Basic Equation Solving

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Solving an equation means to find all values of the variable that make the equation true. Consider the following equations:

1. $\mathbf{5x + 2 = 17}$
2. $\mathbf{2 + x = x + 2}$
3. $\mathbf{x = x + 2}$
4. $\mathbf{x^2 + 3 = 4x}$

In equation #1, $5x + 2 = 17$, if $x = 3$, the equation reads $5(3) + 2 = 17$, which is true. It turns out that for any other value of x, the equation is false. Therefore, the solution to the equation is $x = 3$. In equation #2, $2 + x = x + 2$, since any value of x will make the equation true, it is called an identity, and there are infinitely many solutions. In equation #3, $x = x + 2$, there are no values of x that will make the equation true. (Like, can you think of a number, add 2, and still have the same number?) This means that the equation has "no solution." It is called a contradiction. Sometimes when there is no solution, we write the greek letter "ϕ" (phi, as in "Phi Theta Kappa" Honor Society), which represents the empty set, or no solution. This is *not* saying the solution is 0 (zero)!

Finally, in equation #4, $x^2 + 3 = 4x$, there is an x^2 term. As it turns out, this x^2 term allows the possibility of two solutions. This type of equation, because of the x^2 term, is called a quadratic equation, and it will be discussed in a later topic in algebra. It turns out that both $x = 1$ and $x = 3$ are solutions, since $1^2 + 3 = 4(1)$ and $3^2 + 3 = 4(3)$.

Once a solution has been found, you can always check that solution by substituting into the original equation to see if it actually works. It is important to remember that you always are asked to simplify expressions and to solve equations. You can't solve expressions.

The examples given thus far represent only a few of the many different types of equations that you will ultimately learn to solve. And as you probably could guess, different methods are used to solve different types of equations. The equations to be solved in this section are called linear equations. Linear equations involve only one variable at a time, and the variables are not raised to powers. These equations will be solved using the identity and inverse number properties and the addition and multiplication properties of equations. However, do not let these formalities scare you--it will be as easy as unwrapping presents.

Solving an equation is like "un-wrapping presents!"

Suppose you buy a very nice gift for your best friend. Since it is a very special gift (for a very special friend), you first wrap the gift and then you hide the gift until the time is right to surprise your friend. When the time is right, you tell your friend what you have done. What does the friend do? Is the order of events significant? The friend must first find (or unhide the gift), and then unwrap the gift. Notice that your friend must undo your steps, and in the reverse order that you did them. This story illustrates the principle of opposites from everyday life: wrapping and unwrapping, hiding and finding.

In math we have the same principal of opposites or inverse operations. For example, subtraction (or addition of the negative of a number) is the opposite of addition. Addition is the opposite operation for subtraction. Division (or multiplication by the reciprocal or the inverse of a number) is the opposite of multiplication. Multiplication is the opposite operation for division.

The process of equation solving, like the process of finding and unwrapping the gift, is a series of unwrapping operations. Consider the following equation in Example 1.

Example 1.   Solve for x: $3x + 4 = 34$.

In order to solve for x, you must "undo" everything that has been done to the x, and in reverse order. In order to keep the equation the same or in "balance", you must be sure do the same thing to both sides of the equation. Notice what was done to this x in the equation. First, x was multiplied by 3, and then 4 was added. In order to find x, you must undo these operations in the reverse order, just like finding and unwrapping the present. To undo the 4 that was added you must subtract 4, then to undo the multiplication by 3 you must divide by 3. Whatever you do to one side of the equation, you must do also to the other side of the equation. [This is called the **"golden rule" of equations: "Do unto one side, as ye do unto the other!"]**

Example 1 Solution:

$$3x + 4 = 34$$ First, undo the + 4 by - 4 from both sides

$$\underline{-4 \quad -4}$$

$$\frac{3x}{3} = \frac{30}{3}$$ Second, undo the multiplication by 3 by dividing both sides by 3

$$x = 10$$

Check: $3(10) + 4 = 34$

Each of the following examples will be solved using the principles of opposites as in the previous example.

