

EXAM 1 B DIFFERENTIAL EQUATIONS Dr. Rapalje

1. Find a differential equation (without constants) for the following equations:

a) $x^2 + y^2 - cx = 0$

b) $y^2 = c_1x + c_2$

Solve each of the following differential equations, subject to indicated conditions (if any!) Name the method of solution used in each case. Give all answers in simplest form.

2. $\frac{dy}{dx} = \frac{3x + xy^2}{y + x^2y}$

3. $\frac{dy}{dx} = \frac{xy}{x - y^2}; y(2) = 1$

4. $y^2 dx = (2xy + x^2) dy$

5. $y' = 3x + 2y$

6. $yy'' = y'$

7. $2xy dx + (x^2 + 1) dy = 0; y(1) = -3$

8. $\frac{dr}{d\theta} = \frac{r^2 \sin \theta}{2r \cos \theta - 1}$

9. $(x^2 - y^2) dx + 2xy dy = 0$

10. $y' \sin x = y \cos x + \sin^2 x$

11. $y' = (x+y)^2$ (Hint: let $v = x+y$)

Thought for today: "But God demonstrates His own love for us, in that while we were yet sinners, Christ died for us." Romans 5:8

Differential Equations Exam 1

1a) $x^2 + y^2 - cx = 0 \quad c = \frac{x^2 + y^2}{x}$
 $2x + 2y y' - c = 0$
 $2x + 2y y' - \frac{x^2 + y^2}{x} = 0$
 $2x^2 + 2xy y' - x^2 - y^2 = 0$
 $2xy y' = y^2 - x^2$
 $y' = \frac{y^2 - x^2}{2xy}$

b) $y^2 = c_1 x + c_2$
 $2y y' = c_1$
 $2y y'' + y'(2y') = 0$
 $y y'' + (y')^2 = 0$

2. Variables Separable:

$$\frac{dy}{dx} = \frac{x(3+y^2)}{y(1+x^2)}$$

$$\int \frac{y dy}{3+y^2} = \int \frac{x dx}{1+x^2}$$

$$\frac{1}{2} \ln(3+y^2) = \frac{1}{2} \ln(1+x^2) + \frac{1}{2} \ln C$$

$$\ln(3+y^2) - \ln(1+x^2) = \ln C$$

$$\ln \frac{3+y^2}{1+x^2} = \ln C$$

$$\frac{3+y^2}{1+x^2} = C$$

$$C(1+x^2) = 3+y^2$$

4. $y^2 dx = (2xy + x^2) dy$
 $y^2 dx - (2xy + x^2) dy = 0$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = -2y - 2x$$

Integrating Factor = $e^{\int \frac{2y+2x}{2xy+x^2} dx}$

$$= e^{-\int \frac{2}{x} dx} = x^{-2} = \frac{1}{x^2}$$

$$x^2 y^2 dx - (2xy + 1) dy = 0$$

$$u_1 = \int x^2 y dx = -x^{-1} y^2 + f(y)$$

$$u_2 = \int (-2x^{-1} y - 1) dy = -x^{-1} y^2 - y + f(x)$$

$$-x^{-1} y^2 - y = -C \quad \text{or} \quad y^2 + xy = Cx$$

also Homogeneous.

3. $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2} \quad y(2) = 1$ Integrating factor

$$2xy dx + (y^2 - x^2) dy = 0$$

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = -2x$$

$$IF = e^{\int \frac{-4x}{2xy} dy} = e^{-\int \frac{2}{y} dy} = \frac{1}{y^2}$$

$$2xy^{-1} dx + (1 - x^2 y^{-3}) dy = 0$$

$$u_1 = x^2 y^{-1} + f(y)$$

$$u_2 = y + x^2 y^{-1} + f(x)$$

$$y + x^2 y^{-1} = C$$

$$y^2 + x^2 = Cy$$

$$y(2) = 1$$

$$1 + 4 = C \Rightarrow C = 5$$

$$y^2 + x^2 = 5y$$

5. $y' = 3x + 2y$ Linear

$$y' - 2y = 3x$$

$$IF = e^{\int -2 dx} = e^{-2x}$$

$$y e^{-2x} = \int 3x e^{-2x} dx$$

$$u = 3x \quad dv = e^{-2x}$$

$$du = 3 \quad v = \frac{e^{-2x}}{-2}$$

$$y e^{-2x} = -\frac{3}{2} x e^{-2x} + \frac{3}{2} \int e^{-2x} dx$$

$$= -\frac{3}{2} x e^{-2x} - \frac{3}{4} e^{-2x} + \frac{3}{4}$$

$$4y = -6x - 3 + C e^{2x}$$

$$4y + 6x + 3 = C e^{2x}$$

6. $y y'' = y'$

variable missing.

