4.02 *Solving Quadratic Equations*

(Factoring and Completing the Square)

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

from *Intermediate Algebra: One Step at a Time*

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Remember from **Section 2.02** that a **quadratic equation** is an equation in which the variable is raised to the second power, in the form $aX^2+bX+c=0$. As you have already learned, the best way to solve a quadratic equation is to **set the equation equal to zero** and **factor.** Unfortunately, not all quadratic expressions can be factored. In cases where factoring is not possible, other methods must be used--either **completing the square** or the **quadratic formula.**

EXAMPLE 1. Solve for X:

$$X^2 + 21X = 100$$
$$X^2 + 21X - 100 = 0$$
$$(X + 25)(X - 4) = 0$$
$$X = -25 \text{ or } X = 4$$

EXAMPLE 2. Solve for X:

$$(X-3)(X-4) = 2$$
$$X^2 - 7X + 12 = 2$$
$$X^2 - 7X + 10 = 0$$
$$(X - 5)(X - 2) = 0$$
$$X = 5 \text{ or } X = 2$$

EXERCISES. Solve for X by the method of factoring.

1. $X^2 + X = 12$ 2. $X^2 - 18 = 3X$ 3. $X(X-4) = -2X +8$

4. $2X^2 = 3 + 5X$ 5. $2X^2 = 5X - 3$ 6. $(X - 4)^2 = 2X$

Unfortunately, factoring doesn't always work!! As mentioned, if the equation cannot be factored then you must try another method. The **quadratic formula always works**, whether the equation factors or not! **Completing the square** is important in higher mathematics because it is by completing the square that the quadratic formula is derived.

We begin with some perfect square equations. Perfect square equations (see the next exercises) can be solved by simply using the square root property of equations (i.e., take the square root of both sides of the equation). When you use this property, you must include a "$\pm$" (that is, "+" or "−") in order to obtain both solutions of the equation.

EXERCISES. **Solve the following perfect square equations:**

1. $X^2 = 9$ 2. $X^2 = 25$ 3. $X^2 = 121$ 4. $X^2 = 169$

$X = \pm$ _____ $X =$ _____ $X =$ _____ $X =$ _____

5. $X^2 = 20$ 6. $X^2 = 50$ 7. $X^2 = 72$ 8. $X^2 = 300$

$X = \pm \sqrt{20}$ $X =$ _____ $X =$ _____ $X =$ _____

$X = \pm$ _____ $X =$ _____ $X =$ _____ $X =$ _____

9. $(X+2)^2 = 9$ 10. $(X+2)^2 = 25$ 11. $(X-3)^2 = 121$

$X+2 = \pm 3$ $X+2 = \pm 5$

$X = -2 \pm 3$ $X =$ _____

$X = -2+3$ or $-2-3$ $X =$ _____ or _____

$X =$ _____ or _____

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12. $(X-3)^2 = 169$

13. $(X-5)^2 = 100$

14. $(X+7)^2 = 81$

15. $(X-5)^2 = 7$

$X-5 = \pm\sqrt{7}$

$X =$ _____

16. $(X+5)^2 = 7$

_____ $= \pm\sqrt{}$

$X =$ _____

17. $(X-8)^2 = 13$

18. $(X-5)^2 = 20$

19. $(X+5)^2 = 60$

20. $(X-8)^2 = 27$

21. $X^2 - 4X + 4 = 25$

$(X-2)^2 = 25$

$X-2 = \pm 5$

$X =$ _____

$X =$ _____ or _____

22. $X^2 - 6X + 9 = 36$

23. $X^2 + 6X + 9 = 144$

24. $X^2 - 10X + 25 = 100$

25. $X^2 - 14X + 49 = 5$

26. $X^2 + 8X + 16 = 13$

27. $X^2 + 8X + 16 = 20$

28. $X^2 - 12X + 36 = 50$

More often than not, the equation that is given is <u>not</u> a perfect square equation, as in the previous exercises. In order to "build" perfect square equations, it will help to recognize some **perfect square trinomials.** As examples, consider:

