Step 1: Determine quantity to be maximized or minimized. Express in terms of other variables. (Primary equation) Draw figure.

Step 2: Find a relationship among the variables (secondary equation). Solve for one variable in terms of the other.

Step 3: Use secondary equation to express primary equation in terms of one variable.

Step 4: Take derivative of primary equation, set equal to zero, solve for variable, be sure answer is "reasonable, be sure whether max or min, answer the question.

3. Maximize $P = xy$, given $x + 2y = 24$

$x = 24 - 2y$.

$P = (x)y$

$P = (24-2y)\cdot y$

$P = 24y - 2y^2$

$\frac{dP}{dy} = 24 - 4y = 0$

$y = 6$

$x + 2y = 24$

$x + 2(6) = 24$

$x = 12$

$\frac{d^2P}{dy^2} = -4$ $\therefore$ Max, by 2nd deriv. test.

Maximum Product $= 12 \cdot 6 = 72$

4. Minimize $P = xy$ where $x - y = 5$

$x = y + 5$

$P = (y+50)y$

$P = y^2 + 50y$

$\frac{dP}{dy} = 2y + 50 = 0$

$y = -25$

$x - y = 50$

$x - (-25) = 50$

$x = 25$

$\frac{d^2P}{dy^2} = +2$ $\therefore$ Min. by 2nd de test

Minimum Prod $= 25 \cdot (-25) = -62$

6. Maximize $P = xy$ where $x + 2y = 100$

$x = 100 - 2y$.

$P = (100-2y)y$

$P = 100y - 2y^2$

$\frac{dP}{dy} = 100 - 4y = 0$

$y = 25$

$x + 2y = 100$

$x + 50 = 100$

$x = 50$

8. Minimize $S = x + 3y$,

where $xy = 192 \Rightarrow y = \frac{192}{x}$

$S = x + 3y$

$S = x + 3\left(\frac{192}{x}\right)$

$S = x + 576x^{-1}$

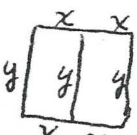
$\frac{dS}{dx} = 1 - 576x^{-2} = 0$

$1 = \frac{576}{x^2}$

$x^2 = 576$

$y = \frac{192}{x} = \frac{192}{24} = 8$

$\frac{d^2S}{dx^2} = 576 \cdot 2x^{-3} > 0$ Rel.

9. 

(See #64 on page 47)

Maximize Area $= 2xy$ where $4x + 3y = 200'$

$x = 50 - \frac{3}{4}y$

$A = 2\left(50 - \frac{3}{4}y\right)y$

$= 100y - \frac{3}{2}y^2$

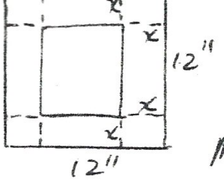
$\frac{dA}{dy} = 100 - 3y = 0$

$y = \frac{100}{3}'$

$x = 50 - \frac{3}{4}\left(\frac{100}{3}\right)$

$x = 50 - 25$

$x = 25'$

17. 

Let x = amt cut from corner.

Base side $= 12-2x$

Height $= x$

Maximize Volume $= lwh$

$$= x(12-2x)^2$$

(see #65 page 47)

$$V = x(144-48x+4x^2)$$

$$V = 144x-48x^2+4x^3$$

$$\frac{dV}{dx} = 144-96x+12x^2 = 0$$

$$12(x^2-8x+12)=0$$

$$12(x-6)(x-2)=0$$

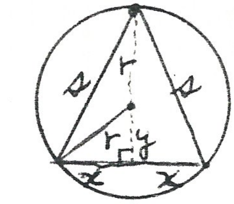
$$x=6 \text{ or } x=2$$

$\boxed{\text{Base} = 12-2x = 0 \quad \text{Reject}}$ $\qquad = 12-4$

Base $= 8$

Volume $= 8\cdot 8\cdot 2 = 128 \text{ cu in}$

16. r = constant



Triangle:

