Ninth Edition

Automatic Control Systems

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Chapter 2

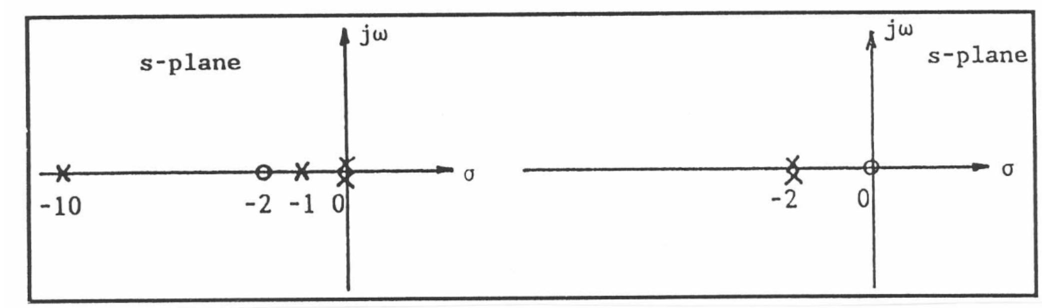
2-1 **(a)** Poles: $s = 0, 0, -1, -10$;

Zeros: $s = -2, \infty, \infty, \infty$.

(b) Poles: $s = -2, -2$;

Zeros: $s = 0$.

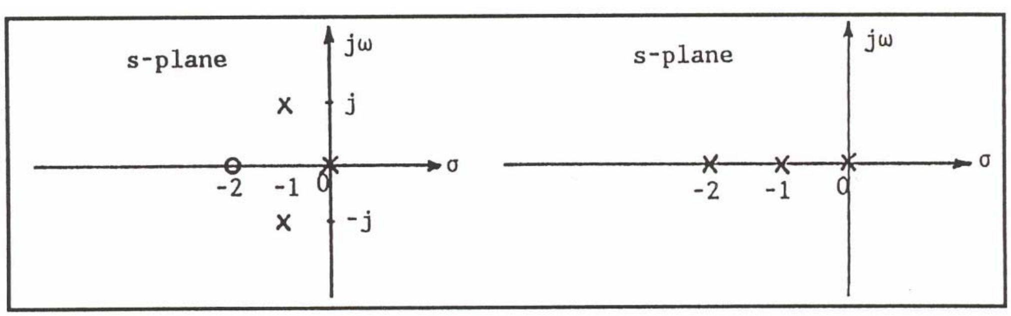
The pole and zero at $s = -1$ cancel each other.



(c) Poles: $s = 0, -1 + j, -1 - j$;

Zeros: $s = -2$.

(d) Poles: $s = 0, -1, -2, \infty$.



2-2) a) $G(s) = \frac{(s+1)}{s(s+2)(s+3)^2}$

b) $G(s) = \frac{s^2}{(s+1)(s+4)}$

c) $G(s) = \frac{s^2-1}{s^2(s+3)(s+1)^2}$

2-3)

MATLAB code:

```
clear all;
s = tf('s')

'Generated transfer function:'
Ga=10*(s+2)/(s^2*(s+1)*(s+10))
'Poles:'
pole(Ga)
'Zeros:'
zero(Ga)

'Generated transfer function:'
Gb=10*s*(s+1)/((s+2)*(s^2+3*s+2))
'Poles:';
pole(Gb)
'Zeros:'
zero(Gb)

'Generated transfer function:'
Gc=10*(s+2)/(s*(s^2+2*s+2))
'Poles:';
pole(Gc)
'Zeros:'
zero(Gc)

'Generated transfer function:'
Gd=pade(exp(-2*s),1)/(10*s*(s+1)*(s+2))
'Poles:';
pole(Gd)
'Zeros:'
zero(Gd)
```

Poles and zeros of the above functions:

(a)

Poles: 0 0 -10 -1

Zeros: -2

(b)

Poles: -2.0000 -2.0000 -1.0000

Zeros: 0 -1

(c)

Poles:

0

-1.0000 + 1.0000i

-1.0000 - 1.0000i

Zeros: -2

Generated transfer function:

(d) using first order Pade approximation for exponential term

Poles:

0

-2.0000

-1.0000 + 0.0000i

-1.0000 - 0.0000i

Zeros:

1

2-4) Mathematical representation:

In all cases substitute $s = j\omega$ and simplify. The use MATLAB to verify.

a)

$$\frac{10(j\omega+2)}{-\omega^2(j\omega+1)(j\omega+10)}$$
$$=\frac{10(j\omega+2)}{-\omega^2(j\omega+1)(j\omega+10)}\times\frac{(-j\omega+1)(-j\omega+10)}{(-j\omega+1)(-j\omega+10)}$$
$$=\frac{10(j\omega+2)(-j\omega+1)(-j\omega+10)}{-\omega^2(\omega^2+1)(\omega^2+100)}$$
$$=R\frac{j\omega+2}{\sqrt{2^2+\omega^2}}\frac{-j\omega+1}{\sqrt{1+\omega^2}}\frac{-j\omega+10}{\sqrt{10^2+\omega^2}}$$
$$=R(e^{j\phi_1}e^{j\phi_2}e^{j\phi_3})$$

$$R=\frac{10\sqrt{2^2+\omega^2}\sqrt{1+\omega^2}\sqrt{10^2+\omega^2}}{-\omega^2(\omega^2+1)(\omega^2+100)};$$

$$\phi_1=\tan^{-1}\frac{\omega}{\sqrt{2^2+\omega^2}}\bigg/\frac{2}{\sqrt{2^2+\omega^2}}$$

$$\phi_2=\tan^{-1}\frac{-\omega}{\sqrt{1+\omega^2}}\bigg/\frac{1}{\sqrt{1+\omega^2}}$$

$$\phi_3=\tan^{-1}\frac{-\omega}{\sqrt{10^2+\omega^2}}\bigg/\frac{10}{\sqrt{10^2+\omega^2}}$$

$$\phi=\phi_1+\phi_2+\phi_3$$

b)

$$\frac{10}{(j\omega+1)^2(j\omega+3)}$$
$$=\frac{10}{(j\omega+1)(j\omega+1)(j\omega+3)}\times\frac{(-j\omega+1)(-j\omega+1)(-j\omega+3)}{(-j\omega+1)(-j\omega+1)(-j\omega+3)}$$
$$=\frac{10(-j\omega+1)(-j\omega+1)(-j\omega+3)}{(\omega^2+1)^2(\omega^2+9)}$$
$$=R\frac{-j\omega+1}{\sqrt{1+\omega^2}}\frac{-j\omega+1}{\sqrt{1+\omega^2}}\frac{-j\omega+3}{\sqrt{9+\omega^2}}$$
$$=R(e^{j\phi_1}e^{j\phi_2}e^{j\phi_3})$$

$$R=\frac{10\sqrt{1+\omega^2}\sqrt{9+\omega^2}}{(\omega^2+1)^2(\omega^2+9)};$$

$$\phi_1=\tan^{-1}\frac{-\omega}{\sqrt{1+\omega^2}}\bigg/\frac{1}{\sqrt{1+\omega^2}}$$

$$\phi_2=\tan^{-1}\frac{-\omega}{\sqrt{1+\omega^2}}\bigg/\frac{1}{\sqrt{1+\omega^2}}$$

$$\phi_3=\tan^{-1}\frac{-\omega}{\sqrt{9+\omega^2}}\bigg/\frac{3}{\sqrt{9+\omega^2}}$$

$$\phi=\phi_1+\phi_2+\phi_3$$

c)
$$\frac{10}{j\omega(j2\omega+2-\omega^2)}$$
$$=\frac{-10j}{\omega(j2\omega+2-\omega^2)}\times\frac{(2-\omega^2-j2\omega)}{(2-\omega^2-j2\omega)}$$
$$=\frac{10(-2\omega-(2-\omega^2)j)}{\omega(4\omega^2+(2-\omega^2)^2)}$$
$$=R\frac{-2\omega-(2-\omega^2)j}{\sqrt{4\omega^2+(2-\omega^2)^2}}$$
$$=R(e^{j\phi})$$

$$R=\frac{10\sqrt{4\omega^2+(2-\omega^2)^2}}{\omega(4\omega^2+(2-\omega^2)^2)}=\frac{10}{\omega\sqrt{4\omega^2+(2-\omega^2)^2}};$$
$$\phi=\tan^{-1}\frac{\frac{-2-\omega^2}{\sqrt{4\omega^2+(2-\omega^2)^2}}}{\frac{-2\omega}{\sqrt{4\omega^2+(2-\omega^2)^2}}}$$

d)
$$\frac{e^{-2j\omega}}{10j\omega(j\omega+1)(j\omega+2)}$$
$$=\frac{-j(-j\omega+1)(-j\omega+2)}{10\omega(\omega^2+1)(\omega^2+2)}e^{-2j\omega}$$
$$=R\frac{-j\omega+2}{\sqrt{2^2+\omega^2}}\frac{-j\omega+1}{\sqrt{1+\omega^2}}e^{-2j\omega-j\pi/2}$$
$$=R(e^{j\phi_1}e^{j\phi_2}e^{j\phi_3})$$

$$R=\frac{1}{10\omega\sqrt{2^2+\omega^2}\sqrt{1+\omega^2}};$$
$$\phi_1=\tan^{-1}\frac{\frac{-\omega}{\sqrt{2^2+\omega^2}}}{\frac{2}{\sqrt{2^2+\omega^2}}}$$
$$\phi_2=\tan^{-1}\frac{\frac{-\omega}{\sqrt{1+\omega^2}}}{\frac{1}{\sqrt{1+\omega^2}}}$$
$$\phi=\phi_1+\phi_2+\phi_3$$

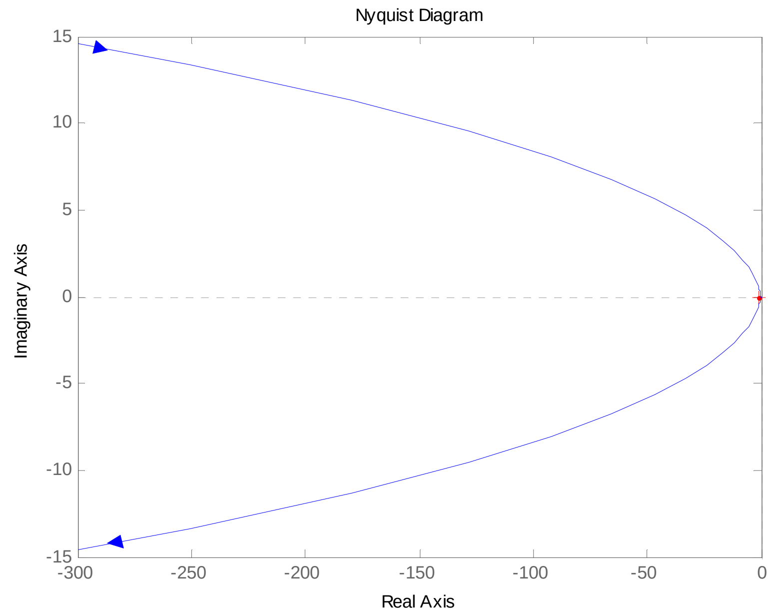
MATLAB code:

```
clear all;

s = tf('s')


'Generated transfer function:'

Ga=10*(s+2)/(s^2*(s+1)*(s+10))

figure(1)
```

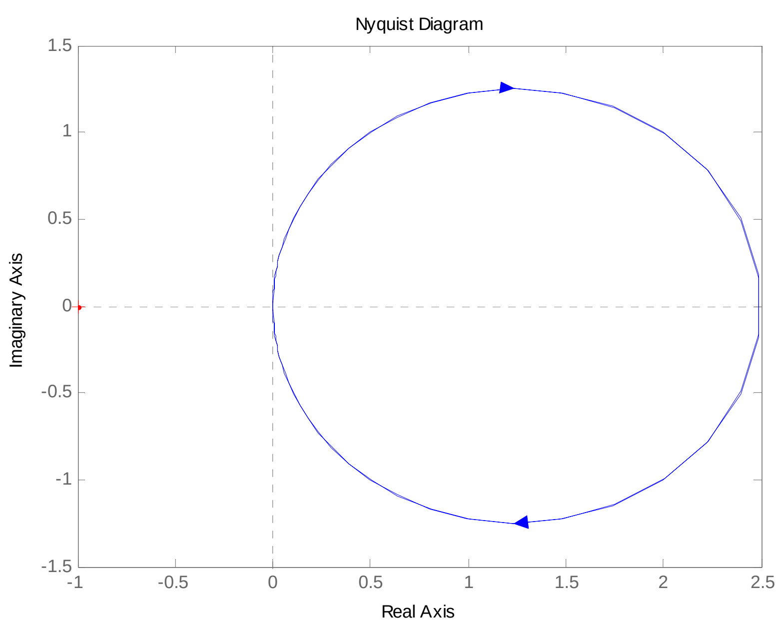
```
Nyquist(Ga)


'Generated transfer function:'

Gb=10*s*(s+1)/((s+2)*(s^2+3*s+2))

figure(2)

Nyquist(Gb)


'Generated transfer function:'

Gc=10*(s+2)/(s*(s^2+2*s+2))

figure(3)

Nyquist(Gc)



'Generated transfer function:'

Gd=pade(exp(-2*s),1)/(10*s*(s+1)*(s+2))

figure(4)

Nyquist(Gd)
```

Nyquist plots (polar plots):

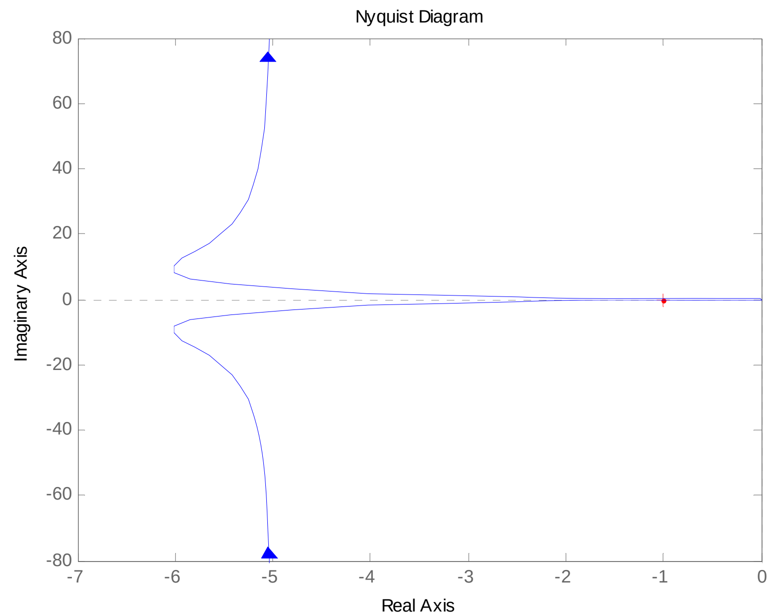
Part(a)


Nyquist Diagram
Imaginary Axis
Real Axis
15
10
5
0
-5
-10
-15
-300
-250
-200
-150
-100
-50
0

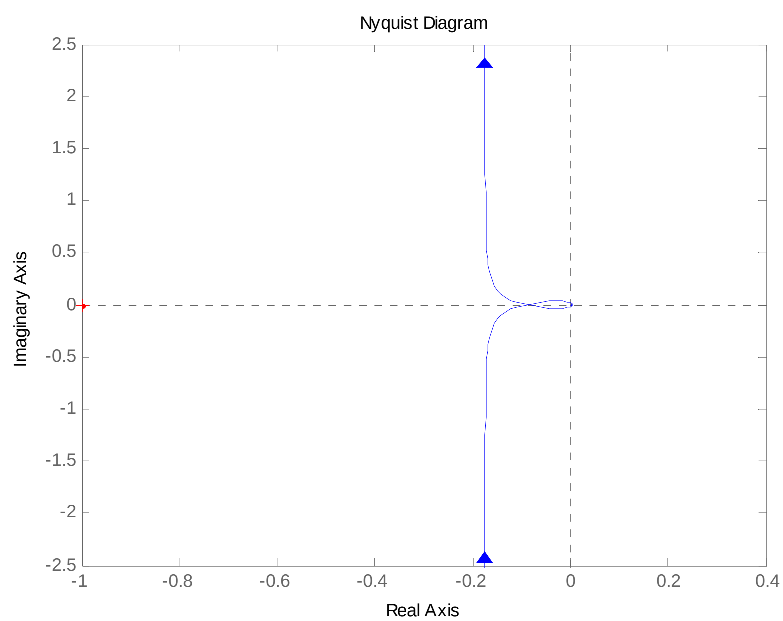
Part(b)


Nyquist Diagram
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

Part(c)


Nyquist Diagram
80
60
40
20
0
-20
-40
-60
-80
-7
-6
-5
-4
-3
-2
-1
0
Imaginary Axis
Real Axis

Part(d)


Nyquist Diagram
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
Imaginary Axis
Real Axis

2-5) In all cases find the real and imaginary axis intersections.

$$G(j\omega)=\frac{10}{(j\omega-2)}=\frac{10(-j\omega+2)}{(\omega^2+4)}=\frac{10}{\sqrt{(\omega^2+4)}}\frac{2-j\omega}{\sqrt{(\omega^2+4)}};$$

$$\mathrm{Re}\{G(j\omega)\}=\cos\phi=\frac{2}{\sqrt{(\omega^2+4)}},$$

a) $$\mathrm{Im}\{G(j\omega)\}=\sin\phi=\frac{-\omega}{\sqrt{(\omega^2+4)}},$$

$$\phi=\tan^{-1}\frac{2}{\sqrt{(\omega^2+4)}}\Big/\frac{-\omega}{\sqrt{(\omega^2+4)}}$$

$$R=\frac{10}{\sqrt{(\omega^2+4)}}$$

$\lim_{\omega\to 0} G(j\omega)=5;\phi=\tan^{-1} 1/-0=-90^\circ$

$\lim_{\omega\to\infty} G(j\omega)=0;\phi=\tan^{-1} 0/-1=-180^\circ$

Real axis intersection @ $j\omega=0$

Imaginary axis int er sec tion does not exist.

b&c) $\lim_{\omega\to 0} G(j\omega)=1\angle 0^\circ$

$\lim_{\omega\to\infty} G(j\omega)=0\angle -180^\circ$

$$G(j\omega)=\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+2\xi\left(j\frac{\omega}{\omega_n}\right)}=\frac{\left(1-\left(\frac{\omega}{\omega_n}\right)^2-2\xi\left(j\frac{\omega}{\omega_n}\right)\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\left(\frac{\omega}{\omega_n}\right)^2}$$

Therefore:

$$\mathrm{Re}\{G(j\omega)\}=\frac{1-\left(\frac{\omega}{\omega_n}\right)^2}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\left(\frac{\omega}{\omega_n}\right)^2}$$

$$\mathrm{Im}\{G(j\omega)\}=-\frac{2\xi\left(j\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\left(\frac{\omega}{\omega_n}\right)^2}$$

If $\mathrm{Re}\{G(j\omega)\} = 0$ ⇨ $\omega = \omega_n$

If $\mathrm{Im}\{G(j\omega)\} = 0$ ⇨ $\begin{cases} \omega = 0 \\ \omega \to 0 \\ \omega \to \infty \end{cases}$

If $\omega = \omega_n$ ⇨ $\begin{cases} G(j\omega_n) = \dfrac{1}{j2\xi} \\ \angle G(j\omega_n) = -90^o \end{cases}$

If $\omega = \omega_n$ and $\xi = 1$ ⇨ $G(j\omega_n) = \dfrac{1}{2j}$

If $\omega = \omega_n$ and $\xi \to 0$ ⇨ $G(j\omega_n) \to \infty$

If $\omega = \omega_n$ and $\xi \to \infty$ ⇨ $G(j\omega_n) \to 0$

d) $G(j\omega) = \dfrac{T\omega - j}{\omega(1+\omega^2T^2)}$

$\lim_{\omega\to 0} G(j\omega) = \infty \angle -90^o$

$\lim_{\omega\to\infty} G(j\omega) = 0 \angle -180^o$

e) $|G(j\omega)| = \left|\dfrac{e^{-j\omega L}}{1+j\omega T}\right| = \dfrac{1}{\sqrt{1+\omega^2T^2}}$

$\angle G(j\omega) = \angle\dfrac{1}{1+j\omega T} + \angle e^{-j\omega L} = \tan^{-1}(\omega T) - \omega L$

2-6

MATLAB code:

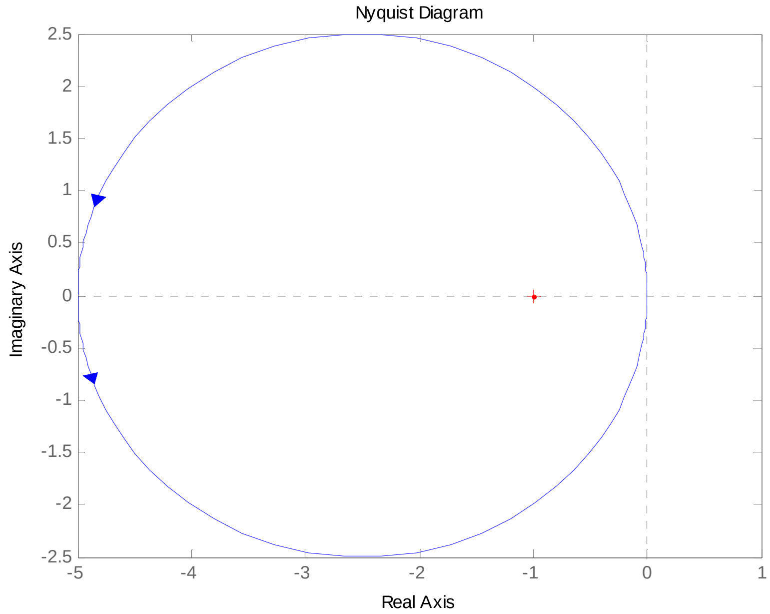
```
clear all;

s = tf('s')


%Part(a)

Ga=10/(s-2)

figure(1)

nyquist(Ga)


%Part(b)

zeta=0.5;  %asuuming a value for zeta <1

wn=2*pi*10 %asuuming a value for wn

Gb=1/(1+2*zeta*s/wn+s^2/wn^2)

figure(2)

nyquist(Gb)


%Part(c)

zeta=1.5;  %asuuming a value for zeta >1

wn=2*pi*10

Gc=1/(1+2*zeta*s/wn+s^2/wn^2)

figure(3)

nyquist(Gc)


%Part(d)

T=3.5 %assuming value for parameter T

Gd=1/(s*(s*T+1))

figure(4)

nyquist(Gd)
```

```
%Part(e)

T=3.5

L=0.5

Ge=pade(exp(-1*s*L),2)/(s*T+1)

figure(5)

hold on;

nyquist(Ge)
```

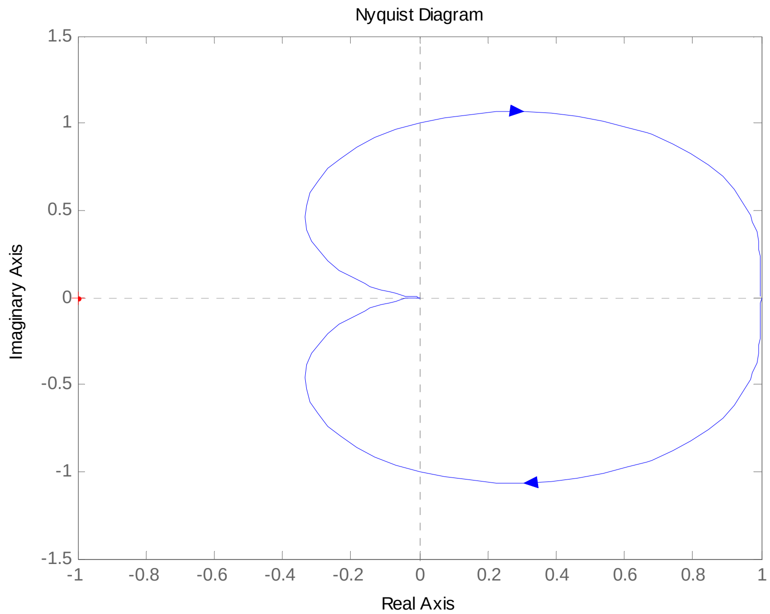
notes: In order to use Matlab Nyquist command, parameters needs to be assigned with values, and Pade approximation needs to be used for exponential term in part (e).

