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NAVAL SEAKEEPING & MANEUVERING - VOL 1 OF 2

Main Category:	Naval Engineering
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Course #:	NAV-121
Course Content:	50 pgs
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OFFICIAL COURSE/EXAM

(SEE INSTRUCTIONS ON NEXT PAGE)

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NAV-121 EXAM PREVIEW

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Exam Preview:

1. For a given sea condition, ship length makes a big difference in behavior. Small ships suffer from large motions (relative to the ship). As the length reduces a ship begins to contour.
    a. True
    b. False
2. Using Figure 1.1: The Six Degrees of Freedom (6DOF) and the surrounding reference material, which of the following motions acts on the Z axis of a ship?
    a. Surge
    b. Sway
    c. Pitch
    d. Yaw
3. The margin line defines the highest permissible location on the side of the vessel of any damaged waterplane in the final condition of sinkage, trim, and heel. It is in no case permitted to be less than 6 inches below the top of the bulkhead deck at the side.
    a. True
    b. False
4. A free surface causes a reduction in the ship's righting arm, range of stability, and dynamic stability. The worst case for a free surface is when the ship's transverse center of gravity is located ___ the centerline.
    a. On
    b. Below
    c. Off
    d. Above

5. According to the reference material, for regular waves in deep water, there is a fixed relationship between the frequency of the wave, the length of the wave, and the speed that the wave travels.
    a. True
    b. False
6. According to the reference material, the list caused by damage shall not exceed __ degrees. This angle is too great for continuous operation of equipment.
    a. 15
    b. 20
    c. 25
    d. 30
7. Using Table 3.1: Nomenclature for describing waves and the surrounding reference material, which of the following terms matches the description: is the angle the water surface makes with the calm water surface?
    a. Wave Slope
    b. Wave Celerity
    c. Wave Period
    d. Wave Steepness
8. According to the reference material, warships, troop transports, and hospital ships over 300 ft long are required to with-stand a hull opening of ___% of the length between perpendiculars.
    a. 7.5
    b. 10
    c. 12.5
    d. 15
9. According to the reference material, water is considered deep when the water particles involved in the wave motion do not detect the bottom.
    a. True
    b. False
10. Using Table 3.1: Nomenclature for describing waves and the surrounding reference material, which of the following terms matches the description: is the speed of the wave traveling over the water surface?
    a. Wave Slope
    b. Wave Celerity
    c. Wave Period
    d. Wave Steepness

Chapter 1

An Introduction to Seakeeping

Learning Objectives:

1. Define seakeeping.
2. List the ship motions that are involved in seakeeping analysis
3. Describe the big picture/general approach to analyzing seakeeping behavior.
4. Identify aspects of ship construction that contribute to seakeeping performance.

The first step to understanding seakeeping is to define it. What is seakeeping? And what makes for a vessel that has *good* seakeeping?

1.1 What is seakeeping?

In the broadest sense, seakeeping is the behavior of ships in waves. Seakeeping is generally considered the ability of a vessel to withstand rough conditions at sea. A more detailed definition is:

> Seakeeping is the study of the motions of a ship or floating structure, when subjected to waves, and the resulting effects on humans, systems, and mission capability.[4]

Gillmer and Johnson differentiate between *seakeeping* and *seakindliness*,

- **seakeeping**: a ship's ability to maintain normal functions at sea
- **seakindliness**: the quality of behaving comfortably and producing easy motions in a seaway

Wikipedia describes *seakeeping ability* as a "measure of how well-suited a vessel is to conditions when underway".

So, how do we describe the seakeeping performance of a vessel, never mind how do we *calculate* or *predict* it? A ship can be viewed as a rigid body (remember those pleasant days of Dynamics classes!) that has six degrees-of-freedom:

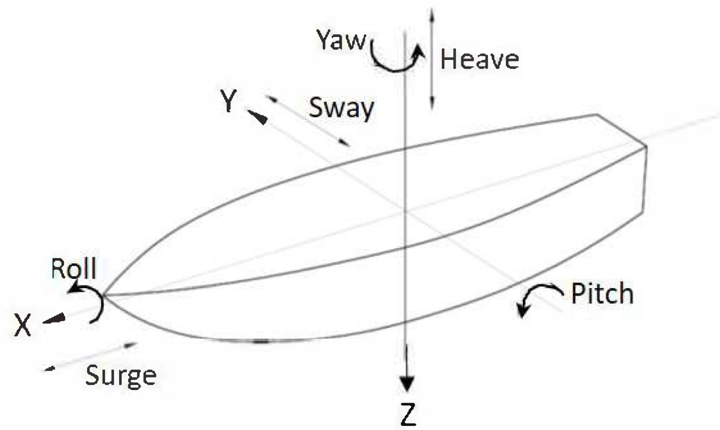

Figure 1.1: The Six Degrees of Freedeom (6DOF)

1. Surge
2. Sway
3. Heave
4. Roll
5. Pitch
6. Yaw

The first three (1 - 3) motions are linear motions and the last three (4 - 6) are rotational motions. All motions are measured relative to the ship itself, as shown in Figure 1.1. Surge describes the forward and back direction (forward is in the direction the bow is pointing and back is in the direction of the stern). Sway is the side-to-side direction, so a vessel moving in its starboard direction is traveling in the positive sway direction. Heave is the vertical direction and, by convention, positive heave is down (toward the water bottom). So, a vessel that is sinking into the water (increasing its draft) is moving in the positive heave direction. Roll is rotational motion about the surge axis. A vessel that has the starboard and port sides moving vertically but in opposite directions (i.e. the starboard side is moving up while the port side is moving down) is rolling. Convention has a positive roll angle when the starboard side is down and the port side is up. Pitch is the rotational motion about the sway axis. When pitching, the bow and stern are moving vertically in opposite directions (i.e. when the bow is moving up and the stern is moving down). Pitch is positive when the bow is up relative to a level ship. Yaw is the rotational motion about the heave axis, or describes the turning motion of the ship. When the bow moves in the starboard direction, we consider that a positive yaw angle.

A ship operating in waves will be moved in all six degrees of freedom, but the motions that cause the greatest problems are the ones that have a restoring force associated with them. If the wave pushes the vessel to the side (a sway motion), it may be inconvenient in terms of navigation, but that is the end of the action. However, if the ship is pushed over so that the starboard deck drops, when the wave moves past the ship will return to its original position upright. It is this kind of response that leads to potential trouble for ships in waves.

Another issue that concerns the description of ship motions is the point of reference. This depends on what we are trying to describe. If we are giving the ship speed and heading we are measuring in reference to a point fixed on the Earth (x_E). If we want to describe the

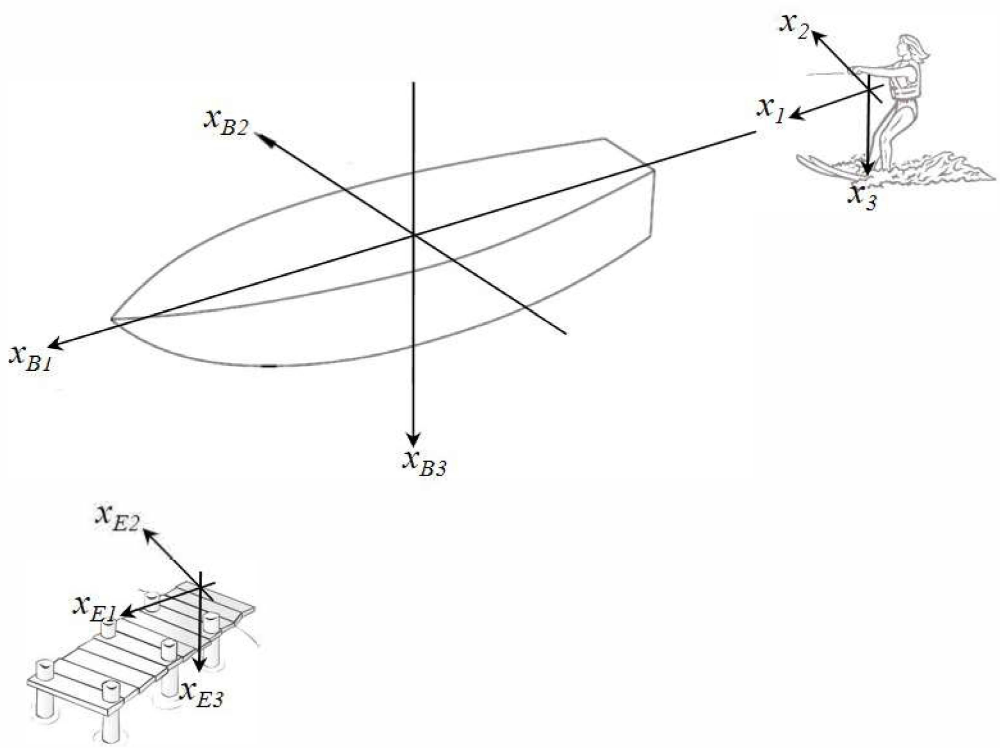

Figure 1.2: Illustration of the three reference frames – x_B Body-Fixed, x_E Earth-Fixed, and x Seakeeping Frame

location of something on the ship - the sink in the galley, for instance - we want to be able to use a reference frame that is fixed to the ship (x_B). For seakeeping motions (roll, heave, and pitch), we want a frame that is co-located with the ship, but does not move with the ship (x_1) in heave, roll, or pitch. One way to visualize these different reference frames is to consider three people's observations of what the ship is doing. The person standing on the pier watching the ship go by is measuring the ship motions from the earth reference frame. He can clearly tell the direction and the speed that the ship is traveling. The person standing on a lower deck in the ship will always tend to consider the direction "up" as towards the deck above her, even if the ship is moving wildly around. Likewise, "forward" is always towards the bow of the ship, regardless of the ship heading. This person is measuring the ship from the body-fixed or ship-fixed reference. The seakeeping reference frame can best be imagined as seen from a water-skier traveling with the ship. The waterskier travels in the same direction and with the same heading as the ship (so, "forward" is towards the bow), but does not move up and down or rotate with the ship. This person has the seakeeping perspective on the ship position with respect to heave, roll, and pitch. Figure 1.2 shows the different reference frames and notation.

1.2 Components of Seakeeping

What we need to discuss in this class, are the important aspects that affect seakeeping. As you might expect, the kinds of seas the ship experiences have an important influence on the ship motions. Larger waves produce, in general, larger motions. But, it is true that different ships experiencing the same waves behave differently, so the ship characteristics are also important. The problem of seakeeping analysis can be broken into three main components:

- Waves (the input to the system)
- Ship characteristics (the system)
- Ship motions (the output of the system)

These notes are organized to address each part in turn. First, we will consider the nature of waves and how, as engineers, we can describe them in a useful way to use as input to our system. Next, we will review rigid body dynamics and look at how we can analyze a ship in waves as a dynamic system. Finally, we will combine the input (waves) to our system (ship) and predict the response (ship motions). At this point, you will be doing seakeeping analysis!

1.3 What features make for a ship that has good seakeeping qualities?

In the interest of BLUF (bottom line up front), what characteristics should a vessel have for it to behave well in waves? This is a good question and, as a colleague once pointed out, the answer to all good questions is, "it depends." However, there are some general guidelines that can be relied upon.

1.3.1 Hull Size

How does the length of a ship, for example, affect its performance in waves? For a given sea condition, ship length makes a big difference in behavior. Small ships suffer from large motions (relative to the ship). As the length reduces a ship begins to contour (travel along the surface of a wave as though it were a road). Thus, if a small ship is in big waves, it moves around a lot compared to its size. However, a big ship will move less in the same set of waves, partly because it is longer relative to the waves, but also because it is more massive than the smaller ship. However, a long ship with a shallow draft runs a higher risk of keel emergence (the keel coming out of the water) when in rough seas.

1.3.2 Hull Form

It turns out that small detail changes (such as reducing the radius of curvature on the bilges) results in little change in ship motions. However, changes to overall ship proportions (such as beam-to-length or beam-to-draft) can have important consequences with respect to the seakeeping qualities of a ship. Therefore, it is important for seakeeping considerations to be addressed early in the design process, before the final proportions and dimensions of the hull have been finalized. For example, reducing the draft of a ship (for a given length and beam) has the effect of reducing motions. However, it also increases the likelihood of keel emergence. Increasing the local beam at the bow results in greater wave excitation action on the ship and, with flared bows, there is a risk of flare slamming. However, a large forward waterplane can reduce overall motions and reduce the probability of keel emergence.

1.3.3 Metacentric Height

As you, hopefully, remember from Hydrostatics and Stability, the larger the metacentric height, the more initial stability a vessel has. However, from a seakeeping perspective, the trouble with too large a metacentric height is that it has too high a natural roll frequency and this is associated with poor motion sickness indices (i.e. lots of people throwing up on your ship). However, if the metacentric height is too small the ship has a "lazy" roll motion and there is the increased risk of capsize. As metacentric height depends strongly on the beam, reducing the beam results in a reduction in GM.

1.3.4 Bales Seakeeping Index

The Bales seakeeping index is one way to assess the relative seakeeping performance of ship designs. It is a relative rating scale from 1 (worst) to 10 (best). It can be used confidently with destroyer-style hull shapes, but may not be very applicable beyond that. The Bales index is as follows:

$$\begin{aligned} R =& 8.422 + 45.104 C_{WF} + 10.078 C_{WA} - 378.465 \frac{T}{L} \\ &+ 1.2736 \frac{c}{L} - 23.501 C_{VPF} - 15.875 C_{VPA} \end{aligned}$$

where

C_{WF}	$=$	$\frac{2A_{WP}}{LB}$	waterplane coefficient fwd of midships
C_{WA}	$=$	$\frac{2A_{WA}}{LB}$	waterplane coefficient aft of midships
C_{VPF}	$=$	$\frac{V_F}{A_{WF}T}$	vertical prismatic coefficient fwd of midships
C_{VPA}	$=$	$\frac{V_A}{A_{WA}T}$	vertical prismatic coefficient aft of midships
T/L			draft to length ratio
c/L			cut-up ratio (c is the distance from the fwd perpendicular to the cut-up point (close to the stern))

V_F and V_A are the underwater volumes for the forward and aft ship sections, respectively. Using this rating system, certain trends in ship characteristics can be shown to improve seakeeping performance. For example, increasing the length relative to the draft, increasing B/T, and reducing C_{VP} will decrease pitching and heaving motions.

So, you now have a clear idea of how to design a ship with good seakeeping qualities, right? More likely you see the difficulties associated with designing a ship with such qualities. About the only general guideline is that the bigger the ship, the better the seakeeping performance. We will spend three chapters looking at what each input and aspect of the system does to contribute to the resulting motions.