Base $= 2x$

Height $= y+r$

Maximize Area $= \frac{1}{2} b\cdot h$; $\quad x^2+y^2=r^2$

$$= \frac{1}{2}(2x)(y+r) \qquad x=\sqrt{r^2-y^2}$$

$$= \left(\sqrt{r^2-y^2}\right)(y+r)$$

$$\frac{dA}{dy} = \sqrt{r^2-y^2}\cdot 1 + (y+r)\tfrac{1}{2}(r^2-y^2)^{-1/2}(-2y)$$

$$= (r^2-y^2)^{-1/2}\left[(r^2-y^2) + -y(y+r)\right]$$

$$= (r^2-y^2)^{-1/2}\left[r^2-y^2-y^2-yr\right]$$

$$= (r^2-y^2)^{-1/2}(r^2-ry-2y^2) = 0$$

$$(r-2y)(r+y) = 0$$

$r = 2y \qquad r=-y$ Reject.

$$y = \frac{r}{2}$$

$$x = \sqrt{r^2-\frac{r^2}{4}} = \sqrt{\frac{3r^2}{4}}$$

$$= \frac{\sqrt{3}\,r}{2}$$

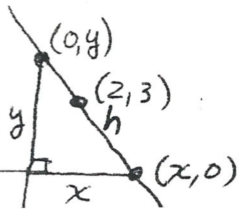
Base $= \sqrt{3}\,r$ $\qquad$ Height $= y+r$

$$h = \frac{3r}{2}$$

Equilateral

$$s = \sqrt{x^2+h^2}$$

$$= \sqrt{\frac{3r^2}{4}+\frac{9r^2}{4}} = \sqrt{3r^2} = \sqrt{3}\,r$$

19. 

Minimize Area $= \dfrac{xy}{2}$, where slope $= \dfrac{y-3}{0-2} = \dfrac{3-0}{2-x}$

$$(y-3)(2-x) = -6$$

$$y-3 = \frac{-6}{2-x}$$

$$y = \frac{6}{x-2}+3$$

$$y = \frac{6+3x-6}{x-2}$$

$$y = \frac{3x}{x-2}$$

$$A = \frac{x}{2}\cdot\frac{3x}{x-2} = \frac{3}{2}\left[\frac{x^2}{x-2}\right]$$

$$\frac{dA}{dx} = \frac{3}{2}\left[\frac{(x-2)\cdot 2x - x^2\cdot 1}{(x-2)^2}\right]$$

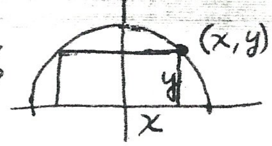
$$= \frac{3}{2}\left[\frac{2x^2-4x-x^2}{(x-2)^2}\right]$$

$$= \frac{3}{2}\left[\frac{x^2-4x}{(x-2)^2}\right] = 0$$

$\cancel{x=0}$ Reject $\quad$ or $x=4$

$$y = \frac{3\cdot 4}{4-2} = 6$$

22. (See #66 p.47)



$x^2+y^2=r^2$, r = constant.

$$y = \sqrt{r^2-x^2}$$

Maximize $A = 2xy$

$$A = 2x\sqrt{r^2-x^2}$$

$$\frac{dA}{dx} = 2\left[x\tfrac{1}{2}(r^2-x^2)^{-1/2}(-2x) + \sqrt{r^2-x^2}\cdot 1\right]$$

$$= 2(r^2-x^2)^{-1/2}\left[-x^2+r^2-x^2\right] = 0$$

$$r^2 = 2x^2$$

$$x = r/\sqrt{2}$$

$$y = \sqrt{r^2-x^2}$$

$$y = \sqrt{r^2-\frac{r^2}{2}}$$

$$= \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}}$$

Base $= 2x = 2\cdot\dfrac{r\sqrt{2}}{\sqrt{2}\sqrt{2}} = r\sqrt{2}$

25. Minimize surface area of cylinder.

$$S.A. = 2\pi r^2 + 2\pi r h, \qquad V = \pi r^2 h = V_0, \text{ where } V_0 = \text{constant}.$$

