2.10 *Theorem of Pythagoras*

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Before introducing the **Theorem of Pythagoras**, we begin with some perfect square equations. Perfect square equations (see the first example and the exercises that follow) can be solved by taking the square root of both sides of the equation. This is called the square root property of equations. When you use this property, you must include a "±" (that is, "+" or "-") in order to obtain both solutions of the equation.

EXAMPLE 1. **Solve the equation $x^2 = 16$.**

Solution: The solution is essentially to answer the question, "What number can be squared (multiplied times itself!) in order to get 16. There are actually two answers: $x = 4$ and also $x = -4$. This answer may be also written as $x = \pm 4$.

EXAMPLE 2. **Solve the equation $x^2 = 5$.**

Solution: Unlike the first example, there is no whole number or integer that you can square in order to get 5. It is possible, however, to take the square root of both sides and write $x = \pm\sqrt{5}$. Using a calculator (see Section 1.04) you can give the decimal approximation which is $x \approx \pm 2.236$ (round off to nearest thousandth. Note: the wavy equal sign "≈" means "approximately equal.")

EXAMPLE 3. **Solve the equation $x^2 + 12^2 = 15^2$.**

Solution:

$x^2 + 12^2 = 15^2$ You know that $12^2 = 144$, and $15^2 = 225$ (or use calculator!)

$x^2 + 144 = 225$ Subtract 144 from each side.

$-144 \quad -144$

$x^2 = 81$ Because of the x^2, you have to have two answers: "±".

$x = \pm 9$.

EXAMPLE 4. **Solve the equation $x^2 + 10^2 = 15^2$.**

Solution:

$x^2 + 10^2 = 15^2$ You know that $10^2 = 100$, and $15^2 = 225$ (or use calculator!)

$x^2 + 100 = 225$ Subtract 100 from each side.

$-100 \quad -100$

$x^2 = 125$ Because there is no "even" answer, use the square root.

$x = \pm\sqrt{125}$ Don't forget the ±, and round off to nearest thousandth.

$x \approx \pm 11.180$.

EXERCISES. **Solve the following perfect square equations. In some, a calculator is needed!**

1. $x^2 = 9$ $x = \pm$_____	**2.** $x^2 = 25$ $x =$ _____	**3.** $x^2 = 49$ $x =$ _____	**4.** $x^2 = 169$ $x =$ _____
5. $x^2 = 81$ $x =$ _____	**6.** $x^2 = 36$ $x =$ _____	**7.** $x^2 = 144$ $x =$ _____	**8.** $x^2 = 121$ $x =$ _____
9. $x^2 = 6$ $x =$ _____	**10.** $x^2 = 30$ $x =$ _____	**11.** $x^2 = 200$ $x =$ _____	**12.** $x^2 = 120$ $x =$ _____
13. $6^2 + 8^2 = x^2$	**14.** $x^2 + 5^2 = 13^2$	**15.** $15^2 + x^2 = 17^2$	**16.** $x^2 = 3^2 + 4^2$
17. $5^2 + 6^2 = x^2$	**18.** $x^2 + 10^2 = 13^2$	**19.** $13^2 + x^2 = 17^2$	**20.** $x^2 = 12^2 + 9^2$
21. $40^2 + 42^2 = x^2$	**22.** $x^2 + 24^2 = 25^2$	**23.** $70^2 + x^2 = 74^2$	**24.** $x^2 = 13^2 + 84^2$

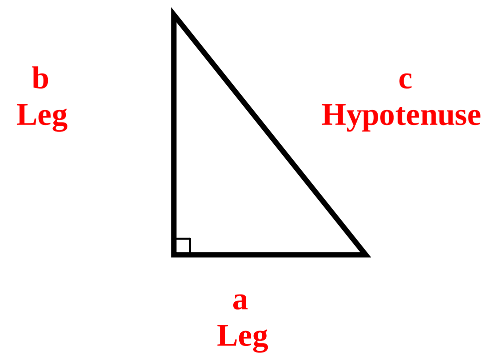
The **Theorem of Pythagoras** is one of the most important formulas in all of mathematics. Although this theorem was known to the Babylonians 1000 years earlier, the credit for the first proof was given to the Greek mathematician **Pythagoras**, 6th century B.C. **The Theorem of Pythagoras deals specifically with right triangles.** In a right triangle, the two sides that are mutually perpendicular are called **legs**, and the third side, always opposite the right angle, and always the longest side, is called the **hypotenuse** of the triangle. **According to the Theorem of Pythagoras, if "a" and "b" are legs, and "c" is the hypotenuse, then $a^2 + b^2 = c^2$.**

Given any two sides of a right triangle, the **Theorem of Pythagoras** can be used to find the third side. The first step is to identify which side is the hypotenuse.