Nyquist diagrams are as follows:

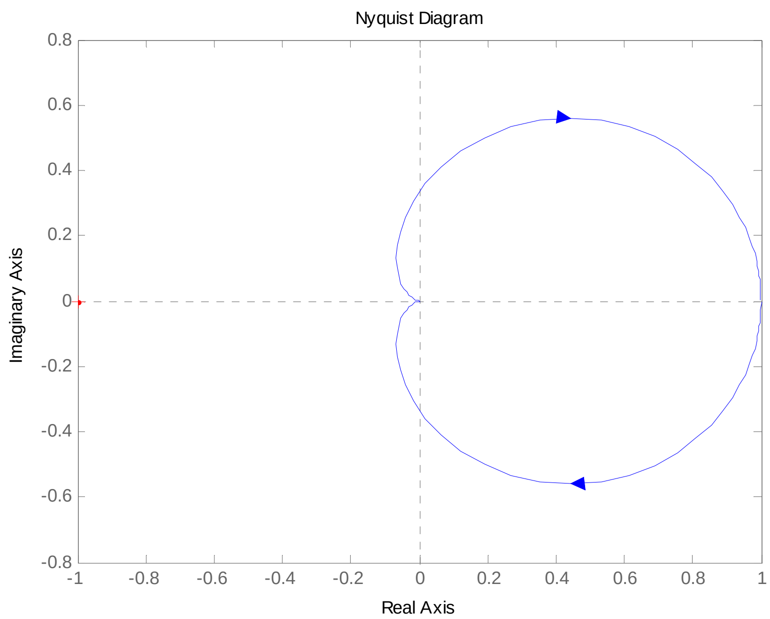
Part(a)


Nyquist Diagram
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
Imaginary Axis
-5
-4
-3
-2
-1
0
1
Real Axis

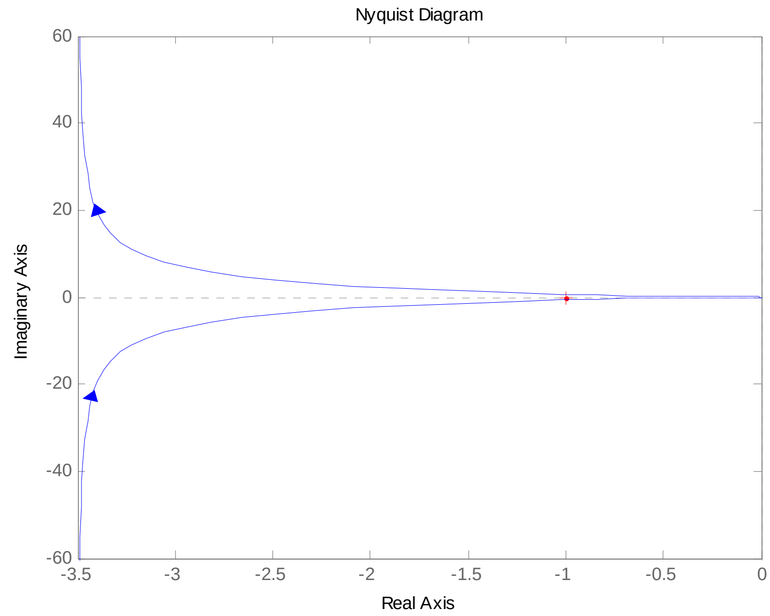
Part(b)


Nyquist Diagram
1.5
1
0.5
0
-0.5
-1
-1.5
Imaginary Axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Real Axis

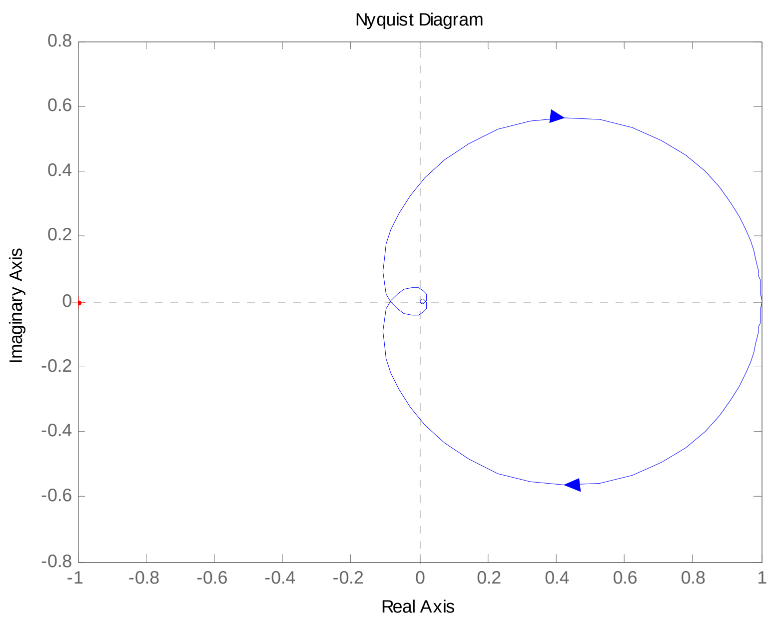
Part(c)


Nyquist Diagram
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
Imaginary Axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Real Axis

Part(d)


Nyquist Diagram
60
40
20
0
-20
-40
-60
Imaginary Axis
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
Real Axis

Part(e)


Nyquist Diagram
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
Imaginary Axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Real Axis

2-7) a) $G(j\omega) = \frac{2(j2\omega+1)}{j\omega(0.1j\omega+1)(0.02j\omega+1)}$

Steps for plotting |G|:

(1) For ω < 0.1, asymptote is $\frac{2}{j\omega}$

Break point: ω = 0.5
Slope = -1 or -20 dB/decade

(2) For 0.5 < ω < 10
Break point: ω = 10
Slope = -1+1 = 0 dB/decade

(3) For 10 < ω < 50:
Break point: ω = 50
Slope = -1 or -20 dB/decade

(4) For ω > 50
Slope = -2 or -40 dB/decade

Steps for plotting ∠ G

(1) $\angle \frac{2}{j\omega} = -90^o$

(2) $\angle \frac{1}{2j\omega+1} = \begin{cases} \omega \to 0: \ \angle \frac{1}{2j\omega+1} \to -90^o \\ \omega \to \infty : \ \angle \frac{1}{2j\omega+1} \to 0^o \end{cases}$

(3) $\angle \frac{1}{0.1j\omega+1} = \begin{cases} \omega \to 0: \ \angle \frac{1}{0.1j\omega+1} \to 0^o \\ \omega \to \infty: \ \angle \frac{1}{0.1j\omega+1} \to -90^o \end{cases}$

(4) $\angle \frac{1}{0.02j\omega+1} = \begin{cases} \omega \to 0: \ \angle \frac{1}{0.02j\omega+1} \to -90^o \\ \omega \to \infty : \ \angle \frac{1}{0.02j\omega+1} \to 0 \end{cases}$

b) Let's convert the transfer function to the following form:

$G(j\omega) = \frac{25}{10j\omega\left(-\frac{2.5}{10}\omega^2+j\frac{\omega}{10}\right)+1} \Rightarrow G(s) = \frac{5}{2}\frac{1}{s\left(\frac{s^2}{4}+0.1s+1\right)}$

Steps for plotting |G|:

(1) Asymptote: ω < 1 $|G(j\omega)| \cong 2.5 / \omega$

Slope: -1 or -20 dB/decade

$|G(j\omega)|_{\omega=1} = 2.5$

(2) $\omega_n = 2$ and $\xi = 0.1$ for second-order pole

break point: $\omega = 2$

slope: -3 or -60 dB/decade

$$|G(j\omega)|_{\omega=2} = \frac{1}{2\xi} = 5$$

Steps for plotting $\angle G(j\omega)$:

(1) for term 1/s the phase starts at -90^o and at $\omega = 2$ the phase will be -180^o

(2) for higher frequencies the phase approaches -270^o

c) Convert the transfer function to the following form:

$$G(j\omega) = \frac{0.01j\omega - \omega^2 + 1}{-\omega^2\left(0.01j\omega - \frac{\omega^2}{9} + 1\right)}$$

for term $\frac{1}{\omega^2}$, slope is -2 (-40 dB/decade) and passes through $|G(j\omega)|_{\omega=1} = 1$

(1) the breakpoint: $\omega = 1$ and slope is zero

(2) the breakpoint: $\omega = 2$ and slope is -2 or -40 dB/decade

$|G(j\omega)|_{\omega = 1} = 2\xi = 0.01$ below the asymptote

$|G(j\omega)|_{\omega = 1} = \frac{1}{2\xi} = \frac{1}{0.02} = 50$ above the asymptote

Steps for plotting $\angle G$:

(1) phase starts from -180^o due to $\frac{1}{s^2}$

(2) $\angle G(j\omega)|_{\omega=1} = 0$

(3) $\angle G(j\omega)|_{\omega=2} = -180^o$

d) $G(j\omega) = \frac{1}{1+2\xi\left(j\frac{\omega}{\omega_n}\right)-\left(\frac{\omega}{\omega_n}\right)^2}$

Steps for plotting the $|G|$:

(1) Asymptote for $\frac{\omega}{\omega_n} < 1$ is zero

(2) Breakpoint: $\frac{\omega}{\omega_n} = 1$, slope = -1 or -10 dB/decade

(3) As ξ is a damping ratio, then the magnitude must be obtained for various ξ when $0 \le \xi \le 1$

The high frequency slope is twice that of the asymptote for the single-pole case

Steps for plotting ∠ G:

(1) The phase starts at 0^o and falls -1 or -20 dB/decade at $\frac{\omega}{\omega_n}$ = 0.2 and approaches -180^o at $\frac{\omega}{\omega_n}$ = 5. For $\frac{\omega}{\omega_n}$ > 5, the phase remains at -180^o.

(2) As ξ is a damping ratio, the phase angles must be obtained for various ξ when $0 \le \xi \le 1$

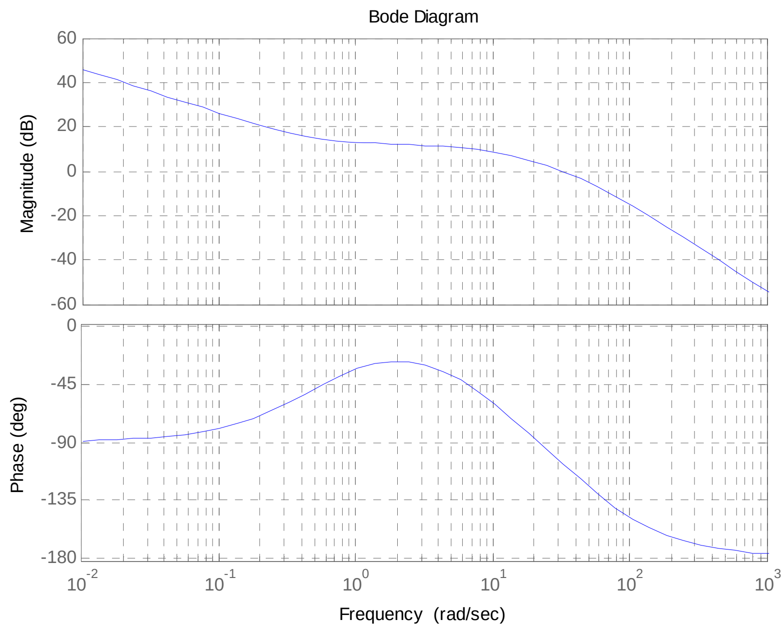
2-8) Use this part to confirm the results from the previous part.

MATLAB code:

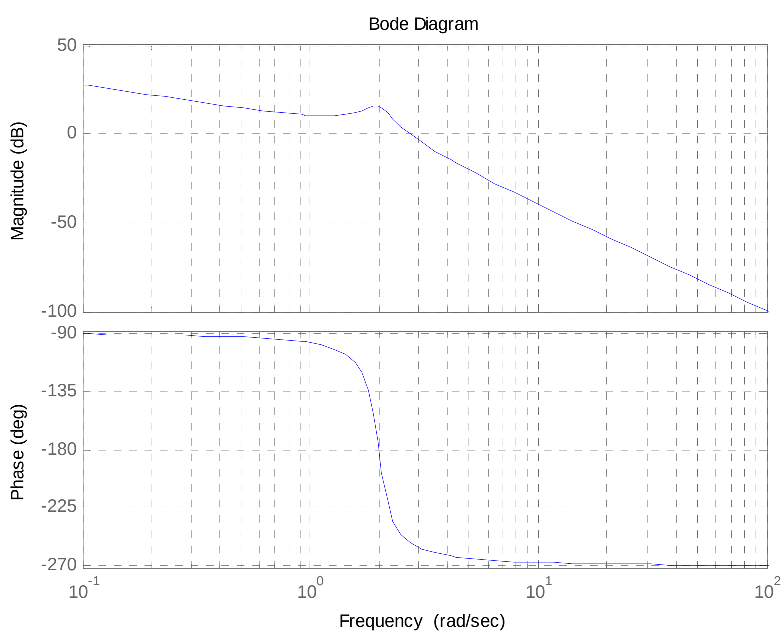
```
s = tf('s')


'Generated transfer function:'

Ga=2000*(s+0.5)/(s*(s+10)*(s+50))

figure(1)

bode(Ga)

grid on;


'Generated transfer function:'

Gb=25/(s*(s+2.5*s^2+10))

figure(2)

bode(Gb)

grid on;


'Generated transfer function:'

Gc=(s+100*s^2+100)/(s^2*(s+25*s^2+100))

figure(3)

bode(Gc)

grid on;
```

```
'Generated transfer function:'
zeta = 0.2
wn=8
Gd=1/(1+2*zeta*s/wn+(s/wn)^2)
figure(4)
bode(Gd)
grid on;

'Generated transfer function:'
t=0.3
'from pade approzimation:'
exp_term=pade(exp(-s*t),1)
Ge=0.03*(exp_term+1)^2/((exp_term-1)*(3*exp_term+1)*(exp_term+0.5))
figure(5)
bode(Ge)
grid on;
```

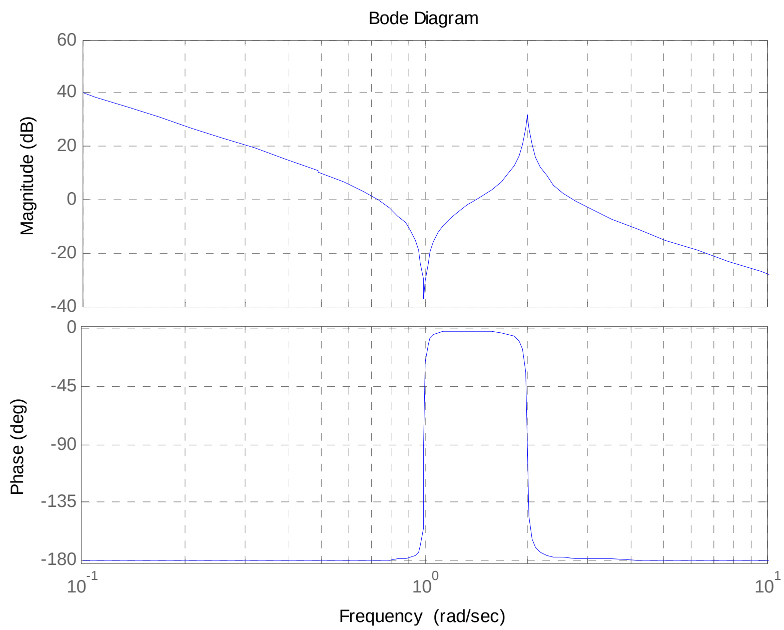
Part(a)


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

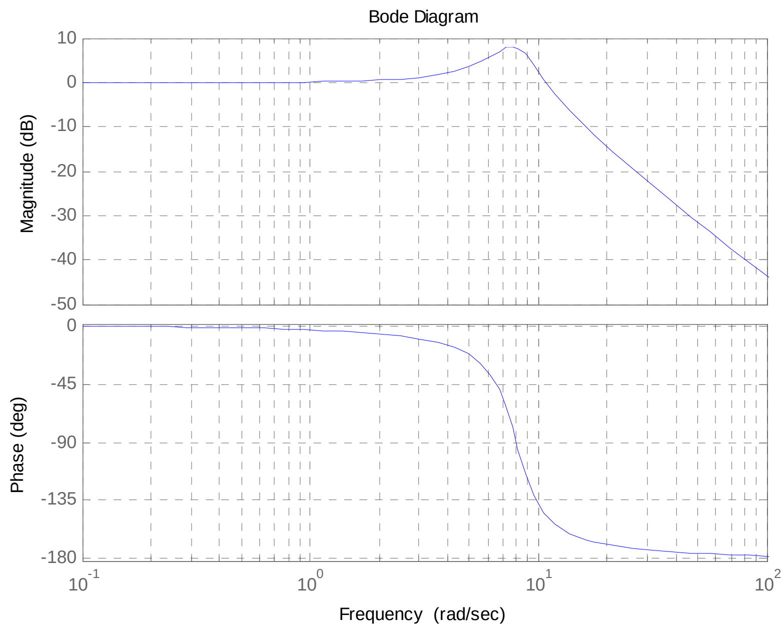
Part(b)


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

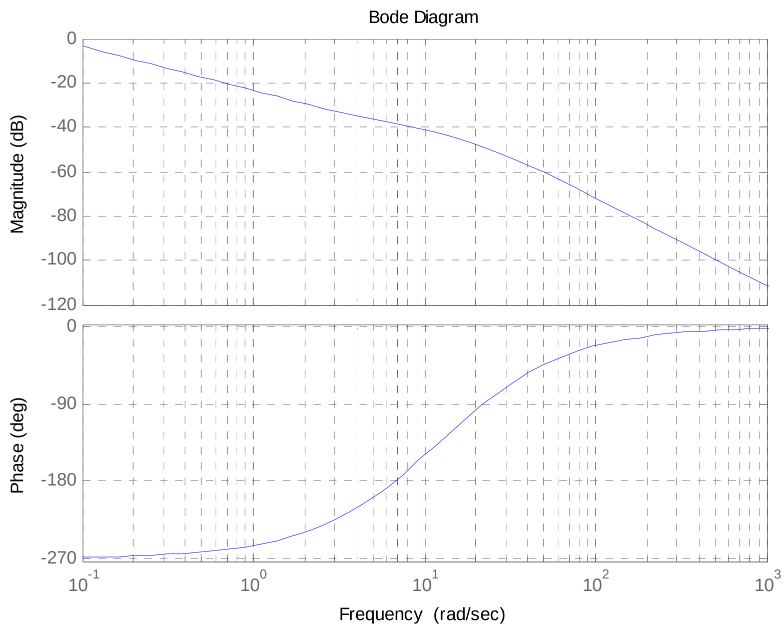
Part(c)


Bode Diagram
Magnitude (dB)
60
40
20
0
-20
-40
Phase (deg)
0
-45
-90
-135
-180
10^-1
10^0
10^1
Frequency (rad/sec)

Part(d)


Bode Diagram
Magnitude (dB)
10
0
-10
-20
-30
-40
-50
Phase (deg)
0
-45
-90
-135
-180
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

Part(e)


Bode Diagram
Magnitude (dB)
0
-20
-40
-60
-80
-100
-120
Phase (deg)
0
-90
-180
-270
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)

2-9)

a)

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 3 \\ -1 & -3 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

b)

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -1 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

2-10) We know that:

$$\begin{cases} G(s) = \int_0^{\infty} g(t)e^{-st}dt & (1) \\ g(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} g(t)e^{-st}dt & (2) \end{cases}$$

Partial integration of equation (1) gives:

$$G(s) = \left[-\frac{g(t)e^{-st}}{s}\right]_0^{\infty} + \frac{1}{s}\int_0^{\infty} g'(t)e^{-st}dt$$

$$\Rightarrow sG(s) = g(0) + \mathcal{L}\{g'(t)\}$$

$$\Rightarrow \mathcal{L}\{g'(t)\} \leftrightarrow sG(s) - g(0)$$

Differentiation of both sides of equation (1) with respect to s gives:

$$\frac{dG(s)}{ds} = \int_{-\infty}^{\infty} -(t)g(t)e^{-st}dt = \int_{-\infty}^{\infty} \left(-tg(t)\right)e^{-st}dt$$

Comparing with equation (1), we conclude that:

$$\mathcal{L}^{-1}\left\{\frac{dG(s)}{ds}\right\} \leftrightarrow -tg(t)$$

2-11) Let g(t) = $\int_{-\infty}^{t} x(\tau)d\tau$ then $x(t) = \frac{dg(t)}{dt}$

Using Laplace transform and differentiation property, we have X(s) = sG(s)