Example 2. Solve for x: $4x + 10 = 30$

Solution: $4x + 10 = 30$ First, undo the "+ 10", with a "-10" to each side of the equation.

$$\underline{-10 \quad -10}$$

$$\frac{4x}{4} = \frac{20}{4}$$ Second, divide both sides by 4.

$$x = 5$$

Example 3. Solve for x: $3x - 17 = 7$

Solution: $3x - 17 = 7$ First, undo the "- 17", with a "+17".

$$\underline{+17 \quad +17}$$

$$\frac{3x}{3} = \frac{24}{3}$$ Second, divide both sides by 3.

$$x = 8$$

Example 4. Solve for x: $13x - 10 = 16$

Solution: $13x - 10 = 16$ First, undo the "- 10", with a "+10" to each side of the equation.

$$\underline{+10 \quad +10}$$

$$\frac{13x}{13} = \frac{26}{13}$$ Second, divide both sides by 13.

$$x = 2$$

Example 5. Solve for x: $9x + 32 = -49$

Solution:

$$\begin{array}{rcl} 9x + 32 &=& -49 \\ \underline{\quad -32} && \underline{-32} \end{array}$$

First, undo the "+ 32", with a "-32".

$$\frac{9x}{9} = \frac{-81}{9}$$

Second, divide both sides by 9.

$$x = -9$$

Example 6. Solve for x: $7x - 10 = -52$

Solution:

$$\begin{array}{rcl} 7x - 10 &=& -52 \\ \underline{\quad +10} && \underline{+10} \end{array}$$

First, undo the "- 10", with a "+10" to each side of the equation..

$$\frac{7x}{7} = \frac{-42}{7}$$

Second, divide both sides by 7.

$$x = -6$$

Example 7. Solve for x: $-3x = 6$

In this equation -3x = 6, what is it that has been done to x in order to get 6? Answer: Multiplication by -3. What would you have to do in order to "undo" the multiplication by -3? Answer: Divide by -3. The number -3 is said to be the coefficient of x. In this equation -3x = 6, you would then want to divide both sides of the equation by the coefficient of x, which is -3. Do you see that if you divide -3x by -3, you will have just 1x? The result looks like this:

Solution: $-3x = 6$ Divide both sides by -3.

$$\frac{-3x}{-3} = \frac{6}{-3}$$

$$x = -2$$

Example 8. Solve for x: $-7x = -56$

Solution: $-7x = -56$ Divide both sides by -7.

$$\frac{-7x}{-7} = \frac{-56}{-7}$$

$$x = 8$$

Example 9. Solve for x: $-x = 6$

Given the equation -x = 6. What is the coefficient of x? Remember that -x means the same as -1x, so the coefficient of x is -1. To solve the equation -x = 6 (which really means -1x = 6) you must divide both sides of the equation by -1:

Example 9 Solution: $-x = 6$ Divide both sides by –1.

$$\frac{-x}{-1} = \frac{6}{-1}$$

$$x = -6$$

Example 10. Solve for x: $-x = -12$

Solution: $-x = -12$ Divide both sides by –1.

$$\frac{-x}{-1} = \frac{-12}{-1}$$

$$x = 12$$

Solving More Complicated Equations

Sometimes there are variable terms on both sides of the equation. For example, consider the equation $4x = 2x + 12$. Notice that in this equation there are three terms. Two of these terms (the 4x and the 2x) contain variables, and the other term, the "12", has no variable--it's just a number term. It would be nice to get all the variable terms together on the same side of the equation. You may accomplish this by adding -2x to both sides of the equation:

Example 11. Solve for x: $4x = 2x + 12$

Solution:

$$\begin{aligned} 4x &= 2x + 12 \\ \underline{-2x} & \;\; \underline{-2x} \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

If there are variable terms and number terms on both sides of the equation, such as $4x - 12 = 2x + 6$, it is important to get all variable terms on one side, and the non-variable or number terms on the other side.