Chapter 2

Review of Intact Statical Stability

Learning Objectives:

1. Explain the concepts of righting arm and righting moment
2. Calculate the righting moment of a ship given the magnitude of the righting arm
3. Read, interpret, and sketch a Curve of Intact Statical Stability (or Righting Arm Curve).
4. Discuss what tenderness and stiffness mean with respect to naval engineering
5. Evaluate the stability of a ship in terms of:
   a. Range of stability
   b. Dynamic stability
   c. Maximum righting arm
   d. Maximum righting moment
   e. Angle at which the maximum righting moment occurs
6. Create a Curve of Intact Statical Stability for a ship at a given displacement and assumed vertical center of gravity, using the Cross Curves of Stability
7. Correct a GZ curve for a shift of the ship's vertical center of gravity.
8. Correct a GZ curve for a shift of the ship's transverse center of gravity.
9. Find the Metacentric Height by using the GZ curve
10. Analyze and discuss damage to ships, including
    a. Use added weight method to calculate ship trim, angle of list and draft
    b. Qualitatively discuss the lost buoyancy method
    c. List the Navy Damage Stability Criteria for ships
11. Analyze and discuss free surface effects, including:
    a. Consequences of free surface on overall ship stability
    b. Ways to limit the effects of free surface
    c. Calculate the effective metacentric height

d. Give the meaning of a negative metacentric height
e. Correct the GZ curve for FSC

Before we dive into the complexity of ship performance in waves, let's have a brief review of ship statical stability that you learned in EN342. This section is going to follow the course notes for *Principles of Ship Performance* (Chapter 4). That chapter is concerned with the ability of the ship to remain upright when external forces are trying to roll it over.

2.1 The Internal Moment for a Heeled Ship

As a ship heels over (due to an external moment), it develops an internal moment. If the ship is stable, the internal moment acts in the opposite rotational direction to the direction of the heel angle. In this situation, the resultant weight of the ship is often not in vertical alignment with the resultant buoyant force so internal moments are produced. Figure 2.1[1] shows the sectional view of a ship that is being heeled over due to an external moment. It shows the relative positions of the center of gravity and the center of buoyancy for a ship that has been properly designed. Notice the perpendicular distance between the lines of action of the resultant weight and resultant buoyant force. This distance is the "righting arm" (GZ). To find the internal righting moment, multiply the righting arm by the magnitude of the resultant weight of the ship (or the magnitude of the resultant buoyant force). Thus, the righting moment is $RM = \bar{GZ} \cdot \Delta$.

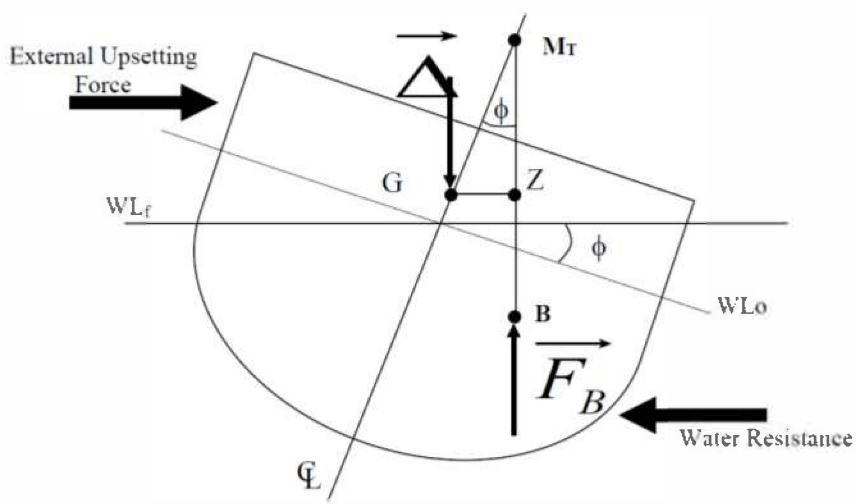

Figure 2.1: Heeled Ship due to an external moment

[1]from EN400 Course Notes

2.2 The Curve of Intact Statical Stability

Figure 2.1 is only a snapshot of the total stability picture. We are really interested in how the relationship between the buoyant force and weight changes as the ship heels over from zero degrees to large enough angles of heel to make the ship capsize. This is best represented with a graph of righting arm (GZ) versus heeling angle (ϕ). The graph is called the "curve of intact statical stability" or the "righting arm curve".

The curve of intact statical stability assumes the ship is being heeled over quasi-statically in calm water. Quasi-static means that the external moment heeling the ship over is doing so in infinitely small steps so that equilibrium is always present. Of course, this is impossible in reality, but it is acceptable as a concept for modeling the ship's stability. Be sure to realize that the predictions made by the curve of intact statical stability can not be directly applied to a rolling ship in a dynamic seaway (i.e. **seakeeping**). The dynamics of such a system, including the application of additional external forces and the presence of rotational momentum, are not considered in the intact statical stability curve. However, the intact statical stability curve is useful for comparison purposes. The stability characteristics of different hull shapes can be compared as well as differences in operating conditions for the same hull.

Figure 2.2[2] shows a typical intact statical stability curve. When the ship is in equilibrium with no outside forces acting on it, the resultant weight of the ship will be vertically aligned with the resultant buoyant force. As an external moment heels the ship to port or starboard, the resultant weight and the resultant buoyant force will become out of vertical alignment, creating the righting arm. The righting arm will obtain a maximum value and then decrease until the resultant weight of the ship and the resultant buoyant force are again in vertical alignment. Heeling any further will cause the ship to capsize. Figure 2.3[3] shows the vector diagrams of the buoyant and weight forces with respect to various points on the GZ curve in Figure 2.2.

Each intact statical stability curve is for a given displacement and given vertical center of gravity. The process of obtaining the actual intact statical stability curve is done by reading values off the "cross curves of stability" for a given displacement of the ship, and then making a sine correction to account for the proper vertical location of the center of gravity of the operating ship, a cosine correction for the correct transverse location of the center of gravity, and a sine correction to account for any free surface effect.

Several overall stability characteristics can be obtained from the curve of intact statical stability.

Range of Stability This is the range of angles for which there exists a righting moment. The range starts at the angle corresponding to the ship's equilibrium position with no external moments applied to it and goes to the angle at which the ship will capsize. For a ship with no initial angle of list the starting angle would be zero degrees. If the ship has a permanent angle of list, then the range is given from that angle of list to the capsizing angle of the heeled side. For example, in Figure 2.2, the range of stability is 0 to 85 degrees. The

[2] image from EN400 notes

[3] from EN400 course notes

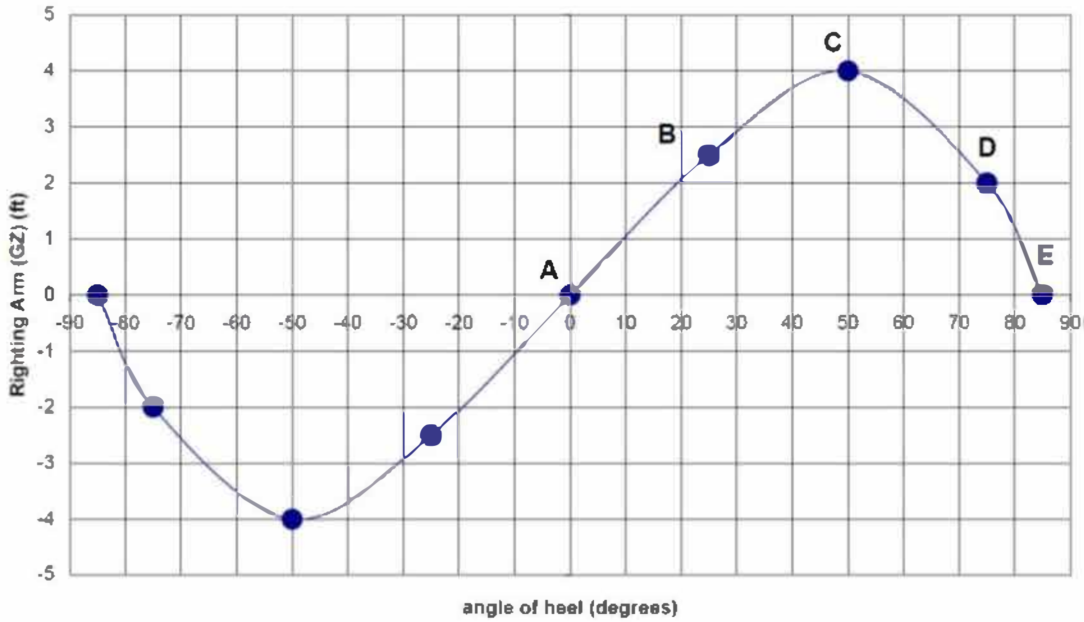

Figure 2.2: Curve of Intact Statical Stability

greater the range of stability, the less likely the ship will capsize. If the ship is heeled to any angle in the range of stability, the ship will exhibit an internal righting moment that will right the ship if the external moment ceases (and no other dynamics are at play).

Maximum Righting Arm (GZ_{max}) This is the largest internal moment arm created by the vertical mis-alignment of the buoyant force and the resultant weight vectors. It is simply measured as the peak of the curve of intact statical stability. In Figure 2.2 the maximum righting arm is 4.1 ft.

Maximum Righting Moment This is the largest static moment the ship can produce. It is simply calculated from the product of the ship's displacement (Δ_s) by the maximum righting arm (GZ_{max}). The standard units are in LT-ft. The larger the value of the maximum righting moment the less likely the ship will capsize. The maximum righting moment can't be shown directly on the curve of statical stability. Only the maximum righting arm is shown.

Angle of GZ_{max} This is the angle of heel at which the maximum righting moment occurs. Beyond this angle the righting moment decreases to zero. In Figure 2.2 the angle of maximum GZ is 50 degrees. It is desirable to have this angle occur at large degrees of heel so that a rolling ship will experience a righting moment that increases in magnitude over a greater range of heeling angles.

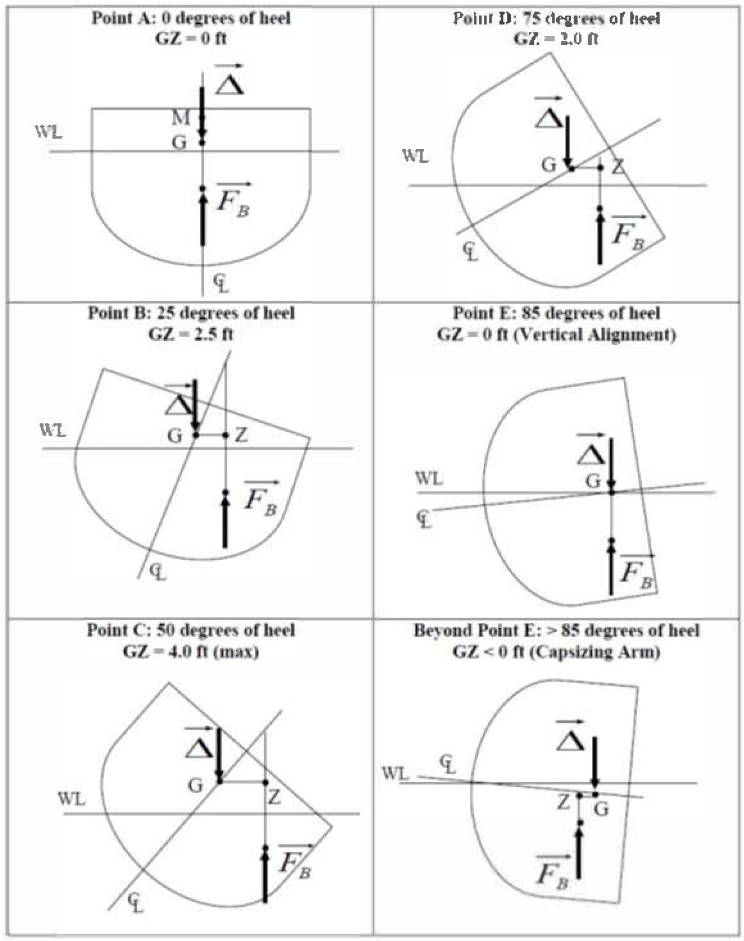

Figure 2.3: Vector Drawings Associated with Figure 2.2

Dynamic Stability This is the work done by quasi-statically (very slowly) rolling the ship through its range of stability to the capsizing angle. Mathematically, this work is

$$\Delta_s \int GZ \mathrm{d}\phi$$

This is the product of the ship's displacement with the area under the curve of intact statical stability. The standard units are LT-ft. The dynamic stability can't be shown directly on the curve of intact statical stability, but the area under the curve can be shown.

A Measure of the Tenderness or Stiffness The initial slope of the intact statical stability curve indicates the rate at which a righting arm is developed as the ship is heeled over. If the initial slope is large, the righting arm develops rapidly as the ship is heeled over and the ship is said to be "stiff". A stiff ship will have a short period of roll and react very strongly to external heeling moments. The ship will try to upright itself very quickly and forcefully. If the ship is too stiff, violent accelerations can damage ship structures and be harmful to personnel. If the initial slope is small, the righting arm develops slowly as the ship is heeled over and the ship is said to be "tender". A tender ship will have a long period of roll and react sluggishly to external heeling moments. Too tender of a ship can compromise stability and leave too little margin for capsizing. This initial slope is equal to the ship's metacentric height.

2.3 Cross Curves of Stability

The cross curves of stability are a series of curves on a single set of axes. The x-axis is the displacement of the ship in LT. The y-axis is the righting arm of the ship in feet. Each curve is for one angle of heel. Typically angles of heel are taken each 5 or 10 degrees. Figure 2.4 is a set of cross curves for the FFG-7.

The entire set of curves assumes an arbitrary location for the vertical center of gravity of the ship. Typically the assumed location of the center of gravity is the keel (KG = 0). The actual location of the assumed value of the center of gravity of the ship will always be marked on the cross curves. The cross curves are made by a series of integrations based on hull geometry. In summary, the intact statical stability curves, for a single displacement, comes from reading values off the cross curves of stability and correcting for the actual location of the center of gravity and any free surface effects that may be involved.