Therefore G(s) = $\frac{1}{s}X(s)$, which means:

$$\mathcal{L}\left\{\int_{-\infty}^{\infty} x(\tau)d\tau\right\} \quad \leftrightarrow \quad \frac{1}{s}X(s)$$

2-12) By Laplace transform definition:

$$\mathcal{L}\{g(t-T)u(t-T)\} = \int_{T}^{\infty} g(t-T)e^{-st}dt$$

Now, consider τ = t - T, then:

$$\mathcal{L}\{g(t-T)\} = \int_{0}^{\infty} g(\tau)e^{-s(\tau+T)}d\tau = e^{-sT}\int_{0}^{\infty} g(\tau)e^{-s\tau}d\tau$$

Which means: $\mathcal{L}\{g(t-T)\} \leftrightarrow e^{-sT}G(s)$

2-13) Consider:

$$f(t) = g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau$$

By Laplace transform definition:

$$F(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau\right]e^{-st}dt$$

$$= \int_{-\infty}^{\infty} g_1(\tau)\left[\int_{-\infty}^{\infty} g_2(t-\tau)e^{-st}dt\right]d\tau$$

By using time shifting theorem, we have:

$$F(s) = \int_{-\infty}^{\infty} g_1(\tau)[e^{-s\,\tau} G_2(s)]d\tau$$

$$= \left[\int_{-\infty}^{\infty} g_1(\tau)e^{-s\tau} d\tau \right] G_2(s) = G_1(s) \cdot G_2(s)$$

Let's consider g(t) = $g_1(t) \cdot g_2(t)$

$$G(s) = \int_0^{\infty} g_1(t)g_2(t)e^{-st}dt$$

By inverse Laplace Transform definition, we have

$$g_1(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} G_1(p)e^{pt}dp$$

Then

$$G(s) = \int_{c-j\infty}^{c+j\infty} G_1(p)dp \int_0^{\infty} f_2(t)e^{-(s-p)t}dt$$

Where

$$G_2(s-p) = \int_0^{\infty} f_2(t)e^{-(s-p)t}dt$$

therefore:

$$G(s) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} G_1(p)G_2(p-s)dp = \mathrm{G}_1(\mathrm{s}) * \mathrm{G}_2(\mathrm{s})$$

2-14) a) We know that

$$\mathcal{L}\left\{\frac{dg(t)}{dt}\right\} = \int_0^{\infty} \frac{dg(t)}{dt} e^{-st}dt = \mathrm{sG(s) + g(0)}$$

When s → ∞ , it can be written as:

$$\lim_{s\to\infty}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt = \lim_{s\to\infty}[sG(s)-g(0)]$$

As

$$lim_{x\to\infty}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt = 0$$

Therefore: $lim_{s\to\infty} sG(s) = g(0)$

b) By Laplace transform differentiation property:

$$\lim_{s\to 0}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt = \lim_{s\to 0}[sG(s)-g(0)]$$

As

$$\lim_{s\to 0}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt = \int_0^\infty \frac{dg(t)}{dt}dt = g(\infty)-g(0)$$

Therefore

$$\lim_{s\to 0}[sG(s)] - g(0) = g(\infty) - g(0)$$

which means:

$$\lim_{s\to 0} sG(s) = g(\infty)$$

2-15)

MATLAB code:

```
clear all;

syms t

s=tf('s')


f1 = (sin(2*t))^2

L1=laplace(f1)
```

```
% f2 = (cos(2*t))^2 = 1-(sin(2*t))^2   ===> L(f2)=1/s-L(f1) ===>
L2= 1/s - 8/s/(s^2+16)


f3 = (cos(2*t))^2
L3=laplace(f3)


'verified as L2 equals L3'
```

MATLAB solution for $L\{\sin^2 2t\}$ is:

8/s/(s^2+16)

Calculating $L\{\cos^2 2t\}$ based on $L\{\sin^2 2t\}$

$L\{\cos^2 2t\}$ = (s^3 + 8 s)/(s^4 + 16 s^2)

verifying $L\{\cos^2 2t\}$:

(8+s^2)/s/(s^2+16)

2-16) (a)

$$G(s)=\frac{5}{(s+5)^2}$$

(b)

$$G(s)=\frac{4s}{(s^2+4)}+\frac{1}{s+2}$$

(c)

$$G(s)=\frac{4}{s^2+4s+8}$$

(d)

$$G(s)=\frac{1}{s^2+4}$$

(e)

$$G(s)=\sum_{k=0}^{\infty} e^{kT(s+5)}=\frac{1}{1-e^{-T(s+5)}}$$

2-17) Note: %section (e) requires assignment of T and a numerical loop calculation

MATLAB code:

```
clear all;
syms t u

f1 = 5*t*exp(-5*t)
L1=laplace(f1)

f2 = t*sin(2*t)+exp(-2*t)
L2=laplace(f2)

f3 = 2*exp(-2*t)*sin(2*t)
L3=laplace(f3)

f4 = sin(2*t)*cos(2*t)
L4=laplace(f4)

%section (e) requires assignment of T and a numerical loop calculation
```

(a) $g(t) = 5te^{-5t}u_s(t)$

Answer: 5/(s+5)^2

(b) $g(t) = (t\sin 2t + e^{-2t})u_s(t)$

Answer: 4*s/(s^2+4)^2+1/(s+2)

(c) $g(t) = 2e^{-2t}\sin 2t\, u_s(t)$

Answer: 4/(s^2+4*s+8)

(d) $g(t) = \sin 2t \cos 2t\, u_s(t)$

Answer: 2/(s^2+16)

(e) $g(t) = \sum_{k=0}^{\infty} e^{-5kT}\, \delta(t-kT)$ where $\delta(t)$ = unit-impulse function

```
%section (e) requires assignment of T and a numerical loop calculation
```

2-18 (a)

$$g(t) = u_s(t) - 2u_s(t-1) + 2u_s(t-2) - 2u_s(t-3) + \cdots$$

$$G(s) = \frac{1}{s}\left(1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + \cdots\right) = \frac{1-e^{-s}}{s\left(1+e^{-s}\right)}$$

$$g_T(t) = u_s(t) - 2u_s(t-1) + u_s(t-2) \qquad 0 \le t \le 2$$

$$G_T(s) = \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{1}{s}\left(1-e^{-s}\right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-2k)u_s(t-2k) \qquad G(s) = \sum_{k=0}^{\infty} \frac{1}{s}(1-e^{-s})^2 e^{-2ks} = \frac{1-e^{-s}}{s(1+e^{-s})}$$

(b)

$$g(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 4(t-1)u_s(t-1) - 4(t-1.5)u_s(t-1.5) + \cdots$$

$$G(s) = \frac{2}{s^2}\left(1 - 2e^{-0.5s} + 2e^{-s} - 2e^{-1.5s} + \cdots\right) = \frac{2\left(1-e^{-0.5s}\right)}{s^2\left(1+e^{-0.5s}\right)}$$

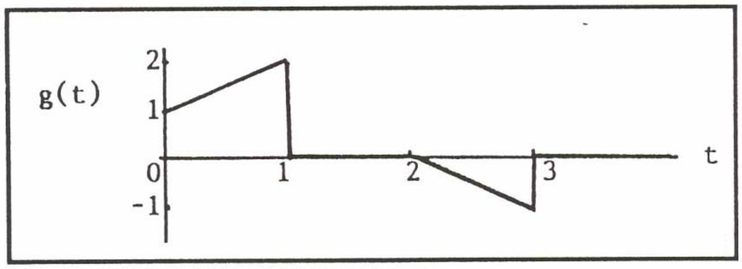
$$g_T(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 2(t-1)u_s(t-1) \qquad 0 \le t \le 1$$

$$G_T(s) = \frac{2}{s^2}\left(1 - 2e^{-0.5s} + e^{-s}\right) = \frac{2}{s^2}\left(1-e^{-0.5s}\right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-k)u_s(t-k) \qquad G(s) = \sum_{k=0}^{\infty} \frac{2}{s^2}\left(1-e^{-0.5s}\right)^2 e^{-ks} = \frac{2\left(1-e^{-0.5s}\right)}{s^2\left(1+e^{-0.5s}\right)}$$

2-19)

$$g(t) = (t+1)u_s(t) - (t-1)u_s(t-1) - 2u_s(t-1) - (t-2)u_s(t-2) + (t-3)u_s(t-3) + u_s(t-3)$$



$$G(s) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-2s} + e^{-3s}\right) + \frac{1}{s}\left(1 - 2e^{-s} + e^{-3s}\right)$$

2-20)

$$\mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt = \int_0^{\frac{T}{2}} e^{-st}dt + \int_{\frac{T}{2}}^{T} (-1)e^{-st}dt$$

$$= \frac{1 - e^{-\frac{Ts}{2}}}{s} + \frac{e^{-Ts} - e^{-\frac{Ts}{2}}}{s} = \frac{1}{s}\left[1 - e^{-\frac{Ts}{2}}\right]^2$$

2-21)

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^L \frac{e^{-st}}{L^2}dt - \int_L^{\frac{L}{2}} \frac{e^{-st}}{L^2}dt$$

$$= \left[-\frac{e^{-st}}{sL^2}\right]_0^L + \left[\frac{e^{-st}}{sL^2}\right]_L^{2L} = \frac{1 - e^{-Lt}}{sL^2} + \frac{e^{-2Lt} - e^{-Lt}}{sL^2}$$

$$= \frac{1}{sL^2}(1 - e^{-Lt})^2$$

2-22) $\mathcal{L}\left\{\frac{d^3y(t)}{dt^2}\right\} = s^3Y(s) - s^2y''(0) - sy'(0) - y(0)$

$\mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s) - sy'(0) - y(0)$

$\mathcal{L}\left\{\frac{dy(t)}{dt^2}\right\} = sY(s) - y(0)$

$\mathcal{L}\{-e^{-t}u_s(t)\} = -\frac{1}{s+1}$

⇨ $s^3Y(s) + s - s + 2s^2Y(s) + 2s - 2 - sY(s) + 2Y(s) = -\frac{1}{s+1}$

⇨ $(s^3 + 2s^2 - s + 2)Y(s) + 2s - 2 = -\frac{1}{s+1}$

⇨ $Y(s) = \frac{2s^2-3}{(s+2)(s^2+1)(s+1)}$

2-23

MATLAB code:

```
clear all;
syms t u s x1 x2 Fs

f1 = exp(-2*t)
L1=laplace(f1)/(s^2+5*s+4);

Eq2=solve('s*x1=1+x2','s*x2=-2*x1-3*x2+1','x1','x2')
f2_x1=Eq2.x1
f2_x2=Eq2.x2

f3=solve('(s^3-s+2*s^2+s+2)*Fs=-1+2-(1/(1+s))','Fs')
```

Here is the solution provided by MATLAB:

Part (a): F(s)=1/(s+2)/(s^2+5*s+4)

Part (b): X1(s)= (4+s)/(2+3*s+s^2)

X2(s)= (s-2)/(2+3*s+s^2)

Part (c): F(s) = s/(1+s)/(s^3+2*s^2+2)

2-24)

MATLAB code:

```
clear all;
syms s Fs
f3=solve('s^2*Fs-Fs=1/(s-1)','Fs')
Answer from MATLAB: Y(s)=1/(s-1)/(s^2-1)
```

2-25)

MATLAB code:

```
clear all;
syms s CA1 CA2 CA3
v1=1000;
v2=1500;
v3=100;
k1=0.1
k2=0.2
k3=0.4

f1='s*CA1=1/v1*(1000+100*CA2-1100*CA1-k1*v1*CA1)'
f2='s*CA2=1/v2*(1100*CA1-1100*CA2-k2*v2*CA2)'
f3='s*CA3=1/v3*(1000*CA2-1000*CA3-k3*v3*CA3)'
Sol=solve(f1,f2,f3,'CA1','CA2','CA3')
CA1=Sol.CA1
CA3=Sol.CA2
CA4=Sol.CA3
```

Solution from MATLAB:

CA1(s) =
1000*(s*v2+1100+k2*v2)/(1100000+s^2*v1*v2+1100*s*v1+s*v1*k2*v2+1100*s*v2+1100*k2*v2+k1*v1*s*v2+1100*k1*v1+k1*v1*k2*v2)

CA3(s) =

1100000/(1100000+s^2*v1*v2+1100*s*v1+s*v1*k2*v2+1100*s*v2+1100*k2*v2+k1*v1*s*v2+1100*k1*v1+k1*v1*k2*v2)

CA4 (s)=

1100000000/(1100000000+1100000*s*v3+1000*s*v1*k2*v2+1100000*s*v1+1000*k1*v1*s*v2+1000*k1*v1*k2*v2+1100*s*v1*k3*v3+1100*s*v2*k3*v3+1100*k2*v2*s*v3+1100*k2*v2*k3*v3+1100*k1*v1*s*v3+1100*k1*v1*k3*v3+1100000*k1*v1+1000*s^2*v1*v2+1100000*s*v2+1100000*k2*v2+1100000*k3*v3+s^3*v1*v2*v3+1100*s^2*v1*v3+1100*s^2*v2*v3+s^2*v1*v2*k3*v3+s^2*v1*k2*v2*v3+s*v1*k2*v2*k3*v3+k1*v1*s^2*v2*v3+k1*v1*s*v2*k3*v3+k1*v1*k2*v2*s*v3+k1*v1*k2*v2*k3*v3)

2-26) (a)

$$G(s)=\frac{1}{3s}-\frac{1}{2(s+2)}+\frac{1}{3(s+3)} \qquad g(t)=\frac{1}{3}-\frac{1}{2}e^{-2t}+\frac{1}{3}e^{-3t} \qquad t\geq 0$$

(b)

$$G(s)=\frac{-2.5}{s+1}+\frac{5}{(s+1)^2}+\frac{2.5}{s+3} \qquad g(t)=-2.5e^{-t}+5te^{-t}+2.5e^{-3t} \qquad t\geq 0$$

(c)

$$G(s)=\left(\frac{50}{s}-\frac{20}{s+1}-\frac{30s+20}{s^2+4}\right)e^{-s} \qquad g(t)=\left[50-20e^{-(t-1)}-30\cos 2(t-1)-5\sin 2(t-1)\right]u_s(t-1)$$

(d)

$$G(s)=\frac{1}{s}-\frac{s-1}{s^2+s+2}=\frac{1}{s}+\frac{1}{s^2+s+2}-\frac{s}{s^2+s+2}$$ Taking the inverse Laplace transform,

$$g(t)=1+1.069e^{-0.5t}\left[\sin 1.323t+\sin\left(1.323t-69.3^{\circ}\right)\right]=1+e^{-0.5t}\left(1.447\sin 1.323t-\cos 1.323t\right) \qquad t\geq 0$$

(e) $g(t)=0.5t^2e^{-t} \qquad t\geq 0$

(f)Try using MATLAB

```
>> b=num*2
```

```
b =

    2     2     2

>>num =

    1     1     1

>> denom1=[1 1]

denom1 =

    1     1

>> denom2=[1 5 5]

denom2 =

    1     5     5

>> num*2

ans =

    2     2     2

>> denom=conv([1 0],conv(denom1,denom2))

denom =

    1     6    10     5     0

>> b=num*2

b =

    2     2     2

>> a=denom

a =

    1     6    10     5     0

>> [r, p, k] = residue(b,a)

r =

  -0.9889
```

2.5889

-2.0000

0.4000

p =

-3.6180

-1.3820

-1.0000

0

k = [ ]

If there are no multiple roots, then

The number of poles n is

$$\frac{b}{a} = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + \ldots + \frac{r_n}{s+p_n} + k$$

In this case, p_1 and k are zero. Hence,

$$G(s) = \frac{0.4}{s} - \frac{0.9889}{s+3.6180} + \frac{2.5889}{s+1.3820} - \frac{2}{s+1}$$

$$g(t) = 0.4 - 0.9889e^{-3.618t} + 1.3820e^{-2.5889t} - 2e^{-t}$$

(g) $G(s) = \frac{2}{(s+1)(s+2)} + \frac{2e^{-s}}{s+1}$

$$= \frac{2}{s+1} - \frac{2}{s+2} + \frac{2e^{-s}}{s+1}$$

⇨ $\mathcal{L}^{-1}\{\mathrm{G(s)}\} = 2\mathrm{e}^{-\mathrm{t}} - 2\mathrm{e}^{-2\mathrm{t}} + 2\mathrm{e}^{-(\mathrm{t}-1)}u(t-1)$

(h) $G(s) = \frac{2s+1}{(s+1)(s+2)(s+3)} = -\frac{\frac{1}{2}}{s+1} + \frac{3}{s+2} - \frac{5}{2(s+3)}$

⇨ $\mathcal{L}^{-1}\{G(s)\} = -\frac{1}{2}e^{-t} + 3e^{-2t} - \frac{5}{2}e^{-3t}$

(i) $G(s) = \frac{3s^3+10s^2+8s+5}{s^{34}+5s^3+7s^2+5s+6} = \frac{1}{s+2} + \frac{1}{s+3} + \frac{s}{s^2+1}$

⇨ $\mathcal{L}^{-1}\{G(s)\} = e^{-2t} + e^{-3t} + cost$

2-27

MATLAB code:

```
clear all;
syms s

f1=1/(s*(s+2)*(s+3))
F1=ilaplace(f1)

f2=10/((s+1)^2*(s+3))
F2=ilaplace(f2)

f3=10*(s+2)/(s*(s^2+4)*(s+1))*exp(-s)
F3=ilaplace(f3)

f4=2*(s+1)/(s*(s^2+s+2))
F4=ilaplace(f4)

f5=1/(s+1)^3
F5=ilaplace(f5)

f6=2*(s^2+s+1)/(s*(s+1.5)*(s^2+5*s+5))
F6=ilaplace(f6)

s=tf('s')
f7=(2+2*s*pade(exp(-1*s),1)+4*pade(exp(-2*s),1))/(s^2+3*s+2) %using Pade command
for exponential term
```

```
[num,den]=tfdata(f7,'v') %extracting the polynomial values

syms s

f7n=(-2*s^3+6*s+12)/(s^4+6*s^3+13*s^2+12*s+4) %generating sumbolic function for ilaplace

F7=ilaplace(f7n)


f8=(2*s+1)/(s^3+6*s^2+11*s+6)

F8=ilaplace(f8)


f9=(3*s^3+10^s^2+8*s+5)/(s^4+5*s^3+7*s^2+5*s+6)

F9=ilaplace(f9)
```

Solution from MATLAB for the Inverse Laplace transforms:

Part (a): $$G(s) = \frac{1}{s(s+2)(s+3)}$$

G(t)=-1/2*exp(-2*t)+1/3*exp(-3*t)+1/6

To simplify:

syms t

digits(3)

vpa(-1/2*exp(-2*t)+1/3*exp(-3*t)+1/6)

ans =-.500*exp(-2.*t)+.333*exp(-3.*t)+.167

Part (b): $$G(s) = \frac{10}{(s+1)^2(s+3)}$$

G(t)= 5/2*exp(-3*t)+5/2*exp(-t)*(-1+2*t)

Part (c): $G(s) = \frac{100(s+2)}{s(s^2+4)(s+1)} e^{-s}$

G(t)=Step(t-1)*(-4*cos(t-1)^2+2*sin(t-1)*cos(t-1)+4*exp(-1/2*t+1/2)*cosh(1/2*t-1/2)-4*exp(-t+1)-cos(2*t-2)-2*sin(2*t-2)+5)

Part (d): $G(s) = \frac{2(s+1)}{s(s^2+s+2)}$

G(t)= 1+1/7*exp(-1/2*t)*(-7*cos(1/2*7^(1/2)*t)+3*7^(1/2)*sin(1/2*7^(1/2)*t))

To simplify:

syms t

digits(3)

vpa(1+1/7*exp(-1/2*t)*(-7*cos(1/2*7^(1/2)*t)+3*7^(1/2)*sin(1/2*7^(1/2)*t)))

ans = 1.+.143*exp(-.500*t)*(-7.*cos(1.32*t)+7.95*sin(1.32*t))

Part (e): $G(s) = \frac{1}{(s+1)^3}$

G(t)= 1/2*t^2*exp(-t)

Part (f): $G(s) = \frac{2(s^2+s+1)}{s(s+1.5)(s^2+5s+5)}$

G(t)= 4/15+28/3*exp(-3/2*t)-16/5*exp(-5/2*t)*(3*cosh(1/2*t*5^(1/2))+5^(1/2)*sinh(1/2*t*5^(1/2)))

Part (g): $G(s) = \frac{2+2se^{-s}+4e^{-2s}}{s^2+3s+2}$

G(t)= 2*exp(-2*t)*(7+8*t)+8*exp(-t)*(-2+t)

Part (h): $$G(s)=\frac{2s+1}{s^3+6s^2+11s+6}$$

G(t)= -1/2*exp(-t)+3*exp(-2*t)-5/2*exp(-3*t)

Part (i): $$G(s)=\frac{3s^3+10s^2+8s+5}{s^4+5s^3+7s^2+5s+6}$$

G(t)= -7*exp(-2*t)+10*exp(-3*t)-

1/10*ilaplace(10^(2*s)/(s^2+1)*s,s,t)+1/10*ilaplace(10^(2*s)/(s^2+1),s,t)+1/10*sin(t)*(10+dirac(t)*(-exp(-

3*t)+2*exp(-2*t)))

2-28) $\frac{dx(t)}{dt} = Ax(t) + Bu(t)$

a)

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_2(t) + 2x_3(t) - u_2(t) \\ \frac{dx_2(t)}{dt} = x_1(t) + x_3(t) + u_1(t) \\ \frac{dx_3(t)}{dt} = -x_1(t) - 2x_2(t) + x_3(t) \end{cases}$$

b)

$$\begin{cases} \frac{dx_1(t)}{dt} = 3x_1(t) + x_2(t) - 2x_3(t) - u(t) \\ \frac{dx_2(t)}{dt} = -x_1(t) + 2x_2(t) + 2x_3(t) \\ \frac{dx_3(t)}{dt} = x_3(t) + 2u(t) \end{cases}$$

2-29) **(a)**

$$\frac{Y(s)}{R(s)} = \frac{3s+1}{s^3 + 2s^2 + 5s + 6}$$

(b)

$$\frac{Y(s)}{R(s)} = \frac{5}{s^4 + 10s^2 + s + 5}$$

(c)

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)}{s^4 + 10s^3 + 2s^2 + s + 2}$$