$$h = \frac{V_0}{\pi r^2}$$

$$= 2\pi r^2 + 2\cancel{\pi} r \cdot \frac{V_0}{\cancel{\pi} r^2}$$

$$= 2\pi r^2 + 2V_0 r^{-1}$$

$$\frac{d(S.A.)}{dr} = 4\pi r - 2V_0 r^{-2} = 0$$

$$r^{-2}(4\pi r^3 - 2V_0) = 0$$

$$r^3 = \frac{V_0}{2\pi}$$

$$r = \sqrt[3]{\frac{V_0}{2\pi}}$$

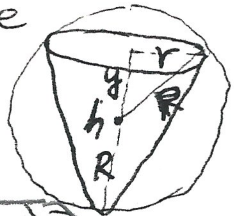
If $V_0 = 12$ oz.

$V_0 = 12 \cdot (1.80469)$ cu. in.

$V_0 = 21.656$ cu. in.

then $r = \sqrt[3]{\frac{21.656}{2\pi}} \doteq 1.51$ in.

29. Maximize Volume of cone



$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (y + R)$$

$$= \frac{1}{3}\pi\left[r^2\left(\sqrt{R^2 - r^2} + R\right)\right]$$

$y^2 + r^2 = R^2$

$y = \sqrt{R^2 - r^2}$

$r^2 = R^2 - y^2$

$$= \frac{1}{3}\pi\left[r^2\sqrt{R^2 - r^2} + r^2 R\right]$$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2 \tfrac{1}{2}(R^2 - r^2)^{-1/2}(-2r) + \sqrt{R^2 - r^2}\cdot 2r + 2rR\right]$$

$$= \frac{1}{3}\pi r\left[\frac{-r^2}{(R^2 - r^2)^{1/2}} + 2(R^2 - r^2)^{1/2} + 2R\right]$$

$$= \frac{1}{3}\pi r\left[\frac{-r^2 + 2(R^2 - r^2) + 2R(R^2 - r^2)^{1/2}}{(R^2 - r^2)^{1/2}}\right] = 0$$

$$-r^2 + 2R^2 - 2r^2 + 2R\sqrt{R^2 - r^2} = 0$$

$$2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$

$$4R^2(R^2 - r^2) = 9r^4 - 12r^2R^2 + 4R^4$$

$$\cancel{4R^4} - 4R^2r^2 = 9r^4 - 12r^2R^2 + \cancel{4R^4}$$

$$0 = 9r^4 - 8r^2R^2$$

$$r^2(9r^2 - 8R^2) = 0$$

$$r^2 = \frac{8R^2}{9}$$

$$r = \frac{2R\sqrt{2}}{3}$$

Hard Way!

$r^2 = \frac{8R^2}{9}$

$h = \frac{4R}{3}$

$V = \frac{1}{3}\pi r^2 h$

$=$

Easier to write $r^2 = R^2 - y^2$

$$V = \frac{1}{3}\pi(R^2 - y^2)(y + R)$$

$$\frac{dV}{dy} = \frac{1}{3}\pi\left[(R^2 - y^2)\cdot 1 + (y + R)(-2y)\right]$$

$$\left[R^2 - y^2 - 2y^2 - 2Ry\right]$$

$$(R^2 - 2Ry - 3y^2) = 0$$

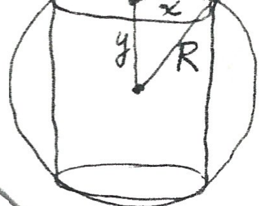
$$(R + y)(R - 3y) = 0$$

$R = 3y \quad y = \frac{R}{3}$

$$y = \sqrt{R^2 - r^2} = \sqrt{R^2 - \frac{8R^2}{9}} = \sqrt{\frac{R^2}{9}} = \frac{R}{3};\ h = \frac{4R}{3}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot \frac{8R^2}{9} \cdot \frac{4R}{3} = \frac{\pi 32R^3}{27}$$

