Miscellaneous Exercises

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We have in this chapter encountered various methods of solving first-order differential equations. It is natural for the student to wonder which of these methods can be expected to arise most often in practice so that appropriate emphasis may be given.

The following list, in which we present those methods of primary importance and those of secondary importance, may serve as a useful guide.

A. *Primary Importance*

1. *Separation of variables:* $f(x)dx + g(y)dy = 0$.
   Integrate to obtain the required solution $\int f(x)dx + \int g(y)dy = c$.
2. *Homogeneous equations:* $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$.
   Let $y = vx$, where v is a new dependent variable depending on x, and thus reduce the equation to the separation of variables type.
3. *Linear equations:* $\dfrac{dy}{dx} + P(x)y = Q(x)$.
   Multiply both sides by the integrating factor $\mu = e^{\int P\,dx}$ so that the equation can be written $\dfrac{d}{dx}(\mu y) = \mu Q$. Then integrate to obtain the required solution $\mu y = \int \mu Q \, dx + c$.
4. *Exact equations:* $M\,dx + N\,dy = 0$, where $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.
   Write the equation as $dU = 0$ and integrate to obtain the required solution $U(x, y) = c$.

B. *Secondary Importance*

1. *Equations which can be made exact:* See pages 48–54. This includes 1 and 3 above.
2. *Transformation of variables:* This involves special devices suggested by the particular form of the differential equation and often falls in the category of "ingenious devices." Method 2 given above is included here.
3. *Miscellaneous techniques*, such as the method of inspection and the Clairaut equation.

The following exercises are intended to serve as a review of the various methods.

MISCELLANEOUS EXERCISES ON CHAPTER 2

A EXERCISES

Solve each of the following differential equations subject to given conditions, if any.

1. $(x^2 + 1)(y^3 - 1)dx = x^2y^2\,dy.$
2. $(y^2 + 2xy)dx + (x^2 + 2xy)dy = 0.$
3. $(x^2 + 2xy)dx + (y^2 + 2xy)dy = 0.$
4. $\dfrac{dy}{dx} + \dfrac{2y}{x} = x^2.$
5. $(3 - y)dx + 2x\,dy = 0;\ y(1) = 1.$
6. $\dfrac{dy}{dx} + 2x = 2.$
7. $s^2t\,ds + (t^2 + 4)dt = 0.$
8. $2xyy' + x^2 + y^2 = 0.$
9. $\dfrac{dy}{dx} = \dfrac{2x^2 - ye^x}{e^x}.$
10. $x^2y' + xy = x + 1.$
11. $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan^{-1}\dfrac{y}{x}.$
12. $\dfrac{dy}{dx} = x + y.$
13. $y' + xy = x^3.$
14. $(3 - x^2y)y' = xy^2 + 4.$
15. $r^2 \sin\phi\, d\phi = (2r\cos\phi + 10)dr.$
16. $y' = x^2 + 2y.$
17. $y' = \dfrac{2xy - y^4}{3x^2}.$
18. $(x^2 + y^2)dx + 2y\,dy = 0;\ y(0) = 2.$
19. $(x^2 + y^2)dx + (2xy - 3)dy = 0.$
20. $y'(2x + y^2) = y.$

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21. $u^2v\,du - (u^3 + v^3)dv = 0.$
22. $(\tan y - \tan^2 y\cos x)dx - x\sec^2 y\,dy = 0.$
23. $\dfrac{dy}{dx} = \dfrac{x + 2y}{y - 2x}.$
24. $y'\sin x = y\cos x + \sin^2 x.$
25. $(x^2 - y^2)dx + 2xy\,dy = 0.$
26. $(2x^2 - ye^x)dx - e^x\,dy = 0.$
27. $(x + y)y' = 1.$
28. $(x + 2y)dx + x\,dy = 0.$
29. $\sin y\,dx + (x\cos y - y)dy = 0.$
30. $y' = e^{y/x} + \dfrac{y}{x}.$
31. $\sin x\cos y\,dx + \cos x\sin y\,dy = 0.$
32. $xy' = x^3 + 2y.$
33. $(3xy^2 + 2)dx + 2x^2y\,dy = 0.$
34. $(2y^2 - x)dy + y\,dx = 0.$
35. $y'' = y' + 2x.$
36. $(1 + y)y' = x\sqrt{y}.$
37. $\tan x\sin y\,dx + 3\,dy = 0.$
38. $x\,dy - y\,dx = x\cos\left(\dfrac{y}{x}\right)dx.$
39. $\dfrac{ds}{dt} = \sqrt{\dfrac{1 - t}{1 - s}};\ s = 0$ where $t = 1.$
40. $(2y + 3x)dx + x\,dy = 0.$
41. $x^2y\,dx + (1 + x^3)dy = 0.$
42. $(\sin y - x)y' = 2x + y;\ y(1) = \dfrac{\pi}{2}.$
43. $\dfrac{dN}{dt} = -\alpha N;\ N = N_0$ at $t = 0.$
44. $\dfrac{dy}{dx} = \dfrac{y(x + y)}{x(x - y)}.$
45. $\dfrac{dI}{dt} + I = e^t.$
46. $xy' + y = x^2;\ y(1) = 2.$
47. $x\,dy - y\,dx = x^2y\,dy.$
48. $\dfrac{dq}{dp} = \dfrac{p}{q}e^{p^2 - q^2}.$