(d)

$$\frac{Y(s)}{R(s)} = \frac{1 + 2e^{-s}}{2s^2 + s + 5}$$

e) $x(t) = y(t+1)$

⇨ $\frac{d^2x(t)}{dt^2} + \frac{4dx(t)}{dt} + 5x(t) = \frac{dr(t)}{dt} + 2r(t) + 2\int_{-\infty}^{t} r(\tau)d\,\tau$

By using Laplace transform, we have:

$$s^2X(s) + 4sX(s) + 5X(s) = sR(s) + 2R(s) + \frac{R(s)}{s}$$

As $X(s) = e^{-s}Y(s)$, then

$$(s^2 + 4s + t)e^{-s}Y(s) = \frac{s^2 + 2s + 1}{s}R(s)$$

Then:

$$\frac{Y(s)}{R(s)} = \frac{(s+1)^2 e^s}{s(s^2+4s+s)}$$

f) By using Laplace transform we have:

$$\left(s^3 + 2s^2 + s + 2 + \frac{2}{s}\right)Y(s) = se^{-s}R(s) + 2e^{-s}R(s)$$

As a result:

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)e^{-s}}{s^4+2s^3+s^2+2s+2}$$

2-30)

After taking the Laplace transform, the equation was solved in terms of Y(s), and consecutively was divided by input R(s) to obtain Y(s)/R(s):

MATLAB code:

```
clear all;

syms Ys Rs s


sol1=solve('s^3*Ys+2*s^2*Ys+5*s*Ys+6*Ys=3*s*Rs+Rs','Ys')

Ys_Rs1=sol1/Rs


sol2=solve('s^4*Ys+10*s^2*Ys+s*Ys+5*Ys=5*Rs','Ys')

Ys_Rs2=sol2/Rs


sol3=solve('s^3*Ys+10*s^2*Ys+2*s*Ys+2*Ys/s=s*Rs+2*Rs','Ys')

Ys_Rs3=sol3/Rs


sol4=solve('2*s^2*Ys+s*Ys+5*Ys=2*Rs*exp(-1*s)','Ys')

Ys_Rs4=sol4/Rs
```

```
 %Note: Parts E&F are too complicated with MATLAB, Laplace of integral is not
executable in MATLAB.....skipped
```

MATLAB Answers:

Part (a): Y(s)/R(s)= (3*s+1)/(5*s+6+s^3+2*s^2);

Part (b): Y(s)/R(s)= 5/(10*s^2+s+5+s^4)

Part (c): Y(s)/R(s)= (s+2)*s/(2*s^2+2+s^4+10*s^3)

Part (d): Y(s)/R(s)= 2*exp(-s)/(2*s^2+s+5)

```
%Note: Parts E&F are too complicated with MATLAB, Laplace of integral is not
executable in MATLAB.....skipped
```

2-31

MATLAB code:

```
clear all;

s=tf('s')


%Part a

Eq=10*(s+1)/(s^2*(s+4)*(s+6));

[num,den]=tfdata(Eq,'v');

[r,p] = residue(num,den)


%Part b

Eq=(s+1)/(s*(s+2)*(s^2+2*s+2));

[num,den]=tfdata(Eq,'v');

[r,p] = residue(num,den)


%Part c

Eq=5*(s+2)/(s^2*(s+1)*(s+5));

[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)


%Part d
Eq=5*(pade(exp(-2*s),1))/(s^2+s+1); %Pade approximation oreder 1 used
[num,den]=tfdata(Eq,'v');
[r,p] = residue(num,den)


%Part e
Eq=100*(s^2+s+3)/(s*(s^2+5*s+3));
[num,den]=tfdata(Eq,'v');
[r,p] = residue(num,den)


%Part f
Eq=1/(s*(s^2+1)*(s+0.5)^2);
[num,den]=tfdata(Eq,'v');
[r,p] = residue(num,den)


%Part g
Eq=(2*s^3+s^2+8*s+6)/((s^2+4)*(s^2+2*s+2));
[num,den]=tfdata(Eq,'v');
[r,p] = residue(num,den)


%Part h
Eq=(2*s^4+9*s^3+15*s^2+s+2)/(s^2*(s+2)*(s+1)^2);
[num,den]=tfdata(Eq,'v');
[r,p] = residue(num,den)
```

The solutions are presented in the form of two vectors, **r** and **p,** where for each case, the partial fraction expansion is equal to:

$$\frac{b(s)}{a(s)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \ldots + \frac{r_n}{s-p_n}$$

Following are **r** and **p** vectors for each part:

Part(a):

r =0.6944

-0.9375

0.2431

0.4167

p =-6.0000

-4.0000

0

0

Part(b):

r =0.2500

-0.2500 - 0.0000i

-0.2500 + 0.0000i

0.2500

p =-2.0000

-1.0000 + 1.0000i

-1.0000 - 1.0000i

0

Part(c):

r =0.1500

1.2500

-1.4000

2.0000

p = -5

-1

0

0

Part(d):

r =10.0000

-5.0000 - 0.0000i

-5.0000 + 0.0000i

p =-1.0000

-0.5000 + 0.8660i

-0.5000 - 0.8660i

Part(e):

r =110.9400

-110.9400

100.0000

p =-4.3028

-0.6972

0

Part(f):

r =0.2400 + 0.3200i

0.2400 - 0.3200i

-4.4800

-1.6000

4.0000

p =-0.0000 + 1.0000i

-0.0000 - 1.0000i

-0.5000

-0.5000

0

Part(g):

r =-0.1000 + 0.0500i

-0.1000 - 0.0500i

1.1000 + 0.3000i

1.1000 - 0.3000i

p =0.0000 + 2.0000i

0.0000 - 2.0000i

-1.0000 + 1.0000i

-1.0000 - 1.0000i

Part(h):

r =5.0000

-1.0000

9.0000

-2.0000

1.0000

p =-2.0000

-1.0000

-1.0000

0

0

2-32)

MATLAB code:

```
clear all;
syms s

%Part a
Eq=10*(s+1)/(s^2*(s+4)*(s+6));
ilaplace(Eq)

%Part b
Eq=(s+1)/(s*(s+2)*(s^2+2*s+2));
ilaplace(Eq)

%Part c
Eq=5*(s+2)/(s^2*(s+1)*(s+5));
ilaplace(Eq)

%Part d
exp_term=(-s+1)/(s+1) %pade approcimation
Eq=5*exp_term/((s+1)*(s^2+s+1));
ilaplace(Eq)

%Part e
Eq=100*(s^2+s+3)/(s*(s^2+5*s+3));
ilaplace(Eq)

%Part f
Eq=1/(s*(s^2+1)*(s+0.5)^2);
ilaplace(Eq)
```

```
%Part g
Eq=(2*s^3+s^2+8*s+6)/((s^2+4)*(s^2+2*s+2));
ilaplace(Eq)

%Part h
Eq=(2*s^4+9*s^3+15*s^2+s+2)/(s^2*(s+2)*(s+1)^2);
ilaplace(Eq)
```

MATLAB Answers:

Part(a):

G(t)= -15/16*exp(-4*t)+25/36*exp(-6*t)+35/144+5/12*t

To simplify:

syms t

digits(3)

vpa(-15/16*exp(-4*t)+25/36*exp(-6*t)+35/144+5/12*t)

ans =-.938*exp(-4.*t)+.694*exp(-6.*t)+.243+.417*tPart(b):

G(t)= 1/4*exp(-2*t)+1/4-1/2*exp(-t)*cos(t)

Part(c):

G(t)= 5/4*exp(-t)-7/5+3/20*exp(-5*t)+2*t

Part(d):

G(t)= -5*exp(-1/2*t)*(cos(1/2*3^(1/2)*t)+3^(1/2)*sin(1/2*3^(1/2)*t))+5*(1+2*t)*exp(-t)

Part(e):

G(t)= 100-800/13*exp(-5/2*t)*13^(1/2)*sinh(1/2*t*13^(1/2))

Part(f):

G(t)= 4+12/25*cos(t)-16/25*sin(t)-8/25*exp(-1/2*t)*(5*t+14)

Part(g):

G(t)= -1/5*cos(2*t)-1/10*sin(2*t)+1/5*(11*cos(t)-3*sin(t))*exp(-t)

Part(h):

G(t)= -2+t+5*exp(-2*t)+(-1+9*t)*exp(-t)

2-33) (a) Poles are at $s = 0, -1.5 + j1.6583, -1.5 - j1.6583$ One poles at $s = 0$. **Marginally stable**.

(b) Poles are at $s = -5, -j\sqrt{2}, j\sqrt{2}$ Two poles on $j\omega$ axis. **Marginally stable**.

(c) Poles are at $s = -0.8688, 0.4344 + j2.3593, 0.4344 - j2.3593$ Two poles in RHP. **Unstable**.

(d) Poles are at $s = -5, -1 + j, -1 - j$ All poles in the LHP. **Stable**.

(e) Poles are at $s = -1.3387, 1.6634 + j2.164, 1.6634 - j2.164$ Two poles in RHP. **Unstable**.

(f) Poles are at $s = -22.8487 \pm j22.6376, 21.3487 \pm j22.6023$ Two poles in RHP. **Unstable**.

2-34) Find the Characteristic equations and then use the roots command.

(a)

p= [1 3 5 0]

sr = roots(p)

p =

1 3 5 0

sr =

0

-1.5000 + 1.6583i

-1.5000 - 1.6583i

(b) p=conv([1 5],[1 0 2])

sr = roots(p)

p =

1 5 2 10

sr =

-5.0000

0.0000 + 1.4142i

0.0000 - 1.4142i

(c)

>> roots([1 5 5])

ans =

-3.6180

-1.3820

(d) roots(conv([1 5],[1 2 2]))

ans =

-5.0000

-1.0000 + 1.0000i

-1.0000 - 1.0000i

(e) roots([1 -2 3 10])

ans =

1.6694 + 2.1640i

1.6694 - 2.1640i

-1.3387

(f) roots([1 3 50 1 10^6])

-22.8487 +22.6376i

-22.8487 -22.6376i

21.3487 +22.6023i

21.3487 -22.6023i

Alternatively

Problem 2-34

MATLAB code:

```
% Question 2-34,
clear all;
s=tf('s')

%Part a
Eq=10*(s+2)/(s^3+3*s^2+5*s);
[num,den]=tfdata(Eq,'v');
roots(den)

%Part b
Eq=(s-1)/((s+5)*(s^2+2));
[num,den]=tfdata(Eq,'v');
roots(den)

%Part c
Eq=1/(s^3+5*s+5);
[num,den]=tfdata(Eq,'v');
roots(den)
```

```
%Part d
Eq=100*(s-1)/((s+5)*(s^2+2*s+2));
[num,den]=tfdata(Eq,'v');
roots(den)

%Part e
Eq=100/(s^3-2*s^2+3*s+10);
[num,den]=tfdata(Eq,'v');
roots(den)

%Part f
Eq=10*(s+12.5)/(s^4+3*s^3+50*s^2+s+10^6);
[num,den]=tfdata(Eq,'v');
roots(den)
```

MATLAB answer:

Part(a)

```
     0
-1.5000 + 1.6583i
-1.5000 - 1.6583i
```

Part(b)

```
-5.0000
```

-0.0000 + 1.4142i

-0.0000 - 1.4142i

Part(c)

0.4344 + 2.3593i

0.4344 - 2.3593i

-0.8688

Part(d)

-5.0000

-1.0000 + 1.0000i

-1.0000 - 1.0000i

Part(e)

1.6694 + 2.1640i

1.6694 - 2.1640i

-1.3387

Part(f)

-22.8487 +22.6376i

-22.8487 -22.6376i

21.3487 +22.6023i

21.3487 -22.6023i

2-35)

(a) $s^3 + 25s^2 + 10s + 450 = 0$ **Roots:** $-25.31,\ 0.1537 + j4.214,\ 0.1537 - 4.214$

Routh Tabulation:

s^3	1	10
s^2	25	450
s^1	$\dfrac{250-450}{25} = -8$	0
s^0	450	

Two sign changes in the first column. Two roots in RHP.

(b) $s^3 + 25s^2 + 10s + 50 = 0$ **Roots:** $-24.6769,\ -0.1616 + j1.4142,\ -0.1616 - j1.4142$

Routh Tabulation:

s^3	1	10
s^2	25	50
s^1	$\dfrac{250-50}{25} = 8$	0
s^0	50	

No sign changes in the first column. No roots in RHP.

(c) $s^3 + 25s^2 + 250s + 10 = 0$ **Roots:** $-0.0402,\ -12.48 + j9.6566,\ -j9.6566$

Routh Tabulation:

s^3	1	250
s^2	25	10
s^1	$\frac{6250-10}{25}=249.6$	0
s^0	10	

No sign changes in the first column. No roots in RHP.

(d) $2s^4+10s^3+5.5s^2+5.5s+10=0$ **Roots:** $-4.466, -1.116, 0.2888+j0.9611, 0.2888-j0.9611$

Routh Tabulation:

s^4	2	5.5	10
s^3	10	5.5	
s^2	$\frac{55-11}{10}=4.4$	10	
s^1	$\frac{24.2-100}{4.4}=-75.8$		
s^0	10		

Two sign changes in the first column. Two roots in RHP.

(e) $s^6+2s^5+8s^4+15s^3+20s^2+16s+16=0$ **Roots:** $-1.222\pm j0.8169, 0.0447\pm j1.153, 0.1776\pm j2.352$

Routh Tabulation:

s^6	1	8	20	16
s^5	2	15	16	
s^4	$\frac{16-15}{2}=0.5$	$\frac{40-16}{2}=12$		
s^3	-33	-48		

s^2	$\frac{-396+24}{-33} = 11.27$	16
s^1	$\frac{-541.1+528}{11.27} = -1.16$	0
s^0	0	

Four sign changes in the first column. Four roots in RHP.

(f) $s^4 + 2s^3 + 10s^2 + 20s + 5 = 0$ **Roots:** $-0.29, -1.788, 0.039 + j3.105, 0.039 - j3.105$

Routh Tabulation:

s^4	1	10	5	
s^3	2	20		
s^2	$\frac{20-20}{2} = 0$	5		
s^2	ε	5		Replace 0 in last row by ε
s^1	$\frac{20\varepsilon - 10}{\varepsilon} \cong -\frac{10}{\varepsilon}$			**Two sign changes in first column. Two roots in RHP.**
s^0	5			

(g)

s^8	1	8	20	16	0
s^7	2	12	16	0	0
s^6	2	12	16	0	0
s^5	0	0	0	0	0

$$A(s) = 2s^6 + 12s^5 + 16s^4$$

$$\frac{dA(s)}{ds} = 12s^5 + 60s^4 + 64s^3$$

s^5	12	60	64	0
s^4	2	$\frac{16}{3}$	0	0
s^3	28	64	0	0
s^2	0.759	0	0	0
s^1	28	0		
s^0	0			

2-36) Use MATLAB roots command

a) roots([1 25 10 450])

```
ans =

 -25.3075
   0.1537 + 4.2140i
   0.1537 - 4.2140i
```

b) roots([1 25 10 50])

```
ans =

 -24.6769
  -0.1616 + 1.4142i
  -0.1616 - 1.4142i
```

c) roots([1 25 250 10])

```
ans =
 -12.4799 + 9.6566i
 -12.4799 - 9.6566i
```

-0.0402

d) roots([2 10 5.5 5.5 10])

ans =

-4.4660

-1.1116

0.2888 + 0.9611i

0.2888 - 0.9611i

e) roots([1 2 8 15 20 16 16])

ans =

0.1776 + 2.3520i

0.1776 - 2.3520i

-1.2224 + 0.8169i

-1.2224 - 0.8169i

0.0447 + 1.1526i

0.0447 - 1.1526i

f) roots([1 2 10 20 5])

ans =

0.0390 + 3.1052i

0.0390 - 3.1052i

-1.7881

-0.2900

g) roots([1 2 8 12 20 16 16])

ans =

0.0000 + 2.0000i

0.0000 - 2.0000i

-1.0000 + 1.0000i

-1.0000 - 1.0000i

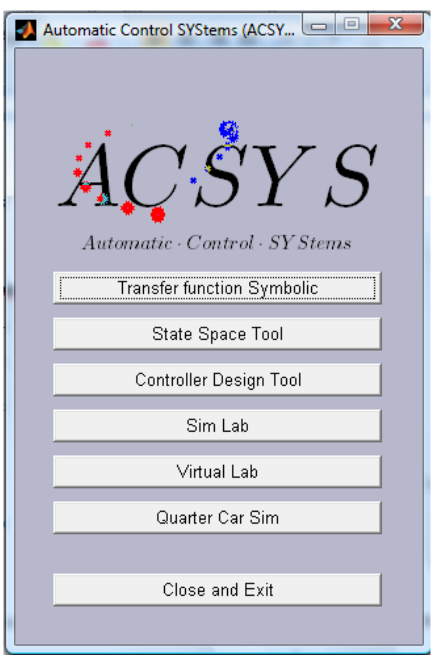
0.0000 + 1.4142i

0.0000 - 1.4142i

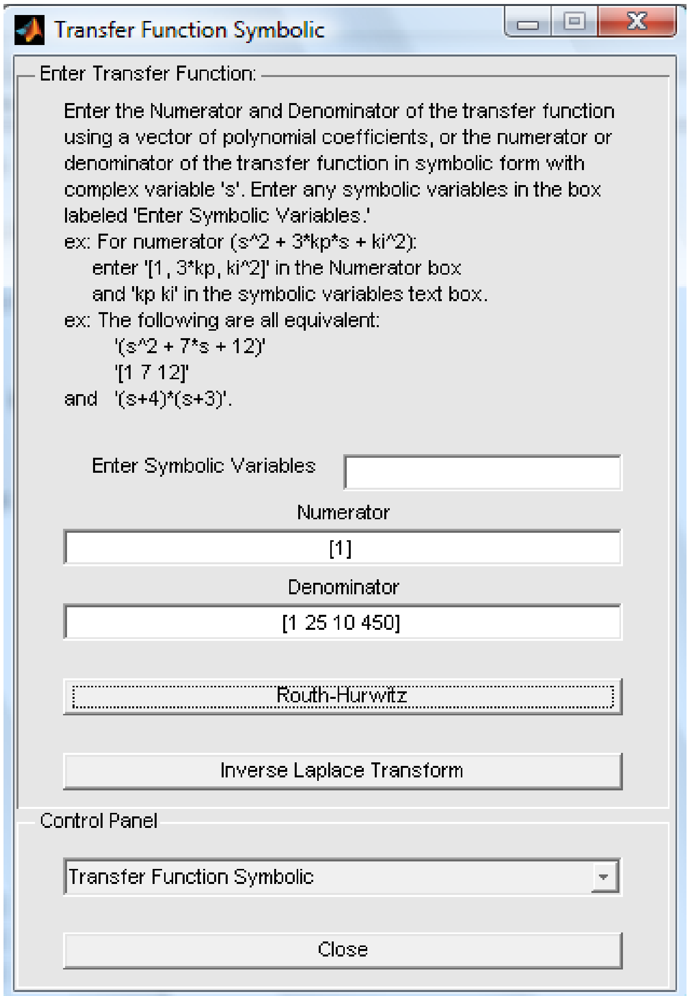
Alternatively use the approach in this Chapter's Section 2-14:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the "transfer function Symbolic button



5. Enter the characteristic equation in the denominator and press the “Routh-Hurwitz” push-button.

RH =

[1, 10]

[25, 450]

[-8, 0]

[450, 0]

Two sign changes in the first column. Two roots in RHP=> UNSTABLE

2-37) Use the MATLAB “roots” command same as in the previous problem.

2-38) To solve using MATLAB, set the value of K in an iterative process and find the roots such that at least one root changes sign from negative to positive. Then increase resolution if desired.

Example: in this case 0<K<12 (increase resolution by changing the loop to: for K=11:.1:12)

```
for K=0:12
K
roots([1 25 15 20 K])
end
```

```
K =

     0

ans =

       0
 -24.4193
  -0.2904 + 0.8572i
  -0.2904 - 0.8572i

K =

     1

ans =
```

-24.4192
-0.2645 + 0.8485i
-0.2645 - 0.8485i
-0.0518

K =

2

ans =

-24.4191
-0.2369 + 0.8419i
-0.2369 - 0.8419i
-0.1071

K =

3

```
ans =

 -24.4191
  -0.2081 + 0.8379i
  -0.2081 - 0.8379i
  -0.1648


K =

     4


ans =

 -24.4190
  -0.1787 + 0.8369i
  -0.1787 - 0.8369i
  -0.2237


K =

     5
```

ans =

-24.4189
-0.1496 + 0.8390i
-0.1496 - 0.8390i
-0.2819

K =

6

ans =

-24.4188
-0.1215 + 0.8438i
-0.1215 - 0.8438i
-0.3381

K =

7

ans =

-24.4188
-0.0951 + 0.8508i
-0.0951 - 0.8508i
-0.3911

K =

8

ans =

-24.4187
-0.0704 + 0.8595i
-0.0704 - 0.8595i
-0.4406

K =

9

ans =

-24.4186
-0.0475 + 0.8692i
-0.0475 - 0.8692i
-0.4864

K =

10

ans =

-24.4186
-0.0263 + 0.8796i
-0.0263 - 0.8796i
-0.5288

K =

11

ans =

-24.4185
-0.0067 + 0.8905i
-0.0067 - 0.8905i
-0.5681

K =

12

ans =

-24.4184
0.0115 + 0.9015i
0.0115 - 0.9015i
-0.6046

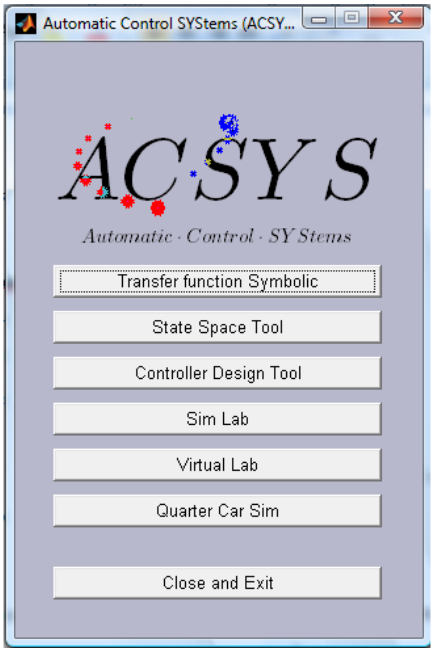
Alternatively use the approach in this Chapter's Section 2-14:

1. Activate MATLAB

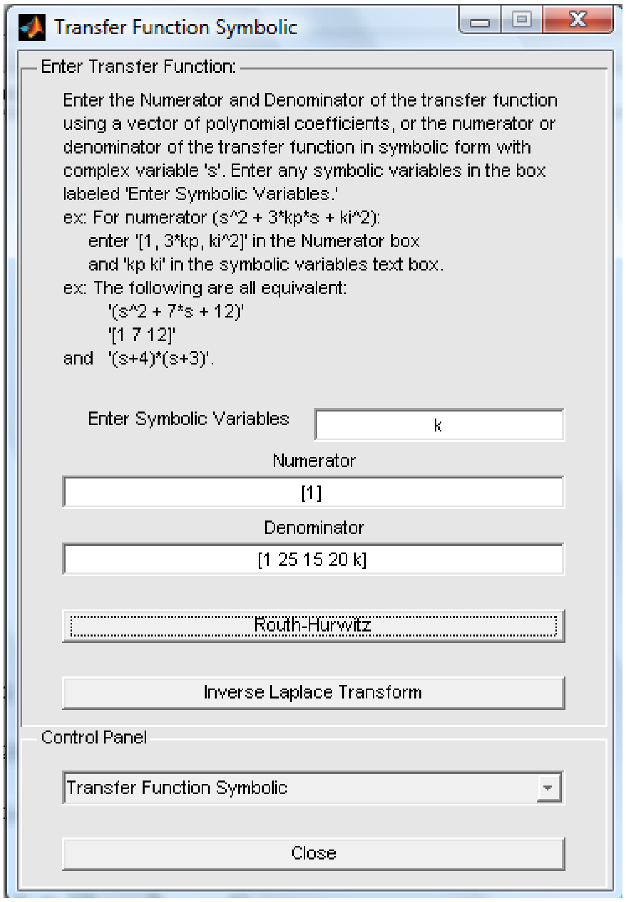
2. Go to the directory containing the ACSYS software.