Example 12. Solve for x: $4x - 12 = 2x + 6$

Solution:

$$\begin{array}{rcrl}
4x - 12 &=& 2x + 6 & \text{Add -2x to both sides.} \\
\underline{-2x} && \underline{-2x} & \\
2x - 12 &=& 6 & \text{Add +12 to both sides.} \\
\underline{+ 12} && \underline{+12} & \\
2x &=& 18 & \text{Divide by 2.} \\
x &=& 9 &
\end{array}$$

Example 13. Solve for x: $6x + 10 = -2x - 46$

Solution:

$$\begin{array}{rcrl}
6x + 10 &=& -2x - 46 & \text{Add +2x to both sides.} \\
\underline{+2x} && \underline{+2x} & \\
8x + 10 &=& - 46 & \text{Add -10 to both sides.} \\
\underline{- 10} && \underline{- 10} & \\
8x &=& - 56 & \text{Divide by 8.} \\
x &=& -7 &
\end{array}$$

Example 14. Solve for x: $-8x + 24 = -2x - 48$

Solution:

$$\begin{array}{rcrl}
-8x + 24 &=& -2x - 48 & \text{Add 2x to both sides.} \\
\underline{+2x} && \underline{+2x} & \\
-6x + 24 &=& - 48 & \text{Add -24 to both sides.} \\
\underline{- 24} && \underline{- 24} & \\
-6x &=& - 72 & \text{Divide by -6.} \\
x &=& 12 &
\end{array}$$

Sometimes the left and/or right side of an equation can be simplified by removing parentheses (distributive property) and combining like terms on a given side. Consider the following examples:

Example 15. Solve for x: $4(x - 1) - 2(x + 3) = 8(5 - x)$

Solution:

$$\begin{array}{rcrl}
4(x - 1) - 2(x + 3) &=& 8(5 - x) & \\
4x - 4 - 2x - 6 &=& 40 - 8x & \text{(Distributive property)} \\
2x - 10 &=& 40 - 8x & \text{(Combine like terms)} \\
\underline{+8x} && \underline{+ 8x} & \text{(Add +8x to each side)} \\
10x - 10 &=& 40 & \text{(Add +10 to each side)} \\
\underline{+ 10} && \underline{+10} & \\
10x &=& 50 & \text{(Divide both sides by 10)} \\
x &=& 5 &
\end{array}$$

The steps required to solve an equation like the previous example may be summarized as follows:

Strategy Summary: Equation Solving

Step 1: **If there are parentheses in the problem, eliminate them by use of the distributive property.**

Step 2: **Combine like terms (if possible) on each side of the equal sign.**

Step 3: **Using the "principle of opposites," get all variable terms to one side of the equation.**

Step 4: **Using the "principle of opposites," get all number terms to the <u>other</u> side of the equation.**

Step 5: **Divide both sides of the equation by the <u>coefficient of the variable</u>--that is, the number times the variable. (Or multiply both sides times the reciprocal of the coefficient.) If the coefficient is positive, divide by a positive number. If the coefficient is negative, divide by a negative number. The coefficient of the variable MUST be a positive one (+1) when you are through.**

Example 16. Solve for x: $-3(2 - x) - 2(3x + 5) = 38 - x$

Solution: $-3(2 - x) - 2(3x + 5) = 38 - x$

Step 1: $-6 + 3x - 6x - 10 = 38 - x$ Distributive property

Step 2: $-3x - 16 = 38 - x$ Combine like terms

$\underline{+x \qquad\qquad +x}$

Step 3: $-2x - 16 = 38$ Variables to one side

$\underline{+16 \qquad +16}$

Step 4: $\dfrac{-2x}{-2} = \dfrac{54}{-2}$ Number terms to other side

Step 5: $x = -27$ Divide by -2

Example 17. Solve for x: $7 - (x + 4) = 8$ [Notice this is not $-7(x+4)$]