30. Maximize $V=\pi r^2 h$



$V=\pi x^2 \cdot 2y$

$= 2\pi x^2\sqrt{R^2-x^2}$

$x^2+y^2=R^2$

$y=\sqrt{R^2-x^2}$

$\frac{dV}{dx}=2\pi\left[x^2\frac{1}{\cancel{2}}(R^2-x^2)^{-1/2}(-\cancel{2}x)+\sqrt{R^2-x^2}\cdot 2x\right]$

$=2\pi(R^2-x^2)^{-1/2}x\left[-x^2+2(R^2-x^2)\right]=0$

$x=0$ Reject.

$-x^2+2R^2-2x^2=0$

$2R^2=3x^2$

$x=\sqrt{\frac{2}{3}}R$ or $\frac{R\sqrt{6}}{3}$

$y=\sqrt{R^2-\frac{2R^2}{3}}$

$=\sqrt{\frac{R^2}{3}}=\frac{R\sqrt{3}}{3}$

HARD WAY!

$V=2\pi x^2 y$

$=2\pi\frac{2R^2}{3}\cdot\frac{R\sqrt{3}}{3}=\frac{4\pi R^3\sqrt{3}}{9}$

31.

$V=\frac{4}{3}\pi x^3+\pi x^2 y=12$

Ends Center.

$4\pi x^3+3\pi x^2 y=36$

$y=\frac{36-4\pi x^3}{3\pi x^2}$

Minimize

$S.A.=4\pi x^2+2\pi x y$

$S.A.=4\pi x^2+2\pi x\left(\frac{36-4\pi x^3}{3\pi x^2}\right)$

$=4\pi x^2+\frac{72\pi x}{3\pi x^2}-\frac{8\pi^2 x^4}{3\pi x^2}$

$=4\pi x^2+24x^{-1}-\frac{8}{3}\pi x^2$

$\frac{d(S.A.)}{dx}=8\pi x-24x^{-2}-\frac{16}{3}\pi x=0$

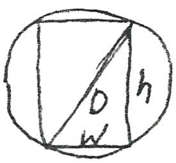
$8x^{-2}\left[\pi x^3-3-\frac{2\pi}{3}x^3\right]=0$

$\frac{\pi x^3}{3}=3$

$x^3=\frac{9}{\pi}$ $x=\sqrt[3]{\frac{9}{\pi}}$

(See #69 p. 48)

37. $S=kwh^2$



$S=kh^2\sqrt{D^2-h^2}$

$w^2+h^2=D^2$

$w=\sqrt{D^2-h^2}$

Same as #30.

$h=\frac{D\sqrt{6}}{3}$ $w=\frac{D\sqrt{3}}{3}$

$h=\frac{24\sqrt{6}}{3}$ $w=\frac{24\sqrt{3}}{3}$

$h=8\sqrt{6}$ $w=8\sqrt{3}$

39. Minimize time

$t=t_{Rowing}+t_{walking}$

$=\frac{D}{r}+\frac{D}{r}$

$=\frac{\sqrt{x^2+4}}{2}+\frac{\sqrt{(3-x)^2+1^2}}{4}$

$=\frac{1}{2}(x^2+4)^{1/2}+\frac{1}{4}(x^2-6x+10)^{1/2}$

$\frac{dt}{dx}=\frac{1}{4}(x^2+4)^{-1/2}2x+\frac{1}{8}(x^2-6x+10)^{-1/2}(2x-6)$

$=\frac{x}{2\sqrt{x^2+4}}+\frac{x-3}{4\sqrt{x^2-6x+10}}=0$

$\frac{-x}{2\sqrt{x^2+4}}=\frac{x-3}{4\sqrt{x^2-6x+10}}$

$\cancel{16}^{4}\cdot\frac{x^2}{\cancel{4}(x^2+4)}=\frac{\cancel{16}\cdot x^2-6x+9}{\cancel{16}(x^2-6x+10)}$