Let $v = y'$

$$y'' = v' = \frac{dv}{dy} \frac{dy}{dx}$$

$$y v' = v$$

$$y \left(\frac{dv}{dy} \cdot v \right) = v$$

$$dv = \frac{dy}{y}$$

$$v = \ln y + C_1$$

$$\frac{dy}{dx} = \ln C_1 y$$

$$\frac{dy}{\ln C_1 y} = dx$$

$$x = \int \frac{dy}{\ln C_1 y} + C_2$$

(I blew it!)

$$7. 2xy \, dx + (x^2+1) \, dy = 0$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \text{ Exact.}$$

$$U_1 = \int 2xy \, dx = x^2y + f(y)$$

$$U_2 = \int (x^2+1) \, dy = x^2y + y + f(x)$$

$$\boxed{x^2y + y = C} \quad \text{or} \quad \boxed{y(x^2+1) = C}$$

$$9. (x^2-y^2) \, dx + 2xy \, dy = 0$$

$$\frac{\partial M}{\partial y} = -2y \quad \frac{\partial N}{\partial x} = 2y$$

Integrating factor or Homogeneous.

$$\text{I.F.} = e^{\int \frac{-2y}{2xy} \, dx} = \frac{1}{x^2}$$

$$(1-x^2y^2) \, dx + 2x^{-1}y \, dy = 0$$

$$U_1 = \int (1-x^2y^2) \, dx = x - \frac{x^3y^2}{3} + f(y)$$

$$U_2 = \int 2x^{-1}y \, dy = x^{-1}y^2 + f(x)$$

$$x - \frac{x^3y^2}{3} = C$$

$$\boxed{x^2 + y^2 = Cx}$$

-OR-

Homogeneous

$$\text{Let } y = Vx$$

$$y' = V + V'x$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

$$V + V'x = \frac{1}{2} \left[\frac{y}{x} - \frac{x}{y} \right]$$

$$V + V'x = \frac{1}{2} \left[V - \frac{1}{V} \right]$$

$$V'x = \frac{1}{2} \left(\frac{V^2-1}{V} \right) - V$$

$$V'x = \frac{V^2-1-2V^2}{2V}$$

$$\frac{2V \, dV}{-1-V^2} = \frac{dx}{x} \quad \text{or} \quad \frac{2V \, dV}{V^2+1} = -\frac{dx}{x}$$

$$\ln(V^2+1) = -\ln x + \ln C$$

$$(V^2+1) = \frac{C}{x}$$

$$\frac{y^2}{x^2} + 1 = \frac{C}{x} \quad \text{or} \quad \boxed{y^2 + x^2 = Cx}$$

$$8. \frac{dr}{d\theta} = \frac{r \sin \theta}{2r \cos \theta - 1}$$

$$r^2 \sin \theta \, d\theta + (1-2r \cos \theta) \, dr = 0$$

$$\frac{\partial M}{\partial r} = 2r \sin \theta = \frac{\partial N}{\partial \theta} \text{ Exact.}$$

$$U_1 = \int r^2 \sin \theta \, d\theta = -r^2 \cos \theta + f(r)$$

$$U_2 = \int (1-2r \cos \theta) \, dr = r - r^2 \cos \theta + f(\theta)$$

$$\boxed{r - r^2 \cos \theta = C}$$

$$10. y' \sin x = y \cos x + \sin^2 x$$

$$y' - (\cot x)y = \sin x \text{ Linear.}$$

$$\text{I.F.} = e^{\int -\cot x \, dx} = e^{-\ln \sin x}$$

$$= \frac{1}{\sin x}$$

$$y \frac{1}{\sin x} = \int (\sin x) \frac{1}{\sin x} \, dx$$

$$= x + C$$

$$\frac{y}{\sin x}$$

$$\boxed{y = x \sin x + C \sin x}$$

[If problem was read " $\sin^3 x$ "

$$\text{then } y' - (\cot x)y = \sin^2 x.$$

$$\text{and } y \frac{1}{\sin x} = \int \sin^2 x \frac{1}{\sin x} \, dx$$

$$y \sec x = -\cos x + C.]$$

$$11. y' = (x+y)^2$$

$$\text{Let } v = x+y$$

$$v' = 1 + y'$$

$$v' - 1 = v^2$$

$$v' = v^2 + 1$$

$$\frac{dv}{v^2+1} = dx$$

$$\text{arctan } v = x + C$$

$$v = \tan(x+C)$$

$$\boxed{x+y = \tan(x+C)}$$