Theorem of Pythagoras

In any right triangle, where "a" and "b" are legs, and "c" is the hypotenuse,

$$a^2 + b^2 = c^2.$$



EXAMPLE 5. **To find the distance across a swamp without getting your feet wet, you can measure a distance of 3 miles, make a 90 degree turn, and measure off a distance of 4 miles, forming a right triangle and going around the swamp as shown in the figure. Find the distance across the swamp.**

Solution: **Let x = unknown distance across the swamp (hypotenuse).**

Equation: $3^2 + 4^2 = x^2$

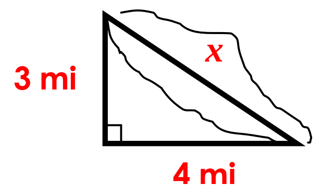
$9 + 16 = x^2$

$x^2 = 25$

$x = \pm 5$ **miles**

Answers: $x = -5$ **is meaningless**

$x = 5$ **miles is distance across swamp**



EXAMPLE 6. **Suppose the sides on the swamp problem (see Example 5) are changed so that the longer leg of the triangle is in the swamp, with the hypotenuse of the right triangle 13 miles, and the shorter leg 5 miles, as shown in the figure below. Find the distance across this swamp.**

Solution: **Let x = unknown distance across the swamp (the other leg).**

Equation: $5^2 + x^2 = 13^2$

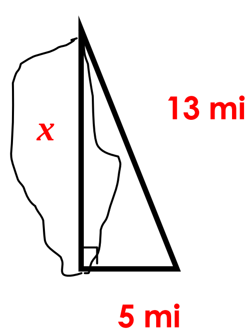
$25 + x^2 = 169$

$x^2 = 144$

$x = \pm 12$ **miles**

Answers: $x = -12$ **is meaningless**

x = 12 miles is the distance across swamp



EXERCISES. **Find the missing side of each triangle. (Solve for x.)**

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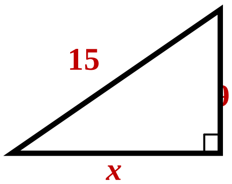
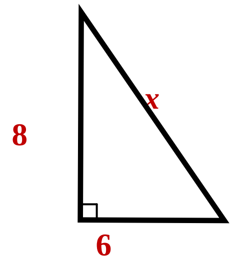


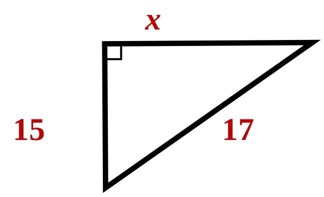
$x^2 + 9^2 = 15^2$

26.



$6^2 + 8^2 = x^2$

27.



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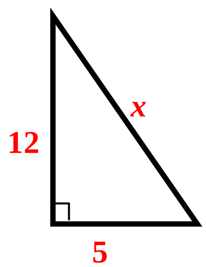

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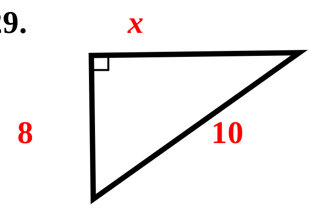

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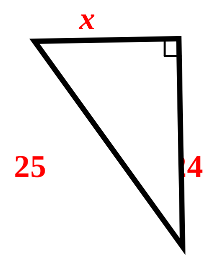


Did you notice that in the swamp examples and the triangle problems so far, all of the sides came out even? Do you think in all such problems, in which you are given two sides of a triangle and asked to find the third side, that the answers come out whole numbers as these did? The truth is that, like the very first examples and exercises of this section, they do not always come out even, and in fact there are really "special" triangles that are like this. Of course, those who make up the exercises (and test questions!) are well aware of these "special" triangles that come out even, and consequently exercises are frequently (usually?) "rigged" to work out.

Perhaps it would be helpful to let you in on these special numbers. They are called **Pythagorean Triples.** Although there are infinitely many such special triangles, only a few have numbers that are small enough to be "reasonable". The two most commonly used are the two from the first two examples: **3,4,5** and **5,12,13.** Two triples that are not as frequent are **8,15,17** (see #27) and **7,24,25** (see #30). In addition to these, any multiple of these numbers is also a "triple". As examples, **6,8,10** or **9,12,15** are multiples of **3,4,5.** Multiples of **5,12,13** are **10,24,26** or **15,36,39**.

PYTHAGOREAN TRIPLES

When three integers a, b, and c are such that $a^2 + b^2 = c^2$, this is called a Pythagorean Triple. The most common are:

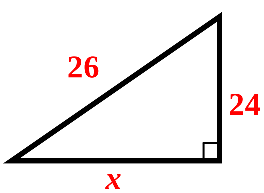
3, 4, 5
5, 12, 13
8, 15, 17
7, 24, 25

or any multiple of the above.

EXERCISES.

Find the missing side of each triangle. (Find *x*.) For answers that do not come out even, use a calculator and round to nearest hundredth. Watch for special triangles.

31.