$4x^2(x^2-6x+10)=(x^2+4)(x^2-6x+9)$

$4x^4-24x^3+40x^2=x^4-6x^3+13x^2-24x+36$

$3x^4-18x^3+27x^2+24x-36=0$

$x^4-6x^3+9x^2+8x-12=0$

1	1	-6	9	8	-12
	↓	1	-5	4	12
	1	-5	4	12	0

$x=1$

CALCULUS WORD PROBLEMS.

p. 216. 4. Let x = 1st no.

y = 2nd no.

Given $xy = 192$

$$y = \frac{192}{x} \text{ or } 192x^{-1}$$

Minimize $f(x) = x + 3y$

$$f(x) = x + 3(192x^{-1})$$

$$f'(x) = 1 - 576x^{-2} = 0$$

$$1 = 576x^{-2} = \frac{576}{x^2}$$

$$x^2 = 576$$

$$x = 24 \qquad y = \frac{192}{24} = 8$$

$$f''(x) = 1152x^{-3} = \frac{1152}{x^3}$$

$f''(24) > 0$ Minimum.

6. Let x = 1st y = 2nd

Given: $x + 2y = 100$

$$x = 100 - 2y$$

Maximize $f(x,y) = xy$

$$f(y) = (100-2y)\cdot y$$

$$= 100y - 2y^2$$

$$f'(y) = 100 - 4y = 0$$

$$y = 25$$

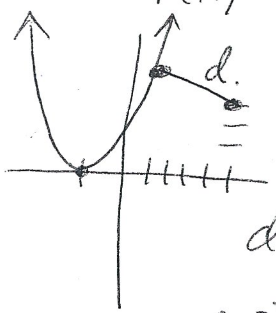
$$x = 100 - 2(25)$$

$$y = 50$$

$f''(y) = -4 < 0$

Maximum.

14. $f(x) = (x+1)^2$ (5, 3)



Minimize d^2 where

d = dist from $(x, (x+1)^2)$ to (5,3)

$$d^2 = (x-5)^2 + \left[(x+1)^2 - 3\right]^2$$

$$(d^2)' = 2(x-5)\cdot 1 + 2\left[(x+1)^2 - 3\right]' 2(x+1)\cdot 1$$

$$= 2x - 10 + 4(x+1)(x^2 + 2x + 1 - 3)$$

$$= 2x - 10 + (4x+4)(x^2 + 2x - 2)$$

$$= 2x - 10 + 4x^3 + 8x^2 - \cancel{8x} + 4x^2 + \cancel{8x} - 8 = 0$$

use calculator techniques to establish that $x = 1$ is a root! $\leftarrow$ $4x^3 + 12x^2 + 2x - 18 = 0$

$$\begin{array}{r|rrrr} 1 & 4 & 12 & 2 & -18 \\ & \downarrow & 4 & 16 & 18 \\ \hline & 4 & 16 & 18 & 0 \end{array}$$

$$4x^2 + 16x + 18 = 0$$

$$2(x^2 + 8x + 9) = 0$$

$$x = 1$$

$$y = (x+1)^2$$

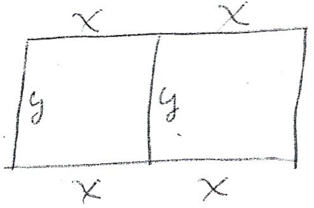
$$y = (1+1)^2$$

$$y = 4$$

$$(1, 4)$$

p246

18.



Given $4x+3y=200$

$y=\frac{200-4x}{3}$

MAXIMIZE Area

$f(x,y)=2xy$

$f(x)=2x\left(\frac{200-4x}{3}\right)$

$=\frac{8}{3}x(50-x)$

$=\frac{8}{3}(50x-x^2)$

$f'(x)=\frac{8}{3}(50-2x)=0$

$x=25$ ft.