$$X^2 + 2X + 1 = (X + 1)^2 \qquad X^2 - 2X + 1 = (X - 1)^2$$
$$X^2 + 4X + 4 = (X + 2)^2 \qquad X^2 - 4X + 4 = (X - 2)^2$$
$$X^2 + 6X + 9 = (X + 3)^2 \qquad X^2 - 6X + 9 = (X - 3)^2$$
$$X^2 + 8X + 16 = (X + 4)^2 \qquad X^2 - 8X + 16 = (X - 4)^2$$
$$X^2 + 10X + 25 = (X + 5)^2 \qquad X^2 - 10X + 25 = (X - 5)^2$$

In the following exercises, what constant term is needed to "complete the square"? [Hint: Figure out the b) part first.]

29a) $X^2 + 2X +$ _____

  b) $(X +$ ____$)^2$

30a) $X^2 + 4X +$ _____

  b) $(X +$ ____$)^2$

31a) $X^2 + 6X +$ _____

  b) $(X +$ ____$)^2$

32a) $X^2 - 8X +$ _____

  b) $(X -$ ____$)^2$

33a) $X^2 - 10X +$ _____

  b) $(X -$ ____$)^2$

34a) $X^2 - 12X +$ _____

  b) $(X -$ ____$)^2$

35. Did you see a pattern? Take b)_____ of the middle term and a)_____ it. Now try some harder ones.

36. $X^2 + 16X + $ _____   37. $X^2 + 24X + $ _____   38. $X^2 + 40X + $ _____

39. $X^2 - 30X + $ _____   40. $X^2 - 80X + $ _____   41. $X^2 + 5X + $ _____

42. $X^2 - 5X + $ _____   43. $X^2 + 9X + $ _____   44. $X^2 + 13X + $ _____

45. $X^2 + X + $ _____   46. $X^2 - X + $ _____   47. $X^2 + bX + $ _____

48. $X^2 - bX + $ _____   49. $X^2 - \pi X + $ _____   50. $X^2 + \pi X + $ _____

RULE: When completing the square for $X^2 + bX + $ _____, take **half** the coefficient of X and **square**.

EXAMPLE 3. Solve for X:

$X^2 + 6X - 7 = 0$ Add +7 to both sides to express in the form $X^2 + 6X + $ ____.

$X^2 + 6X + $ ____ $ = 7 + $ ____   Take half of 6 and square to get 9. Add +9 to both sides of equation.

$X^2 + 6X + \underline{9} = 7 + \underline{9}$ Rewrite with perfect square on left.

$(X + 3)^2 = 16$ Square root property of equations. **(Don't forget the ± sign!)**

$X + 3 = \pm 4$ Add −3 to both sides.

$X = -3 \pm 4$

$X = -3 + 4$ or $X = -3 - 4$

$X = 1$ or $X = -7$

Check by factoring: $X^2 + 6X - 7 = 0$
                     $(X - 1)(X + 7) = 0$
                     $X = 1 ; X = -7$

[Note: Factoring <u>is</u> easier, but it doesn't always work!]

EXAMPLE 4.   Solve for X:

    $X^2 + 6X - 8 = 0$ Add +8 to both sides of equation.

    $X^2 + 6X + \underline{\qquad} = 8 + \underline{\qquad}$ Take half of 6 and square to get 9.
                                              Add +9 to both sides of equation.

    $X^2 + 6X + \underline{9} = 8 + \underline{9}$ Rewrite with perfect square on left.

    $(X + 3)^2 = 17$ Take square root of both sides.
                                      (Don't forget the ± sign!)

    $X + 3 = \pm \sqrt{17}$ Add −3 to both sides.

    $X = -3 \pm \sqrt{17}$ Answer does not simplify.

Note: Because of the radical, this problem cannot be solved
       by factoring.

EXERCISES.   Solve the equations by method of completing the square.

