DIFF. EQ. EXAM 2 D

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1. Find the general solutions:

(8 pts) a) $(D^3 - 6D^2 + 9D)y = 0$

(8 pts) b) $(D^4 + 2D^2 - 8)y = 0$

(8 pts) c) A diff. eq. with roots $m = -1, -1, 1 \pm 2i, 1 \pm 2i$

(10 pts) d) $(D^2 + 2D + 1)y = 2\cos 2x + 3e^x$

2. Find the assumed form of y_c and y_p. Do NOT SOLVE.

(8 pts) a) $(D^2 - 2D + 1)y = 2\cos 2x + 3e^x$

(8 pts) b) $(D^4 + D^2)y = 3x^2 - 4e^x$

(8 pts) c) $(D^2 + 1)y = e^{-x}\cos x + 2x$

3. Given a family of curves $y^3 = c_1 x$, determine the differential
(10 pts) equation for the family, and the equation of the family of curves that is orthogonal to it.

4. A paratrooper and his parachute together weigh 160 lb. The air
(10 pts) resistance is twice the velocity at any time t. If he begins from rest, find the equation of velocity and position. Find the limiting velocity.

5. The population of a colony of bacteria grows at a rate proportional to the number present. If y = population at any time t, and
(10 pts) A_0 is the population when $t = 0$, a) show that $y = A_0 e^{kt}$, (where k is the proportionality constant). b) How long will it take the population to triple if it doubles in 3 hours.

6. A 100 gallon tank is filled with brine containing 60 lb. of salt. Salt water containing 3 lb. of salt per gallon enters the tank at
(10 pts) 4 gallons per minute. The brine, kept well stirred, runs out at the same rate.

a) Find the equation for the amount of salt at any time t.

b) How much salt will be in the tank after 1 hour?

1a) $(D^3-6D^2+9D)y=0$

$m(m-3)^2=0$

$y=c_1+c_2e^{3x}+c_3xe^{3x}$

b) $(D^4+2D^2-8)y=0$

$(m^2+4)(m^2-2)=0$

$m=\pm 2i \quad m=\pm\sqrt{2}$

$y=c_1\sin 2x+c_2\cos 2x+c_3e^{\sqrt{2}x}+c_4e^{-\sqrt{2}x}$

c) $m=-1,-1,1\pm 2i,\ 1\pm 2i$

$y=c_1e^{-x}+c_2xe^{-x}+e^x(c_3\sin 2x+c_4\cos 2x)+xe^x(c_5\sin 2x+c_6\cos 2x)$

d) $(D^2+2D+1)y=2\cos 2x+3e^x$

$(m+1)^2=0$

$y_c=c_1e^{-x}+c_2xe^{-x}$

$y_p=a\cos 2x+b\sin 2x+fe^x$

$y_p'=-2a\sin 2x+2b\cos 2x+fe^x$

$y_p''=-4a\cos 2x-4b\sin 2x+fe^x$

$$\begin{array}{l}-4a\cos 2x-4b\sin 2x+fe^x\\ -4a\sin 2x+4b\cos 2x+2fe^x\\ +a\cos 2x+b\sin 2x+fe^x=2\cos 2x+3e^x\end{array}$$

$=(-3a+4b)\cos 2x+(-3b-4a)\sin 2x+4fe^x$

$-3a+4b=2 \qquad 4f=3$

$-4a-3b=0 \qquad f=3/4$

$a=-\frac{6}{25},\ b=\frac{8}{25}$

$y=c_1e^{-x}+c_2xe^{-x}+\frac{3}{4}e^x-\frac{6}{25}\cos 2x+\frac{8}{25}\sin 2x$

2a) $(D^2-2D+1)y=2\cos 2x+3e^x$

$y_c=c_1e^x+c_2xe^x$

$y_p=ae^x+b\cos 2x+c\sin 2x$

$y_p=ax^2e^x+b\cos 2x+c\sin 2x$

b) $(D^4+D^2)y=3x^2-4e^x$

$m^2(m^2+1)=0$

$y_c=c_1+c_2x+c_3\sin x+c_4\cos x$

$y_p=ax^2+bx+c+fe^x$

$y_p=ax^4+bx^3+cx^2+fe^x$

c) $(D^2+1)y=e^{-x}\cos x+2x$

$y_c=c_1\sin x+c_2\cos x$

$y_p=Ae^{-x}\cos x+Be^{-x}\sin x+Fx+G$ (optimal)

3. $y^3=c_1x$

$3y^2dy=c_1dx$

$3y^2y'=c_1$

$3y^2y'=\frac{y^3}{x}$

$y'=\frac{y}{3x}$

$y'_{O.T.}=-\frac{3x}{y}$

$ydy=-3xdx$

$\frac{y^2}{2}=-\frac{3x^2}{2}+\frac{c}{2}$

$3x^2+y^2=c$

4. $m\frac{dv}{dt}=mg-kv \qquad \left(m=\frac{w}{g}\right)$

$\frac{160}{32}\frac{dv}{dt}=160-2v$

$5\frac{dv}{dt}=160-2v$

$\frac{dv}{dt}+\frac{2}{5}v=32$

Linear I.F. $=e^{\frac{2}{5}t}$

$ve^{\frac{2}{5}t}=\int 32e^{\frac{2}{5}t}dt$

$ve^{\frac{2}{5}t}=32\cdot\frac{5}{2}e^{\frac{2}{5}t}+c_1$

$ve^{\frac{2}{5}t}=80e^{\frac{2}{5}t}+c_1$

$v=0$ at $t=0$, so $c_1=-80$

$v=80-80e^{-\frac{2}{5}t}$

$x=80t-80\left(-\frac{5}{2}\right)e^{-\frac{2}{5}t}+c_2$

$0=0+200+c_2 \qquad c_2=-200$

$x=80t+200e^{-\frac{2}{5}t}-200$ ft.

$\lim_{t\to\infty} v=80$ ft/sec.

5a) Given $\frac{dy}{dt}=ky$

$\frac{dy}{y}=k\,dt$

$\ln y=kt+c$

$y=e^{kt+c}$

$y=e^{kt}\cdot e^c$

$y=A_0$ when $t=0$.

$A_0=e^0e^c$ so $e^c=A_0$

$y=A_0e^{kt}$

b) $y=2A_0$ at $t=3$ hrs.

$2A_0=A_0e^{3k}$

$2=e^{3k}$

$\ln 2=\ln e^{3k}=3k$

$k=\frac{\ln 2}{3}=.231$

$3A_0=A_0e^{.231t}$

$\ln 3=.231t$

$t=\frac{\ln 3}{.231}=4.75$ hrs.

6. $\frac{dA}{dt}=(\text{Rate in})-(\text{Rate out})$

Rate in $=\frac{3\text{ lb}}{\text{gal}}\cdot\frac{4\text{ gal}}{\text{min}}=12$ lb/min.

Rate out $=\frac{A\text{ lb}}{100\text{ gal}}\cdot\frac{4\text{ gal}}{\text{min}}=\frac{A}{25}$ lb/min.

$\frac{dA}{dt}=12-\frac{A}{25}$

$\frac{dA}{dt}+\frac{A}{25}=12$

$Ae^{\frac{1}{25}t}=12\int e^{\frac{1}{25}t}dt$

$Ae^{\frac{1}{25}t}=300e^{\frac{1}{25}t}+c$

$t=0,\ A=60 \Rightarrow 60=300+c$

$A=300-240e^{-\frac{1}{25}t}$

at $t=1$ hr. $=60$ minutes.

$A=300-240e^{-\frac{1}{25}(60)}$

$A=278.23$ lb.