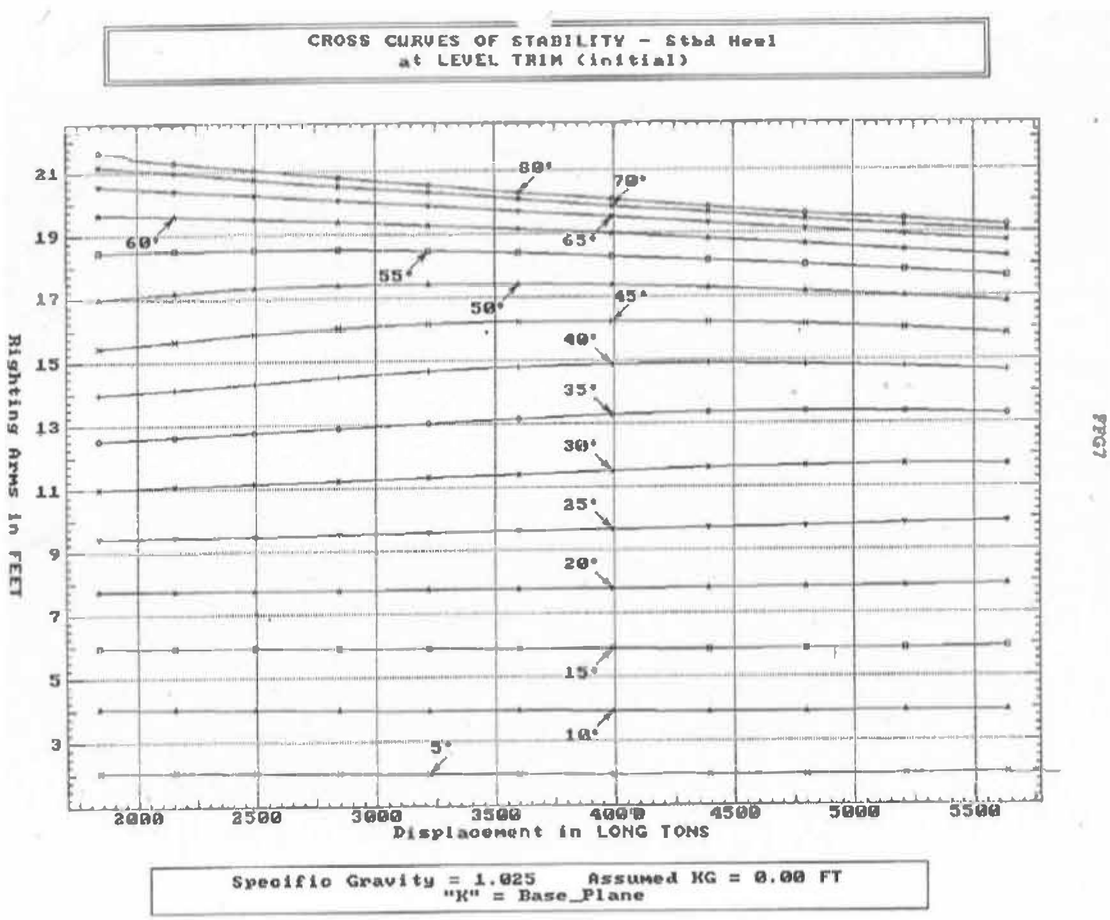

Figure 2.4: Cross Curves of Stability for the FFG-7

2.4 Correcting for Center of Gravity Location

To correct the righting arm values from the cross curves of stability, the vertical and transverse position of the center of gravity must be known. The righting arm can be corrected for the vertical position of the center of gravity using a sine correction and the transverse position of the center of gravity using a cosine correction.

If G_v is the final vertical location of the center of gravity, and G_0 is the initial location, then the value of the righting arm, G_vZ_v at each angle of heel may be found using the following relationship:

$$G_vZ_v = G_0Z_0 - G_0G_v \sin\phi$$

where ϕ is the heel angle in question, G_0G_v is the vertical distance between G_0 and G_v, and G_0Z_0 is the righting arm at the initial center of gravity location. If the center of gravity moves down, the distance G_0G_v would be negative. If the hull were heeling to port, ϕ would be negative.

If G_t is the final transverse location of the center of gravity, and G_v is the initial location, then the value of G_tZ_t at each angle of heel may be found using the following relationship:

$$G_tZ_t = G_vZ_v - G_vG_t \cos\phi$$

where ϕ is the heel angle in question, G_vG_t is the transverse distance between G_v and G_t, and G_vZ_v is the righting arm at the initial center of gravity location. If the center of gravity moves to port, the distance G_vG_t would be negative. If the hull were heeling to port, ϕ would be negative.

To combine into a single equation, consider a condition with an initial center of gravity position at G_0 (if starting at the cross curves of stability, this would be a center of gravity on the centerline at the keel). First we consider the vertical shift from G_0 to G_v and then the transverse shift from G_v to G_t:

$$GZ = G_0Z_0 - G_0G_v \sin\phi - G_vG_t \cos\phi.$$

2.5 Free Surface Correction (at small angles of heel)

A free surface is fluid that is allowed to move freely, such as water in a partially filled tank. As the ship lists, the fluid in the tank moves. The fluid movement acts like a weight shift, causing the center of gravity of the fluid to move, which causes the ship's center of gravity to shift in both the vertical and horizontal directions. The effect of the vertical shift is negligible at small angles ($\phi = 5$ to 7 degrees) but the transverse shift causes a decrease in the righting arm (GZ).

It is shown graphically in Figure 2.5[4] that a vertical rise in the center of gravity also causes a shortened righting arm. The distance the center of gravity would have to rise to cause a reduction in the righting arm equivalent to that caused by the actual transverse shift is called the *Free Surface Correction* (FSC). The position of this new center of gravity is called the "virtual" center of gravity (G_v). The distance from the virtual center of gravity to the metacenter is called the *Effective Metacentric Height* (GM_{eff}).

[4] from EN400 course notes

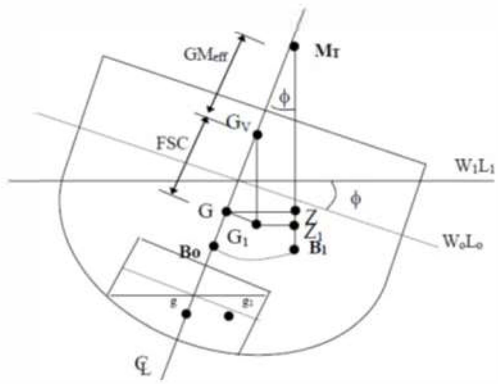

Figure 2.5: The Free Surface Correction (list angle in drawing is exaggerated to show geometry)

2.5.1 Static Effects

The static effects of a free surface are adverse resulting in a virtual rise in the center of gravity, a smaller range of stability, a smaller maximum righting arm, a smaller angle at which the maximum righting arm occurs, and an exaggerated list and trim if the ship is listing or trimming.

2.5.2 Dynamic Effects

It should be noted that the preceding analysis is referring to the static effects of a free surface. It has nothing to do with the dynamic effects of the water rushing back and forth. This effect is also detrimental, but is not described by the free surface correction. It is a common misconception to mix the dynamic effects of a free surface with the static analysis and the FSC. Baffles are a good way to minimize the dynamic effects of free surface.

2.5.3 Calculating the FSC and $\mathrm{GM}_{\mathrm{eff}}$

The free surface correction (FSC) created by a tank within a ship is given by the following equation:

$$\mathrm{FSC} = \frac{\rho_i i_t}{\rho_s \nabla_s}$$

where ρ_i is the density of the fluid in the tank, ρ_s is the density of the water the ship is floating in, ∇_s is the underwater volume of the ship, and i_t is the transverse second moment of area of the tank's free surface area.

The formula for the second moment of area of a rectangle is given by the following equation:

$$i_t = \frac{(\text{length})(\text{width})^3}{12} = \frac{l \cdot b^3}{12}.$$

The free surface correction is applied to the original metacentric height to find the effective metacentric height:

$$GM_{eff} = GM - FSC = KM - KG - FSC.$$

If any of the terms in this equation are not familiar to you, be sure to go back to your EN342 notes/book and review!

2.5.4 Effect of a Free Surface on GZ

A free surface causes a reduction in the ship's righting arm, range of stability, and dynamic stability. With a free surface, the ship now behaves as if the center of gravity were located at the virtual center of gravity. To calculate the effective righting arm of a ship with a free surface, the original righting arm must be corrected for the virtual rise in G caused by the free surface.

$$G_1Z_1 = GZ - FSC \sin\phi.$$

The worst case for a free surface is when the ship's transverse center of gravity is located off the centerline. Not only has the overall stability been reduced by the transverse location of G, but the effective rise in G due to the free surface further reduces the righting arm, range of stability, and dynamic stability. To correct the righting arm curve for a free surface and a transverse change in G, one must first correct GZ for the virtual rise in G caused by the free surface using the sine correction, then correct GZ for the transverse location of G using the cosine correction. This correction is given by the following equation:

$$G_1Z_1 = GZ - FSC \sin\phi - TCG \cos\phi.$$

2.5.5 Damage Stability

Naval ships are intended to go in harms way. When the shooting starts the object is to do harm to others, but sometimes damage to your ship is unavoidable. If the watertight portion of the hull is breached and water pours into the ship, the draft will increase, the trim will change, a permanent angle of list will result, and stability will be affected. In extreme circumstances the ship could be lost. This section discusses the fundamental behavior of a damaged ship and reviews 2 techniques that allow for its analysis:

- The Lost Buoyancy Method
- The Added Weight Method

Lost Buoyancy Method

In the lost buoyancy method we analyze changes in buoyancy rather than the center of gravity or displacement. Simply stated, the center of gravity remains the same (the ship weight, metal, etc. is constant) and any changes due to damage effect the distribution of the buoyancy volume. The total buoyant volume must remain constant since the weight of the ship is not changing. The draft will increase and the ship will list and trim until the buoyant volume is regained.

The lost buoyancy method allows a damaged ship to be modeled mathematically so that the final drafts, list, and trim can be determined from assessed damage. The engineer can analyze every conceivable damage scenario and produce a damage stability handbook that may be used by the crew in the event of flooding. Using the lost buoyancy method allows "a priori" knowledge of the resulting stability condition of the ship so that appropriate procedures can be written and followed in the event of a breach in the ship's hull.

Added Weight Method

As the name suggests, in the added weight method the ship is assumed undamaged but part of it is filled with the water the ship is floating in. This is equivalent to a weight addition. Provided the volume of the damaged compartment, its average location from the centerline, keel, and midships, and the water density is known, the change in the ship's center of gravity can be predicted along with the consequences of this shift upon the draft, trim, and list of the ship.

Permeability An added complication to the analysis of a damaged ship is the space available in a damaged compartment for the water to fill. When a compartment is flooded, it is rare for the total volume of this compartment to be completely filled with water. This is because the compartment will already contain certain equipment or stores depending upon its use. The ratio of the volume that can be occupied by water to the total gross volume is called the "permeability":

$$\text{Permeability} = \frac{\text{volume available for flooding}}{\text{total gross volume}} = \mu$$

US Navy Damage Stability Design Criteria

There are quite a few Navy Intact Stability Design criteria, but here we review some of the damage criteria:

Margin Line The margin line defines the highest permissible location on the side of the vessel of any damaged waterplane in the final condition of sinkage, trim, and heel. It is in no case permitted to be less than 3 inches (0.075 m) below the top of the bulkhead deck at the side.

List The list caused by damage shall not exceed 20 degrees. This angle is too great for continuous operation of equipment. Naval machinery is designed to operate indefinitely at a permanent list of 15 degrees, although most equipment will probably remain functional up to about 25 degrees for at least a few hours. Personnel can continue damage control efforts effectively at a permanent list of 20 degrees. At a permanent list of 20 degrees, the ship will possess adequate stability against wind and waves to be towed, at the very least.

Extent of Damage to the Hull

1. Ships less than 100 ft long are required to withstand flooding in one compartment.

2. Ships 100-300 ft long are required to withstand flooding in any two adjacent compartments.

3. Warships, troop transports, and hospital ships over 300 ft long are required to withstand a hull opening of 15% of the length between perpendiculars.

4. Any other ship over 300 ft long is required to withstand a hull opening of 12.5% of the length between perpendiculars.

Chapter 3

The Input: Waves

Learning Objectives:

1. **For Regular Waves:**
   (a) Identify wave length, wave frequency, and wave celerity given the water elevation measurements over time.
   (b) Explain the distinction between wave celerity and group velocity.
   (c) Calculate the theoretical wave celerity and group speed given wave frequency and water depth.
   (d) Write the equation for regular sinusoidal water elevation as a function of time given the wave frequency (ω) or wave length (λ) and the wave height or amplitude (H_0 or ζ_0).
   (e) Calculate the average energy per unit area of the wave surface.
   (f) Calculate the pressure at a point beneath the water surface, accounting for wave motion.
   (g) Describe the particle motion below a water wave.

2. **For Irregular Waves:**
   (a) Find the mean water elevation, wave height, zero-crossing period, and peak period for a given irregular wave time history.
   (b) Find the variance of a water elevation time history.
   (c) Explain what the significant wave height is and how to find it from the irregular *wave time history.*
   (d) Explain the concept of linear superposition and how it applies to modeling irregular waves.
   (e) Describe how the Fourier Transform can be used to find the frequency content of an irregular wave time history.
   (f) Describe how to use a Fourier Transform in finding the wave energy spectrum of an irregular wave time history.
   (g) State what information is shown in a wave energy spectrum.

(h) State the equation for the spectral ordinate of the wave energy spectrum and how the spectral ordinate relates to the wave amplitude.

(i) Explain how to find a spectral moment and what the zeroth, second, and fourth spectral moments correspond to with respect to the water surface.

(j) Find the mean water elevation, wave height, zero-crossing period, and peak period for a given irregular wave energy spectrum.

(k) Explain what the significant wave height is and how to find it from the irregular *wave energy spectrum*.

(l) State the three idealized wave spectra used in this course, what parameters each depends on, and their equations.

(m) Identify what ocean conditions each idealized wave spectrum is most suited to model.

(n) Explain what a "probability of exceedance" is and be able to calculate it for a given sea condition.

(o) State which probability distribution can be used to describe probabilities related to **water elevation** and **wave heights**.

3. **Laboratory Objectives:**

(a) Calculate the wave length and frequency of a regular wave using wave elevation data (using both digital signal processing techniques and equations)

(b) Calculate the wave celerity and group velocity for regular waves using wave elevation time history at two different, known locations.

(c) Identify the effect of water depth on wave celerity and group velocity.

(d) Describe the purpose of the discrete Fourier Transform (DFT) for analyzing wave measurements.

(e) Explain the connection between an irregular wave time history and the related wave energy spectrum.

(f) Create, measure, and analyze complex waves using a FFT.

What are water waves? How do they move? How do we describe them? To tackle this topic, I am going to start with the simple (yet unrealistic) concept of "regular" waves. Once we have an understanding of how these hypothetical waves (that can be accurately produced in the laboratory) work, we can move to discussing how to deal with more realistic wave conditions.

3.1 Regular Waves

Regular waves are shaped like a sine wave moving along the surface of the water. We are going to start this discussion with long-crested, deep water waves. This type of wave is *periodic*, meaning it has a consistent frequency or period of occurrence. Table 3.1 gives the

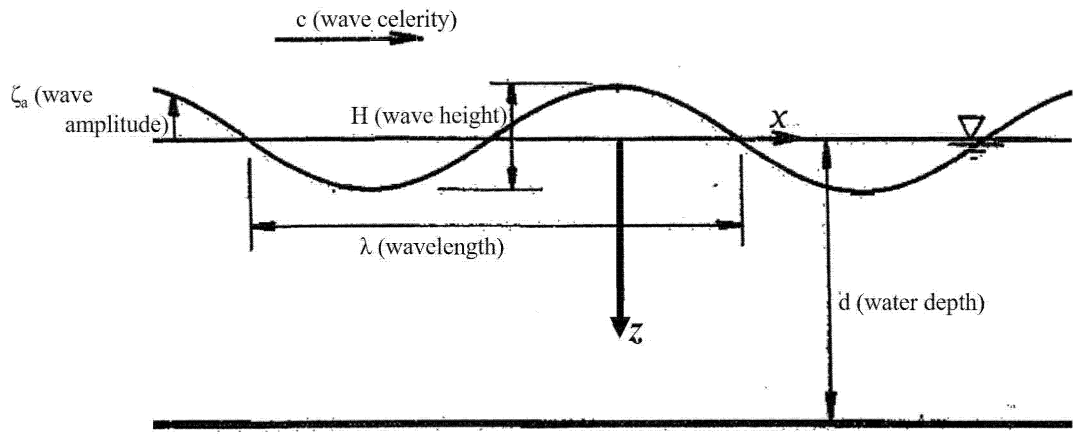

Figure 3.1: Regular Wave Characteristics

notation we will use to describe the characteristics of our regular wave and Figure 3.1 shows a wave with the components labeled. Our wave is a progressive wave, meaning it moves horizontally over the water surface (a wave that merely oscillates up and down is considered a standing wave). The shape of each wave passing by looks the same and the whole wave train can be viewed as an advancing rigid corrugated sheet.