49. $(3y\cos x + 2)y' = y^2 \sin x;\ y(0) = -4.$

50. $(x + x\cos y)dy - (y + \sin y)dx = 0.$

51. $y' = 3x + 2y.$

52. $y^2\,dx = (2xy + x^2)dy.$

53. $\dfrac{dr}{d\phi} = \dfrac{r(1 + \ln\phi)}{\phi(1 + \ln r)};\ \phi = e^2$ where $r = e.$

54. $\dfrac{dU}{dt} = -a(U - 100t);\ U(0) = 0.$

55. $(uv - 2v)du + (u - u^2)dv = 0.$

56. $\dfrac{dI}{dt} + 3I = 10\sin t;\ I(0) = 0.$

57. $\dfrac{ds}{dt} = \dfrac{1}{s + t + 1}.$

58. $yy'' + (y')^2 = 0.$

59. $x\sqrt{1 - y^2} + yy'\sqrt{1 - x^2} = 0.$

60. $y' + (\cot x)y = \cos x.$

61. $y' = \left(\dfrac{y + 3}{2x}\right)^2.$

62. $xy' - 3y = x^4 e^{-x}.$

63. $y' = \sin x \tan y.$

64. $y' = \dfrac{x}{y} + \dfrac{y}{x}.$

65. $x\,dy - y\,dx = 2x^2y^2\,dy.$

66. $xy' + y\ln x = y\ln y + y.$

67. $y' = 2 - \dfrac{y}{x}.$

68. $xy'' + y' = 1.$

69. $\dfrac{dI}{dt} = \dfrac{It^2}{t^3 - I^3}.$

70. $(e^y + x + 3)y' = 1.$

71. $\dfrac{dr}{d\phi} = e^\phi - 3r;\ r = 1$ at $\phi = 0.$

72. $yy'' = (y')^2.$

73. $x^4y''' + 1 = 0.$

74. $\dfrac{dy}{dx} = \dfrac{x + 3y}{x - 3y}.$

75. $y'\cos x = y - \sin 2x.$

76. $e^{2x-y}\,dx + e^{y-2x}\,dy = 0.$

77. $r^3\dfrac{dr}{d\phi} = \sqrt{a^8 - r^8}.$

78. $(2x^2 - ye^x)dx - e^x\,dy = 0.$

79. $x\,dy + 2y\,dx - x\cos x\,dx = 0.$

80. $\sqrt{1 + x^3}\,\dfrac{dy}{dx} = x^2y + x^2.$

81. $(3y^2 + 4xy)dx + (2xy + x^2)dy = 0.$

82. $y' = y(x + y).$

83. $y' = x(x + y).$

84. $\dfrac{d^2U}{dr^2} + \dfrac{1}{r}\dfrac{dU}{dr} = 4(1 - r);\ U = 15, \dfrac{dU}{dr} = 0$ at $r = 1.$

85. $\dfrac{dy}{dx} = 1 - (x - y)^2;\ y(0) = 1.$

86. $\dfrac{dy}{dx} = \dfrac{e^{x-y}}{y}.$

B EXERCISES

1. Solve $xyy' + y^2 = \sin x$ by letting $y^2 = u$.

2. Show that $\sin^{-1} x + \sin^{-1} y = c_1$ and $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = c_2$ are general solutions of

$$\sqrt{1 - y^2}\,dx + \sqrt{1 - x^2}\,dy = 0$$

Can one of these solutions be obtained from the other?

3. (a) Solve $(1 + x^2)dy + (1 + y^2)dx = 0$, given that $y(0) = 1$. (b) Show that $y = (1 - x)/(1 + x)$ is a solution. Reconcile this with the solution obtained in (a).

