MATHEMATICAL LOGIC

by

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LOGIC NOTES

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I. A **SIMPLE STATEMENT** is a statement that is either **"TRUE"** or **"FALSE"**. We frequently use letters such as p and q to represent simple statements.

II. Two or more simple statements may be combined using connective words such as **"AND"** and **"OR."** The symbol **"$\wedge$"** represents **"AND"** and is called a **"CONJUNCTION."** In a truth table, **"$p \wedge q$" is true only if "p" and "q" are <u>BOTH</u> true.** The symbol **"$\vee$"** represents **"OR"** and is called a **"DISJUNCTION."** In a truth table, **"$p \vee q$" is true if either "p" or "q" or "both" are true.**

III. The symbol **"$\sim$"** is used to represent the **negation** of a statement. "It is NOT true that . . ."

IV. Frequently one statement is a consequence of another statement. In this situation, the symbol **"$\rightarrow$" represents an "IMPLICATION."** For example, **"$p \rightarrow q$" means "p implies q"** or **"if p, then q."** In this case, **"p" is the "HYPOTHESIS" or the "ANTECEDENT,"** and **"q" is the "CONCLUSION" or the "CONSEQUENT." In a truth table, "$p \rightarrow q$" is always TRUE unless "p" is true <u>AND</u> "q" is false.**

In the case of $p \rightarrow q$, much the same as the idea of innocent unless proven guilty, we define $p \rightarrow q$ to be true unless it is proven to be false. Consider the statement:
"If you make an A in this course, then I will give you $100."
Relative to this statement, there are four cases:

Case 1: You make an A, and I give you $100. Obviously, in this case the statement is TRUE.
Case 2: You make an A, and I do NOT give you $100. You met the condition, yet I did not keep my promise to give you the $100. The statement is FALSE.
Case 3: You do not make an A, and I give you $100 anyway.
Case 4: You do not make an A, and I do not give you $100.

In these last two cases, you did not make an A, so I did not go back on my promise. The statement did not prove to be false, so it is considered true by default.

V. **BICONDITIONAL: "$p \leftrightarrow q$" means that p and q are equivalent statements.** In a truth table, $p \leftrightarrow q$ is true if p and q have the same truth value, false if they have the opposite value. **$p \leftrightarrow q$** may be read **"p is equivalent to q," "p if and only if q,"** or **"p iff q."**

VI. NEGATIONS OF "ALL, SOME, NONE" STATEMENTS:

Statement		Negation
All are	←------→	Some are not
None are	←------→	Some are

Examples: Statement	Negation
A. All teachers are nice.	Some teachers are not nice.
B. Some teachers are nice.	No teachers are nice.
C. Some students are not happy.	All students are happy.
D. No students are happy.	Some students are not happy.

VII. DE MORGAN'S LAWS:

$$\sim(p \wedge q) = \sim p \vee \sim q$$
$$\sim(p \vee q) = \sim p \wedge \sim q$$

VIII. FORMS OF CONDITIONAL STATEMENTS: Given "$p \to q$"

A. CONVERSE $q \to p$ NOT equivalent to $p \to q$

B. INVERSE $\sim p \to \sim q$ NOT equivalent to $p \to q$

C. CONTRAPOSITIVE $\sim q \to \sim p$ IS EQUIVALENT to $p \to q$

IX. SUMMARY OF TRUTH VALUES.

A. $\sim p$ = Opposite truth value of p

B. $p \wedge q$ = True only in the case that both "p" and "q" are true. Otherwise, $p \wedge q$ is false.

C. $p \vee q$ = True if "p" or "q" or both are true. False only in the case that both "p" and "q" are false.

D. $p \to q$ = True, except for the case in which p is true and q is false.

E. $p \leftrightarrow q$ = True if "p" and "q" are both true or both false. False if one is true and the other is false.

X. "DISJUNCTION" TO "IF-THEN"

With "p or q," negating one implies the other.

$p \vee q = \sim p \to q$ or $p \vee q = \sim q \to p$.

XI. "IF-THEN" TO "DISJUNCTION"

With "if-then," negating the "if" part (hypothesis), requires the consequence.

EXAMPLE: "If I get up, then my back will hurt." $p \to q$

"I must NOT get up OR my back will hurt." $\sim p \vee q$

$p \to q = \sim p \vee q$ $\sim p \to q = p \vee q$

XII. NEGATING AN "IF-THEN" $\sim(p \to q) = p \wedge \sim q$

An "IF-THEN" is negated if the hypothesis happens, yet the consequence does NOT follow.

EXAMPLE: "If you pass the CLAST, then I will give you $100." You negate the statement (make me a liar!), by "passing the CLAST AND NOT receiving the $100.

SUMMARY OF PRINCIPLES

1. Modus Ponens
   Law of Detachment
   (Obvious)

   $p \to q$

   $\underline{p}$

   $\therefore q$

2. Modus Tollens .
   Law of the Contrapositive

   $p \to q$

   $\underline{\sim q}$

   $\therefore \sim p$

3. Disjunctive Syllogism
   Law of Disjunction

   $p \vee q$ $\quad$ $p \vee q$

   $\underline{\sim p}$ $\quad$ or $\quad$ $\underline{\sim q}$

   $\therefore q$ $\quad$ $\therefore p$

4. Transitive Property
   Syllogism

   $p \to q$

   $\underline{q \to r}$

   $\therefore p \to r$

Remember, the validity of an argument depends upon these principles. It does not depend upon whether the premises and conclusions are true or false. An argument may be fallacious even if its conclusion is true. However, an argument may be valid even though its conclusion is not true (because of false assumptions).

ne following are <u>invalid</u> arguments or <u>logical fallacies</u>:

Fallacy of the converse

$p \to q$

$\underline{q}$

$\therefore p$

2. Fallacy of the inverse

$p \to q$

$\underline{\sim p}$

$\therefore \sim q$

3. Fallacy of the Disjunction

$p \vee q$ $\quad$ $p \vee q$

$\underline{p}$ $\quad$ or $\quad$ $\underline{q}$

$\therefore \sim q$ $\quad$ $\therefore \sim p$

4. Fallacy of Ambiguity -- results from using a word in more than one meaning. For example, if a dog has fleas, he scratches. If he scratches on the door, it means he wants to go out. Therefore, if a dog has fleas, he wants to go out.

5. Fallacy of False Authority -- results from assuming that because an individual is an authority in one field, he or she will be an authority in some other field. For example, Dr. Smith, the great cardiac surgeon, drinks a particular brand of coffee. Therefore, that brand of coffee must be the best.

6. Fallacy of Composition -- results from assuming that a characteristic of a group applies also to the individuals or vice-versa. Good examples of this include: behavior of children, mob psychology, and tendencies to follow the crowd.

7. Fallacy of the Complex Question -- a question to which either a positive or negative response is incriminating. For example, "Have you stopped cheating on tests?"