The following notation (following reference 2) is used for describing water waves:

ζ	the instantaneous depression of the water surface below the mean level ($y = 0$)
ζ_0	the wave amplitude from the mean level ($y = 0$) to a crest or trough
H	wave height (always twice the wave amplitude)
λ	wave length (distance from one crest - or trough - to the next)
c	wave celerity
T	wave period (time interval between successive crests - or troughs - passing a fixed point)
α	the instantaneous wave slope (gradient of the surface profile)
α_0	maximum wave slope or wave slope amplitude
H/λ	wave steepness

Table 3.1: Nomenclature for describing waves

If we were to consider only a single point in space and describe the water surface elevation at that point as the wave moves past, we have the following mathematical expression:

$$\zeta(t) = \zeta_0 \sin(\omega t - \epsilon)$$

In this expression, t is the variable for time, ζ_0 is the wave amplitude, ϵ is the phase angle (the degrees the shape is different from a perfect sine wave) and ω is the wave frequency (in radians/second) - i.e. a measure of the oscillations that pass this point in one second. If we instead consider the entire wave train in space, but only for a single moment in *time*, the mathematical expression is:

$$\zeta(x) = \zeta_0 \sin(kx)$$

In this expression x is the variable for position and k is the **wave number** (in m^{-1} or ft^{-1}). The wave number represents the frequency as a function of wavelength - the number of cycles that occur over a unit of length. The expression for wave number is

$$k = \frac{2\pi}{\lambda}.$$

But water waves exist and change in both *time* and *space* - you can stay with a wave and move through space in time **or** you can stay at one location and see the wave move past in time. So the equation for the water elevation must account for the point at which we are measuring (where are you standing?) and the time the measurement is made (what time are you looking?).

$$\zeta(x,t) = \zeta_0 \sin(kx - \omega t - \epsilon)$$

For regular waves in deep water, there is a fixed relationship between the frequency of the wave, the length of the wave, and the speed that the wave travels. For a high frequency wave, there is only a short time between peaks and, therefore, the wave length is very short. For a low frequency wave, there is a long time between peaks and the wave length is long. *For deep water*, these relationships are as follows:

$$T = \frac{2\pi}{\omega}$$
$$\lambda = \frac{2\pi g}{\omega^2} = \frac{gT^2}{2\pi}$$

As the water depth becomes shallower, the relationship between wave length and wave frequency changes. In shallow water the wave length depends only on the water depth. So, *for shallow water*:

$$\lambda = 2\pi d$$

where d is the water depth.

The general relationship between wave frequency, wave length, and water depth is given by the **"Dispersion Equation"**, which includes a hyperbolic tanh:

$$\omega^2 = gk \tanh kd.$$

Hyperbolic Functions

Hyperbolic functions are combinations of exponentials:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

For water waves, we are using the kd term (or $2\pi d/\lambda$) in the hyperbolic function. When the water depth is very large relative to the wave length, kd is large. When depth is small relative to the wave length, kd becomes very small. We can simplify the hyperbolic expression in these situations:

Function	Large kd	Small kd
$\sinh kd$	$e^{kd}/2$	1
$\cosh kd$	$e^{kd}/2$	kd
$\tanh kd$	1	kd

Using these simplifications of the hyperbolic functions, we can show that in deep water the wave frequency depends only on the wave length,

$$\omega = \sqrt{gk}$$

while in shallow water the wave frequency also depends on the water depth,

$$\omega = \sqrt{gk^2 d}.$$

Wave Celerity

The wave celerity, that is the speed of the wave traveling over the water surface, is given by:

$$c = \sqrt{\frac{g}{k}\tanh kd}.$$

As with the wave length, the equation for the velocity of the wave in deep and shallow water can be found by simplifying this equation. In deep water the wave celerity is:

$$c = \sqrt{\frac{g}{k}}$$

or

$$c = \frac{g}{\omega}.$$

In shallow water the wave celerity is:

$$c = \sqrt{gd}.$$

This relationship helps explain why tsunamis are difficult to observe out in the open ocean yet develop into towering waves as they approach the shore. When in deep water the tsunami wave has a very long wave length and is traveling extremely fast. However, as the wave approaches the shore the wave speed (and length) becomes determined by the water depth and the wave has to slow down. Thus, the energy in the wave that was stored in the speed (kinetic energy) is transformed into energy stored in the wave amplitude (potential energy).

Energy Transmission and Group Velocity

While celerity can give you the velocity of the crest of a wave moving over the water surface, the group velocity gives the velocity of the energy associated with the wave. This is a strange concept, but it can be demonstrated in a wave tank or tow tank. When the wave maker sends the first wave of a regular wave train down the tank, you can witness the height of the wave decreasing as it travels and eventually that first wave disappears. This is because the energy in the wave (which is seen in the wave height) is traveling half as fast as the wave crest (for waves in deep water). How does this work? Consider one wave in the middle of a train of waves, all nominally the same height and wavelength. This one wave moves one wavelength forward in one wave period, but it only takes 1/2 of the energy with it. However, the wave directly in front also took only 1/2 of its energy, so our wave gets the left over energy (returning to full energy level) and maintains its current wave amplitude. Now, what happens to the wave at the front of the wave train? There is no wave in front of it to leave any energy. So, each wavelength this wave moves forward decreases its energy level by 1/2, which results in a reduced wave amplitude! And what happens to the wave at the back of the wave train? When it travels forward it leaves 1/2 of its energy behind, but there is no wave behind to make use of that energy. So, the 1/2 energy of the last wave that is left behind creates a new wave at a lower amplitude! An observer thus sees waves disappear from the front of a wave train and appear at the back. Be sure to observe this phenomenon when we have the regular waves lab!

For any depth water, the mathematical expression for group velocity is:

$$u_G = \frac{c}{2}\left(1 + \frac{2kd}{\sinh 2kd}\right).$$

Just as for the wave celerity, we can simplify this expression for considering just deep or shallow water. For deep water the group velocity is equal to half the wave celerity:

$$u_G = \frac{c}{2} = \frac{g}{2\omega}.$$

In shallow water the group velocity is equal to the wave celerity (the wave crest and the energy of the wave travel at the same speed):

$$u_G = c = \frac{g}{\omega}.$$

The energy associated with a train of regular waves includes contributions from both potential and kinetic energy. Consider a vertical chunk of wave that has a height of ζ. The center of gravity of this chunk is located in the middle $\zeta/2$, and has a mass of $\rho g \delta \mathrm{x}$. The potential energy of this chunk (mgh) is thus,

$$(\rho g \zeta \delta \mathrm{x})\frac{\zeta}{2} = \frac{\rho g \zeta^2 \delta \mathrm{x}}{2}.$$

If we integrate this energy over the entire wavelength, we get the potential energy of the wave (per unit width),

$$E_{PE} = \frac{\rho g \lambda \zeta_0^2}{4}.$$

The wave also has energy stored as kinetic energy ($\frac{1}{2}mv^2$). If the total velocity of a segment is q, the kinetic energy of a segment is

$$\frac{\rho q^2 \delta \mathrm{x} \delta \mathrm{z}}{2}.$$

Integrating this over the full wavelength gives the kinetic energy of the wave (per unit width) and using the relationship between wave speed and wavelength,

$$E_{KE} = \frac{\rho g \lambda \zeta_0^2}{4}.$$

So, adding the potential and kinetic energy contributions together provides the total energy for a wave per unit area of sea surface,

$$\bar{E} = \frac{\rho g \zeta_0^2}{2}$$

Deep Water

How do we determine if water depth is "deep" or not? Water is deep when the water particles involved in the wave motion do not detect the bottom. For deep water waves, the water particles move in a *circular* motion. This means that the particles are NOT traveling with the wave, but the wave passes along while the particles stay in pretty much the same spot. You have experienced this if you have ever bounced up and down in the ocean as waves pass you by. Although the waves move you up and down, there is very little sideways motion. The particles near the surface of the water make large circular motions, but as you go deeper in the water the particle motion decreases in amplitude (see Figure 3.2). The motion of water particles at a depth of z is given by,

$$\zeta(x, z, \omega, t) = e^{-kz}\zeta_0 \sin(kx - \omega t - \epsilon)$$

Eventually the circular motion becomes so small that the water particles don't move as the wave passes by. You have experienced something like this if you have every tried to avoid a wave by diving deep as it travels past. If you had stayed on the surface you would have been moved all over by the breaking wave, but by diving deep you feel only a slight push or pull as the wave travels past. The water depth is considered deep if the particles near the bottom don't react to the wave moving past. If the water particles near the bottom move back and forth as the wave passes by, the water depth cannot be considered deep. Typically, we can assume water depth is deep if the water depth is greater than half the wave length:

$$d > \lambda/2 \text{ for deep water.}$$

For comparison, we consider truly shallow water to be when the water depth is $1/20^{\text{th}}$ of the wavelength.

Table 3.2 shows the information given in reference 2 relating the different wave characteristics to each other for deep water waves. The water is considered deep when it is greater than half the wave length.

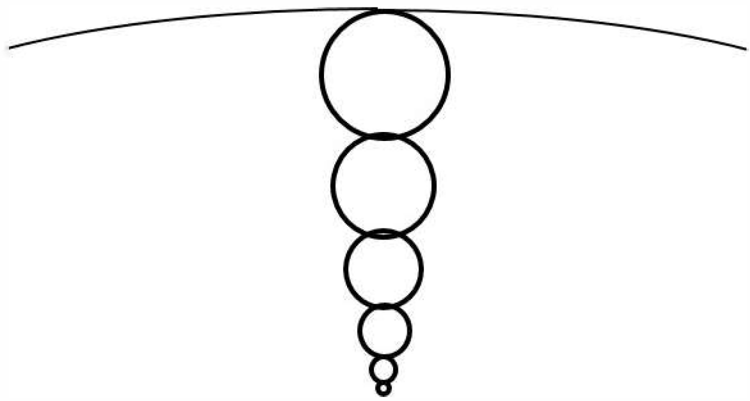

Figure 3.2: Decreasing Particle Motion as a Function of Depth

-	ω	**T**	**k**	λ	**c**	$\mathbf{u_G}$
ω	-	$T = \frac{2\pi}{\omega}$	$k = \frac{\omega^2}{g}$	$\lambda = \frac{2\pi g}{\omega^2}$	$c = \frac{g}{\omega}$	$u_G = \frac{g}{2\omega}$
T	$\omega = \frac{2\pi}{T}$	-	$k = \frac{4\pi^2}{gT^2}$	$\lambda = \frac{gT^2}{2\pi}$	$c = \frac{gT}{2\pi}$	$u_G = \frac{gT}{4\pi}$
k	$\omega = \sqrt{gk}$	$T = \frac{2\pi}{\sqrt{gk}}$	-	$\lambda = \frac{2\pi}{k}$	$c = \sqrt{\frac{g}{k}}$	$u_G = \sqrt{\frac{g}{4k}}$
λ	$\omega = \sqrt{\frac{2\pi g}{\lambda}}$	$T = \sqrt{\frac{2\pi\lambda}{g}}$	$k = \frac{2\pi}{\lambda}$	-	$c = \sqrt{\frac{g\lambda}{2\pi}}$	$u_G = \sqrt{\frac{g\lambda}{8\pi}}$
c	$\omega = \frac{g}{c}$	$T = \frac{2\pi c}{g}$	$k = \frac{g}{c^2}$	$\lambda = \frac{2\pi c^2}{g}$	-	$u_G = \frac{c}{2}$
$\mathbf{u_G}$	$\omega = \frac{g}{2u_G}$	$T = \frac{4\pi u_G}{g}$	$k = \frac{g}{4u_G^2}$	$\lambda = \frac{8\pi u_G^2}{g}$	$c = 2u_G$	-

Table 3.2: Linear Wave Relationships

Wave Slope

The *slope* of a wave surface, α, is the angle the water surface makes with the calm water surface. The slope changes as you move along the wave reaching a maximum as you travel from the trough to the crest and at a minimum when measured at the crest or trough. The maximum wave slope is,

$$\alpha_0 = \frac{2\pi}{\lambda}\zeta_0.$$

Pressure under a Wave

What is the pressure at a point under the wave? If the water surface was level (calm water), the pressure at a point beneath that calm surface would depend on the depth, z_B,

$$P_B = \rho g z_B.$$

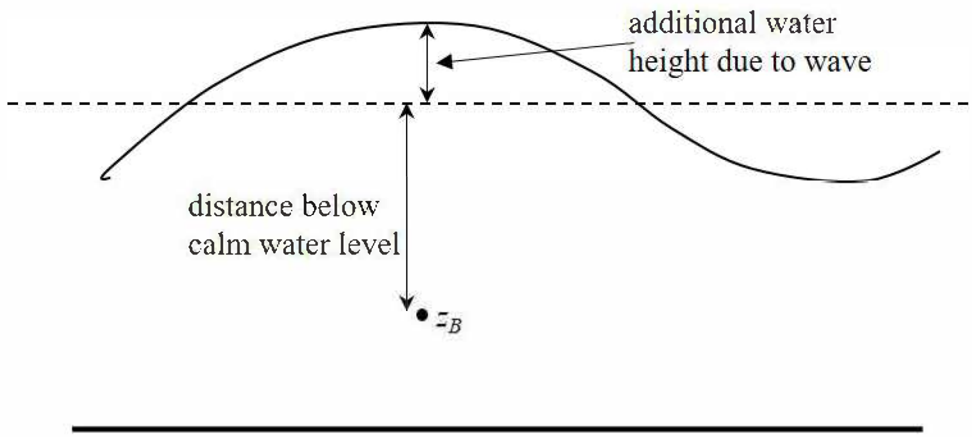

Figure 3.3: Figure showing a point z_B beneath a wave

The wave height would appear to add an additional hydrostatic component based on the water surface elevation above (or below) the calm water level (see Figure 3.3) However, due to the dynamic nature of the wave and the decreasing effect of the particle motion, the pressure fluctuation due to the wave is **less** than the additional hydrostatic contribution! The hydrostatic contribution due to the wave is actually

$$\tilde{P} = -(\rho g \zeta_0)e^{-kz_B}\sin(kx - \omega t - \epsilon).$$

where the e^{-kz_B} term accounts for the decreasing effect of the wave with depth. Adding this to the calm surface hydrostatic pressure gives the pressure at a point under a wave as

$$P = \rho g z_B - (\rho g \zeta_0)e^{-kz_B}\sin(kx - \omega t - \epsilon).$$

3.2 Irregular Waves

We are now going to move on to waves that have characteristics more similar to naturally occurring "real" waves. Waves are formed from wind blowing over the surface of the water. This energy transfer continues after the waves have formed. The energy absorption of the waves is countered by wave breaking and viscosity. Consider the time-history of water elevation as measured by a buoy in the ocean shown in Figure 3.4. A histogram can be used to evaluate the range of water elevation variation. To create such a histogram, water surface measurements at a particular location are made at regular intervals (say, every 1 minute). The measurements are then grouped into elevation ranges. For example, the number of measurements between 0 ft (the calm water surface) and 0.25 ft are recorded, then the number of measurements between 0.25 ft and 0.5 ft are recorded, and so on for all measurements. If we divide the number of measurements in each group by the total number of measurements, it gives a percentage of measurements (or occurrences) within each elevation range. Figure 3.5 shows a typical histogram for water elevation measurements made in a seaway. As is generally the case, the histogram of water elevation has the shape of a Gaussian (or normal) curve. If, however, instead of plotting all the water elevation measurements, only the *wave heights* are

