_n} = \left|\frac{[2(n+1)]!}{(n+1)^{100}}\cdot\frac{n^{100}}{(2n)!}\right|$

$= \left|\frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)^{100}}\cdot\frac{n^{100}}{\cancel{(2n)!}}\right|$

$\lim_{n\to\infty}\left|(2n+2)(2n+1)\left(\frac{n}{n+1}\right)^{100}\right| = \infty$

$\boxed{\text{Divergent}}$

5. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{3n+2}$

Alternating series test:

$\lim_{n\to\infty} u_n = \frac{1/n}{3} = 0$

Conditionally convergent.

Test for absolute convergence:

$\sum\frac{\ln n}{3n+2} \geq \sum\frac{1}{3n+2}$ Comparison test.

$\int_1^{\infty}\frac{1}{3x+2}\,dx = \frac{1}{3}\ln|3x+2|\Big]_1^{\infty}$

= Divergent

Hence $\boxed{\text{conditionally convergent}}$

6. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2} \leq \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ (Comparison test.)

$\boxed{\text{Absolutely convergent}}$ by p series.

7. $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ Ratio test:

$\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$

$= \left|\frac{2}{n+1}\right| = 0$

$\boxed{\text{Absolutely convergent.}}$

8. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^n}{n^3}$: $\lim_{n\to\infty}|u_n| = \frac{e^n}{n^3}$

$= \lim_{n\to\infty}\frac{e^n}{3n^2} = \lim_{n\to\infty}\frac{e^n}{6n}$

$= \lim_{n\to\infty}\frac{e^n}{6} = \infty$

$\boxed{\text{Diverges.}}$ $u_n \neq 0$.

9. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^{n+1}}{2^{4n}}$ Ratio test:

$\frac{u_{n+1}}{u_n} = \left|\frac{3^{n+2}}{2^{4n+4}}\cdot\frac{2^{4n}}{3^{n+1}}\right|$

$= \left|\frac{3}{2^4}\right| = \frac{3}{16} < 1$

$\boxed{\text{Absolutely convergent.}}$

10. $\sum_{n=1}^{\infty}\frac{(x+1)^n}{\sqrt{n}}$ Ratio test:

$\frac{u_{n+1}}{u_n} = \left|\frac{(x+1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x+1)^n}\right| < 1$

$\left(\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 1\right)$

$|x+1| < 1$

$-1 < x+1 < 1$

$-2 < x < 0$

Test $x = 0$: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ Divergent by p series.

$x = -2$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ Convergent by Alt. series Theorem.

Convergent for $\boxed{-2 \leq x < 0}$

11. $f(x) = \ln x$ $f(a) = \ln 2$

$f'(x) = 1/x$ $f'(a) = 1/2$

$f''(x) = -1/x^2$ $f''(a) = -1/2^2$

$f'''(x) = +2/x^3$ $f'''(a) = +2/2^3$

$f^{IV}(x) = -\frac{3!}{x^4}$ $f^{IV}(a) = -3!/2^4$

$f^{V}(x) = 4!/x^5$ $f^{n}(a) = \frac{(n-1)!(-1)^{n-1}}{2^n}$

Taylor: $\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n-1)!}{2^n}\cdot\frac{(x-2)^n}{n!}$

$= \boxed{\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-2)^n}{n\,2^n}}$

11. con't. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n\,2^n}$

Ratio test: $\left| \dfrac{(x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \dfrac{n\,2^n}{(x-2)^n} \right| < 1$

$$= \left| \frac{(x-2)}{2} \right| < 1$$

$$= |x-2| < 2$$

$$-2 < x-2 < 2$$

$$0 < x < 4$$

End points: $x=0$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-2)^n}{n(2^n)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$ Does not alternate. Diverges by p series.

$x=4$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ = Alternating series. $\lim u_n = 0$, converges, by alt. series thm.

$$\boxed{0 < x \le 4}$$

12. $f(x) = e^{-x}$ Let $b = 1$, $a = 0$

$$f(b) = f(1) = e^{-1} = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f^n(a)}{n!}(b-a)^n, \ldots$$

$f(x) = e^{-x}$ $\qquad f(a) = 1$

$f'(x) = -e^{-x}$ $\qquad f'(a) = -1$

$f''(x) = e^{-x}$ $\qquad f''(a) = 1$

$f'''(x) = -e^{-x}$ $\qquad f'''(a) = -1$

$f^n(x) = (-1)^n e^{-x}$ $\qquad \vdots$

$(b-a)^n = 1$

1.0000 00	1
−1.0000 00	1
.5000 00	1/2
−.1666 67	1/6
.0416 67	1/24
−.0083 33	1/120
.0013 89	1/720
−.0001 98	1/5040
.0000 24	1/40320
.5430 80	
−.1751 98	
.3678 82	

$$\boxed{= .3679}$$