1. $X^2 + 6X - 9 = 0$ 2. $X^2 + 6X + 5 = 0$

    $X^2 + 6X + \underline{\quad} = 9 + \underline{\quad}$ $X^2 + 6X + \underline{\quad} = -5 + \underline{\quad}$

    $(X + \underline{\quad})^2 = \underline{\qquad}$ $(X + \underline{\quad})^2 = \underline{\quad}$

    $X + \underline{\quad} = \pm \underline{\quad}$ $X + \underline{\quad} = \pm \underline{\quad}$

    $X = \underline{\quad} \pm \underline{\quad}$ $X = \underline{\quad} \pm \underline{\quad}$

    $X = \underline{\qquad\qquad}$ $X = \underline{\qquad\qquad}$

3. $X^2 - 6X - 9 = 0$ 4. $X^2 - 2X - 8 = 0$

5. $X^2 + 2X - 48 = 0$ 6. $X^2 - 12X - 64 = 0$

7. $X^2 - 10X - 15 = 0$ 8. $X^2 - 8X + 8 = 0$

In 9 - 14, watch out for complex numbers.

9. $X^2 + 4X + 5 = 0$ 10. $X^2 - 6X + 13 = 0$

11. $X^2 - 10X + 50 = 0$ 12. $X^2 + 8X + 52 = 0$

13. $X^2 + 8X + 40 = 0$ 14. $X^2 - 6X + 36 = 0$

EXTRA CHALLENGE:

If the coefficient of X^2 is not **1**, then divide both sides of the equation by that coefficient.

15. $2X^2 + 7X + 6 = 0$ 16. $2X^2 - 9X + 10 = 0$

$$X^2 + \frac{7}{2}X + 3 = 0$$

Half of $\frac{7}{2}$ is $\frac{7}{4}$, and $(\frac{7}{4})^2$ is $\frac{49}{16}$

$$X^2 + \frac{7}{2}X + \quad = -3 +$$

$$(X + \frac{7}{4})^2 = -3 + \frac{49}{16}$$

$$(X + \frac{7}{4})^2 = \frac{-48 + 49}{16}$$

$$(X + \frac{7}{4})^2 = \frac{1}{16}$$

$$X + \frac{7}{4} = \pm \frac{1}{4}$$

$$X = -\frac{7}{4} \pm \frac{1}{4}$$

X = _____ or _____

X = _____ or _____

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    1. -4,3; **2.** 6,-3; **3.** 4,-2; **4.** -1/2, 3; **5.** 3/2, 1; **6.** 8,2.

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    1. ±3; **2.** ±5; **3.** ±11; **4.** ±13; **5.** ±2√5; **6.** ±5√2; **7.** ±6√2;
    8. ±10√3; **9.** 1,-5; **10.** 3,-7; **11.** 14,-8; **12.** 16,-10;
    13. 15,-5; **14.** 2,-16; **15.** 5±√7; **16.** -5±√7; **17.** 8±√13;
    18. 5±2√5; **19.** -5±2√15; **20.** 8±3√3; **21.** 7,-3; **22.** 9,-3;
    23. 9,-15; **24.** 15,-5; **25.** 7±√5; **26.** -4±√13; **27.** -4±2√5;
    28. 6±5√2; **29a)** 1 b)1; **30a)** 4 b)2; **31a)** 9, b)3; **32a)** 16, b)4;
    33a) 25, b)5; **34a)** 36, b)6; **35b)** half a) square; **36.** 64;
    37. 144; **38.** 400; **39.** 225; **40.** 1600; **41.** 25/4 or 6.25;
    42. 25/4 or 6.25; **43.** 81/4 or 20.25; **44.** 169/4 or 42.25;
    45. 1/4 or 0.25; **46.** 1/4 or 0.25; **47.** $b^2/4$; **48.** $b^2/4$;
    49. $\pi^2/4$; **50.** $\pi^2/4$.

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    1. -3±3√2; **2.** -1,-5; **3.** 3±3√2; **4.** 4,-2; **5.** -8,6; **6.** 16,-4;
    7. 5±2√10; **8.** 4±2√2; **9.** -2±i; **10.** 3±2i; **11.** 5±5i; **12.** -4±6i;
    13. -4±2i√6; **14.** 3±3i√3; **15.** -3/2, -2; **16.** 5/2, 2.