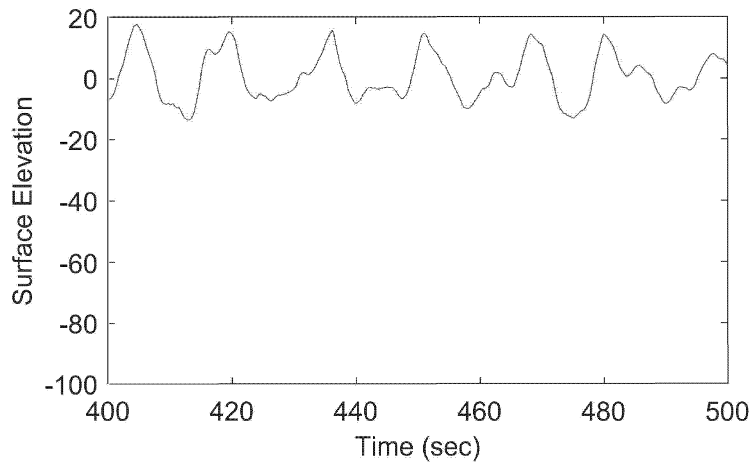

Figure 3.4: Water Surface Elevation from Hurricane Andrew (1992)

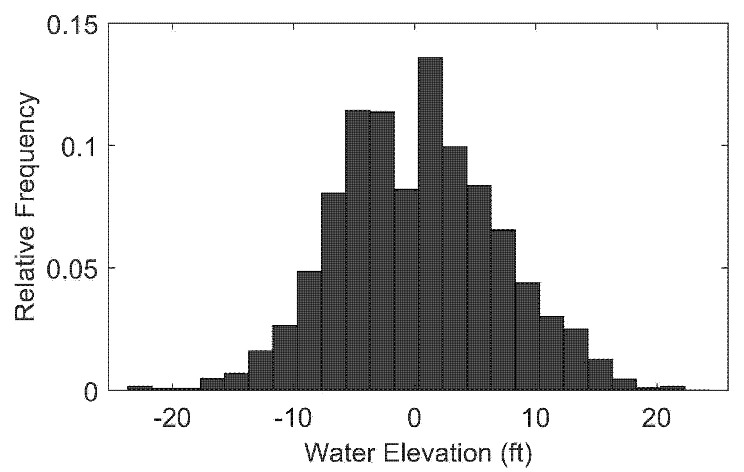

Figure 3.5: Hurricane Andrew Water Elevation Histogram

made into a histogram, the shape of the histogram is of a Rayleigh curve. Figure 3.6 shows the wave height histogram for the same data set as in Figure 3.5.

Unlike with regular waves, there are no global parameters that make describing the shape of an irregular sea straight-forward. The shape is generally sinusoidal, but each oscillation has a different amplitude and different period. Some of the oscillations don't even go below the calm water surface, but ride on other waves! To be able to summarize such a system, we need to use the tools provided by *statistics*.

Useful Statistical Measures

The easiest statistical measures to work with are means or averages. There are different characteristics of the wave signal that we can average. For example, we can average all the measurements of the water elevation, ζ. This will give us the mean water level. For notation, averages are written with a bar, so the mean water level is written as $\bar{\zeta}$. The mean water

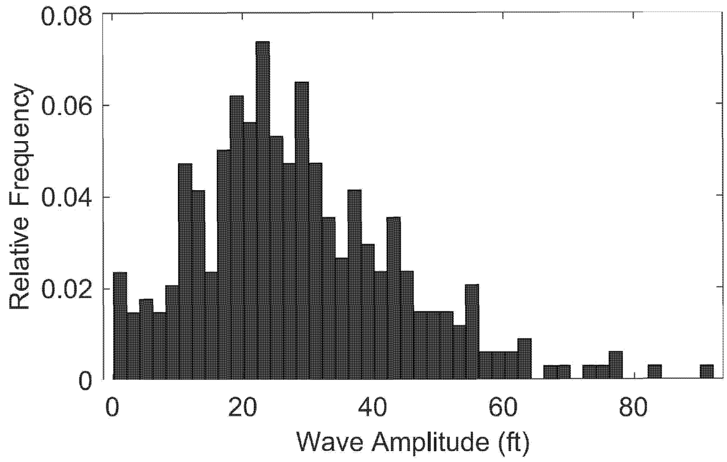

Figure 3.6: Hurricane Andrew Wave Height Histogram

level for the data shown in Figure 3.4 is 0.25 ft. We can also measure the peak amplitude for every wave in the signal, ζ_a. The mean of all the peaks would be the mean wave amplitude, $\bar{\zeta}_a$. The mean wave height would be twice the mean wave amplitude,

$$\bar{H}_a = 2\bar{\zeta}_a.$$

The time between each peak can also be measured, T_p and averaged, giving the mean period of the peaks, $\bar{T}_p$. The time between zero-crossings (the time between the water surface passing up through the nominal zero water level) can be measured for all waves, T_z and averaged, $\bar{T}_z$. The mean of any N set of numbers (x) is given by

$$\bar{x} = \sum_{n=1}^{N} \frac{x_n}{N}. \tag{3.1}$$

The average value is also referred to as the "expected" value. So, the mean water elevation is

$$\bar{\zeta} = \sum_{n=1}^{N} \frac{\zeta_n}{N}$$

where N is the number of measurements and ζ_n is each measurement of the water surface.

The variance of a set of numbers is a measure of how spread out the data is (i.e. how far the numbers lie from the mean). The variance of the water elevation is given by

$$m_0 = \sum_{n=1}^{N} \frac{(\zeta_n - \bar{\zeta})^2}{N}. \tag{3.2}$$

The standard deviation is another measure of the dispersion of the data. If the standard deviation is small, the data points are close to the mean. If the standard deviation is large, the data points are spread out over a large range of values. The standard deviation is equal to the square root of the variance,

$$\sigma_0 = \sqrt{m_0}.$$

For meaningful results, the data must include **AT LEAST** 100 pairs of peaks and troughs.

Another useful concept when explaining waves is the value of the *significant wave height*. This value is the average of the highest 1/3rd of the wave heights in the measurements. To find the significant wave height, you measure all of the wave amplitudes and order them by magnitude. For example, from the data shown in Figure 3.4 the largest 10 wave amplitudes are 31.4, 31.8, 33.6, 34.8, 36.4, 37.6, 38.6, 38.6, 41.6, and 45.2. Next, identify the set of amplitudes that consist of the top third and take the average. For the data set shown in Figure 3.4, the average of the top $1/3^{\text{rd}}$ of the wave amplitudes is 23.2 ft. To find the significant wave height, we double this number,

$$\bar{H}_{1/3} = 2\bar{\zeta}_{1/3},$$

so for our data set the significant wave height is 46.4 ft.

3.2.1 Superposition and Fourier Analysis

Another way to describe the waves shown in Figure 3.4 is to look at the frequency information in the signal. The frequency components of the wave exist because of the principle of superposition.

Principle of Superposition

Consider two waves traveling past the same point ($x = 0$) that have the same amplitude, ζ_0, but different frequencies, ω_1 and ω_2:

$$\zeta_1(t) = \zeta_0 \sin(\omega_1 t)$$
$$\zeta_2(t) = \zeta_0 \sin(\omega_2 t).$$

The water elevation at this point will be the sum of the two waves traveling past:

$$\zeta(t) = \zeta_1(t) + \zeta_2(t)$$
$$\zeta(t) = \zeta_0 \sin(\omega_1 t) + \zeta_0 \sin(\omega_2 t).$$

The top image in Figure 3.7 shows how two waves both with a wave amplitude of 1 ft but different wave frequencies of 2π rad/sec and 3π rad/sec, respectively, combine to form a new wave. If we also change the wave amplitude (the second wave now has an amplitude of 0.5 ft) and add a phase angle (the second wave now has a phase angle of $\epsilon = \pi/2$), the combined wave changes. This is shown in the bottom image of Figure 3.7. The more waves of different frequencies, amplitudes, and phase angles are used, the more complex the resulting water elevation becomes. Fiqure 3.8 shows a wave consisting of 50 component frequencies.

Just as we can create an irregular wave train by combining many components of different frequencies, we can identify the different frequency components in a given wave train using

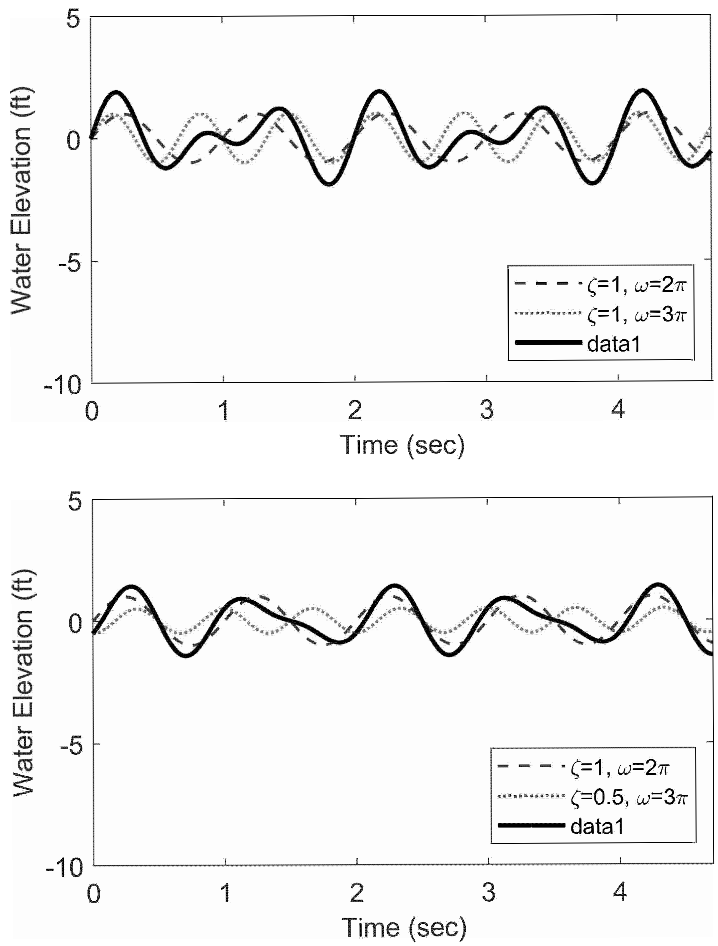

Figure 3.7: Superposition of two waves

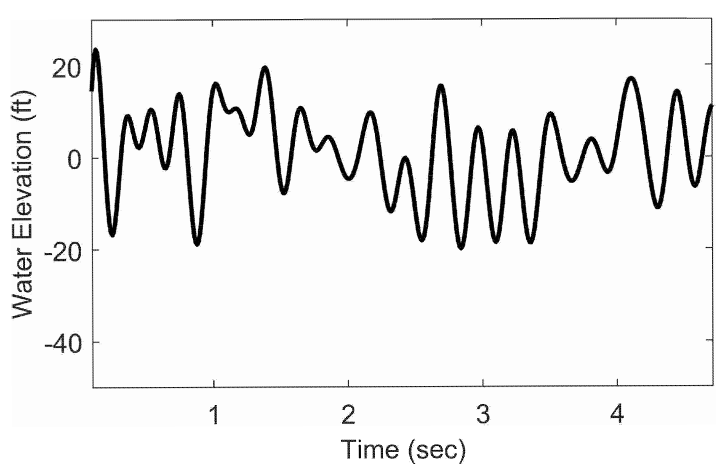

Figure 3.8: Superposition of 50 components

a Fourier Transform. The Fourier Transform identifies the amplitude and phase angle associated with each frequency component in the wave. We can consider the wave time history to be represented by the sum of all these components added to the mean water elevation,

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^{N} \zeta_{n0} \sin(\omega_n t + \epsilon_n)$$

where ω_n represents each wave frequency in the signal and ζ_{n0} and ϵ_n are the associated wave amplitude and phase angle for each frequency.

Discrete Fourier Transform

If a signal consists of a single sinusoid, finding the frequency content of that signal will result in a single number. We typically represent the frequency content as a plot of amplitude versus frequency, so for this case there would be a single spike with a height equal to the wave amplitude and the spike occurring at the frequency of the sinusoid. However, most wave signals do not look like perfect sinusoids and measured systems of all kinds typically are contaminated by some amount of noise.

Our data always consists of a sequence of measurements, X_1, X_2, X_3, etc..... The *discrete fourier transform* (DFT) converts a sequence of values corresponding to certain times into a sequence of values corresponding to specific frequencies. Computers are very good at this and the most commonly used algorithm for computing a DFT quickly is called the Fast Fourier Transform (FFT). The FFT is so common that the terms DFT and FFT are often used interchangeably.

The **time domain** consists of a set of numbers (x_0, x_1, x_2,...) each measured at a particular time (t_0, t_1, t_2,...). The DFT provides the **frequency domain** information in the form of numbers (X_0, X_1, X_2,...) where each X_k represents a portion of the signal that occurs at f_k. By convention, n refers to the time domain data point and k refers to the frequency domain data point. The frequency values X_k are typically complex numbers (expressed as real and imaginary components).

If we have N data points in time (x_n), we will only have $N/2$ points in the frequency domain. This is because for each frequency we have two pieces of information - the magnitude and phase - while at each time we only have one piece of information: magnitude. Since we can't create new information out of nothing, with N points in the time domain we can only have enough information for $N/2$ points in the frequency domain.

In the time domain, the data has an associated sampling frequency (samples/second). This is related to the time interval between data points (Δt),

$$\text{sample frequency} = \frac{1}{\Delta t}.$$

For example, if we take 100 regularly spaced measurements in 0.5 sec we have a sample of 100 data points ($N = 100$). The time between samples is found by dividing the total time by the total number of samples. We are taking a sample every 0.005 seconds (0.5 seconds divided by 100 samples):

$$\Delta t = \frac{0.5 \text{ sec}}{100 \text{ samples}} = 0.005 \text{ seconds/sample}.$$

Our sampling frequency is, therefore,

$$f_s = \frac{1}{0.005} = 200 \text{ samples/sec or Hz.}$$

The resolution with which we can get frequency information from the DFT depends on the sampling frequency **AND** the total number of points

$$\Delta f = \frac{f_s}{N}$$

where Δf is the frequency resolution. So, in this example we have a frequency resolution of 2 Hz (=200 Hz/100 samples).

