EXAM 4C DIFFERENTIAL EQUATIONS Dr. Rapalje

Show all work on separate paper.
You may use "EXAM 2 Formula Sheet", Integral Tables handout, Integral formulas and Laplace Transform tables in back of text.

$$\frac{W}{g}\frac{d^2X}{dt^2} + \beta\frac{dx}{dt} + kX = 0$$

1. A 10 lb. weight stretches a spring 6 inches. The weight is pulled 4 inches below its equilibrium position and released. Find the equation of motion, the amplitude, and the frequency. (Assume no damping)

2. A spring has a spring constant of $\frac{75}{4}$, a damping force equal to 3 times the instantaneous velocity, and a 6 lb. weight on it. The weight is lifted 6 inches above the equilibrium position and given an upward velocity of 2 ft/sec.
   a) Find the equation of position X in terms of t.
   b) Find the time varying amplitude and quasi period.
   c) Is the motion oscillatory, critically damped, or overdamped.
   d) Give an example of a force F(t) which would cause the system to resonate.

3. A spring, with a 6 lb. weight on it, has a damping force 1.5 times the instantaneous velocity. a) For what spring constant would there be critical damping? b) Using that spring constant, find the equation of motion. c) What type of motion would result if the spring constant were increased? d) What motion would result if the damping force were increased?

4. A resistor of 50 ohms, an inductor of 2 henries, and a .005 farad capacitor are in series with an emf of e^{-2t}. At t=0, the charge is 4 coulombs and current is 0. Find the equations for Q and I in terms of t.

DIFFERENTIAL EQUATIONS EXAM 4 C Solutions Dr. Rapalje

1. $f = kd$

$10 = k\frac{1}{2}$

$k = 20$

$x(0) = \frac{1}{3}$

$v(0) = 0$

$\frac{10}{32}\frac{d^2x}{dt^2} + 20x = 0$

$\frac{d^2x}{dt^2} + 64x = 0$

$x = c_1 \cos 8t + c_2 \sin 8t$

$\frac{1}{3} = c_1$

$v = -8c_1 \sin 8t + 8c_2 \cos 8t$

$0 = 0 + 8c_2$

$c_2 = 0$

$f = \frac{8}{2\pi} = \frac{4}{\pi}$

$x = \frac{1}{3}\cos 8t$

$Amp = \frac{1}{3}$

2. $\frac{6}{32}x'' + 3x' + \frac{75}{4}x = 0 \qquad x(0) = -\frac{1}{2} \quad v(0) = -2$

$3x'' + 48x' + 300x = 0$

$3(m^2 + 16m + 100) = 0$

$m^2 + 16m + 64 + 36 = 0$

$(m+8)^2 = -36$

$m = -8 \pm 6i$

$x = e^{-8t}(c_1 \sin 6t + c_2 \cos 6t)$

$-\frac{1}{2} = c_2$

$v = e^{-8t}(6c_1 \cos 6t - 6c_2 \sin 6t) - 8e^{-8t}(c_1 \sin 6t + c_2 \cos 6t)$

$-2 = 6c_1 - 8(-\frac{1}{2}) \qquad c_1 = -1$

a) $x = -e^{-8t}(\sin 6t + \frac{1}{2}\cos 6t)$

$x = -e^{-8t}\sqrt{1^2 + \frac{1}{2}^2}\sin(6t + \phi)$

$= -\frac{\sqrt{5}}{2}e^{-8t}\sin(6t + \phi)$

b) time varying

$amp = \frac{\sqrt{5}}{2}e^{-8t}$

$QP = \frac{2\pi}{6} = \frac{\pi}{3}$

c) Oscillatory

d) $F(t) = e^{-8t}\sin 6t$ or $F(t) = e^{-8t}\cos 6t$, etc.

3. $\frac{6}{32}x'' + \frac{3}{2}x' + kx = 0$

$3x'' + 24x' + 16kx = 0$

$3m^2 + 24m + 16k = 0$

$m = \frac{-24 \pm \sqrt{24^2 - 4(3)(16k)}}{6}$

a) critical damping occurs if $b^2 - 4ac = 0$

$24^2 - 192k = 0$

$k = 3$

b) $m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$x = c_1 e^{-4t} + c_2 t e^{-4t}$

or $x = e^{-4t}(c_1 + c_2 t)$

c) if $k > 3$, the discriminant would be negative; complex roots and hence oscillatory motion.

d) if damping force increases real roots and overdamped motion result.

4. $R = 50 \qquad L = 2 \qquad C = .005$

$\frac{1}{C} = \frac{1}{.005} = 200$

$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E$

$2\frac{d^2Q}{dt^2} + 50\frac{dQ}{dt} + 200Q = e^{-2t}$

$2m^2 + 50m + 200 = 0$

$m^2 + 25m + 100 = 0$

$(m+20)(m+5) = 0$

$m = -20 \quad m = -5$

$Q_c = c_1 e^{-20t} + c_2 e^{-5t}$

$Q_p = ae^{-2t}$

$Q_p' = -2ae^{-2t}$

$Q_p'' = 4ae^{-2t}$

$8ae^{-2t} - 100ae^{-2t} + 200ae^{-2t} = e^{-2t}$

$108ae^{-2t} = e^{-2t}$

$a = 1/108$

$Q_p = \frac{1}{108}e^{-2t}$

$Q = c_1 e^{-20t} + c_2 e^{-5t} + \frac{1}{108}e^{-2t}$

$Q(0) = 4 = c_1 + c_2 + \frac{1}{108}$

$Q' = -20c_1 e^{-20t} - 5c_2 e^{-5t} - \frac{1}{54}e^{-2t}$

$Q'(0) = 0 = -20c_1 - 5c_2 - \frac{1}{54}$

$20 = 5c_1 + 5c_2 + \frac{5}{108}$

$0 = -20c_1 - 5c_2 - \frac{2}{108}$

$20 = -15c_1 + 3/108$

$\frac{2157}{108(-15)} = c_1 = -\frac{719}{540} = -1.33$

$c_1 + c_2 = \frac{431}{108}$

$c_2 = \frac{431}{108} + \frac{719}{540} = 5.32$

$Q(t) = -1.33e^{-20t} + 5.32e^{-5t} + .01e^{-2t}$

$I(t) = 26.6e^{-20t} - 26.6e^{-5t} - .02e^{-2t}$