3. Type in

Acsys



4. Then press the “transfer function Symbolic button



5. Enter the characteristic equation in the denominator and press the “Routh-Hurwitz” push-button.

RH =

[1, 15, k]

[25, 20, 0]

[71/5, k, 0]

[-125/71*k+20, 0, 0]

[k, 0, 0]

6. Find the values of K to make the system unstable following the next steps.

Alternative Problem 2-36

Using ACSYS toolbar under "Transfer Function Symbolic", the Routh-Hurwitz option can be used to generate RH matrix based on denominator polynhomial. The system is stable if and only if the first column of this matrix contains NO negative values.

MATLAB code: to calculate the number of right hand side poles

```
%Part a
den_a=[1 25 10 450]
roots(den_a)

%Part b
den_b=[1 25 10 50]
roots(den_b)

%Part c
den_c=[1 25 250 10]
roots(den_c)

%Part d
den_d=[2 10 5.5 5.5 10]
roots(den_d)

%Part e
den_e=[1 2 8 15 20 16 16]
roots(den_e)

%Part f
den_f=[1 2 10 20 5]
roots(den_f)
```

```
%Part g
den_g=[1 2 8 12 20 16 16 0 0]
roots(den_g)
```

using ACSYS, the denominator polynomial can be inserted, and by clicking on the “Routh-Hurwitz” button, the R-H chart can be observed in the main MATLAB command window:

Part(a): for the transfer function in part (a), this chart is:

RH chart =

[1, 10]

[25, 450]

[-8, 0]

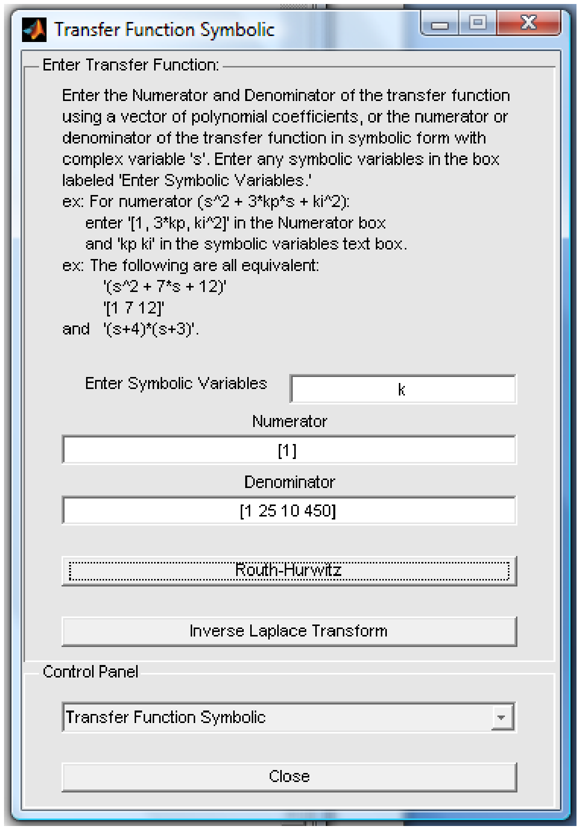
[450, 0]

Unstable system due to -8 on the 3rd row.

2 complex conjugate poles on right hand side. All the poles are:

-25.3075

0.1537 + 4.2140i and 0.1537 - 4.2140i



Part (b):

RH chart:

[1, 10]

[25, 50]

[8, 0]

[50, 0]

Stable system >> No right hand side pole

Part (c):

RH chart:

[1, 250]

[25, 10]

[1248/5, 0]

[10, 0]

Stable system >> No right hand side pole

Part (d):

RH chart:

[2, 11/2, 10]

[10, 11/2, 0]

[22/5, 10, 0]

[-379/22, 0, 0]

[10, 0, 0]

Unstable system due to -379/22 on the 4th row.

2 complex conjugate poles on right hand side. All the poles are:

-4.4660

-1.1116

0.2888 + 0.9611i

0.2888 - 0.9611i

Part (e):

RH chart:

[1, 8, 20, 16]

[2, 15, 16, 0]

[1/2, 12, 16, 0]

[-33, -48, 0, 0]

[124/11, 16, 0, 0]

[-36/31, 0, 0, 0]

[16, 0, 0, 0]

Unstable system due to -33 and -36/31 on the 4th and 6th row.

4 complex conjugate poles on right hand side. All the poles are:

0.1776 + 2.3520i

0.1776 - 2.3520i

-1.2224 + 0.8169i

-1.2224 - 0.8169i

0.0447 + 1.1526i

0.0447 - 1.1526i

Part (f):

RH chart:

[1, 10, 5]

[2, 20, 0]

[eps, 5, 0]

[(-10+20*eps)/eps, 0, 0]

[5, 0, 0]

Unstable system due to ((-10+20*eps)/eps) on the 4th.

2 complex conjugate poles slightly on right hand side. All the poles are:

0.0390 + 3.1052i

0.0390 - 3.1052i

-1.7881

-0.2900

Part (g):

RH chart:

[1, 8, 20, 16, 0]

[2, 12, 16, 0, 0]

[2, 12, 16, 0, 0]

[12, 48, 32, 0, 0]

[4, 32/3, 0, 0, 0]

[16, 32, 0, 0, 0]

[8/3, 0, 0, 0, 0]

[32, 0, 0, 0, 0]

[0, 0, 0, 0, 0]

Stable system >> No right hand side pole

6 poles wt zero real part:

0

0

0.0000 + 2.0000i

0.0000 - 2.0000i

-1.0000 + 1.0000i

-1.0000 - 1.0000i

0.0000 + 1.4142i

0.0000 - 1.4142i

(a) $s^4 + 25s^3 + 15s^2 + 20s + K = 0$

Routh Tabulation:

s^4	1	15	K
s^3	25	20	
s^2	$\frac{375-20}{25} = 14.2$	K	
s^1	$\frac{284-25K}{14.2} = 20 - 1.76K$		$20 - 1.76K > 0$ or $K < 11.36$
s^0	K		$K > 0$

Thus, the system is stable for 0 < *K* < 11.36. When *K* = 11.36, the system is marginally stable. The auxiliary equation is $A(s) = 14.2s^2 + 11.36 = 0.$ The solution of *A*(*s*) = 0 is $s^2 = -0.8.$ The frequency of oscillation is 0.894 rad/sec.

(b) $s^4 + Ks^3 + 2s^2 + (K+1)s + 10 = 0$

Routh Tabulation:

s^4	1	2	10	
s^3	K	$K+1$		$K > 0$
s^2	$\frac{2K-K-1}{K} = \frac{K-1}{K}$	10		$K > 1$
s^1	$\frac{-9K^2-1}{K-1}$			$-9K^2 - 1 > 0$
s^0	10			

The conditions for stability are: *K* > 0, *K* > 1, and $-9K^2 - 1 > 0$. Since K^2 is always positive, the last condition cannot be met by any real value of *K*. Thus, the system is unstable for all values of *K*.

(c) $s^3+(K+2)s^2+2Ks+10=0$

Routh Tabulation:

s^3	1	$2K$	
s^2	$K+2$	10	$K>-2$
s^1	$\dfrac{2K^2+4K-10}{K+2}$		$K^2+2K-5>0$
s^0	10		

The conditions for stability are: $K > -2$ and $K^2+2K-5>0$ or $(K+3.4495)(K-1.4495) > 0$, or $K > 1.4495$. Thus, the condition for stability is $K > 1.4495$. When $K = 1.4495$ the system is marginally stable. The auxiliary equation is $A(s)=3.4495s^2+10=0$. The solution is $s^2=-2.899$. The frequency of oscillation is 1.7026 rad/sec.

(d) $s^3+20s^2+5s+10K=0$

Routh Tabulation:

s^3	1	5	
s^2	20	$10K$	
s^1	$\dfrac{100-10K}{20}=5-0.5K$		$5-0.5K>0$ or $K<10$
s^0	$10K$		$K>0$

The conditions for stability are: $K > 0$ and $K < 10$. Thus, $0 < K < 10$. When $K = 10$, the system is marginally stable. The auxiliary equation is $A(s)=20s^2+100=0$. The solution of the auxiliary equation is $s^2=-5$. The frequency of oscillation is 2.236 rad/sec.

(e) $s^4 + Ks^3 + 5s^2 + 10s + 10K = 0$

Routh Tabulation:

s^4	1	5	$10K$	
s^3	K	10		$K > 0$
s^2	$\frac{5K-10}{K}$	$10K$		$5K - 10 > 0$ or $K > 2$
s^1	$\frac{\frac{50K-100}{K} - 10K^2}{\frac{5K-10}{K}} = \frac{50K - 100 - 10K^3}{5K-10}$			$5K - 10 - K^3 > 0$
s^0	$10K$			$K > 0$

The conditions for stability are: $K > 0$, $K > 2$, and $5K - 10 - K^3 > 0$.

Use Matlab to solve for k from last condition

```
>> syms k

>> kval=solve(5*k-10+k^3,k);

>> eval(kval)

kval =

   1.4233

  -0.7117 + 2.5533i

  -0.7117 - 2.5533i
```

So K>1.4233.

Thus, the conditions for stability is: $K > 2$

(f) $s^4 + 12.5s^3 + s^2 + 5s + K = 0$

Routh Tabulation:

s^4	1	1	K
s^3	12.5	5	
s^2	$\frac{12.5-5}{12.5} = 0.6$	K	
s^1	$\frac{3-12.5K}{0.6} = 5 - 20.83K$		$5 - 20.83K > 0$ or $K < 0.24$
s^0	K		$K > 0$

The condition for stability is $0 < K < 0.24$. When $K = 0.24$ the system is marginally stable. The auxiliary equation is $A(s) = 0.6s^2 + 0.24 = 0$. The solution of the auxiliary equation is $s^2 = -0.4$. The frequency of oscillation is 0.632 rad/sec.

2-39)

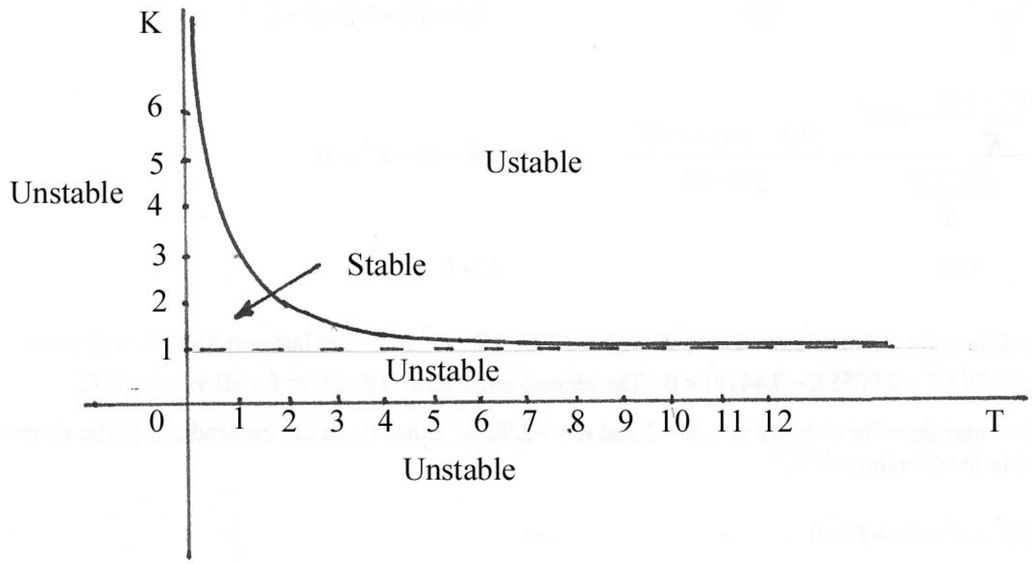
The characteristic equation is $Ts^3 + (2T+1)s^2 + (2+K)s + 5K = 0$

Routh Tabulation:

s^3	T	$K+2$	$T > 0$
s^2	$2T+1$	$5K$	$T > -1/2$
s^1	$\frac{(2T+1)(K+2) - 5KT}{2T+1}$		$K(1-3T) + 4T + 2 > 0$
s^0	$5K$		$K > 0$

The conditions for stability are: $T > 0$, $K > 0$, and $K < \frac{4T+2}{3T-1}$. The regions of stability in the

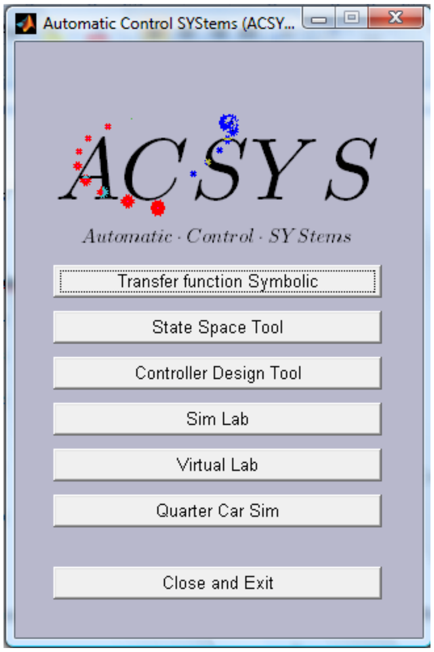
T-versus-*K* parameter plane is shown below.



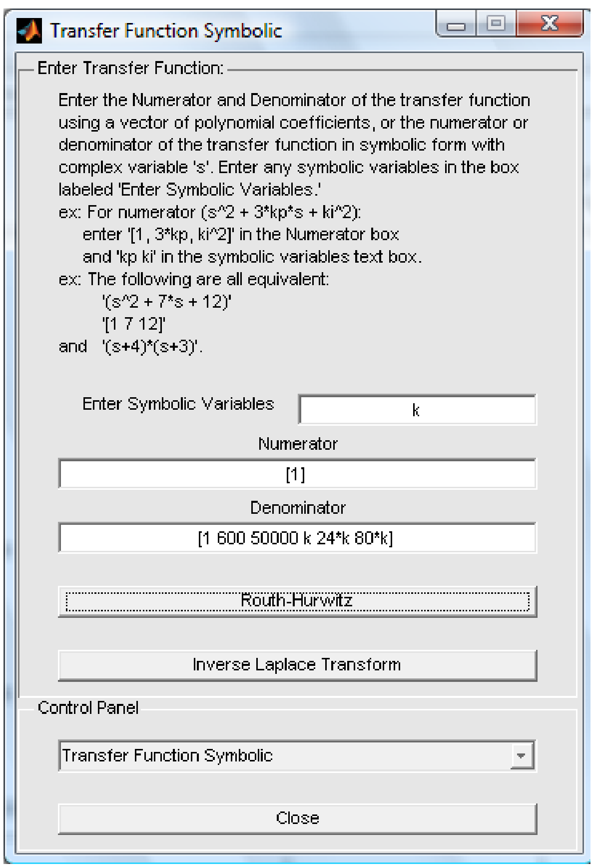
2-40 Use the approach in this Chapter's Section 2-14:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the "transfer function Symbolic button."



5. Enter the characteristic equation in the denominator and press the "Routh-Hurwitz" push-button.

RH =

```
[1,                                        50000,       24*k]
[600,                                          k,       80*k]
[-1/600*k+50000,                        358/15*k,          0]
[ (35680*k-1/600*k^2)/(-1/600*k+50000),     80*k,          0]
[ 24*k*(k^2-21622400*k+5000000000000)/(k-30000000)/(35680*k-1/600*k^2)*(-1/600*k+50000),
0,                                             0]
[80*k,                                         0,          0]
```

6. Find the values of K to make the system unstable following the next steps.

(a) Characteristic equation: $s^5 + 600s^4 + 50000s^3 + Ks^2 + 24Ks + 80K = 0$

Routh Tabulation:

s^5	1	50000	$24K$	
s^4	600	K	$80K$	
s^3	$\dfrac{3\times10^7 - K}{600}$	$\dfrac{14320K}{600}$		$K < 3\times10^7$
s^2	$\dfrac{21408000K - K^2}{3\times10^7 - K}$	$80K$		$K < 21408000$
s^1	$\dfrac{-7.2\times10^{16} + 3.113256\times10^{11}K - 14400K^2}{600(21408000 - K)}$			$K^2 - 2.162\times10^7 K + 5\times10^{12} < 0$
s^0	$80K$			$K > 0$

Conditions for stability:

From the s^3 row: $K < 3 \times 10^7$

From the s^2 row: $K < 2.1408 \times 10^7$

From the s^1 row: $K^2 - 2.162 \times 10^7 K + 5 \times 10^{12} < 0$ or $(K - 2.34 \times 10^5)(K - 2.1386 \times 10^7) < 0$

Thus, $2.34 \times 10^5 < K < 2.1386 \times 10^7$

From the s^0 row: $K > 0$

Thus, the final condition for stability is: $2.34 \times 10^5 < K < 2.1386 \times 10^7$

When $K = 2.34 \times 10^5$ $\omega = 10.6$ rad/sec.

When $K = 2.1386 \times 10^7$ $\omega = 188.59$ rad/sec.

(b) Characteristic equation: $s^3 + (K+2)s^2 + 30Ks + 200K = 0$

Routh tabulation:

s^3	1	$30K$	
s^2	$K+2$	$200K$	$K > -2$
s^1	$\dfrac{30K^2 - 140K}{K+2}$		$K > 4.6667$
s^0	$200K$		$K > 0$

Stability Condition: $K > 4.6667$

When K = 4.6667, the auxiliary equation is $A(s) = 6.6667s^2 + 933.333 = 0$. The solution is $s^2 = -140$.

The frequency of oscillation is 11.832 rad/sec.

(c) Characteristic equation: $s^3 + 30s^2 + 200s + K = 0$

Routh tabulation:

s^3	1	200	
s^2	30	K	
s^1	$\dfrac{6000-K}{30}$		$K < 6000$
s^0	K		$K > 0$

Stabililty Condition: $0 < K < 6000$

When *K* = 6000, the auxiliary equation is $A(s) = 30s^2 + 6000 = 0.$ The solution is $s^2 = -200.$

The frequency of oscillation is 14.142 rad/sec.

(d) Characteristic equation: $s^3 + 2s^2 + (K+3)s + K + 1 = 0$

Routh tabulation:

s^3	1	$K+3$	
s^2	2	$K\ +1$	
s^1	$\dfrac{K+5}{30}$		$K > -5$
s^0	$K\ +1$		$K > -1$

Stability condition: *K* > –1. When *K* = –1 the zero element occurs in the first element of the s^0 row. Thus, there is no auxiliary equation. When *K* = –1, the system is marginally stable, and one of the three characteristic equation roots is at *s* = 0. There is no oscillation. The system response would increase monotonically.

2-42 **State equation:** **Open-loop system:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 10 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Closed-loop system: $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t)$

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 1 & -2 \\ 10 - k_1 & -k_2 \end{bmatrix}$$

Characteristic equation of the closed-loop system:

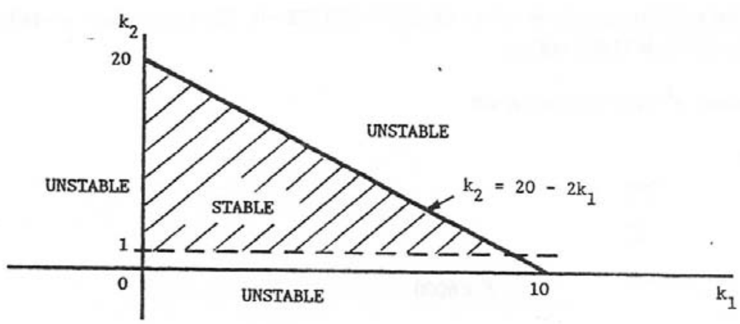
$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{BK} \right| = \begin{vmatrix} s-1 & 2 \\ -10 + k_1 & s + k_2 \end{vmatrix} = s^2 + \left(k_2 - 1\right)s + 20 - 2k_1 - k_2 = 0$$

Stability requirements:

$$k_2 - 1 > 0 \text{ or } k_2 > 1$$

$$20 - 2k_1 - k_2 > 0 \text{ or } k_2 < 20 - 2k_1$$

Parameter plane:



2-43) **Characteristic equation of closed-loop system:**

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{BK} \right| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_1 & k_2 + 4 & s + k_3 + 3 \end{vmatrix} = s^3 + \left(k_3 + 3\right)s^2 + \left(k_2 + 4\right)s + k_1 = 0$$

Routh Tabulation:

s^3	1	k_2+4	
s^2	k_3+3	k_1	$k_3+3>0$ or $k_3>-3$
s^1	$\dfrac{(k_3+3)(k_2+4)-k_1}{k_3+3}$		$(k_3+3)(k_2+4)-k_1>0$
s^0	k_1		$k_1>0$

Stability Requirements:

$$k_3>-3, \quad k_1>0, \quad (k_3+3)(k_2+4)-k_1>0$$

2-44 (a) Since **A** is a diagonal matrix with distinct eigenvalues, the states are decoupled from each other. The second row of **B** is zero; thus, the second state variable, x_2 is uncontrollable. Since the uncontrollable state has the eigenvalue at –3 which is stable, and the unstable state x_3 with the eigenvalue at –2 is controllable, the system is stabilizable.

