MS 235 Diff Equations Exam 3A Dr. Rapalje

Find the general solution for the differential equations:

1. $y''' - y' = 0$

2. $(D^3 - 2D^2 - D + 2)y = 0$

3. $(4D^3 - 3D + 1)y = 0$

4. $y'' - 6y' + 25y = 0$

5. $y''' - y'' - 2y' = e^{-x}$

6. $y'' + y = \sec x$

7. $y'' + 4y = x^2 + \cos x$

8. $(D^3 - 2D^2 - 3D)y = 3x^2 + \sin x$

9. $x^2y'' + xy' + 4y = x$
(Let $x = e^z$)

E.C. (1) $D^6 y = 729y$

(2) $(D-1)^2 y = \dfrac{e^x}{1-x^2}$

Thought for today: This know also, that in the last days perilous times shall come. For men shall be lovers of their own selves, lovers of money, boasters, proud, blasphemers, disobedient to parents, unthankful, unholy, ... lovers of pleasure more than lovers of God; having a form of godliness, but denying the power thereof. II Tim 3:1-5.

1. $y''' - y' = 0$

$m^3 - m = 0$

$m(m^2-1) = 0$

$m(m-1)(m+1) = 0$

$\boxed{y = c_1 + c_2 e^{x} + c_3 e^{-x}}$

2. $(D^3 - 2D^2 - D + 2)y = 0$

$m^2(m-2) - (m-2) = 0$

$(m^2-1)(m-2) = 0$

$(m-1)(m+1)(m-2) = 0$

$\boxed{y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x}}$

3. $(4D^3 - 3D + 1)y = 0$

$$\begin{array}{cccc|l} 4 & 0 & -3 & 1 & \tfrac{1}{2} \\ & 2 & 1 & -1 & \\ \hline 4 & 2 & -2 & 0 & \end{array}$$

$4m^2 + 2m - 2 = 0$

$2m^2 + m - 1 = 0$

$(2m-1)(m+1) = 0$

$m = \tfrac{1}{2} \quad m = -1$

$\boxed{y = c_1 e^{-x} + c_2 e^{\frac{1}{2}x} + c_3 x e^{\frac{1}{2}x}}$

4. $y'' - 6y' + 25y = 0$

$m^2 - 6m + 25 = 0$

$m^2 - 6m + 9 = -16$

$(m-3) = \pm 4i$

$m = 3 \pm 4i$

$\boxed{y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x)}$

5. $y''' - y'' - 2y' = e^{-x}$

$(m^3 - m^2 - 2m) = 0$

$m(m^2 - m - 2) = 0$

$m(m-2)(m+1) = 0$

$m = 0 \quad m = 2 \quad m = -1$

$y_c = c_1 + c_2 e^{-x} + c_3 e^{2x}$

$y_p = Axe^{-x}$

$y_p' = -Axe^{-x} + Ae^{-x}$

$y_p'' = +Axe^{-x} - Ae^{-x} - Ae^{-x}$

$= Axe^{-x} - 2Ae^{-x}$

$y_p''' = -Axe^{-x} + Ae^{-x} + 2Ae^{-x}$

$= -Axe^{-x} + 3Ae^{-x}$

$\cancel{-Axe^{-x}} + 3Ae^{-x} - \cancel{Axe^{-x}} + \cancel{2Ae^{-x}} + \cancel{2Axe^{-x}} - \cancel{2Ae^{-x}} = e^{-x}$

$A = \tfrac{1}{3}$

$\boxed{y = c_1 + c_2 e^{-x} + c_3 e^{2x} + \tfrac{1}{3} x e^{-x}}$

6. $y'' + y = \sec x$

$y_c = c_1 \sin x + c_2 \cos x$

Let $y = A(x)\sin x + B(x)\cos x$

$y' = A(x)\cos x + A'(x)\sin x - B(x)\sin x + B'(x)\cos x$

$A'(x)\sin x + B'(x)\cos x = 0$

$y'' = -A(x)\sin x + A'(x)\cos x - B(x)\cos x - B'(x)\sin x$

$\cancel{-A\sin x} + A'\cos x - \cancel{B\cos x} - B'\sin x + \cancel{A\sin x} + \cancel{B\cos x} = \sec x$

$A'(x)\sin x + B'(x)\cos x = 0$ mult. by $\sin x$

$A'(x)\cos x - B'(x)\sin x = \sec x$ mult by $\cos x$

$A'(x)\sin^2 x + A'(x)\cos^2 x = 1$

$A'(x) = 1$

$A(x) = x + c_1$

Then $\sin x + B'(x)\cos x = 0$

$B'(x) = -\tan x$

$B(x) = -\int \tan x\, dx$

$= \ln|\cos x| + c_2$

$\boxed{y = c_1 \sin x + c_2 \cos x + x\sin x + \cos x \ln|\cos x|}$

7. $y'' + 4y = x^2 + \cos x$

$y_c = c_1 \sin 2x + c_2 \cos 2x$

$y_p = ax^2 + bx + c + f\cos x + g\sin x$

$y_p' = 2ax + b + (-f\sin x) + g\cos x$

$y_p'' = 2a - f\cos x - g\sin x$

$2a - f\cos x - g\sin x + \underline{4ax^2} + \underline{4bx} + 4c + 4f\cos x + 4g\sin x = \underline{x^2} + \cos x$