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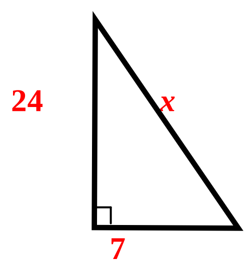

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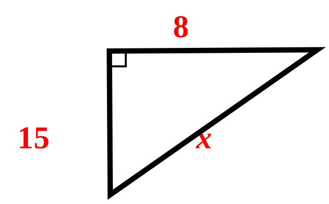

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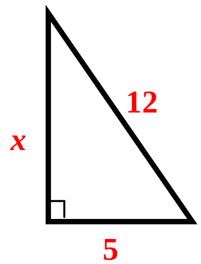

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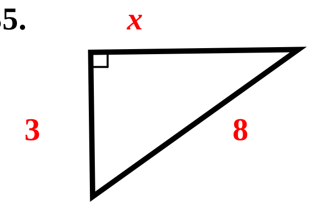

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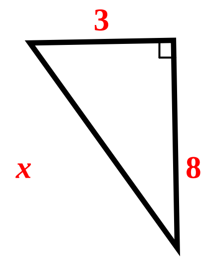
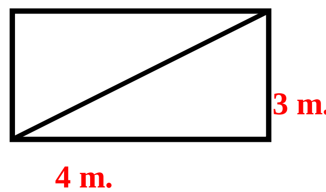


37. Notice in the figure at the right that the diagonal of a rectangle divides the rectangle into two triangles. Use this to find the diagonal if the width is 3 m. and the length is 4 m.



38. Find the diagonal of a rectangle whose width is 6 ft. and whose length is 8 ft.

39. Find the diagonal of a rectangle whose width is 12 cm. and whose length is 16 cm.

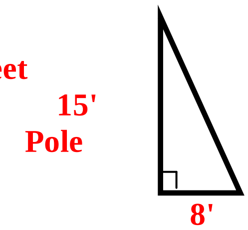
40. Find the length of a rectangle whose width is 8 ft. and whose diagonal is 17 ft.

41. Find the width of a rectangle whose diagonal is 25 cm. and length is 24 cm.

42. Find the width of a rectangle whose diagonal is 29 cm. and whose length is 21 cm.

43. Find the diagonal of a rectangle whose width is 13 cm. and whose length is 84 cm.

44. A guy wire to the top of a 15 foot pole reaches the ground 8 feet from the base of the pole. How long is the wire?



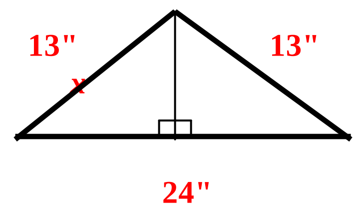
45. A guy wire to the top of a 35-foot pole reaches the ground 18 feet from the base of the pole. How long is the wire?

46. A guy wire to the top of a pole is 35 feet long. It reaches the ground 18 feet from the base of the pole. How tall is the pole?

47. A guy wire to the top of a pole is 73 feet long. It reaches the ground 48 feet from the base of the pole. How tall is the pole?

In the next exercises, it will be helpful to know that an isosceles triangle is a triangle with exactly two equal sides. Also, the height of the triangle is always perpendicular to the base and it cuts the base in half to form two equal triangles as shown in the illustration for #48.

48. Find the height of an isosceles triangle whose base is 24 inches and whose equal sides are each 13 inches.



49. Find the height of an isosceles triangle whose base is 140 inches and whose equal sides are each 74 inches.

50. An isosceles triangle has a base of 10 cm. and a height of 12 cm. How long are the equal sides?

51. An isosceles triangle has a base of 48 cm. and a height of 70 cm. How long are the equal sides?

52. An isosceles triangle has a base of 64 cm. and a height of 126 cm. How long are the equal sides?

2.10 ANSWERS

p. 193-200:

1. ± 3; **2.** ± 5; **3.** ± 7; **4.** ± 13; **5.** ± 9; **6.** ± 6; **7.** ± 12; **8.** ± 11; **9.** $\pm\sqrt{6}$, ±2.45;
10. $\pm\sqrt{30}$, ±5.48; **11.** $\pm\sqrt{200}$, ±14.14; **12.** $\pm\sqrt{120}$, ±10.95; **13.** ± 10;
14. ±12; **15.** ± 8; **16.** ± 5; **17.** $\pm\sqrt{61}$, ±7.81; **18.** $\pm\sqrt{69}$, ±8.31;
19. $\pm\sqrt{120}$, ±10.95; **20.** ± 15; **21.** ± 58; **22.** ± 7; **23.** ± 24; **24.** ± 85; **25.** 12;
26. 10; **27.** 8; **28.** 13; **29.** 6; **30.** 7; **31.** 10; **32.** 25; **33.** 17; **34.** $\sqrt{119}$, 10.91;
35. $\sqrt{55}$, 7.42; **36.** $\sqrt{73}$, 8.54; **37.** 5 m; **38.** 10 ft; **39.** 20 cm; **40.** 15 ft;
41. 7 cm; **42.** 20 cm; **43.** 85 cm; **44.** 17 ft; **45.** 39.36 ft; **46.** 30.02 ft; **47.** 55 ft;
48. 5 in; **49.** 24 in; **50.** 13 cm; **51.** 74 cm; **52.** 130 cm.