4. Solve $y' = 2/(x + 2y - 3)$ by letting $x + 2y - 3 = v$.

5. Solve $y' = \sqrt{y + \sin x} - \cos x$. (*Hint:* Let $\sqrt{y + \sin x} = v$.)

6. Solve $y' = \tan(x + y)$.

7. Solve $y' = e^{x+3y} + 1$.

MISCELLANEOUS EXERCISES, CHAPTER TWO

A EXERCISES, p. 65

1. $x - x^{-1} = \frac{1}{3}\ln(y^3 - 1) + c.$
2. $x^2y + y^2x = c.$
3. $x^2 + xy + y^2 = c(x + y).$
4. $5x^2y = x^5 + c.$
5. $(3 - y)^2 = 4x.$
6. $y = 2x - x^2 + c.$
7. $2s^3 + 3t^2 + 24\ln t = c.$
8. $x^3 + 3xy^2 = c.$
9. $3ye^x - 2x^3 = c.$
10. $xy = x + \ln x + c.$
11. $\ln x = \int \frac{dv}{\tan^{-1} v} + c$, where $v = \frac{y}{x}.$
12. $x + y + 1 = ce^x.$
13. $y = x^2 - 2 + ce^{-x^2/2}.$
14. $\frac{1}{2}x^2y^2 + 4x - 3y = c.$
15. $r^2\cos\phi + 10r = c.$
16. $y = ce^{2x} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}.$
17. $x^2y^{-3} - x = c.$
18. $x^2 + y^2 - 2x + 2 = 6e^{-x}.$
19. $3xy^2 + x^3 - 9y = c.$
20. $y^2\ln y - x = cy^2.$
21. $u^3v^{-3} - 3\ln v = c.$
22. $x\cot y - \sin x = c.$
23. $x^2 - y^2 + 4xy = c.$
24. $y = (x + c)\sin x.$
25. $x^2 + y^2 = cx.$
26. $2x^3 - 3ye^x = c.$
27. $y - \ln(x + y + 1) = c.$
28. $x^3 + 3x^2y = c.$
29. $y^2 - 2x\sin y = c.$
30. $e^{-y/x} + \ln x = c.$
31. $\cos x\cos y = c.$
32. $y = x^3 + cx^2.$
33. $x^3y^2 + x^2 = c.$
34. $xy^{-1} + 2y = c.$
35. $y = c_1e^x - x^2 - 2x + c_2.$
36. $2y^{1/2} + \frac{2}{3}y^{3/2} = \frac{x^2}{2} + c.$
37. $\sec x(\csc y - \cot y)^3 = c.$
38. $\sec\frac{y}{x} + \tan\frac{y}{x} = cx.$
39. $(1 - s)^{3/2} - (1 - t)^{3/2} = 1.$
40. $x^2y + x^3 = c.$
41. $y^3(1 + x^3) = c.$
42. $2xy + 2\cos y + 2x^2 = \pi + 2.$
43. $N = N_0e^{-at}.$
44. $xye^{x/y} = c.$
45. $I = ce^{-t} + \frac{1}{2}e^t.$
46. $3xy = x^3 + 5.$
47. $xy^2 - 2y = cx.$
48. $e^{p^2} - e^{q^2} = c.$
49. $y^3\cos x + y^2 + 48 = 0.$
50. $y + \sin y = cx.$
51. $y = ce^{2x} - \frac{3}{2}x - \frac{3}{4}.$
52. $y^2 = x(c - y).$
53. $2\ln r + (\ln r)^2 = 2\ln\phi + (\ln\phi)^2 - 5.$
54. $U = 100t - \frac{100}{a}(1 - e^{-at}).$
55. $(u - 1)v = cu^2.$
56. $I = 3\sin t - \cos t + e^{-3t}.$
57. $s - \ln(s + t + 2) = c.$
58. $y^2 = c_1x + c_2.$
59. $\sqrt{1 - x^2} + \sqrt{1 - y^2} = c.$
60. $2y\sin x + \cos^2 x = c.$
61. $(y + 3)^{-1} = (4x)^{-1} + c.$
62. $y = cx^3 - x^3e^{-x}.$
63. $\ln\sin y + \cos x = c.$
64. $y^2 = 2x^2\ln x + cx^2.$
65. $2xy^3 - 3y = cx.$
66. $\ln y = cx + \ln x.$
67. $xy - x^2 = c.$
68. $y = x + c_1\ln x + c_2.$
69. $t^3I^{-3} + 3\ln I = c.$
70. $x = ye^y - 3 + ce^y.$
71. $r = \frac{1}{4}e^{\phi} + \frac{3}{4}e^{-3\phi}.$
72. $y = ae^{bx}.$
73. $6xy = c_1x^3 + c_2x^2 + c_3x + 1.$
74. $\ln(x^2 + 2xy + 3y^2) = 2\sqrt{2}\tan^{-1}\left(\frac{x + 3y}{x\sqrt{2}}\right) + c.$
75. $y(\sec x - \tan x) = 2\ln(1 + \sin x) - 2\sin x + c.$
76. $e^{4x} + 2e^{2y} = c.$
77. $r^4 = a^4\sin(4\phi + c).$
78. $2x^3 - 3ye^x = c.$
79. $x^2y = (x^2 - 2)\sin x + 2x\cos x + c.$
80. $2\sqrt{1 + x^3} = 3\ln(y + 1) + c.$
81. $x^3y^2 + x^4y = c.$
82. $y^{-1} = -e^{-x^2/2}\int e^{x^2/2}\,dx + ce^{-x^2/2}.$
83. $y = e^{x^2/2}\int x^2e^{-x^2/2}\,dx + ce^{x^2/2}.$

Ans. 84. $U = 12 + 4r - r^2 - 2\ln r.$

85. $(y - x)(x + 1) = 1.$
86. $(y - 1)e^y - e^x = c.$

B EXERCISES, p. 67

1. $x^2y^2 = 2\sin x - 2x\cos x + c.$
4. $x + 2y + 1 = ce^{y/2}.$
5. $\sqrt{y + \sin x} = \frac{x}{2} + c.$
6. $\sin(x + y) + \cos(x + y) = ce^{x-y}.$