Solution: It may be helpful to re-write the equation

$7 - 1(x + 4) = 8$ Distributive property

$7 - x - 4 = 8$ Combine like terms

$3 - x = 8$ Add -3 to each side

$-x = 5$ Divide each side by -1

$x = -5$

If an equation involves an x^2 term, it is called a quadratic equation, and it can't be solved in this section. However, if the x^2 term subtracts itself out, then you have a linear equation that can be easily solved.

Example 18. Solve for x: $x(x + 3) = x^2 - 5x - 16$

Solution:

	$x(x + 3) = x^2 - 5x - 16$	Distributive property
	$x^2 + 3x = x^2 - 5x - 16$	Add $-x^2$ to each side
	$\underline{-x^2 \qquad\quad -x^2}$	
	$3x = \quad -5x - 16$	Add 5x to each side
	$\underline{+5x \qquad + 5x}$	
	$8x = -16$	Divide by 8
	$x = -2$	

Sometimes the Solution, Like Real Life, Doesn't Come Out Even!

The variable in an equation can be any symbol or letter--it is not always x. Also, the answers in real life don't always come out even, as in the first examples of this section on equation solving. When expressing fractions, such as $-\frac{3}{5}$, remember that $-\frac{3}{5}$, $\frac{-3}{5}$, and $\frac{3}{-5}$ are all equivalent. The tradition in math is to avoid negative denominators, so the first two forms are preferred over the last. Remember that fractions such as $\frac{12}{5}$ in which the numerator is larger than the denominator are called improper fractions. Such answers can also be written as mixed fractions, in this case $2\frac{2}{5}$. In higher math, the improper fraction is much preferred. Mixed fractions are seldom used in algebra. Nevertheless, either form is usually acceptable. Of course, be careful to reduce the fraction completely. For example, $\frac{12}{8}$ is an improper fraction but it should be reduced. Notice that both numerator and denominator are divisible by 4. Dividing numerator and denominator by 4 gives $\frac{3}{2}$. You may also reduce the fraction $\frac{12}{8}$ by dividing the 8 into 12 to obtain $1\frac{4}{8}$. This reduces to $1\frac{1}{2}$ which is equivalent to $\frac{3}{2}$, so the result is the same!

Example 19. Solve for y: $7y = 30 - 5y$

Solution:

$$\begin{array}{rcl} 7y &=& 30 - 5y \\ \underline{+5y} && \underline{+\ 5y} \end{array} \quad \text{Add +5y to each side.}$$

$$\frac{12y}{12} = \frac{30}{12} \quad \text{Divide by 12.}$$

$$y = \frac{30}{12} \quad \text{Divide numerator and denominator by 6.}$$

$$y = \frac{5}{2} \text{ or } 2\frac{1}{2} \text{ or } 2.5$$

Example 20. Solve for p: $p + 8 - 13(p - 4) = -2(p + 2) + 8p$

Solution:

$p + 8 - 13(p - 4) = -2(p + 2) + 8p$ Distributive property.

$p + 8 - 13p + 52 = -2p - 4 + 8p$ Combine like terms.

$$\begin{array}{rcl} -12p + 60 &=& 6p - 4 \\ \underline{-\ 6p\qquad} && \underline{-\ 6p} \end{array} \quad \text{Add -6p to each side.}$$

$$\begin{array}{rcl} -18p + 60 &=& -\ 4 \\ \underline{-\ 60} && \underline{-60} \end{array} \quad \text{Add -60 to each side.}$$

$$\frac{-18p}{-18} = \frac{-64}{-18} \quad \text{Divide each side by -18.}$$

$$p = \frac{64}{18} \text{ or } \frac{32}{9} \quad \text{Divide numerator and denominator by 2.}$$

EXERCISES.

