IFFERENTIAL EQUATIONS EXAM 4 B Dr. Rapalje

1. A spring is stretched 3 inches by an 8 lb. weight.
A 4 lb. weight is placed on the spring, the spring is pulled
(12) 6 inches below the equilibrium position and given an upward initial velocity of 3 ft/sec. Assuming no damping force, find the equation of motion.

2. A spring has differential equation $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+25X=\sin t$.
(6) a) Without solving for the constants, give the general form (i.e., X_c+X_p) of the equation of motion.
(3) b) What type of motion is this? Does damping occur?
(2) c) Identify transient and steady state solutions.

3. Given a weight of 6 lb. on a spring with spring constant 8, and a damping force 3.5 V, where $X(0)=\frac{1}{3}$ and $X'(0)=0$.
(9) a) Find the equation of motion.
(3) b) What type of motion occurs?
(5) 3 c) What damping force would have caused critical damping?

4. A spring has differential equation $\frac{d^2X}{dt^2}+6X=F(t)$ where $F(t)$ is an external force that produces resonance.
(3) a) Give an example of a force $F(t)$ that would produce resonance.
(7) b) Without solving for constants, give X_c and X_p.

5. A spring has equation of motion $X=\frac{1}{3}\sin 3t-\frac{1}{4}\cos 3t$.
(8) Find the amplitude, period, and frequency.

6. In an electric circuit, an inductor of .1 henry, a resistor of 4 ohms, and a .02 farad capacitor are in series with an emf of $10\sin 5t$.
(3) a) Write a differential equation for charge Q at any time t.
(6) b) Without solving for constants, give the general form of the equation Q
(2) c) Identify steady state and transient terms.
(3) d) What emf would have produced resonance in the circuit?

(12) 7. Solve by Euler method: $x^2y''-3xy'+3y=\ln x$

(12) 8. Show that if $x=e^z$, then a) $x\frac{dy}{dx}=\frac{dy}{dz}$ and b) $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$.
c) What is $x^3\frac{d^3y}{dx^3}$?

(4) 9. There are many parallels between the oscillating spring problems and electric circuit problems. Match the following: capacitor, inductor, battery, resistor
mass, damping constant, spring constant, external force,

(100 total)

Diff. Eq. Exam 4 B Solutions — Dr. Rapalje

1. $f = kd$
$8 = k\frac{1}{4}$
$k = 32$
$x(0) = \frac{1}{2}$
$x'(0) = -3$

$\frac{W}{g}x'' + kx = 0$
$\frac{4}{32}x'' + 32x = 0$
$x'' + 256x = 0$
$m^2 + 256 = 0$
$m = \pm 16i$
$x = c_1 \sin 16t + c_2 \cos 16t$
$\frac{1}{2} = c_1 \cdot 0 + c_2 \quad c_2 = \frac{1}{2}$
$x' = 16c_1 \cos 16t - 16c_2 \sin 16t$
$-3 = 16c_1 + 0 \quad c_1 = -\frac{3}{16}$
$x = -\frac{3}{16}\sin 16t + \frac{1}{2}\cos 16t$

3. $\frac{W}{g}x'' + \beta x' + kx = 0$
a) $\frac{3}{16}x'' + \frac{7}{2}x' + 8x = 0$
$3x'' + 56x' + 128x = 0$
$3m^2 + 56m + 128 = 0$
$(3m+8)(m+16) = 0$
$m = -8/3 \quad m = -16$
$x = c_1 e^{-8/3 t} + c_2 e^{-16t}$
$\frac{1}{3} = c_1 + c_2$
$x' = -8/3\, c_1 e^{-8/3 t} - 16c_2 e^{-16t}$
$0 = -8/3\, c_1 - 16c_2$
$\frac{16}{3} = 16c_1 + 16c_2$
$\frac{16}{3} = \frac{40}{3}c_1; \quad c_1 = \frac{2}{5}$
$c_2 = \frac{1}{3} - \frac{2}{5} = -\frac{1}{15}$
$x = \frac{2}{5}e^{-8/3 t} - \frac{1}{15}e^{-16t}$

b) Motion is overdamped.

c) $\frac{3}{16}m^2 + \beta m + 8 = 0$
where β = damping force.
Critical damping occurs where $b^2 - 4ac = 0$ and causes a double root.
$b^2 - 4ac = \beta^2 - 4 \cdot \frac{3}{16} \cdot 8 = 0$
$\beta^2 - 6 = 0$
$\beta = \pm\sqrt{6}$

2. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = \sin t$.
$m^2 + 6m + 25 = 0$
$m^2 + 6m + 9 = -16$
$m = -3 \pm 4i$
$x = \underbrace{e^{-3t}(c_1 \sin 4t + c_2 \cos 4t)}_{\text{Transient}} + \underbrace{a \sin t + b \cos t}_{\text{steady state}}$
Motion is oscillatory, but damped.