The raw form of the frequency information that the FFT delivers is not in a physically meaningful form (for example, the values are just complex numbers). We need to scale the results to find the amplitudes of the peaks (power spectrum) and the phase information. To find the amplitude of response at a particular frequency, we need to take the absolute magnitude of the complex number, multiply it by 2, and divide by the total number of points.

$$\text{Magnitude} = 2\frac{|X_k|}{N}$$

where X_k is the complex number at frequency k. The phase angle information is determined from taking the tangent of the real and complex parts of the FFT output. The process for the FFT is mathematically complicated, but algorithms exist using software like MATLAB that we can use to get X_k. The scaling to find the magnitude (and also the phase information) must be done manually.

The fastest oscillation (i.e. the highest frequency) that can be observed depends on how rapidly the data is sampled. The time resolution dictates the highest observable frequency. Remember, N samples in the time domain only gives frequency information up to $N/2$. Therefore, the highest frequency that can be measured is

$$f_{\max} = \frac{N}{2} \cdot \Delta f$$

which is known as the Nyquist frequency. To summarize, the highest frequency you can observe depends on the total number of points. So, if you want to observe 100 Hz responses and only are currently able to observe 50 Hz responses you need to collect more points. You can do this by collecting data for a longer total time or for the same amount of time, but at a higher sampling frequency. If you want to be able to resolve your frequency more (for example, you only have a resolution of 2 Hz but want to be able to distinguish results at a resolution of 0.5 Hz), you must increase the total number of time domain data points by increasing the total time you take take measurements. If I currently have 100 samples taken at a 200 Hz sampling frequency, my frequency resolution is 2 Hz and my Nyquist frequency is 100 Hz. If I want to increase my Nyquist frequency to 200 Hz, I could keep my total time the same and double my sampling frequency (to 400 Hz). My frequency resolution would then by 400 Hz/200 samples = 2 Hz (same as before), but my Nyquist frequency is now (200 samples/2)·(2 Hz) = 200 Hz. If I want to have a finer frequency resolution, I need to take more data points at the same sampling frequency. If I continue at my 400 Hz sampling frequency for a full second (as opposed to the previous 0.5 second), I will get $N = 400$ samples. Now my frequency resolution is 400 Hz/400 samples = 1 Hz and my Nyquist frequency is (400 samples/2)·(1 Hz) = 200 Hz (same as before).

3.2.2 Wave Energy Spectrum

Using the principle of superposition, an irregular wave pattern can be created as a large number of sinusoidal waves of different frequencies and heights that are superimposed on each other. One way to summarize a set of waves is to determine the total energy. This is found by adding together the energies of all of the waves that produced the irregular wave pattern. The roughness of the seaway is then decided by the total energy content of all the waves present. As shown above when considering regular waves, the energy content for a sinusoidal wave is $\frac{1}{2}\rho g\zeta_a^2$ per square area of sea surface. To account for all the sinusoidal waves superimposed, every wave amplitude must be accounted for, so

$$E_T = \frac{1}{2}\rho g(\zeta_{a1}^2 + \zeta_{a2}^2 + ... + \zeta_{an}^2).$$

In essence, the total energy is made up of the energy contribution of each wave component. Each wave component (the wave amplitude for a particular wave frequency) has an energy component. The frequency distribution of this energy is called the *energy spectrum* of the seaway.

Wave Energy Spectrum Example [1]

Consider an irregular seaway composed of four wave components:

Wave component	1	2	3	4
Length (ft)	1265	562	316	202
Freq (rad/sec)	0.4	0.6	0.8	1.0
Height (ft)	3	5	4	2
Amplitude (ft)	1.5	2.5	2	1

The total energy per square foot of the wave surface is

$$\begin{aligned} E_T &= \frac{\rho g}{2}(\zeta_{a1}^2 + \zeta_{a2}^2 + \zeta_{a3}^2 + \zeta_{a4}^2) \\ &= \frac{64}{2}(1.5^2 + 2.5^2 + 2^2 + 1^2) \\ &= 432 \text{ lb/ft} \end{aligned}$$

The individual energy components can be graphed by taking the energy contribution for each component and dividing by the frequency spacing. For example, the first wave component has an energy of 72 lb/ft. The ordinate on the wave energy spectrum plot (see Figure 3.9) is (72 lb/ft)/(0.2 rad/sec) = 360 lb-sec/ft. Figure 3.9 shows the energy component contributions. The area under this 'curve' is the total energy of the irregular seaway, E_T. The Wave energy spectrum curve, Figure 3.10, shows the Wave Energy Spectrum curve with the spectral ordinate (equation 3.4) on the vertical axis. The area under this curve is proportional to the total energy: total energy = (water specific gravity) · (area under curve).

[1] taken from reference 3

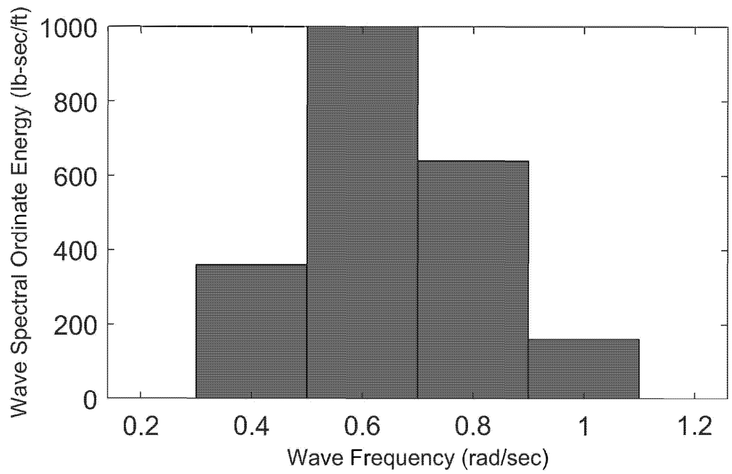

Figure 3.9: Wave Energy Ordinates Spectrum example

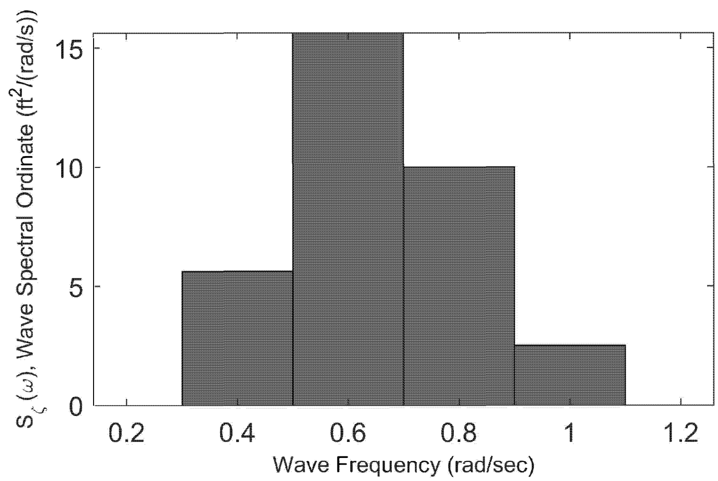

Figure 3.10: Wave Spectral Ordinates Spectrum example

The relative importance of the wave components (each a sinusoidal wave) making up an irregular wave pattern (usually recorded as a time history) may be quantified in terms of a wave amplitude energy density spectrum, also called a **wave energy spectrum**. The time history of the water elevation can be expressed as

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^{\infty} \zeta_{i0} \sin(\omega_i t + \epsilon_i) \tag{3.3}$$

where the magnitude of ζ_{i0} represents the significance of each frequency component amplitude. Figure 3.11 shows a wave energy curve when the number of components is high and the frequency spacing approaches zero. The spectral ordinate ($S_\zeta(\omega_n)$) is the value on the vertical axis. The area under a segment of the curve equals the energy of that frequency component wave, so the spectral ordinate for each frequency is

$$\rho g S_\zeta(\omega_i)\delta\omega = \frac{\rho g \zeta_{i0}^2}{2}$$
$$S_\zeta(\omega_i) = \frac{\zeta_{i0}^2}{2\delta\omega}. \quad (3.4)$$

If the energy spectrum is known, it is possible to reverse the spectral analysis process and generate a corresponding time history by adding a large number of component sine waves (see equation 3.3). In principle, an infinite number of sine wave components is required. However, a limited number (≈ 50) will work since we can generally ignore components that have small spectral ordinate values. Manipulating the spectral ordinate equation (3.4) gives the wave amplitude for each frequency component:

$$\zeta_{i0} = \sqrt{2S_\zeta(\omega_i)\delta\omega}. \quad (3.5)$$

To recreate an irregular wave energy time history we also need the phase angle for each component (ϵ_i). If the phase angles for each component are chosen at random, the recreated time history will not match the original (from which the wave energy spectrum was generated), but the seaway will have the same energy as the original waves.

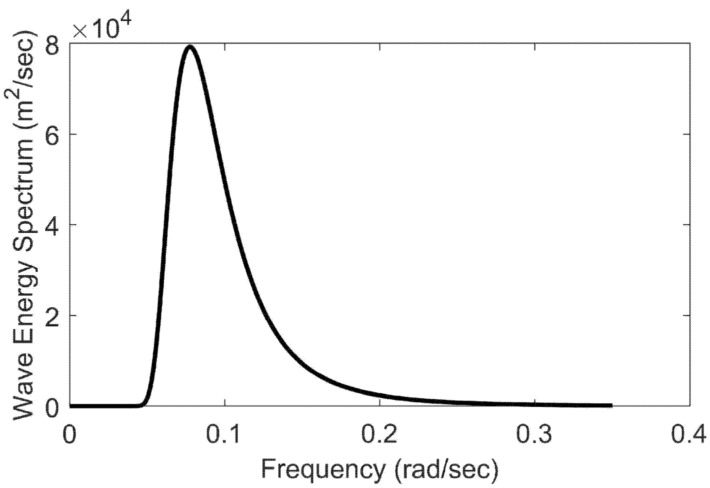

Figure 3.11: Wave Energy Spectrum as number of components approaches ∞

Relationship between Water Elevation Time History and the Wave Energy Spectrum

It is important to understand that the measured time history of the water surface elevation and the wave energy spectrum are both representations of the same information - the seaway. We can get the same information from either source, although the calculation method is different. Consider the **variance** of the water surface time history. The variance is a measure of the degree of "spread" in the wave surface and the wave amplitudes are a measure of the wave energy. For our water elevation, the larger the waves, the larger the variance and the higher the energy in the seaway. The equation for the variance (represented as m_0) is given in equation 3.2 ($m_0 = \sum_{i=1}^{N} \frac{(\zeta_i - \bar{\zeta})^2}{N}$). It turns out that the variance of the time history is also equal to the area under the wave energy spectrum:

$$m_0 = \int_0^\infty S_\zeta(\omega)\mathrm{d}\omega.$$

To restate, **the variance of the irregular wave time history is equal to the area under the wave energy spectrum**.

The **wave energy spectrum**, $S_\zeta(\omega)$, is determined from the wave amplitudes and frequencies from a DFT of the wave time history. Therefore, the **wave velocity spectrum**, $S_{\dot{\zeta}}(\omega)$, and the **wave acceleration spectrum**, $S_{\ddot{\zeta}}(\omega)$, can be found using the velocity and acceleration amplitudes and frequencies.

As shown before, the water surface elevation (position) is given by equation 3.3. The velocity ($\dot{\zeta}$) and acceleration ($\ddot{\zeta}$) of the water surface can be found from the derivatives of the water surface elevation. So,

$$|\dot{\zeta}(t)| = \sum_{i=1}^{\infty} \zeta_{i0}\omega_i \cos(\omega_i t + \epsilon_i)$$
$$|\ddot{\zeta}(t)| = \sum_{i=1}^{\infty} \zeta_{i0}\omega_i^2 \sin(\omega_i t + \epsilon_i).$$

Similarly, the velocity and acceleration spectra ordinates can be related to the wave energy ordinates:

$$S_{\dot{\zeta}}(\omega_i) = \frac{\omega_i^2 \zeta_{i0}}{2\delta\omega} = \omega_i^2 S_\zeta(\omega_i)$$
$$S_{\ddot{\zeta}}(\omega_i) = \frac{\omega_i^4 \zeta_{i0}}{2\delta\omega} = \omega_i^4 S_\zeta(\omega_i).$$

Notice the patterns with respect to the frequency (ω_i) terms. The velocity and acceleration spectral ordinates can be obtained by multiplying the position spectral ordinates by the 2$^{\text{nd}}$ and 4$^{\text{th}}$ powers of the frequency. As for the wave energy spectrum, the area under the velocity and acceleration spectra is equivalent to the variances of these time histories, respectively,

$$m_2 = \int_0^\infty S_{\dot{\zeta}}(\omega)\mathrm{d}\omega = \int_0^\infty \omega^2 S_\zeta(\omega)\mathrm{d}\omega$$
$$m_4 = \int_0^\infty S_{\ddot{\zeta}}(\omega)\mathrm{d}\omega = \int_0^\infty \omega^4 S_\zeta(\omega)\mathrm{d}\omega.$$

These areas (m_0, m_2, and m_4) are called **spectral moments**. In general, the relationship between a spectral moment and the wave energy spectrum curve is

$$m_n = \int_0^\infty \omega^n S_\zeta(\omega)\mathrm{d}\omega. \tag{3.6}$$

These spectral moments can be used to link the spectra to statistical characteristics of the time history such as the mean wave period ($\bar{T}$) and frequency ($\bar{\omega}$), zero-crossing mean period ($\bar{T}_z$), and peak mean period ($\bar{T}_p$). These characteristics can be determined directly from the time history of the water surface elevation or from the wave energy spectrum for that time history using the spectral moments.

mean frequency: $\bar{\omega} = \frac{m_1}{m_0}$

mean period: $\bar{T} = \frac{2\pi}{\bar{\omega}} = 2\pi \frac{m_0}{m_1}$

mean peak period: $\bar{T}_p = 2\pi \sqrt{\frac{m_2}{m_4}}$

mean zero-crossing period: $\bar{T}_z = 2\pi \sqrt{\frac{m_0}{m_2}}$

Spectrum Bandwidth The spectrum bandwidth describes the relative width of the wave energy spectrum compared to the height. A **Narrow Band** spectrum has a sharp spike that covers only a small range of frequencies, see Figure 3.12. A narrow-banded water surface elevation time history can loosely be described as a sine wave of varying amplitude. The wave energy is concentrated in a narrow band of frequencies. Essentially, for every peak or trough there is a zero-crossing. So, $\bar{T}_p \approx \bar{T}_z$.

A **Wide Band** spectrum has a flatter curve that covers a large range of frequencies, see Figure 3.13. A wide-banded water surface elevation time history has many peaks and troughs not followed immediately by a zero crossing. The wave energy is spread over a wide band of frequencies. Essentially, there are many local peaks and troughs between each zero-crossing. So, $\bar{T}_p << \bar{T}_z$.

The ratio between the average period of the peaks and the average zero-crossing period can be regarded as a measure of the "narrow-bandedness." This is quantified by the **Bandwidth Parameter**.

Bandwidth Parameter

$$\epsilon = \sqrt{1 - \frac{\bar{T}_p^2}{\bar{T}_z^2}} = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}. \tag{3.7}$$

As the bandwidth parameter approaches zero ($\epsilon = 0$), the spectrum is very narrow-banded ($\bar{T}_z = \bar{T}_p$) and the wave surface approaches a regular wave. In general, the spectra associated with waves and ship motions are narrow-banded. As the bandwidth parameter approaches 1 ($\epsilon = 1$), the spectrum is very wide-banded ($\bar{T}_p \approx 0$).