In 1 – 46, solve the equations for x.

1. $3x + 4 = 34$
2. $5x + 12 = 47$
3. $4x + 10 = 30$
4. $3x - 8 = 7$
5. $13x - 10 = 16$
6. $7x - 12 = 44$
7. $5x + 13 = 68$
8. $8x + 34 = 2$
9. $6x + 32 = 20$
10. $6x + 32 = -22$
11. $9x + 32 = -49$
12. $5x - 22 = -7$
13. $7x - 10 = -52$
14. $-5x = 25$
15. $-3x = 30$
16. $-8x = 32$
17. $-7x = -21$
18. $-6x = -36$
19. $-12x = -60$
20. $-x = 5$
21. $-x = 9$
22. $-x = -5$
23. $-x = -9$
25. $4x - 12 = 2x + 6$
26. $6x + 10 = 2x + 50$
27. $6x + 10 = -2x - 46$
28. $-6x + 10 = -2x + 50$
29. $-8x + 24 = -2x - 30$
30. $4(x - 1) - 2(x + 3) = 8(5 - x)$

31. $8(x+2) - 7x = 3(x-2) + 2$

32. $-3(2 - x) + 2(3x + 5) = 31$

33. $4(2-3x) + 4(2x-3) = 4(x+1)$

34. $3(x - 6) - 5(x - 10) = 24$

35. $3x - 5(2x - 6) = 9(2 - x)$

36. $8(2x - 5) - 5(2 - x) = -4x$

37. $9 - 2(x + 4) = 17$

38. $7 - 5(2x - 8) = -13$

39. $9 + 2(x - 4) = -17$

40. $7 + 5(2x - 8) = -13$

41. $7 - (x + 4) = 8$

42. $4 - (x + 7) = 8$

43. $7 - (x - 4) = 8$

44. $7 - (4 - x) = 8$

45. $x(x + 3) = x^2 - 5x - 16$

46. $x(x - 3) = x^2 + 3x - 18$

In 47 - 58, solve for the variable. Reduce all fractions completely.

47. $7y = 15 - 3y$

48. $8d + 4 = 2d$

49. $3c - 5c = 9 + 4c$

50. $4z - (z-8) = 0$

51. $5 - 3(f-4) = 13$

52. $s-(3-s) = 5-(2s+6)$

53. $p + 5 - 3(p - 4) = 2(p + 2)$

54. $b - 3 -(2b + 3) = 3b$

55. $b - 3 -(2b + 3) = 3b - 6$

56. $3(7-2q) = 14 - 8 (q - 1)$

57. $j(j + 3) = 4 - j(2 - j)$

58. $w(w + 2) - 15 = w(w - 2)$

ANSWERS TO EXERCISES

1. 10; **2.** 7; **3.** 5; **4.** 5; **5.** 2; **6.** 8; **7.** 11; **8.** -4; **9.** –2; **10.** -9; **11.** -9; **12.** 3; **13.** -6; **14.** -5; **15.** -10; **16.** -4; **17.** 3; **18.** 6; **19.** 5; **20.** -5; **21.** -9; **22.** 5; **23.** 9; **24.** 6; **25.** 9; **26.** 10; **27.** -7; **28.** -10; **29.** 9; **30.** 5; **31.** 10; **32.** 3; **33.** -1; **34.** 4; **35.** -6; **36.** 2; **37.** -8; **38.** 6; **39.** -9; **40.** 2; **41.** -5; **42.** -11; **43.** 3; **44.** 5; **45.** -2; **46.** 3; **47.** 3/2; **48.** –2/3; **49.** -3/2 or -1.5; **50.** -8/3; **51.** 4/3; **52.** 1/2; **53.** 13/4; **54.** -3/2; **55.** 0; **56.** 1/2; **57.** 4/5; **58.** 15/4;