(b) Since the uncontrollable state x_1 has an unstable eigenvalue at 1, the system is no stabilizable.

2-45) a)

$$G(s)=\frac{Y(s)}{F(s)}$$

If $\frac{d^2y}{dt}-\frac{g}{l}y=z$, then $s^2Y(s)-\frac{g}{l}Y(s)=Z(s)$ or $Y(s)=\frac{Z(s)}{s^2-\frac{g}{l}}$

If $f(t)=\frac{\tau d\tau}{dt}+z$, then $F(s)=(\tau s+1)Z(s)$. As a result:

$$G(s)=\frac{\dfrac{Z(s)}{s^2-\frac{g}{l}}}{(\tau s+1)Z(s)}=\frac{1}{\left(s^2-\frac{g}{l}\right)(\tau s+1)}$$

b) $\begin{cases} F(s) = (\tau s + 1)Z(s) \\ F(s) = \left(K_p + K_d s\right)E(s) \end{cases} \Rightarrow \; Z(s) = \frac{K_p + K_d(s)}{\tau s + 1}E(s)$

As a result:

$$\frac{Y(s)}{E(s)} = G(s)H(s) = \frac{K_p + K_d s}{(\tau s + 1)\left(s^2 - \frac{g}{l}\right)}$$

$$\frac{Y(s)}{X(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{K_p + \; K_d s}{(\tau s + 1)\left(s^2 - \frac{g}{l}\right) + K_p + K_d s}$$

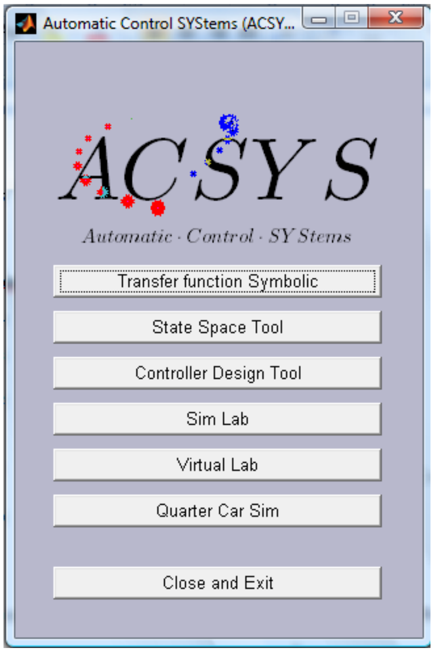
$$\frac{Y(s)}{X(s)} = \frac{G(s)H(s)}{(1 + G(s)H(s))} = \frac{(K_p + K_d s)}{((\tau s + 1)(s^2 - g/l) + K_p + K_d s)}$$
$$= \frac{(K_p + K_d s)}{(\tau s^3 + (\tau(-g/l) + 1)s^2 + K_d s - g/l + K_p)}$$

c) lets choose $\frac{g}{l} = 10 \; and \; \tau = 0.1.$

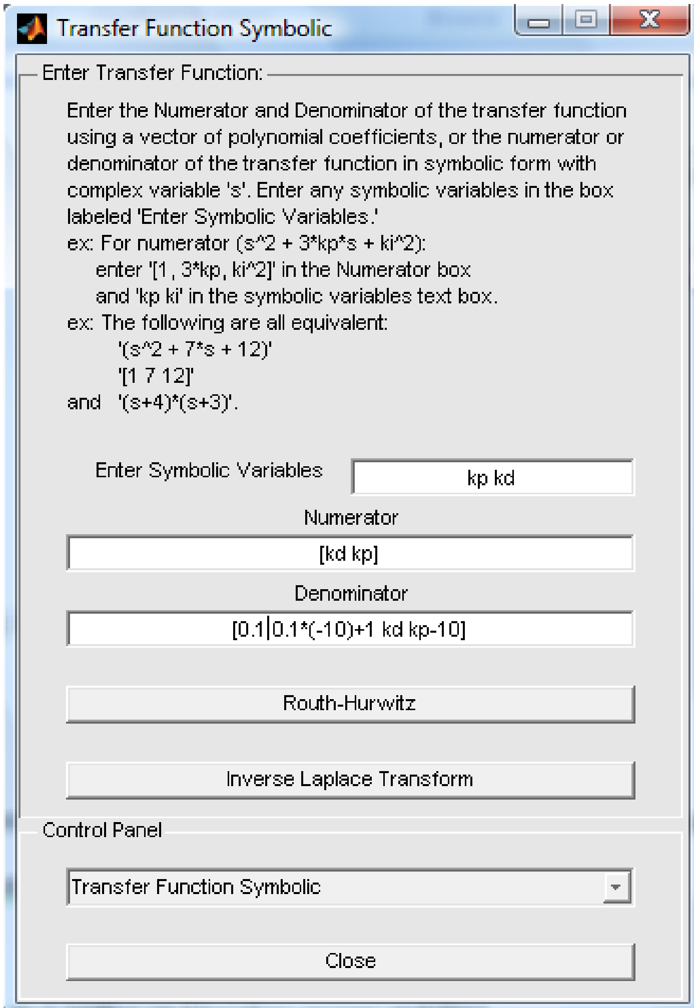
Use the approach in this Chapter's Section 2-14:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the “transfer function Symbolic button.”



5. Enter the characteristic equation in the denominator and press the "Routh-Hurwitz" push-button.

RH =

```
[                1/10,                kd]
[                 eps,             kp-10]
[ (-1/10*kp+1+kd*eps)/eps,             0]
[               kp-10,                 0]
```

For the choice of g/l or τ the system will be unstable. The quantity $\tau\, g/l$ must be >1.

Increase $\tau\, g/l$ to 1.1 and repeat the process.

d) Use the ACSYS toolbox as in section 2-14 to find the inverse Laplace transform. Then plot the time response by selecting the parameter values. **Or use toolbox 2-6-1.**

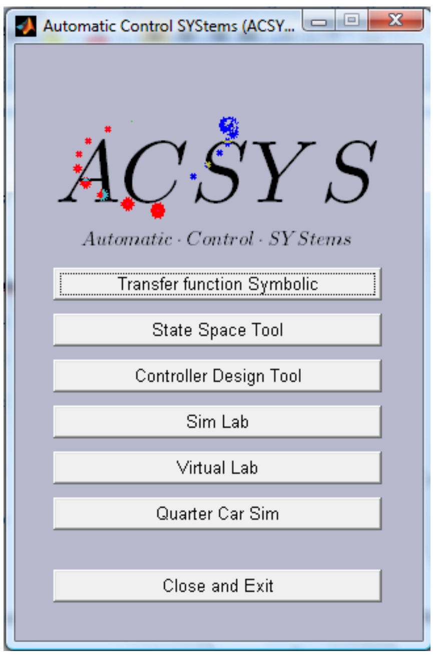
Use the approach in this Chapter's Section 2-14:

1. Activate MATLAB

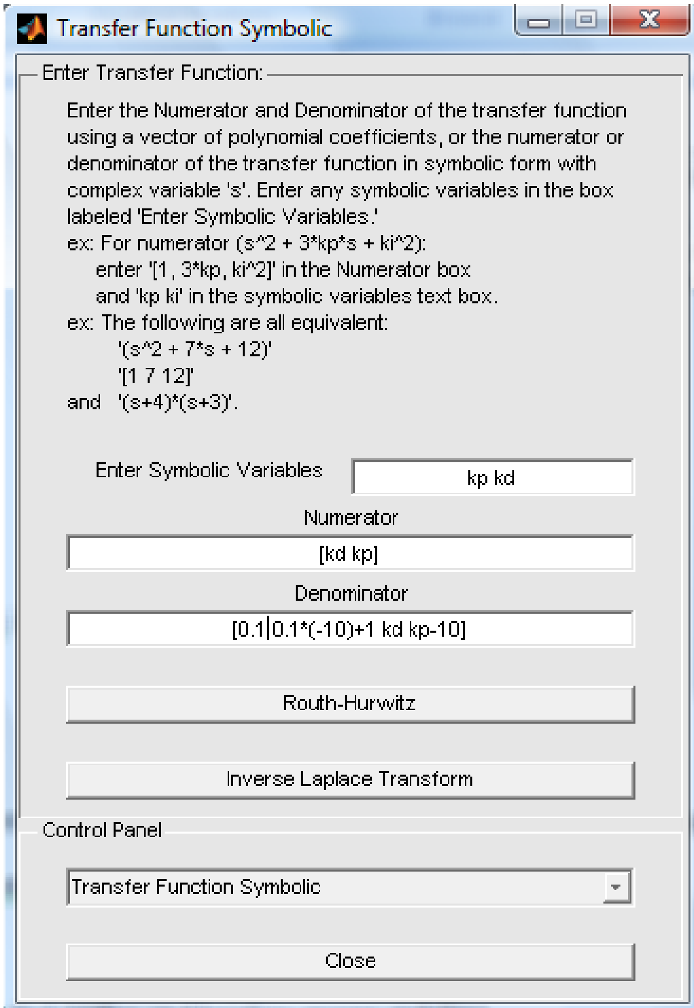
2. Go to the directory containing the ACSYS software.

3. Type in

Acsys



4. Then press the "transfer function Symbolic button."



5. Enter the characteristic equation in the denominator and press the "Inverse Laplace Transform" push-button.

Inverse Laplace Transform

G(s) =

```
[          kd                        kp              ]
[------------------------    ------------------------------]
[          3                         3               ]
[1/10 s  + s kd + kp - 10    1/10 s  + s kd + kp - 10]
```

G(s) factored:

```
[              kd                          kp           ]
[10 --------------------------    10 --------------------------]
[    3                                  3               ]
[   s  + 10 s kd + 10 kp - 100       s  + 10 s kd + 10 kp - 100]
```

Inverse Laplace Transform:

g(t) =
matrix([[10*kd*sum(1/(3*_alpha^2+10*kd)*exp(_alpha*t),_alpha=RootOf(_Z^3+10*_Z*kd+10*kp-100)),10*kp*sum(1/(3*_alpha^2+10*kd)*exp(_alpha*t),_alpha=RootOf(_Z^3+10*_Z*kd+10*kp-100))]])

While MATLAB is having a hard time with this problem, **it is easy to see the solution will be unstable for all values of Kp and Kd.** Stability of a **linear** system is independent of its initial conditions. For different values of g/l and τ, you may solve the problem similarly – assign all values (including Kp and Kd) and then find the inverse Laplace transform of the system. Find the time response and apply the initial conditions.

Lets chose g/l=1 and keep τ=0.1, take Kd=1 and Kp=10.

$$\frac{Y(s)}{X(s)}=\frac{G(s)H(s)}{(1+G(s)H(s))}=\frac{(K_p+K_d s)}{((\tau s+1)(s^2-g/l)+K_p+K_d s)}$$
$$=\frac{(10+s)}{(0.1s^3+(0.1(-1)+1)s^2+s-1+10)}=\frac{(10+s)}{(0.1s^3+0.9s^2+s+9)}$$

Using ACSYS:

RH =

[1/10, 1]

[9/10, 9]

[9/5, 0]

[9, 0]

Hence the system is **stable**

Inverse Laplace Transform

G(s) =

```
         s + 10
  -------------------------
         3        2
  1/10 s  + 9/10 s  + s + 9
```

G factored:

Zero/pole/gain:

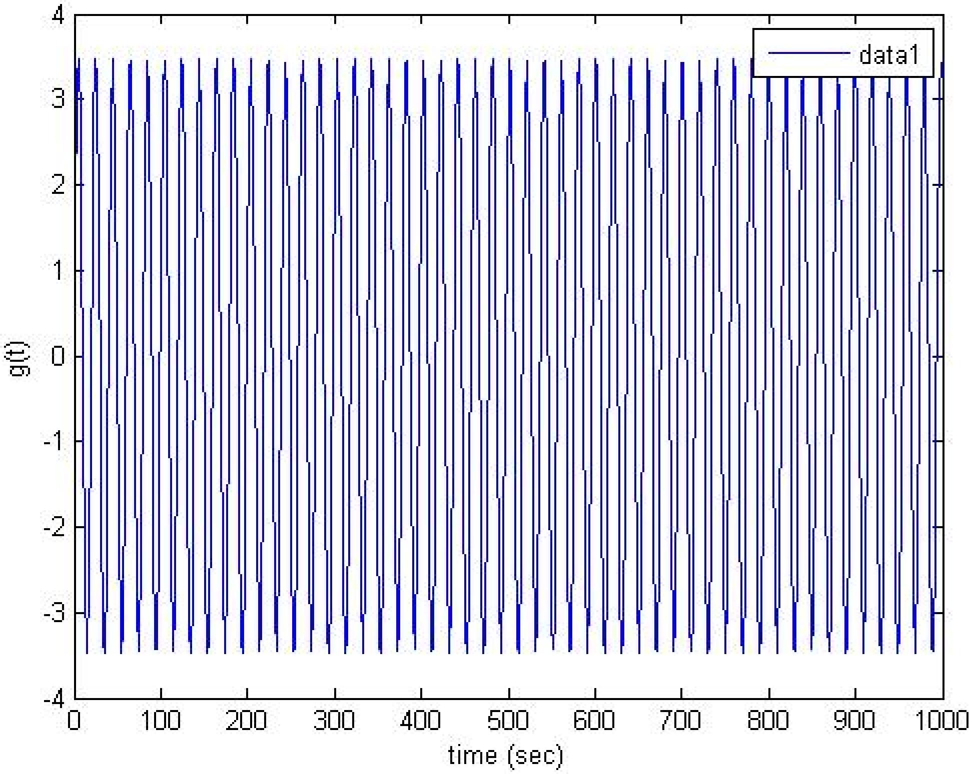
```
   10 (s+10)
-----------------
(s+9) (s^2  + 10)
```

Inverse Laplace Transform:

g(t) = -10989/100000*exp(-2251801791980457/40564819207303340847894502572032*t)*cos(79057/25000*t)+868757373/250000000*exp(-2251801791980457/40564819207303340847894502572032*t)*sin(79057/25000*t)+10989/100000*exp(-9*t)

Use this MATLAB code to plot the time response:

```
for i=1:1000
t=0.1*i;
tf(i)=-10989/100000*exp(-2251801791980457/40564819207303340847894502572032*t)*cos(79057/25000*t)+868757373/250000000*exp(-2251801791980457/40564819207303340847894502572032*t)*sin(79057/25000*t)+10989/100000*exp(-9*t);
end
figure(3)
plot(1:1000,tf)
```



2-52) USE MATLAB

```
syms t

f=5+2*exp(-2*t)*sin(2*t+pi/4)-4*exp(-2*t)*cos(2*t-pi/2)+3*exp(-4*t)

F=laplace(f)

cltF=F/(1+F)

 f =

 5+2*exp(-2*t)*sin(2*t+1/4*pi)-4*exp(-2*t)*sin(2*t)+3*exp(-4*t)


F =

(8*s^3+44*s^2+112*s+160+8*2^(1/2)*s^2+16*2^(1/2)*s+2^(1/2)*s^3)/s/(s^2+4*s+8)/(s+4)

cltF =
```

(8*s^3+44*s^2+112*s+160+8*2^(1/2)*s^2+16*2^(1/2)*s+2^(1/2)*s^3)/s/(s^2+4*s+8)/(s+4)/(1+(8*s^3+44*s^2+112*s+160+8*2^(1/2)*s^2+16*2^(1/2)*s+2^(1/2)*s^3)/s/(s^2+4*s+8)/(s+4))

syms s

cltFsimp=simplify(cltF)

Next type the denominator into ACSYS Routh-Hurwitz program.

char=collect(s^4+16*s^3+68*s^2+144*s+160+8*2^(1/2)*s^2+16*2^(1/2)*s+2^(1/2)*s^3)

char =

160+s^4+(16+2^(1/2))*s^3+(8*2^(1/2)+68)*s^2+(16*2^(1/2)+144)*s

>> eval(char)

ans =

160+s^4+4901665356903357/281474976710656*s^3+2790603031573437/35184372088832*s^2+2931340519928765/17592186044416*s

>> sym2poly(ans)

ans =

1.0000 17.4142 79.3137 166.6274 160.0000

Hence the Characteristic equation is:

$$\Delta = s^4 + 17.4142s^3 + 79.3137s^2 + 166.6274s + 160$$

USE ACSYS Routh-Hurwitz tool as described in previous problems and this Chapter's section 2-14.

RH =

```
[          1,                     5581205465083989*2^(-46),          160]
[87071/5000,                      5862680441794645*2^(-45),            0]
[427334336632381556219/6127076924293382144,     160,                   0]
```

[23808343891282712794360268040124483340З/1879436288300987963959490983755776000,
0, 0]

[160, 0, 0]

The first column is all positive, and the system is **STABLE.**

For the other section

syms s

G=(s+1)/(s*(s+2)*(s^2+2*s+2))

g=ilaplace(G)

G =

(s+1)/s/(s+2)/(s^2+2*s+2)

g =

1/4-1/2*exp(-t)*cos(t)+1/4*exp(-2*t)

cltG=G/(1+G)

cltG =

(s+1)/s/(s+2)/(s^2+2*s+2)/(1+(s+1)/s/(s+2)/(s^2+2*s+2))

cltGsimp=simplify(cltG)

cltGsimp =

(s+1)/(s^4+4*s^3+6*s^2+5*s+1)

Next type the denominator into ACSYS Routh-Hurwitz program.

RH =

[1, 6, 1]

[4, 5, 0]

[19/4, 1, 0]

[79/19, 0, 0]

[1, 0, 0]

STABLE

Chapter 3

3-1) a) $G(s)H(s) = \left[\frac{K_p}{s(s+p)}\right] K_D s = \frac{K_p K_D}{s+p}$

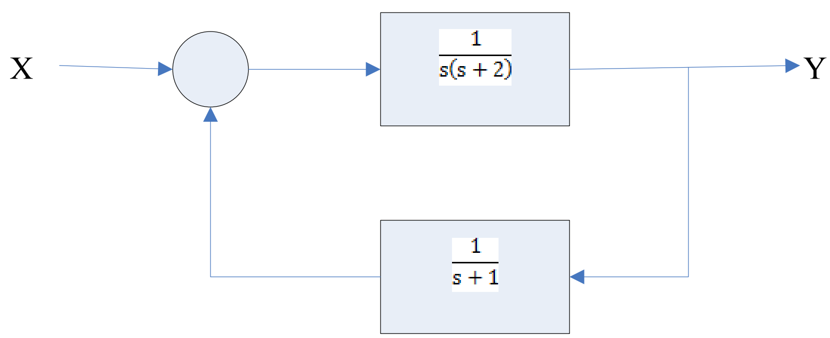
b) $G(s) = \frac{K_p}{s(s+p)}$

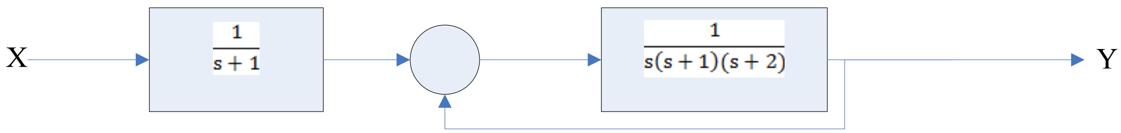
c) $\frac{E(s)}{R(s)} = \frac{1}{1-G(s)H(s)} = \frac{s+p}{s+p-K_p K_D}$

d) Feedback ratio $= \frac{G(s)H(s)}{1-G(s)H(s)} = \frac{K_p K_D}{s+p-K_p K_D}$

e) $\frac{Y(s)}{X(s)} = \frac{G(s)}{1-G(s)H(s)} = \frac{K_p}{s(s+p-K_D K_p)}$

3-2)