The bandwidth parameter together with the spectral moments allows the significant wave height of the irregular seaway to be calculated. Remember that the *significant wave height* is the average of the highest one-third of all wave heights in the seaway. Using the bandwidth parameter, ϵ, and the spectral moment m_0, the significant wave height is equal to

$$\bar{H}_{1/3} = 4.00\sqrt{m_0}\sqrt{1 - \frac{\epsilon^2}{2}}.$$

If the wave spectrum is narrow-banded, such that ϵ approaches 0, the significant wave height can be found from

$$\bar{H}_{1/3} = 4.00\sqrt{m_0}. \tag{3.8}$$

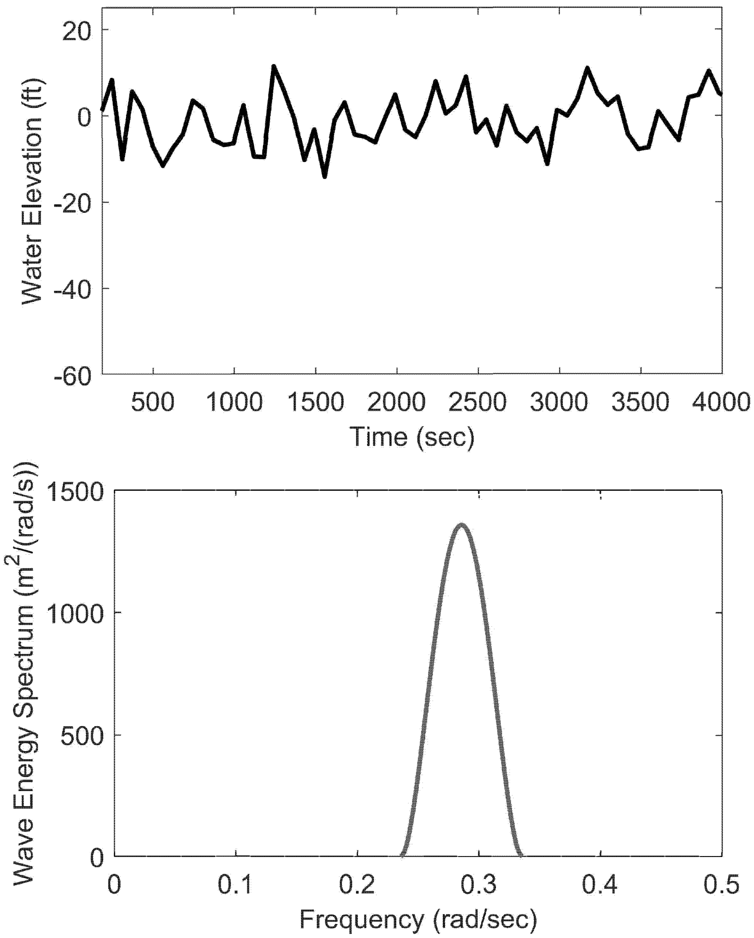

Figure 3.12: Sample Narrow-Banded Wave Energy Spectrum and Water Surface Time History

If the wave spectrum is wide-banded, such that ϵ approaches 1, the significant wave height can be found from

$$\bar{H}_{1/3} = 2.83\sqrt{m_0}.$$

Since m_0 is equal to the area under the wave energy spectrum curve, the significant wave height can be estimated by integrating $S_\zeta(\omega)$.

3.2.3 Idealized Wave Spectra

Sea waves are primarily the result of wind transferring energy to the sea surface. The kinetic energy of the wind (wind speed) creates potential energy of the water (waves). The height and length of the generated waves depends on the wind velocity, the length of time the wind blows over the water surface, and the *fetch*. Fetch is the distance of water over which the wind blows before reaching land. Open ocean has effectively infinite fetch, while bays, lakes, and coastal areas are considered "limited fetch" areas. Fetch-limited areas have waves

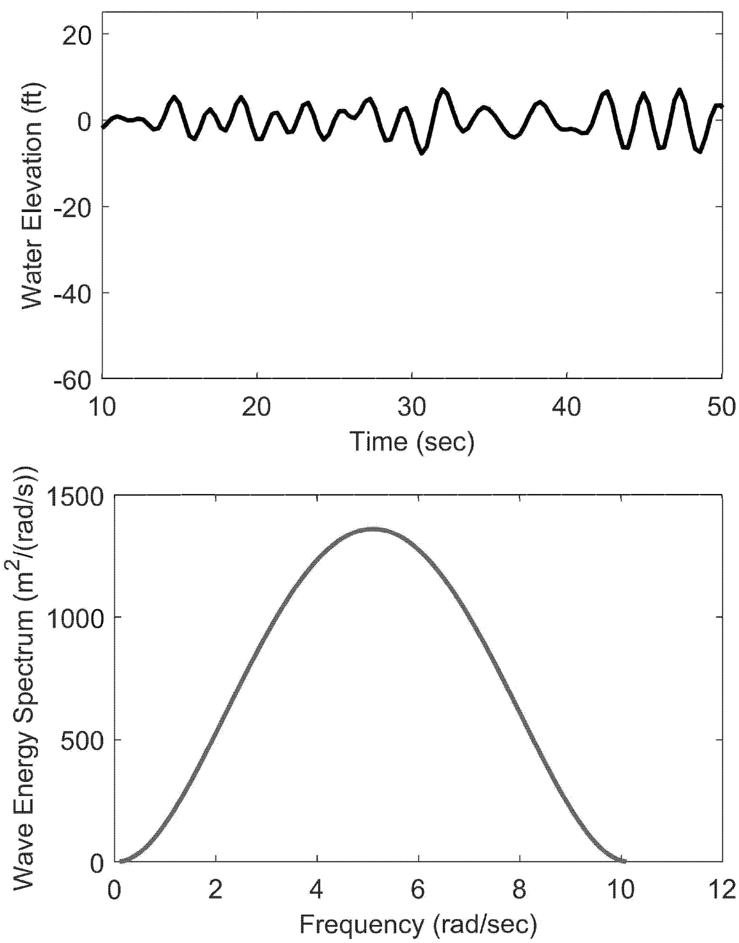

Figure 3.13: Sample Wide-Banded Wave Energy Spectrum and Water Surface Time History

that are shorter (higher frequency) and choppier (steeper). The waves often have white caps and there is not as much of an underlying swell like you see in open ocean waves. If there are no fetch limitations, the waves eventually reach an equilibrium where the amount of energy transferred from the wind maintains the wave heights, but the dissipation of the water (viscous and wave breaking) prevents additional amplitude growth. A sea in this condition is considered *fully developed*.

For ship design purposes, we use different formulae to represent open ocean and coastal (limited fetch) wave conditions. There are two types of open ocean spectra that we will be considering. The first is the Pierson-Moskowitz (1964) one parameter spectrum. This wave field has only one input and that is the wind speed. The spectrum assumes there is plenty of area for the wind/water interface and that the wind has been blowing for long enough that the wave field has reached equilibrium. The Pierson-Moskowitz spectrum was based off of extensive measurements in the North Atlantic Ocean and is intended to represent the spectrum of a fully developed sea.

Pierson-Moskowitz Wave Spectrum - one parameter spectum, wind speed

$$S_\zeta(\omega) = \frac{0.0081 g^2}{\omega^5} e^{-0.74(\frac{g}{W_{19.5}\omega})^4} \tag{3.9}$$

where $W_{19.5}$ is the average wind speed (in m/s) at 19.5 meters above the sea surface. The units for this spectrum are m^2-sec. Since 19.5 meters is not a typical height for wind measurements, this can be related to the more standard 10 meters by $W_{19.5} = 1.026 W_{10}$. Figure 3.14 shows several generated Pierson-Moskowitz spectra for different wind speeds.

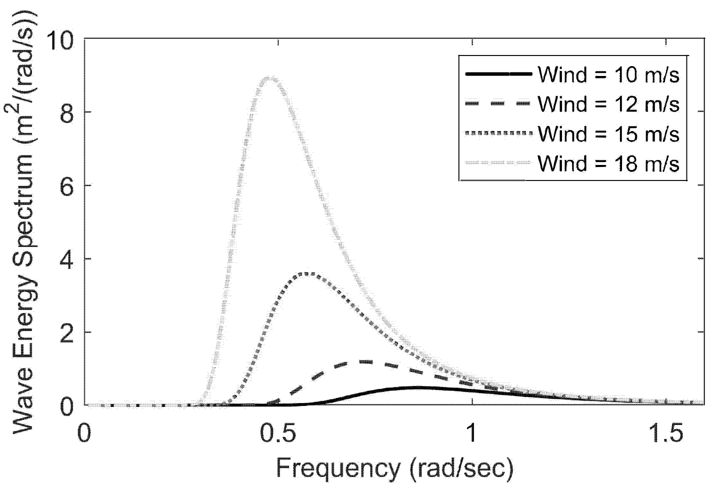

Figure 3.14: Example Pierson Moskowitz spectra for different wind speeds

Another open-ocean wave spectrum is the ITTC (International Towing Tank Conference) 1973 wave spectrum. This is also known as the **Bretschneider Wave Spectrum**. This is a two-parameter spectrum and depends on the given significant wave height and modal period. The **modal period** is the wave period that coincides with the peak of the wave energy spectrum. The ITTC wave spectrum can exactly replicate the Pierson-Moskowitz spectrum if the significant wave height and modal period that results from the wind speed are used.

ITTC 1973 (or Bretschneider) Wave Spectrum - two parameter spectum, significant wave height and modal wave period

$$S_\zeta(\omega) = \frac{1.25}{4} (\frac{\omega_0}{\omega})^4 \frac{\bar{H}_{1/3}^2}{\omega} e^{-1.25(\frac{\omega_0}{\omega})^4} \tag{3.10}$$

where $\bar{H}_{1/3}$ is the significant wave height and ω_0 is the modal wave *frequency* ($\omega_0 = \frac{2\pi}{T_0}$ where T_0 is the modal wave period). Figure 3.15 shows several generated ITTC spectra for different wind speeds.

For limited fetch conditions, we typically use the JONSWAP (Joint North Sea Wave Observation Project) spectrum. This spectrum is intended to represent open wave conditions, but where the fetch is limited (i.e., like the North Sea). The spectrum is more narrow (peakier) than the pure open ocean spectra.

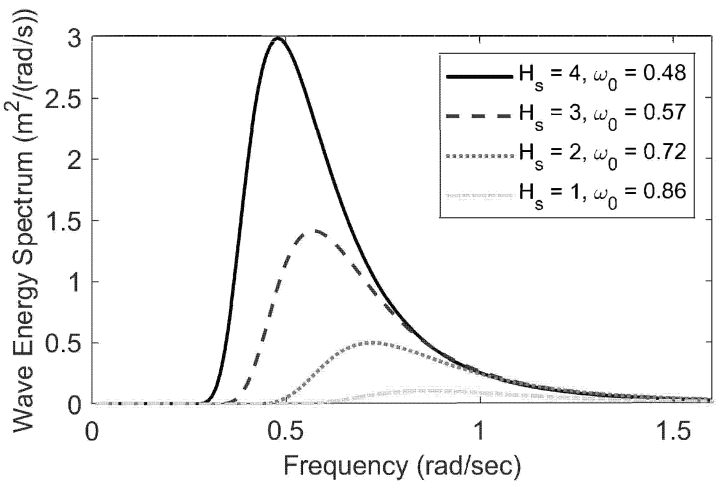

Figure 3.15: Example ITTC (Bretschneider) spectra for different significant wave height/modal period combinations.

JONSWAP Wave Spectrum - three parameter spectum, wind speed and fetch and peakness factor (γ)

$$S_\zeta(\omega) = \frac{\alpha g^2}{\omega^5} e^{-(\frac{5\omega_0^4}{4\omega^4})}\gamma^r \tag{3.11}$$

where α depends on the wind speed (W_{10}) and fetch (F) and equals $\alpha = 0.076(\frac{W_{10}^2}{Fg})^{0.22}$, the modal wave frequency (ω_0) depends on the wind speed and fetch and equals

$$\omega_0 = 22(\frac{g^2}{W_{10}F})^{1/3}$$

and r depends on the frequencies and a σ parameter and equals

$$r = e^{-\left[\frac{(\omega-\omega_0)^2}{2\sigma^2\omega_0^2}\right]}$$

The σ parameter depends on whether the spectral ordinate being calculated is less than or greater than the wave modal frequency,

$$\sigma = \begin{cases} 0.07 & \text{if } \omega \leq \omega_0 \\ 0.09 & \text{if } \omega > \omega_0 \end{cases}$$

The parameter γ is the "peakness factor" and can be modified to meet the needs of the sea conditions. It ranges from about 1 to about 7, but a typical value for γ is 3.3. The JONSWAP spectrum can also be expressed as a function of significant wave height, modal frequency and the peakness factor. This formulation makes it easier to see how the limited fetch changes the wave energy distribution compared with the fully open ocean ITTC spectrum. This JONSWAP formula is

$$S_\zeta(\omega) = B_J \bar{H}_{1/3}^2 \frac{2\pi}{\omega}(\frac{\omega_0}{\omega})^4 e^{-\left[\frac{5}{4}\left(\frac{\omega_0}{\omega}\right)^4\right]}\gamma^r$$

where

$$B_J = \frac{0.06238}{0.230 + 0.0336\gamma - \frac{0.0185}{1.9+\gamma}}[1.094 - 0.01915 \ln \gamma]$$

and r and γ are the same as defined above. Figure 3.16 shows several generated JONSWAP spectra for the same significant wave heights and modal wave periods as used for the ITTC spectra (Figure 3.15) and with a peakness factor of 2.

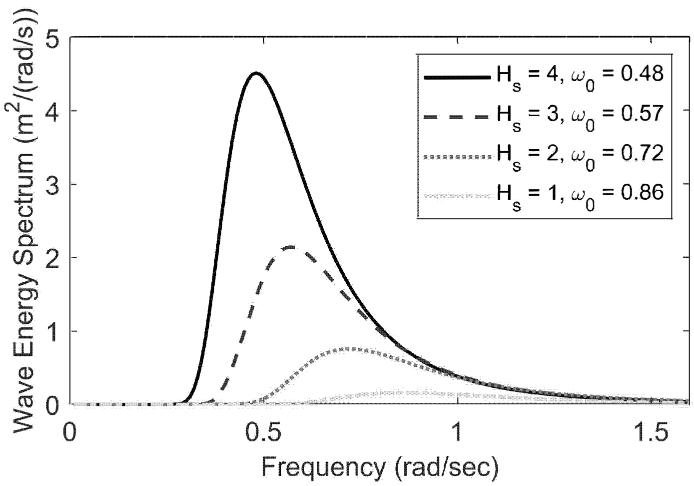

Figure 3.16: Example JONSWAP spectra for different wind speed and fetch combinations.

Recap

Okay, so we have covered a LOT of material so far in this chapter. Before we continue, let's do a short review.

We started with a discussion of **regular waves**. These waves have a single amplitude and period and we can calculate the wave celerity and group velocity as well as the energy of such a wave. We also learned how to calculate the pressure of a point under such a wave.