$y=\frac{200-4x}{3}$

$y=\frac{100}{3}$ ft.

$f''(x)=\frac{8}{3}(-2)<0$ MAX!

24. $y=\frac{6-x}{2}$

Maximum Area $=xy$

$f(x)=x\left(\frac{6-x}{2}\right)$

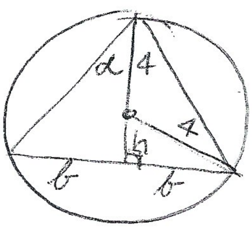
$=\frac{6x-x^2}{2}=3x-\frac{1}{2}x^2$

$f'(x)=3-x=0$

$x=3$ $\quad y=\frac{6-3}{2}=\frac{3}{2}$

$f''(x)=-1<0$ Max!

26.



$r=4$

$b^2+h^2=16$

$b^2=16-h^2$

$b=\pm\sqrt{16-h^2}$

a) $A=\frac{1}{2}\underset{\text{Base}}{(2b)}\cdot\underset{\text{Height}}{(h+4)}$

$=\frac{1}{\cancel{2}}\cancel{2}\sqrt{16-h^2}(h+4)$

$=(h+4)\sqrt{16-h^2}$

$A'=(h+4)\frac{1}{\cancel{2}}(16-h^2)^{-1/2}\cdot(-\cancel{2}h)+\sqrt{16-h^2}\cdot 1$

$=(16-h^2)^{-1/2}\left[-h(h+4)+16-h^2\right]$

$=(16-h^2)^{-1/2}\left[-h^2-4h+16-h^2\right]$

$=(16-h^2)^{-1/2}(-2)(h^2+2h-8)$ $\rightarrow (h+4)(h-2)=0$

$h=2$ $\quad b=\sqrt{16-4}$

$A=\frac{1}{2}$ Base$\cdot$Height $\quad 3\sqrt{3}$

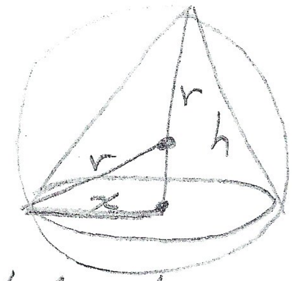
$=\frac{1}{2}(2\cdot 3\sqrt{2})\cdot 6=18\sqrt{3}$

p.218.

35. $V = \frac{4}{3}\pi r^3$ (Sphere)

Maximize

$V = \frac{1}{3}\pi x^2 h$



$V = \frac{1}{3}\pi h\left[r^2 - (h-r)^2\right]$

$= \frac{1}{3}\pi h(2rh - h^2)$

$= \frac{1}{3}\pi(2rh^2 - h^3)$

$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2) = 0$

$\frac{1}{3}\pi h(4r - 3h) = 0$

$h = 0 \quad h = \frac{4r}{3}$

$V_{CONE} = \frac{1}{3}\pi x^2 h$

$= \frac{1}{3}\pi \cdot \frac{8r^2}{9} \cdot \frac{4r}{3}$

$= \frac{32\pi r^3}{81}$

Relationship between x + h:

$x^2 + (h-r)^2 = r^2$

$x^2 = r^2 - (h-r)^2$

$= r^2 - (h^2 - 2rh + r^2)$

$= \cancel{r^2} - h^2 + 2rh - \cancel{r^2}$

$x^2 = -\frac{16}{9}r^2 + 2 \cdot r \cdot \frac{4r}{3}$

$= -\frac{16r^2}{9} + \frac{8r^2 \cdot 3}{3 \cdot 3} = \frac{8r^2}{9}$

35. $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right)\right] = \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0$$

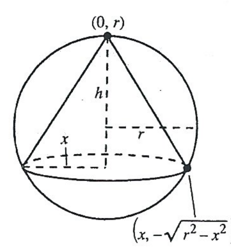
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$



By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$