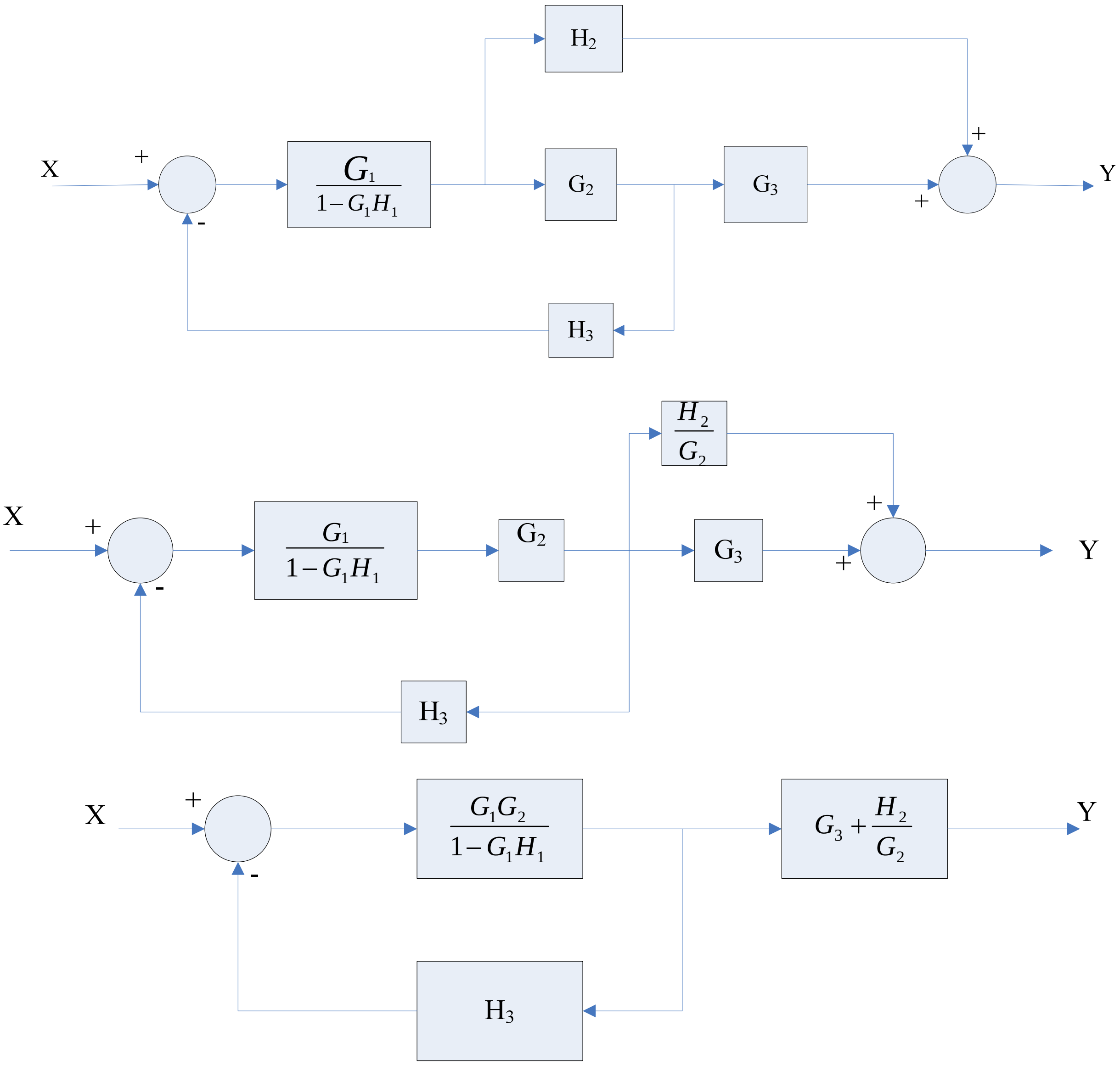




Characteristic equation: $s(s+1)(s+2) + 1 = 0$

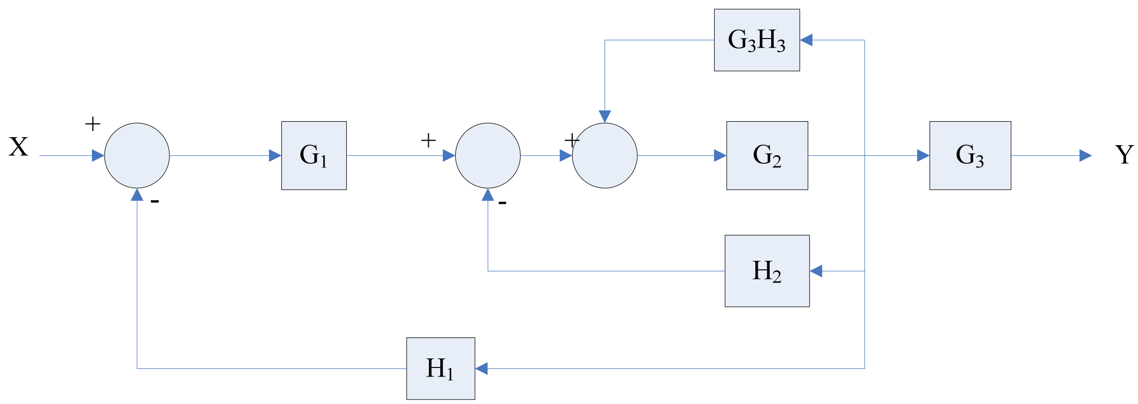
$$\Rightarrow \; s^3 + 3s^2 + 2s + 1 = 0$$

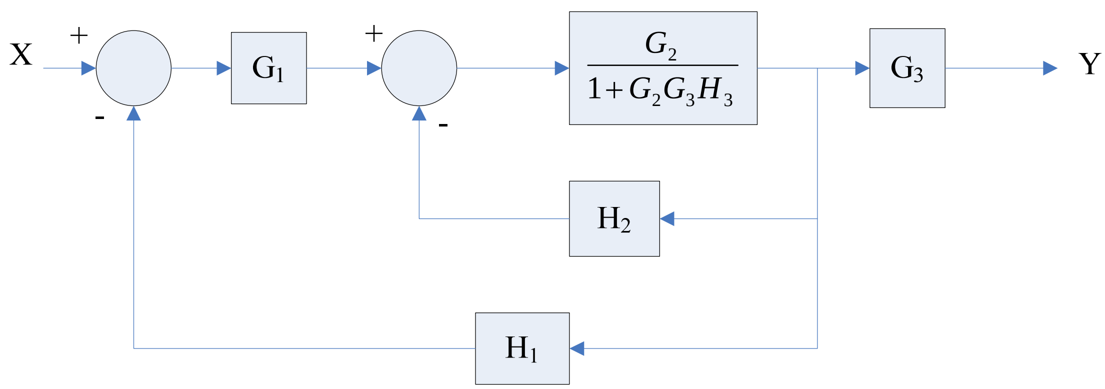
3-3)

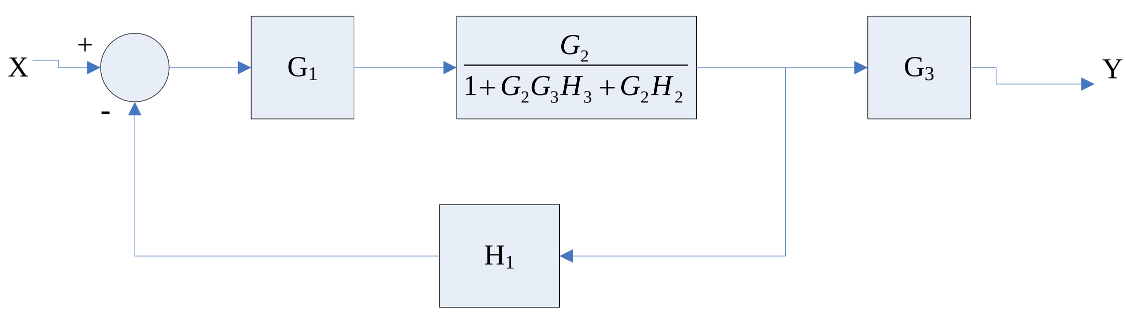


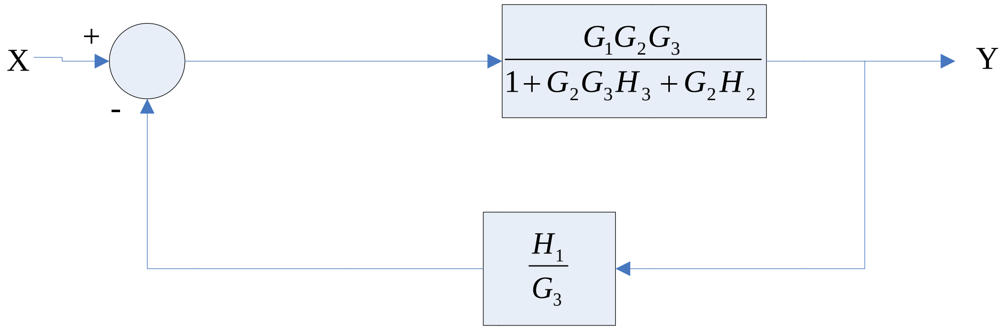
$$\frac{Y(s)}{X(s)} = \frac{\frac{G_1G_2}{1-G_1H_1}}{1+\frac{G_1G_2H_3}{1-G_1H_1}}\left(G_3+\frac{H_2}{G_2}\right) = \frac{G_1G_2G_3+\ G_1H_2}{1-G_1H_1+G_1G_2H_3}$$

3-4)


X
+
-
G1
+
-
+
G3H3
G2
G3
Y
H2
H1


X
+
-
G1
+
-
$\frac{G_2}{1+G_2G_3H_3}$
G3
Y
H2
H1


X
+
-
G1
$\frac{G_2}{1+G_2G_3H_3+G_2H_2}$
G3
Y
H1



$$\frac{Y(s)}{X(s)} = \frac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2H_2 + G_2G_3H_3}$$

3-5)

$$\mathbf{Y}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s)\mathbf{R}(s) = \mathbf{M}(s)\mathbf{R}(s)$$

$$\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{s(s+2)} & 10 \\ \frac{5}{s} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{s^2+2s+2}{s(s+2)} & 10 \\ \frac{5}{s} & \frac{s+2}{s+1} \end{bmatrix}$$

$$[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1} = \frac{1}{\Delta}\begin{bmatrix} \frac{s+2}{s+1} & -10 \\ \frac{-5}{s} & \frac{s^2+2s+2}{s(s+2)} \end{bmatrix} \qquad \Delta(s) = \frac{s^2 - 48s - 48}{s(s+1)}$$

$$\mathbf{M}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s) = \frac{1}{\Delta}\begin{bmatrix} \frac{s+2}{s+1} & -10 \\ \frac{-5}{s} & \frac{s^2+2s+2}{s(s+2)} \end{bmatrix}\begin{bmatrix} \frac{2}{s(s+2)} & 10 \\ \frac{5}{s} & \frac{1}{s+1} \end{bmatrix}$$

$$\mathbf{M}(s) = \frac{1}{\Delta}\begin{bmatrix} \frac{-50s-48}{s(s+1)} & 10 \\ \frac{5}{s} & \frac{-49s^2-148s-98}{s(s+1)(s+2)} \end{bmatrix}$$

3-6) MATLAB

```
syms s
G=[2/(s*(s+2)),10;5/s,1/(s+1)]
H=[1,0;0,1]
A=eye(2)+G*H
B=inv(A)
Clp=simplify(B*G)
```

```
G =

[ 2/s/(s+2),        10]
[       5/s,   1/(s+1)]

H =
     1     0
     0     1


A =

[ 1+2/s/(s+2),          10]
[         5/s,   1+1/(s+1)]

B =

[                  s*(s+2)/(s^2-48*s-48),             -10/(s^2-48*s-48)*(s+1)*s]
[                 -5/(s^2-48*s-48)*(s+1), (s^2+2*s+2)*(s+1)/(s+2)/(s^2-48*s-48)]

Clp =

[             -2*(24+25*s)/(s^2-48*s-48),               10/(s^2-48*s-48)*(s+1)*s]
[                   5/(s^2-48*s-48)*(s+1), -(49*s^2+148*s+98)/(s+2)/(s^2-48*s-48)]
```

3-7)

(a) Open-loop transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_sK_1K_iN}{s\left[L_aJ_ts^2 + \left(L_aB_t + R_aJ_t + K_1K_2J_t\right)s + R_aB_t + K_iK_b + KK_1K_iK_t + K_1K_2B_t\right]}$$

(b) System transfer function:

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_sK_1K_iN}{\left[L_aJ_ts^3 + \left(L_aB_t + R_aJ_t + K_1K_2J_t\right)s^2 + \left(R_aB_t + K_iK_b + KK_1K_iK_t + K_1K_2B_t\right)s + KK_sK_1K_iN\right]}$$

3-8)

(a)

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{\dfrac{10(s+4)}{s(s+1)}}{1+\dfrac{5s}{s(s+1)}+\dfrac{10(s+2)}{s(s+1)}} = \frac{10(s+4)}{s^2+16s+20}$$

(b)

$$\left.\frac{Y(s)}{E(s)}\right|_{N=0} = \left.\frac{Y(s)/R(s)}{E(s)/R(s)}\right|_{N=0} = \frac{\dfrac{10(s+4)}{s(s+1)}}{1+\dfrac{5s}{s(s+1)}-\dfrac{20}{s(s+1)}} = \frac{10(s+4)}{s^2+6s-20}$$

(c)

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1}{1+\dfrac{5s}{s(s+1)}+\dfrac{10(s+2)}{s(s+1)}} = \frac{s(s+1)}{s^2+16s+20}$$

(d)

$$Y(s) = \left.\frac{Y(s)}{R(s)}\right|_{N=0} R(s) + \left.\frac{Y(s)}{N(s)}\right|_{R=0} N(s)$$

3-9)

(a)

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_1(s)G_2(s)G_3(s)+G_4(s)}{\Delta} \qquad \left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1+G_1(s)G_2(s)H_1(s)}{\Delta}$$

$$\Delta = 1+G_1(s)G_2(s)H_1(s)+G_2(s)G_3(s)H_2(s)+G_4(s)-G_2(s)G_4(s)H_1(s)H_2(s)$$

$$Y(s) = \left.\frac{Y(s)}{R(s)}\right|_{N=0} R(s) + \left.\frac{Y(s)}{N(s)}\right|_{R=0} N(s)$$

(b) When $1+G_1(s)G_2(s)H_1(s)=0$ $Y(s)$ is not affected by $N(s)$.

3-10)

Set $$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1-\dfrac{10(s+5)}{s(s+5)(s+10)}G_d(s)}{\Delta} = 0 \qquad \text{Then, } G_d(s) = \frac{s(s+10)}{10}$$

3-11)

(a)

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1+G(s)H(s)}{\Delta} = 0 \qquad H(s) = \frac{-1}{G(s)} = -\frac{s(s+1)(s+2)}{K(s+3)}$$

(b)

$N=0.$ $$E(s) = \frac{R(s)}{1+G(s)+G(s)H(s)} = \frac{R(s)}{G(s)} = \frac{s(s+1)(s+2)}{K(s+3)}R(s) \qquad R(s) = \frac{1}{s^2}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s(s+1)(s+2)}{Ks(s+3)} = \frac{2}{3K} = 0.1 \qquad K = 6.67$$

3-12)

(a) Controller transfer function:

$$\frac{F(s)}{sE_c(s)} = \frac{100}{s} - \frac{30}{s+6} - \frac{70}{s+10} = \frac{880(s+6.818)}{s(s+6)(s+10)} \qquad G_c(s) = \frac{F(s)}{E_c(s)} = \frac{880(s+6.818)}{(s+6)(s+10)}$$

(b) Open-loop transfer function:

$$\frac{V(s)}{E(s)} = \frac{K}{Ms}G_c(s) = \frac{880K(s+6.818)}{30000s(s+6)(s+10)} = \frac{0.0293K(s+6.818)}{s(s+6)(s+10)}$$

(c) System transfer function:

$$\frac{V(s)}{E_r(s)} = \frac{KG_c(s)/Ms}{1+KK_tG_c(s)/Ms} = \frac{KG_c(s)}{Ms+KK_tG_c(s)} = \frac{0.0293K(s+6.818)}{s^3+16s^2+(0.0044K+60)s+0.03K}$$

(d) Steady-state speed: $E_r = 1\text{V}, \quad E_r(s) = E_r/s = 1/s$

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = \lim_{s\to 0}\frac{0.0293K(s+6.818)}{s^3+16s^2+(0.0044K+60)s+0.03K} = 6.66 \text{ ft/sec}$$

3-13)

```
syms t
f=100*(1-0.3*exp(-6*t)-0.7*exp(-10*t))
F=laplace(f)
syms s
F=eval(F)
Gc=F*s
M=30000
syms K
Olp=simplify(K*Gc/M/s)
Kt=0.15
Clp= simplify(Olp/(1+Olp*Kt))
s=0
Ess=eval(Clp)
```

```
f =
100-30*exp(-6*t)-70*exp(-10*t)

F =
80*(11*s+75)/s/(s+6)/(s+10)

ans =
```

(880*s+6000)/s/(s+6)/(s+10)

Gc =
(880*s+6000)/(s+6)/(s+10)

M =
30000

Olp =
1/375*K*(11*s+75)/s/(s+6)/(s+10)

Kt =
0.1500

Clp =
20/3*K*(11*s+75)/(2500*s^3+40000*s^2+150000*s+11*K*s+75*K)

s =
0

Ess =
20/3

3-14)

(a) Controller transfer function:

$$\frac{F(s)}{sE_c(s)} = \left(\frac{100}{s} - \frac{30}{s+6}\right)e^{-0.5s} = \frac{70(s+8.5714)}{s(s+6)}e^{-0.5s}$$

$$G_c(s) = \frac{F(s)}{E_c(s)} = \frac{70(s+8.5714)}{s+6}e^{-0.5s}$$

(b) Open-loop transfer function:

$$\frac{V(s)}{E_r(s)} = \frac{K}{Ms}G_c(s) = \frac{70K(s+8.5714)}{30000s(s+6)}e^{-0.5s} = \frac{0.002333K(s+8.5714)}{s(s+6)}e^{-0.5s}$$

(c) System transfer function:

$$\frac{V(s)}{E_r(s)} = \frac{KG_c(s)/Ms}{1+KK_tG_c(s)/Ms} = \frac{0.002333K(s+8.5714)e^{-0.5s}}{s^2+6s+0.00035K(s+8.5714)e^{-0.5s}}$$

(d) Steady-state speed: $E_r = 1\text{ V}$, $E_r(s) = E_r/s = 1/s$

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = \lim_{s\to 0}\frac{0.002333K(s+8.5714)e^{-0.5s}}{s^2+6s+0.00035K(s+8.5714)e^{-0.5s}} = 6.66 \text{ ft/sec}$$

3-15)

Note: If $\mathcal{L}^{-1}G(s) = g(t)$, then $\mathcal{L}^{-1}\{e^{-as}G(s)\} = u(t - a) \bullet g(t - a)$

```
syms t s
f=100*(1-0.3*exp(-6*(t-0.5)))
F=laplace(f)*exp(-0.5*s)
F=eval(F)
Gc=F*s
M=30000
syms K
Olp=simplify(K*Gc/M/s)
Kt=0.15
Clp= simplify(Olp/(1+Olp*Kt))
s=0
Ess=eval(Clp)
digits (2)
Fsimp=simplify(expand(vpa(F)))
Gcsimp=simplify(expand(vpa(Gc)))
Olpsimp=simplify(expand(vpa(Olp)))
Clpsimp=simplify(expand(vpa(Clp)))

f =
100-30*exp(-6*t+3)

F =
(100/s-30*exp(3)/(s+6))*exp(-1/2*s)

F =
(100/s-2650113767660283/4398046511104/(s+6))*exp(-1/2*s)

Gc =
(100/s-2650113767660283/4398046511104/(s+6))*exp(-1/2*s)*s

M =
     30000

Olp =
-1/131941395333120000*K*(2210309116549883*s-2638827906662400)/s/(s+6)*exp(-1/2*s)

Kt =
   0.1500

Clp =
```

20/3*K*(2210309116549883*s-2638827906662400)*exp(-1/2*s)/(-879609302220800000*s^2-5277655813324800000*s+2210309116549883*K*exp(-1/2*s)*s-2638827906662400*K*exp(-1/2*s))

s =
0

Ess =
20/3

Fsimp =
-.10e3*exp(-.50*s)*(5.*s-6.)/s/(s+6.)

Gcsimp =
-.10e3*exp(-.50*s)*(5.*s-6.)/(s+6.)

Olpsimp =
-.10e-2*K*exp(-.50*s)*(17.*s-20.)/s/(s+6.)

Clpsimp =
5.*K*exp(-.50*s)*(15.*s-17.)/(-.44e4*s^2-.26e5*s+11.*K*exp(-.50*s)*s-13.*K*exp(-.50*s))

3-16)

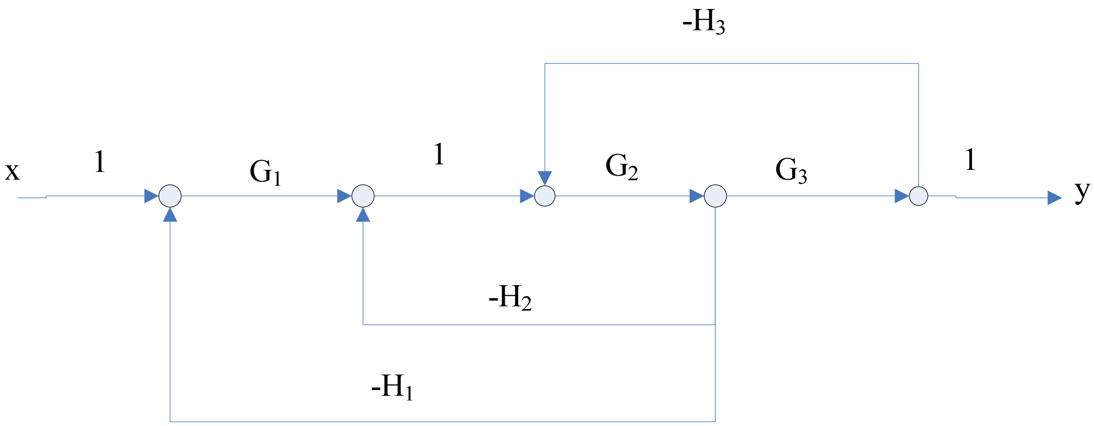
Taking the Laplace transform of the differential equations and expressing in matrix form, the following matrix equations are obtained. All the initial conditions are set to zero.

$$\begin{bmatrix} s(s+2) & 3 \\ 3s+1 & s^2-1 \end{bmatrix}\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ s & 1 \end{bmatrix}\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} \qquad \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} s^2-3s-1 & s^2-4 \\ s^3+2s^2-3s-1 & s^2-s-1 \end{bmatrix}\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$$

$$\Delta(s) = s^4 + 2s^3 - s^2 - 11s - 3$$

$$\left.\frac{Y_1(s)}{R_1(s)}\right|_{R_2=0} = \frac{s^2-3s-1}{\Delta} \qquad \left.\frac{Y_2(s)}{R_1(s)}\right|_{R_2=0} = \frac{s^3+2s^2-3s-1}{\Delta} \qquad \left.\frac{Y_1(s)}{R_2(s)}\right|_{R_1=0} = \frac{s^2-4}{\Delta} \qquad \left.\frac{Y_2(s)}{R_2(s)}\right|_{R_1=0} = \frac{s^2-s-1}{\Delta}$$

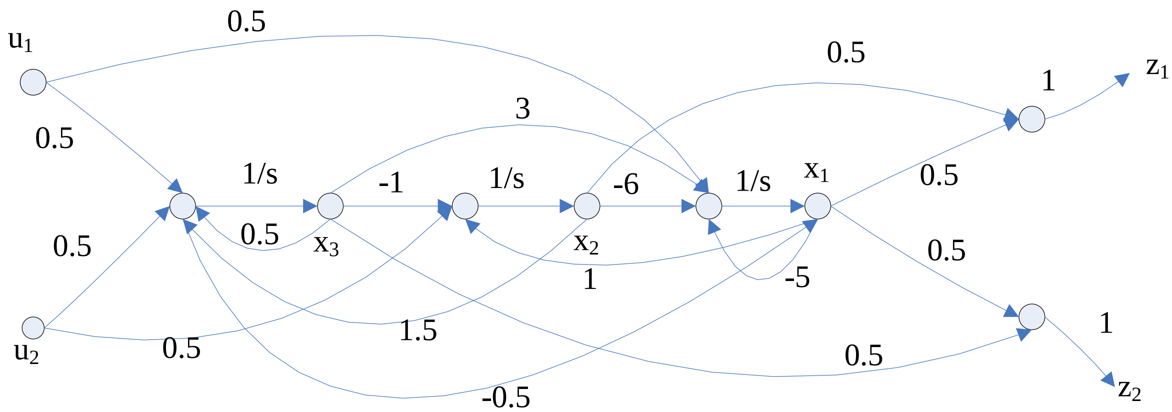
3-17)