4. $\frac{d^2x}{dt^2} + 6x = F(t)$
$m^2 + 6 = 0$
$x_c = c_1 \sin\sqrt{6}t + c_2 \cos\sqrt{6}t$
Resonance by $F(t) = a \sin\sqrt{6}t$
or $F(t) = a\cos\sqrt{6}t$
$x_p = at \sin\sqrt{6}t + bt\cos\sqrt{6}t$

5. $x = \frac{1}{3}\sin 3t - \frac{1}{4}\cos 3t$
$x = c \sin(3t + \phi)$
where $c = \sqrt{a^2 + b^2}$ = Amplitude.
$= \sqrt{\frac{1}{9} + \frac{1}{16}}$
$= \sqrt{\frac{25}{9 \cdot 16}}$ $\quad \frac{5}{12}$ = Amp.
Period $= \frac{2\pi}{3}$ $\quad f = \frac{3}{2\pi}$

6. $.1Q'' + 4Q' + \frac{1}{.02}Q = 10 \sin 5t$
$.1Q'' + 4Q' + 50Q = 10 \sin 5t$
$Q'' + 40Q' + 500Q = 100 \sin 5t$
$m^2 + 40m + 400 + 100 = 0$
$(m+20)^2 = -100$
$m = -20 \pm 10i$
$Q_c = e^{-20t}(c_1 \sin 10t + c_2 \cos 10t)$

$Q_p = a \sin 5t + b \cos 5t$
$Q_p' = 5a \cos 5t - 5b \sin 5t$
$Q_p'' = -25a \sin 5t - 25b \cos 5t$

$-25a \sin 5t - 25b\cos 5t + 200a \cos 5t - 200b \sin 5t + 500a \sin 5t + 500b \cos 5t = 100 \sin 5t$
$475a - 200b = 100 \quad 17(17a - 8b = 100)$
$200a + 475b = 0 \quad 8(8a + 17b = 0)$
$225a = 1700$
$a = \frac{68}{9} \quad b = -\frac{8a}{17}$
$= -\frac{8}{17} \cdot \frac{68}{9} = -\frac{32}{9}$
$Q = \overbrace{e^{-20t}(c_1 \sin 10t + c_2 \cos 10t)}^{\text{Transient}} + \underbrace{\frac{68}{9}\sin 5t - \frac{32}{9}\cos 5t}_{\text{Steady}}$

d) EMF $= e^{-20t}\sin 10t$ or $e^{-20t}\cos 10t$ produces resonance.

7. $x^2y'' - 3xy' + 3y = \ln x$
$x = e^z \quad z = \ln x$
$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 3\frac{dy}{dz} + 3y = z$
$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 3y = z$.
$m^2 - 4m + 3 = 0$
$m = 3, m = 1$
$y_c = c_1 e^{3z} + c_2 e^z$
$y_c = c_1x^3 + c_2x$

$y_p = az + b$
$y_p' = a$
$y_p'' = 0$
$-4a + 3az + 3b = z$
$3a = 1 \quad -4a + 3b = 0$
$a = 1/3 \quad 3b = 4a$
$b = 4/9$

$y = c_1x^3 + c_2x + \frac{1}{3}\ln x + 4/9$

8. $x = e^z$
a) $dx = e^z dz$
$\frac{dz}{dx} = e^{-z} = \frac{1}{x}$
$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$
$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$
$x\frac{dy}{dx} = \frac{dy}{dz}$

b) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$
$= \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx}$
$= \frac{d}{dz}\left(\frac{dy}{dz}e^{-z}\right)e^{-z}$
$= \left[\frac{dy}{dz}(-e^{-z}) + e^{-z}\frac{d^2y}{dz^2}\right]e^{-z}$
$= \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)e^{-2z} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)$
$= \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2$

9. inductor = mass; resistor = damping
capacitor = spring const; battery = ext. force