DIFFERENTIAL EQUATIONS EXAM 2C Dr. Rapalje

Show all work as necessary on separate paper.

In 1-5, solve the differential equations, solving for constants as necessary.

1. $y'' - 4y = 0$, $y(0) = 1$ $y'(0) = 4$

2. $(D^2 + 4D + 5)y = 0$

3. $y''' + 5y'' - y' - 5y = 0$

4. $y'' + 4y = 2e^x + 4x^2$

5. $y'' + 4y = 4 \sin 2x$

6. Give $(D^3 - 5D^2 - 2D + 24)y = x^2 e^{3x}$, what form should be assumed for y_c? What form should be assumed for y_p? DO NOT SOLVE.

7. Find a family of curves which is orthogonal to $y = cx^3$.

8a) A bacteria population grows at a rate that is proportional to the population P at any time t. Show that the formula for P is $P = P_0 e^{kt}$ where P_0 is the initial population and k is a constant.

b) A population is known to double every 3 hours. If the population initially is 1000 bacteria, when will the population be 10,000?

9. A tank holds 50 gallons of salt water with 10 lb. of salt initially. A solution with 2 lb. of salt per gallon enters at the rate of 5 gal. per minute. The mixture is well-stirred and leaves at the same rate. When will there be 50 lb. of salt in the tank?

10. A body weighs 96 lb. and falls due to gravity opposed by air resistance that is six times the instantaneous velocity. If $V(0) = 20$ ft/sec, find the equations for velocity and position, and the limiting velocity.

DIFFERENTIAL EQUATIONS EXAM 2C Solutions Dr. Rapalje

1. $y''-4y=0 \quad y(0)=1 \quad y'(0)=4$

$m^2-4=0$

$m=\pm 2$

$y_c=c_1e^{2x}+c_2e^{-2x} \Rightarrow 1=c_1+c_2$

$y_c'=2c_1e^{2x}-2c_2e^{-2x} \Rightarrow 4=2c_1-2c_2$

$6=4c_1$

$c_1=3/2 \quad c_2=-1/2$

$\boxed{y=\frac{3}{2}e^{2x}-\frac{1}{2}e^{-2x}}$

2. $(D^2+4D+5)y=0$

$m^2+4m+5=0$

$m=-2\pm i$

$\boxed{y=e^{-2x}(c_1\sin x+c_2\cos x)}$

3. $m^3+5m^2-m-5=0$

$m^2(m+5)-1(m+5)=0$

$(m+5)(m^2-1)=0$

$m=-5 \quad m=\pm 1$

$\boxed{y=c_1e^{-5x}+c_2e^{x}+c_3e^{-x}}$

4. $y''+4y=2e^x+4x^2$

$m^2+4=0$

$y_c=c_1\sin 2x+c_2\cos 2x$

$y_p=ae^x+bx^2+cx+f$

$y_p'=ae^x+2bx+c$

$y_p''=ae^x+2b$

$ae^x+2b+4ae^x+4bx^2+4cx+4f=2e^x+4x^2$

$5ae^x+4bx^2+4cx+2b+4f=2e^x+4x^2$

$5a=2 \quad 4b=4 \quad 4c=0 \quad 2b+4f=0$

$a=2/5 \quad b=1 \quad c=0 \quad 2+4f=0$

$f=-1/2$

$\boxed{y=c_1\sin 2x+c_2\cos 2x+\frac{2}{5}e^x+x^2-1/2}$

5. $(D^2+4)y=4\sin 2x$

$y_c=c_1\sin 2x+c_2\cos 2x$

$y_p=(a\sin 2x+b\cos 2x)x$

$y_p'=(a\sin 2x+b\cos 2x)\cdot 1+x(2a\cos 2x-2b\sin 2x)$

$y_p''=2a\cos 2x-2b\sin 2x+x(-4a\sin 2x-4b\cos 2x)$

$+2a\cos 2x-2b\sin 2x$

$4a\cos 2x-4b\sin 2x-4ax\sin 2x-4bx\cos 2x+4ax\sin 2x$

$+4bx\cos 2x=4\sin 2x$

$4a=0 \quad -4b=4$

$a=0 \quad b=-1$

$\boxed{y=c_1\sin 2x+c_2\cos 2x-x\cos 2x}$

6. $(D^3-5D^2-2D+24)y=x^2e^{3x}$

$m^3-5m^2-2m+24=0$

$m=3$ | $1 \quad -5 \quad -2 \quad 24$ / $3 \quad -6 \quad -24$ / $1 \quad -2 \quad -8 \quad 0$