$4a = 1 \quad 4b = 0 \quad 2a + 4c = 0 \quad 3f = 1 \quad 3g = 0$

$a = \tfrac{1}{4} \quad b = 0 \quad \tfrac{1}{2} + 4c = 0 \quad f = \tfrac{1}{3} \quad g = 0$

$c = -\tfrac{1}{8}$

$\boxed{y = c_1 \sin 2x + c_2 \cos 2x + \tfrac{1}{4}x^2 - \tfrac{1}{8} + \tfrac{1}{3}\cos x}$

8. $(D^3-2D^2-3D)y = 3x^2 + \sin x$

$m(m-3)(m+1)=0$

$y_c = c_1 + c_2e^{3x} + c_3e^{-x}$

$y_p = ax^2 + Bx + C + f\sin x + g\cos x$

Avoid repetition, so

$y_p = ax^3 + Bx^2 + Cx + f\sin x + g\cos x$

$y_p' = 3ax^2 + 2Bx + C + f\cos x - g\sin x$

$y_p'' = 6ax + 2B - f\sin x - g\cos x$

$y_p''' = 6a - f\cos x + g\sin x$

$6a - f\cos x + g\sin x - 12ax - 4b + 2f\sin x + 2g\cos x - 9ax^2 - 6bx - 3c - 3f\cos x + 3g\sin x = 3x^2 + \sin x$

$-9a = 3$ $\quad a = -\frac{1}{3}$

$-12a - 6b = 0$ $\quad 4 - 6b = 0$ $\quad b = \frac{2}{3}$

$6a - 4b - 3c = 0$ $\quad -2 - \frac{8}{3} - 3c = 0$ $\quad -\frac{14}{3} = 3c$ $\quad c = -\frac{14}{9}$

$\left.\begin{array}{l} -4f + 2g = 0 \\ 2f + 4g = 1 \end{array}\right\}$ $\quad -2f + g = 0$, $\quad 2f + 4g = 1$, $\quad 5g = 1$, $\quad g = \frac{1}{5}$

$2f = g$ $\quad f = \frac{1}{10}$

$$\boxed{y = c_1 + c_2e^{3x} + c_3e^{-x} - \tfrac{1}{3}x^3 + \tfrac{2}{3}x^2 - \tfrac{14}{9}x + \tfrac{1}{5}\cos x + \tfrac{1}{10}\sin x.}$$

9. $x^2y'' + xy' + 4y = x$ $\qquad x = e^z,\ xy' = \frac{dy}{dz}$ $\quad x^2y'' = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ $\quad z = \ln x$

$\frac{d^2y}{dz^2} - \frac{dy}{dz} + \frac{dy}{dz} + 4y = e^z$

$\frac{d^2y}{dz^2} + 4y = e^z$

$y_c = c_1\cos 2z + c_2\sin 2z$

$y_p = Ae^z$

$y_p' = Ae^z$

$y_p'' = Ae^z$

$Ae^z + 4Ae^z = e^z$

$5Ae^z = e^z$

$A = \frac{1}{5}$

$$\boxed{y = c_1\cos(2\ln x) + c_2\sin(2\ln x) + \tfrac{1}{5}x}$$

E.C. 1. $D^6y - 729y = 0$

$m^6 - 729 = 0$

$(m^3-27)(m^3+27) = 0$

$(m-3)(m^2+3m+9)(m+3)(m^2-3m+9) = 0$

$m = \pm 3$, $\quad m = \frac{-3\pm\sqrt{9-36}}{2}$ $\qquad m = \frac{3\pm\sqrt{9-36}}{2}$

$m = -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$ $\qquad m = \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$

$y = c_1e^{3x} + c_2e^{-3x} + e^{-3/2x}\left(c_3\cos\frac{3\sqrt{3}}{2}x + c_4\sin\frac{3\sqrt{3}}{2}x\right) + e^{3/2x}\left(c_5\cos\frac{3\sqrt{3}}{2}x + c_6\sin\frac{3\sqrt{3}}{2}x\right)$

2. $(D-1)^2y = \frac{e^x}{1-x^2}$

$y_c = c_1e^x + c_2xe^x$ $\qquad A' + B'x = 0$

$y = Ae^x + Bxe^x$

$y' = Ae^x + A'e^x + Bxe^x + Be^x + B'xe^x$

$y'' = Ae^x + A'e^x + Bxe^x + Be^x + B'xe^x + Be^x + B'e^x$

$= (A+2B)e^x + Bxe^x + B'e^x$

$(D^2-2D+1)y = \frac{e^x}{1-x^2}$

$(\cancel{A} + \cancel{2B} + \cancel{Bx} + B' - \cancel{2A} - \cancel{2Bx} - \cancel{2B} + \cancel{A} + \cancel{Bx})e^x = \frac{e^x}{1-x^2}$

$B' = \frac{1}{1-x^2}$ $\qquad A' = -B'x = \frac{x}{x^2-1}$

$B = \frac{1}{2}\log\frac{1+x}{1-x} + c_2$ $\quad A = \frac{1}{2}\log|x^2-1| + c_1$

$$\boxed{y = c_1e^x + c_2xe^x + \tfrac{1}{2}\log|x^2-1|e^x + \tfrac{1}{2}x\log\tfrac{1+x}{1-x}e^x}$$