Next we used the principle of superposition to create a set of (more realistic) **irregular waves**. These waves have varying wave amplitudes and periods and, in a real seaway, do not have any repeating patterns. Although we can't create an infinite discrete irregular wave field, we can create a wave field that approximates the chaotic environment typically found in the ocean. These waves can't be described using a single amplitude and period, so we turn to averages. The important quantities we take averages of include wave frequency, the zero-crossing wave period, the peak wave period, the wave amplitude, and the wave height. We also can find **significant wave amplitude** or **significant wave height**, which are the average of the 1/3$^{\text{rd}}$ highest wave amplitudes and heights, respectively.

Using a **Fourier Transform**, we can transform the irregular wave time history into a **wave energy spectrum**. This shows the energy distribution of the irregular waves over a range of frequencies. The area under this curve, also known as the **zeroth spectral moment** or m_0 is equal to the variance (σ^2) of the irregular wave time history. Using other spectral moments and a bandwidth parameter we can use the wave energy spectrum to find the statistical wave properties mentioned above (mean wave frequency, the zero-crossing wave period, the peak wave period, the wave amplitude, the wave height, and the significant wave height).

Okay, so why did we just work our way in a circle (start with finding these properties from the irregular wave time history, transform the time history into a wave energy spectrum, then find these properties from the wave energy spectrum)? Because, as ship designers, we want to create realistic random sea states to design from and to do this we use **idealized wave spectra** such as the Pierson-Moskowitz, ITTC (Bretschneider), or JONSWAP spectra. In the next chapter we are going to learn about how ships respond to regular waves and then put that together with these realistic sea environments to make predictions about how a ship may respond when encountering realistic sea conditions.

Before we move on to discussing ship responses, however, we have one more wave related issue to tackle. All of these energy wave spectra give information about what frequencies contain the main energy of the sea surface. What we don't know is what the specific waves the ship will encounter will be. Since the sea is random, we must approach this from a probabilistic perspective. So, to finish out the chapter on waves, we will now do a review of the relevant Probability and Statistics topics you have, hopefully, seen before!

3.2.4 Review of Probability and Statistics with Marine Applications

We have shown that the irregular time histories of waves can be characterized in terms of energy spectra and various statistical quantities. Seakeeping studies, however, often demand a more intimate knowledge of waves. In particular, we need to be able to answer questions like "What is the likelihood of a particular wave height being exceeded?" We can use wave energy spectra and probability distributions to answer this type of question.

Probability Density Function (PDF)

The probability density function is defined such that the area enclosed by the PDF curve over a bin is equal to the probability of the measurement falling within that bin. So, the probability of the x-axis value falling between a and b is equal to the area under the curve from a to b. Figure 3.17 shows the area from a to b for a normal probability distribution curve. The probability the water elevation falls between these two limits is equal to the shaded area on the plot. The area under the entire probability density function equals one, since there is 100% probability that any measurement falls within the set of collected measurements.

Water elevation typically follows a Gaussian or normal distribution. This is the typical "bell" curve, see Figure 3.18. The empirical rule states that there is about a 68% probability a measurement will fall between $\pm\sigma$ (one standard deviation), there is about a 95% probability a measurement will fall between $\pm 2\sigma$, and a 99% probability any measurement will fall between $\pm 3\sigma$.

While *water elevation* typically follows a Gaussian distribution, *wave heights* (and amplitudes) follow a Rayleigh distribution for narrow-banded wave spectra. The probability for

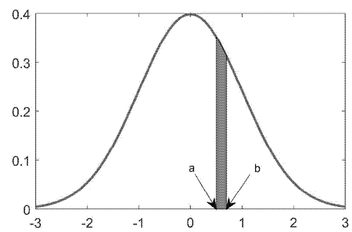

Figure 3.17: The probability of the wave elevation falling between a and b equals the area under the pdf curve between these two values

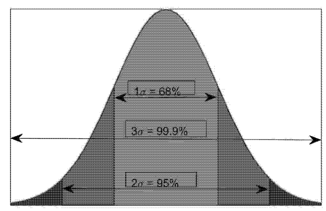

Figure 3.18: Gaussian or Normal Probability Distribution

the wave amplitudes depends on the variance of the water elevation. Figure 3.19 shows a typical Rayleigh distribution. The probability a wave amplitude would fall between two heights is equal to the area under the curve between those two points. The Rayleigh probability distribution equals

$$f = \frac{\zeta_a}{m_0} e^{-\frac{\zeta_a^2}{2m_0}}$$

where ζ_a is the wave amplitude and m_0 is the *variance from the water elevation time history* or area under the wave energy spectrum curve.

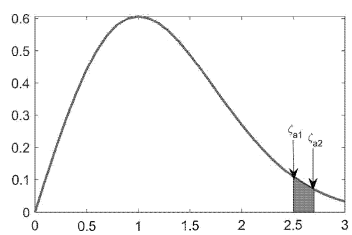

Figure 3.19: Rayleigh Probability Distribution

Considering the wave amplitudes (Rayleigh probability distribution), the probability that an amplitude ζ_a will exceed a specific amplitude, ζ_{A1} is

$$P(\zeta_a > \zeta_{A1}) = e^{-\frac{\zeta_{A1}^2}{2m_0}}. \tag{3.12}$$

The probability that the wave amplitude will fall *between* amplitudes ζ_{A1} and ζ_{A2} is

$$P(\zeta_{A1} < \zeta_a < \zeta_{A2}) = e^{-\frac{\zeta_{A2}^2}{2m_0}} - e^{-\frac{\zeta_{A1}^2}{2m_0}},$$

i.e. the probability of exceeding ζ_{A2} minus the probability of exceeding ζ_{A1}.

Significant Wave Height and Related Statistics

The significant wave height is the mean of the highest $1/3^{rd}$ of the heights recorded in a wave time history. It closely correlates with the average wave height estimated visually by an experienced observer. It is expected that the experienced sailor's estimates of "average" wave heights might be similar to the significant wave height. The Rayleigh formula for the mean value of the highest $1/n^{th}$ of all observations is

$$\zeta_{\frac{1}{n}} = \sqrt{-2m_0 \ln \frac{1}{n}}.$$

So, for $n = 1$, the mean of all amplitudes, $\bar{\zeta}_a = 1.25\sigma_0$ where σ_0 is the standard deviation from the water surface elevation ($\sigma_0 = \sqrt{m_0}$). For the significant wave amplitude,

$$\bar{\zeta}_{1/3} = 2.00\sigma_0.$$

Remember from earlier that the significant wave height, $\bar{H}_{1/3}$ equaled $4.00\sqrt{m_0}$. This is the same as saying that the significant wave *height* is equal to twice the significant wave *amplitude*. These results are widely assumed to apply to all wave records. However, this is only **strictly** true if the Rayleigh formula applies. Table 3.3 shows the values that can be multiplied by the water elevation standard deviation (σ_0) to determine the average of the highest $1/n^{th}$ amplitudes.

n	$\frac{\zeta_{1/n}}{\sigma_0}$	n	$\frac{\zeta_{1/n}}{\sigma_0}$
1	1.25	10	2.54
2	1.77	100	3.34
3	2.00	1000	3.72

Table 3.3: Mean of the highest $1/n$ amplitudes (Rayleigh formula)

Probability of Exceedance

So, what is the procedure for predicting the probability of the wave amplitude exceeding a particular value (ζ_B) in a specific sea state? First we have to decide which idealized spectra we want to use to create our sea state. Since the NATO sea state conditions provide a mean significant wave height and most probable modal wave period, using the ITTC (Bretschneider) spectrum makes sense.

1. First we build an ITTC wave energy spectrum for our sea state (using the mean significant wave height and most probable modal period). We will need to convert the modal period, T_0, into modal frequency, $\omega_0 = 2\pi/T_0$.

$$S_\zeta(\omega) = \frac{1.25}{4}(\frac{\omega_0}{\omega})^4 \frac{\bar{H}_{1/3}^2}{\omega} e^{-1.25(\frac{\omega_0}{\omega})^4}.$$

2. We can find the variance, m_0 from this wave energy spectrum.

$$m_0 = \int_0^{\infty} S_\zeta(\omega) \mathrm{d}\omega.$$

3. Then, we use the variance and the value of interest to calculate the probability of exceedance.

$$P(\zeta_a > \zeta_B) = e^{-\frac{\zeta_B^2}{2m_0}}.$$

Calculation of Probability of Exceedance Example Consider the ocean spectrum for a Bretschneider sea state 6. For the time history recorded (a total of 23.5 minutes), the variance of the water elevation was 16.81 ft^2. Find the significant wave height and the probability of a wave height exceeding 25 ft. Find the probability of exceeding the significant wave height.

How did we get m_0 (variance of the water elevation)? It was found either by taking the variance of the time history (as in this problem) or by finding the area under the wave energy spectrum (as explained in the procedure above). Once we have it we can find the significant wave height directly

$$\bar{H}_{1/3} = 4.00\sqrt{m_0} = 4.00\sqrt{16.81} = 16.4 \text{ ft.}$$

To find the probability of exceedance we plug this into the equation 3.12. This equation requires us to use the wave amplitude. Since we want the probability of exceedance for a wave *height* of 25 ft, the corresponding wave amplitude is $25/2 = 12.5$ ft.

$$P(\zeta_a > 12.5) = e^{-\frac{12.5^2}{2(16.81)}} = 0.0096 = 0.96\%$$

So, there is a 0.96% probability that we will encounter a wave height over 25 ft. The probability we will exceed the significant wave height of 16.4 ft (amplitude of 8.2 ft) is

$$P(\zeta_a > 8.2) = e^{-\frac{8.2^2}{2(16.81)}} = 0.1353 = 13.53\%.$$

Seakeeping Notation

Dynamics Review		Added Mass & Damping	
m	mass	DFT	Discrete Fourier Transform
b	damping coefficient	FFT	Fast Fourier Transform
c	stiffness coefficient	a	added mass
x	position	T	period (sec)
$\dot{x}$	velocity	f	frequency (Hz)
$\ddot{x}$	acceleration	D	draft
F_0	force amplitude	T_n	natural period (sec)
X	motion amplitude	**Regular Waves**	
ω	wave frequency (rad/s)	ζ_0	wave amplitude
ω_e	excitation (or encounter) frequency (rad/s)	c	wave celerity
ϕ	phase angle	u_g	group velocity
Λ	Tuning Factor, ω_e/ω_n	k	wave number
η	Damping Factor, $b/[2(m+a)\omega_n]$	α_0	wave slope
$X/(F_0/c)$	Magnification Factor	d	water depth
ω_n	natural frequency (rad/sec)	λ	wavelength
Dynamic Ballasting		**DDG-51 in Head Seas**	
I_5	pitch mass moment of inertia	ω_{n3}	heave natural frequency (rad/sec)
k_5	pitch gyradius	ω_{n5}	pitch natural frequency (rad/sec)
I_6	yaw mass moment of inertia	X_3/ζ_0	Heave Transfer Function
k_6	yaw gyradius	X_5/α_0	Pitch Transfer Function
R	model scale ratio	U	ship speed
Irregular Waves			
ϵ	bandwidth parameter	$S_\zeta(\omega)$	wave energy spectra

Subscripts

1	Surge (linear)
2	Sway (linear)
3	Heave (linear)
4	Roll (rotational)
5	Pitch (rotational)
6	Yaw (rotational)
M	model
S	ship
e	excitation or encounter
n	natural (as in natural frequency)

Maneuvering Notation

Reference Frame Maneuvering Notation

$\boldsymbol{x_{0G}}$	CG x-position from Earth Reference Frame
$\boldsymbol{y_{0G}}$	CG y-position from Earth Reference Frame
$\boldsymbol{\beta}$	drift angle
$\boldsymbol{\psi}$	yaw angle
$\boldsymbol{\ddot{x}_{0G}}$	CG x-direction acceleration from Earth Reference Frame
$\boldsymbol{\ddot{y}_{0G}}$	CG y-direction acceleration from Earth Reference Frame
$\boldsymbol{\dot{\psi}}$	angular yaw velocity
$\boldsymbol{r}$	angular yaw velocity
$\boldsymbol{\ddot{\psi}}$	angular yaw acceleration
$\boldsymbol{\dot{r}}$	angular yaw acceleration
$\boldsymbol{x}$	direction through ship bow
$\boldsymbol{y}$	direction to ship starboard
$\boldsymbol{u}$	velocity in x-direction
$\boldsymbol{v}$	velocity in y-direction
$\boldsymbol{\dot{u}}$	acceleration in x-direction
$\boldsymbol{\dot{v}}$	acceleration in y-direction
$\boldsymbol{X_0}$	Total force in Earth x-direction
$\boldsymbol{Y_0}$	Total force in Earth y-direction
$\boldsymbol{N}$	Moment in yaw
$\boldsymbol{I_z}$	yaw mass moment of inertia

SNAME (1952) Derivative and Non-dimensional Notation

$\boldsymbol{X_u}$	$\frac{\partial X}{\partial u}$		$\boldsymbol{X_{\dot{u}}}$	$\frac{\partial X}{\partial \dot{u}}$
$\boldsymbol{Y_v}$	$\frac{\partial Y}{\partial v}$		$\boldsymbol{Y_{\dot{v}}}$	$\frac{\partial Y}{\partial \dot{v}}$
$\boldsymbol{N_v}$	$\frac{\partial N}{\partial v}$		$\boldsymbol{N_{\dot{v}}}$	$\frac{\partial N}{\partial \dot{v}}$
$\boldsymbol{Y_r}$	$\frac{\partial Y}{\partial r}$		$\boldsymbol{Y_{\dot{r}}}$	$\frac{\partial Y}{\partial \dot{r}}$
$\boldsymbol{N_r}$	$\frac{\partial N}{\partial r}$		$\boldsymbol{N_{\dot{r}}}$	$\frac{\partial N}{\partial \dot{r}}$
$\boldsymbol{m'_\Delta}$	$\frac{m_\Delta}{1/2\rho L^3}$		$\boldsymbol{I'_z}$	$\frac{I_z}{1/2\rho L^5}$
$\boldsymbol{v'}$	$\frac{v}{U}$		$\boldsymbol{r'}$	$\frac{rL}{U}$
$\boldsymbol{\dot{v}'}$	$\frac{\dot{v}L}{U^2}$		$\boldsymbol{\dot{r}'}$	$\frac{\dot{r}L^2}{U^2}$
$\boldsymbol{Y'_v}$	$\frac{Y_v}{1/2\rho L^2 U}$		$\boldsymbol{Y'_r}$	$\frac{Y_r}{1/2\rho L^3 U}$
$\boldsymbol{N'_v}$	$\frac{N_v}{1/2\rho L^3 U}$		$\boldsymbol{N'_r}$	$\frac{N_r}{1/2\rho L^4 U}$
$\boldsymbol{Y'_{\dot{v}}}$	$\frac{Y_{\dot{v}}}{1/2\rho L^3}$		$\boldsymbol{Y'_{\dot{r}}}$	$\frac{Y_{\dot{r}}}{1/2\rho L^4}$
$\boldsymbol{N'_{\dot{v}}}$	$\frac{N_{\dot{v}}}{1/2\rho L^4}$		$\boldsymbol{N'_{\dot{r}}}$	$\frac{N_{\dot{r}}}{1/2\rho L^5}$