$m^2-2m-8=0$

$(m-4)(m+2)=0$

$m=4 \quad m=-2$

$y_c=c_1e^{3x}+c_2e^{4x}+c_3e^{-2x}$

If $y_p=ax^2e^{3x}+bxe^{3x}+ce^{3x}$ overlap.

So $\boxed{y_p=cxe^{3x}+bx^2e^{3x}+ax^3e^{3x}}$

7. $y^2=cx^3 \quad c=\frac{y^2}{x^3}$

$2yy'=3cx^2$

$2yy'=3\frac{y^2}{x^3}x^2$

$y'=\frac{3}{2}\frac{y}{x}$

$y'_{o.t.}=-\frac{2x}{3y}=\frac{dy}{dx}$

$-2x\,dx=3y\,dy$

$-x^2=\frac{3}{2}y^2+\frac{c_1}{2}$

$\boxed{2x^2+3y^2+c_1=0}$

8a) $\frac{dP}{dt}=kP$

$\int\frac{dP}{P}=\int k\,dt$

$\ln P=kt+C$

$P=e^{kt+C}$

$P=e^{kt}\cdot e^C$

$\text{At } t=0, P=P_0, \text{ so}$

$P_0=e^C$

$\boxed{P=P_0e^{kt}}$

b) $2P_0=P_0e^{3k}$

$e^{3k}=2$

$3k=\ln 2$

$k=\frac{1}{3}\ln 2$

$=.231$

$P=P_0e^{.231t}$

$10{,}000=1000e^{.231t}$

$10=e^{.231t}$

$.231t=\ln 10$

$t=\frac{\ln 10}{.231}=9.97 \text{ hrs}$

9. $\frac{dA}{dt}=(\text{salt in})-(\text{salt out})$

$=\frac{2\text{lb}}{\text{gal}}\cdot\frac{5\text{gal}}{\text{min}}-\frac{A\text{lb}}{50\text{gal}}\cdot\frac{5\text{gal}}{\text{min}}$

$=10-\frac{1}{10}A$

$\frac{dA}{dt}+\frac{1}{10}A=10 \quad \text{(Linear) IF}=e^{\int\frac{1}{10}dt}=e^{t/10}$

$Ae^{t/10}=\int 10e^{t/10}dt$

$Ae^{t/10}=100e^{t/10}+C$

$A(0)=10 \Rightarrow 10=100+C$

$C=-90$

$\boxed{A=100-90e^{-t/10}}$

$50=100-90e^{-t/10}$

$-50=-90e^{-t/10}$

$\frac{5}{9}=e^{-t/10}$

$\frac{9}{5}=e^{t/10}$

$t/10=\ln 9/5 \Rightarrow t=10\ln 9/5=5.88\text{min.}$

10. $W = 96$ lbs. $R = 6V$

$F_{Total} = F_{Gravity} - F_{AIR\ RESIST.}$

$ma = W - R$

$\frac{W}{g}\frac{dV}{dt} = 96 - 6V$

$\frac{96}{32}\frac{dV}{dt} = 96 - 6V$

$3\frac{dV}{dt} = 96 - 6V$

$\frac{dV}{dt} + 2V = 32$. Linear.

$I.F. = e^{2t}$

$Ve^{2t} = \int 32e^{2t}\,dt$

$Ve^{2t} = 16e^{2t} + C_1$

$V(0) = 20$

$20 = 16 + C_1$, so $C_1 = 4$

$Ve^{2t} = 16e^{2t} + 4$

$V = 16 + 4e^{-2t}$ limiting $V = \lim_{t\to\infty} V = 16$

$\frac{dx}{dt} = V = 16 + 4e^{-2t}$

$x = 16t - 2e^{-2t} + C_2$

If $x(0) = 0$, then,

$x = 0 = 0 - 2 + C_2$

$C_2 = 2$

$x = 16t - 2e^{-2t} + 2$