3-18)

$$\begin{cases} X_1(s) = \dfrac{1}{s}[-5X_1(s) - 6X_2(s) + 3X_3(s) + 0.5U_1(s)] \\ X_2(s) = \dfrac{1}{s}[X_1(s) - X_3(s) + 0.5U_2(s)] \\ X_3(s) = \dfrac{1}{s}[-0.5X_1(s) + 1.5X_2(s) + 0.5X_3(s) + 0.5U_1(s) + 0.5\,U_2(s)] \end{cases}$$

$$\begin{cases} Z_1(s) = 0.5X_1(s) + 0.5X_2(s) \\ Z_2(s) = 0.5X_1(s) + 0.5X_3(s) \end{cases}$$



3-19)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{B_1 s + B_0}{s^2 + A_1 s + A_0}$$

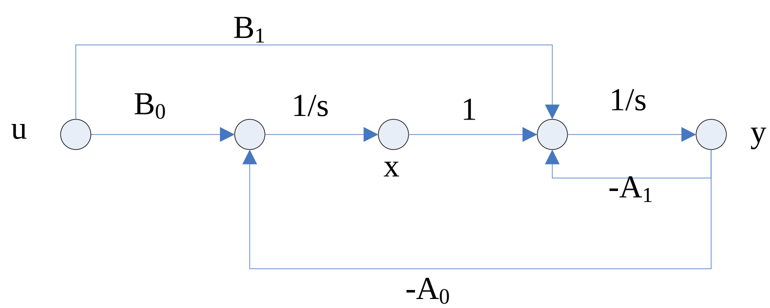
⇨ $(s + A_1 s + A_0)Y(s) = (B_1 s + B_0)U(s)$

⇨ $\left(s + A_1 + \frac{A_0}{s}\right)Y(s) = B_1 U(s) + \frac{B_0}{s}U(s)$

⇨ $\begin{cases} sY(s) = -A_1 Y(s) + X(s) + B_1 U(s) \\ X(s) = -\frac{A_0}{s}Y(s) + \frac{B_0}{s}U(s) \end{cases}$

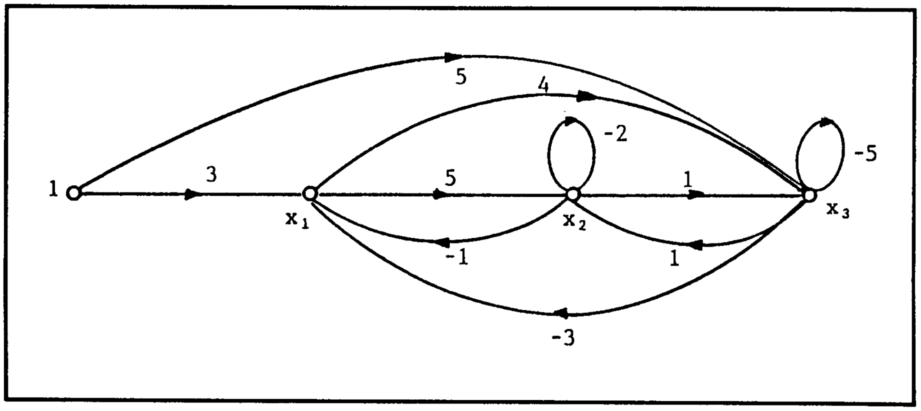
⇨ $\begin{cases} sY(s) = -A_1 Y(s) + X(s) + B_1 U(s) \\ sX(s) = -A_0 Y(s) + B_0 U(s) \end{cases}$

⇨ $\begin{cases} \dot{y} = -Ay + x + B_1 u(t) \\ \dot{x} = -A_0 y + B_0 u(t) \end{cases}$



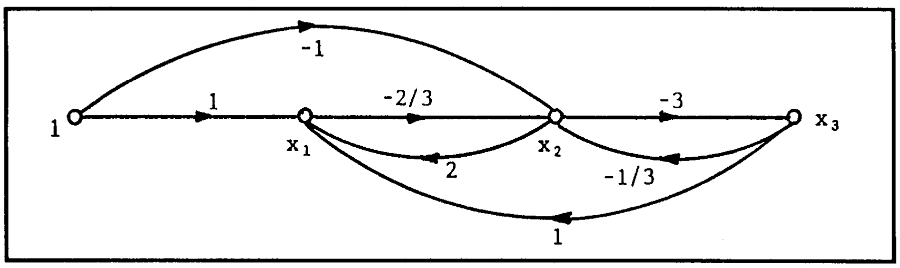
3-20)

(a)

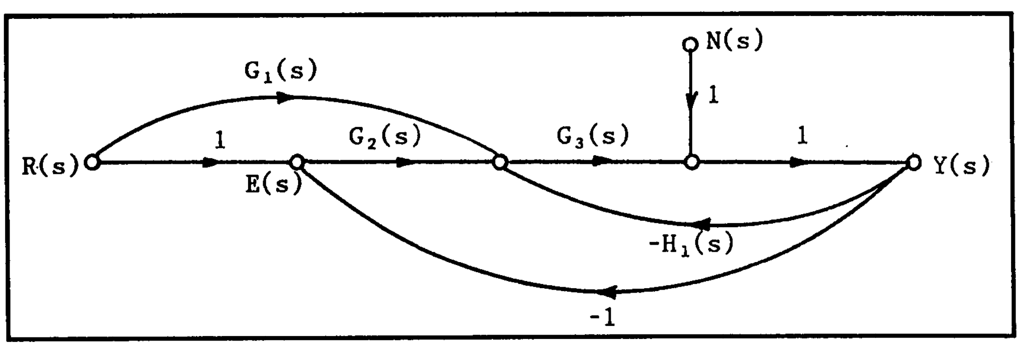


(b) Rewrite the equations as (This is not unique):

$x_1 = 2x_2 + x_3 + 1 \qquad x_2 = (-2/3)x_1 - (1/3)x_3 - 1 \qquad x_3 = -3x_2$



3-21)



$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_1(s)G_3(s) + G_2(s)G_3(s)}{\Delta} \qquad \left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1}{\Delta} \qquad \left.\frac{E(s)}{N(s)}\right|_{R=0} = \frac{-1}{\Delta}$$

$$\left.\frac{E(s)}{R(s)}\right|_{N=0} = \frac{1 + G_3(s)H_1(s) - G_1(s)G_3(s)}{\Delta} \qquad \Delta = 1 + G_2(s)G_3(s) + G_3(s)H_1(s)$$

3-22)

(a)

$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3+G_3G_4}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_3H_2}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3+G_3G_4}{1+G_3H_2}$$

$$\Delta=1+G_1H_1+G_3H_2+G_3G_4H_3+G_1G_2G_3H_3+G_1G_3H_1H_2$$

(b)

$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3+G_3G_4}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_3H_2+H_4}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3+G_3G_4}{1+G_3H_2+H_4}$$

$$\Delta=1+G_1H_1+G_3H_2+G_3G_4H_3+G_1G_2G_3H_3+H_4+G_1G_3H_1H_2+G_1H_1H_4$$

(c)

$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3+G_4}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_2G_3H_3}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3+G_3G_4}{1+G_2G_3H_3}$$

$$\Delta=1+G_1H_1+G_2G_3H_3+G_1G_2H_2-G_2G_4H_2H_3$$

(d)

$$\frac{Y_5}{Y_1}=\frac{G_3G_4+G_1G_2G_3}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_2H_2}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_3G_4+G_1G_2G_3}{1+G_2H_2}$$

$$\Delta=1+G_1H_1+G_2H_2+G_3G_4H_3+G_1G_2G_3H_3-G_4H_1H_2$$

(e)

$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3(1+H_4)+G_4G_5(1+G_2H_1)}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_2H_1+G_3H_2+H_4+G_2H_1H_4+G_3H_2H_4}{\Delta}$$

$$\frac{Y_5}{Y^2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3(1+H_4)+G_4G_5(1+G_2H_{1)}}{1+G_2H_1+G_3H_2+H_4+G_2H_1H_4+G_3H_2H_4}$$

$$\Delta=1+G_2H_1+G_3H_2+H_4+G_4G_5H_3+G_1G_2G_3H_3+G_2H_1H_4+G_3H_2H_4+G_1G_2G_3H_3H_4+G_2G_4H_1H_3$$

3-23)

(a)

$$\frac{Y_7}{Y_1}=\frac{G_1G_2G_3G_4G_5+G_5G_6(1+G_2H_2+G_3H_3)}{\Delta}$$

$$\frac{Y_2}{Y_1}=\frac{1+G_2H_2+G_3H_3+G_4G_5H_4+H_6+G_2G_3G_4G_5H_5+G_2H_2G_4G_5H_4+G_2H_2H_6+G_2H_3H_6}{\Delta}$$

$$\Delta=1+G_1H_1+G_2H_2+G_3H_3+G_4G_5H_4+H_6+G_2G_3G_4G_5H_5-G_5G_6H_1H_5-G_5G_6H_1H_2H_3H_4+G_1G_3H_1H_3$$
$$+G_1G_4G_5H_1H_4+G_1H_1H_6+G_2G_4G_5H_2H_4+G_2H_2H_6+G_3H_3H_6-G_3G_5G_6H_1H_3H_5+G_1G_3H_1H_3H_6$$

(b)

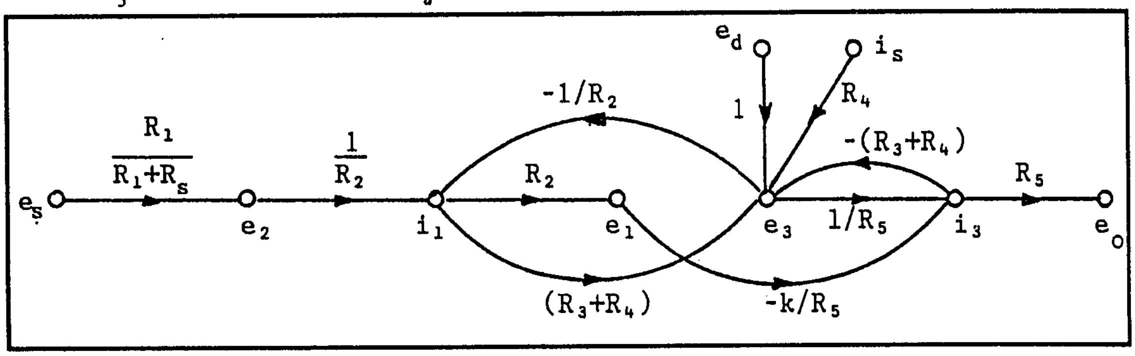
$$\frac{Y_7}{Y_1}=\frac{G_1G_2G_3G_4G_5+G_6(1+G_3H_2+G_4H_3)}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_3H_2+G_4H_3+G_2G_3G_4G_5H_4}{\Delta}$$

$$\Delta=1+G_1G_2H_1+G_3H_2+G_4H_3+G_2G_3G_4G_5H_4-G_2G_6H_1H_4+G_1G_2G_4H_1H_3-G_2G_4G_6H_1H_3H_4$$

3-24)

$e_2 = \frac{R_1}{R_1 + R_s} e_s$ $\quad i_1 = \frac{e_2 - e_3}{R_2}$ $\quad e_1 = R_2 i_1$ $\quad e_3 = e_d + R_3(i_1 - i_3) + (i_s + i_1 - i_3)R_4$

$i_3 = \frac{e_3 - k e_1}{R_5}$ $\quad e_o = R_5 i_3$ $\quad \frac{e_o}{e_d} = \frac{1+k}{\Delta} = 0$ $\quad k = -1$



3-25)

(a)

$$\frac{Y_3}{Y_1} = \frac{G}{1+GH}$$

(b)

$$\frac{Y_3}{Y_1} = \frac{G}{1+GH}$$

3-26)

(a) The three loops are not in touch.

$$\Delta = 1 + G_1H_1 + G_2H_2 + G_3H_3 + G_1G_2H_1H_2 + G_2G_3H_2H_3 + G_1G_3H_1H_3 + G_1G_2G_3H_1H_2H_3$$

(b) The three loops are in touch. $\Delta = 1 + G_1H_1 + G_2H_2 + G_3H_3 + G_1G_3H_1H_3$

3-27)

(a)

$$\left.\frac{Y_6}{Y_1}\right|_{Y_7=0} = \frac{G_1G_2G_3G_4 + G_3G_4G_5}{\Delta} \qquad \left.\frac{Y_6}{Y_7}\right|_{Y_1=0} = \frac{1+G_2H_1}{\Delta}$$

$$\Delta = 1 + G_2H_1 + G_4H_2 + G_1G_2G_3G_4H_3 + G_3G_4G_5H_3 + G_2G_4H_1H_2$$

(b)

$$\left.\frac{Y_6}{Y_1}\right|_{Y_7=0} = \frac{G_1G_2G_3G_4 + G_3G_4G_5}{\Delta} \qquad \left.\frac{Y_6}{Y_7}\right|_{Y_1=0} = \frac{1+G_1H_1 + G_3H_2 + G_1G_3H_1H_2}{\Delta}$$

$$\Delta = 1 + G_1H_1 + G_3H_2 + G_3G_4G_5H_4 + G_1G_2G_3G_4H_4 + G_1G_3H_1H_2 + G_1G_3G_4H_1H_3$$

3-28)

(a)

$$\left.\frac{Y_7}{Y_1}\right|_{Y_8=0} = \frac{G_1G_2G_3G_4G_5 + G_3G_4G_5G_6}{\Delta}$$

$$\Delta = 1 + G_2H_1 + G_5H_2 + G_1G_2G_3G_4G_5H_3 + G_3G_4G_5G_6H_3 + G_2G_5H_1H_2$$

(b)

$$\left.\frac{Y_7}{Y_8}\right|_{Y_1=0} = \frac{G_4G_5(1+G_2H_1)}{\Delta}$$

(c)

$$\left.\frac{Y_7}{Y_4}\right|_{Y_8=0} = \left.\frac{Y_7/Y_1}{Y_4/Y_1}\right|_{Y_8=0} = \frac{G_1G_2G_3G_4G_5 + G_3G_4G_5G_6}{(G_1G_2 + G_6)(1+G_5H_2)}$$

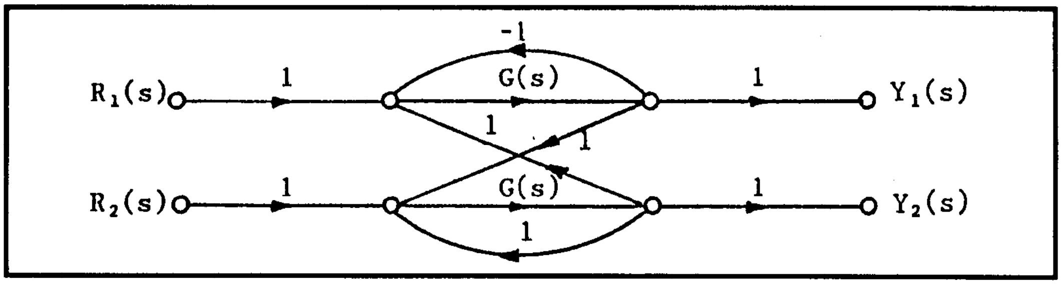
(d)

$$\left.\frac{Y_7}{Y_4}\right|_{Y_1=0} = \left.\frac{Y_7/Y_8}{Y_4/Y_8}\right|_{Y_1=0} = \frac{-G_4G_5(1+G_2H_1)}{G_4G_5H_3(G_6 + G_1G_2)}$$

The results in (c) and (d) are different because different inputs are used.

3-29)

(a) Equivalent SFG:



(b) $\Delta = 1 - 2[G(s)]^2$

(c)

$$\left.\frac{Y_1(s)}{R_1(s)}\right|_{R_2=0} = \frac{G(s)[1-G(s)]}{\Delta} \qquad \left.\frac{Y_1(s)}{R_2(s)}\right|_{R_1=0} = \frac{[G(s)]^2}{\Delta}$$

$$\left.\frac{Y_2(s)}{R_1(s)}\right|_{R_2=0} = \frac{[G(s)]^2}{\Delta} \qquad \left.\frac{Y_2(s)}{R_2(s)}\right|_{R_1=0} = \frac{G(s)[1+G(s)]}{\Delta}$$

(d) Transfer function in matrix form: $\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{R}(s)$

$$G(s) = \frac{1}{\Delta}\begin{bmatrix} G(s)[1-G(s)] & [G(s)]^2 \\ [G(s)]^2 & G(s)[1+G(s)] \end{bmatrix}$$

3-30) Use Mason's formula:

(a)

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_p(s)\left[1+G_c(s)H(s)\right]}{1+G_p(s)H(s)} \qquad \left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{G_p(s)}{1+G_p(s)H(s)}$$

When $G_c(s) = G_p(s)$ $\quad \left.\frac{Y(s)}{R(s)}\right|_{N=0} = G_p(s)$

(b)

$$G_p(s) = G_c(s) = \frac{100}{(s+1)(s+5)} \qquad \left.\frac{Y(s)}{R(s)}\right|_{N=0} = G_p(s) = \frac{100}{(s+1)(s+5)}$$

$$R(s) = \frac{1}{s} \qquad Y(s) = \frac{100}{s(s+1)(s+5)} = \frac{20}{s} - \frac{25}{s+1} + \frac{5}{s+5} \qquad y(t) = \left(20 - 25e^{-t} + 5e^{-5t}\right)u_s(t)$$

(c)

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{G_p(s)}{1+G_p(s)H(s)} = \frac{100}{(s+1)(s+5)+100H(s)} \qquad N(s) = \frac{1}{s} \qquad \left.G(s)\right|_{R=0} = \frac{100}{s(s+1)(s+5)+100s}$$

$H(s)$ must have a pole at $s = 0$, but the system must be stable.

$$H(s) = \frac{K}{s}. \qquad \Delta = s(s+1)(s+5)+100K$$

K must be selected so that the system is stable.

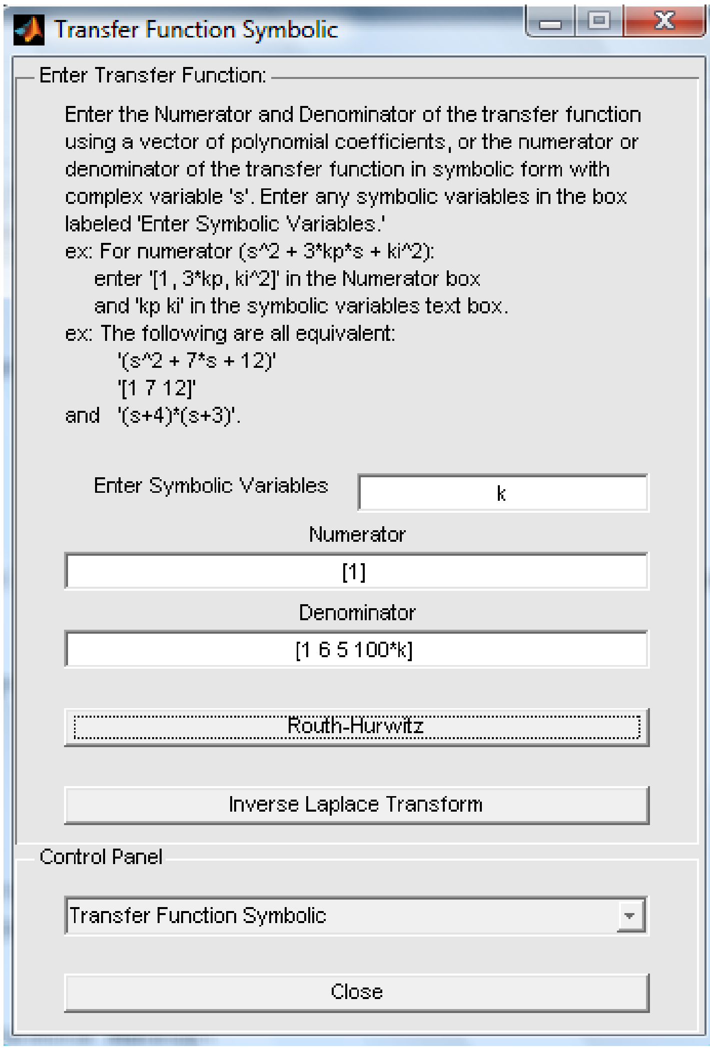
3-31) MATLAB

```
syms s K
G=100/(s+1)/(s+5)
g=ilaplace(G/s)
H=K/s
YN=simplify(G/(1+G*H))
Yn=ilaplace(YN/s)

G =
 100/(s+1)/(s+5)

g =
 -25*exp(-t)+5*exp(-5*t)+20

H =
K/s
```

YN =

100*s/(s^3+6*s^2+5*s+100*K)

Apply Routh-Hurwitz within Symbolic tool of ACSYS (see chapter 3)



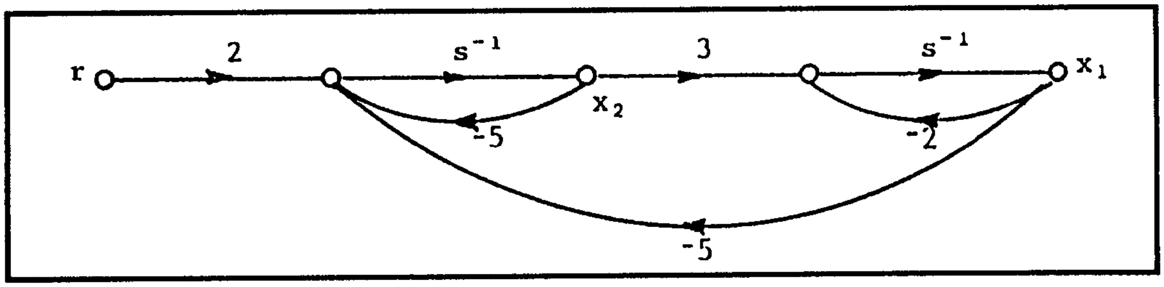
RH =

```
[       1,       5]
[       6,   100*k]
[ -50/3*k+5,       0]
[   100*k,       0]
```

Stability requires: 0<k<3/10.

3-32)

(a) State diagram:



(b) Characteristic equation: $\Delta = 1+2s^{-1}+5s^{-1}+15s^{-1}+10s^{-2}=0$ $\qquad s^2+7s+25=0$

(c) Transfer functions:

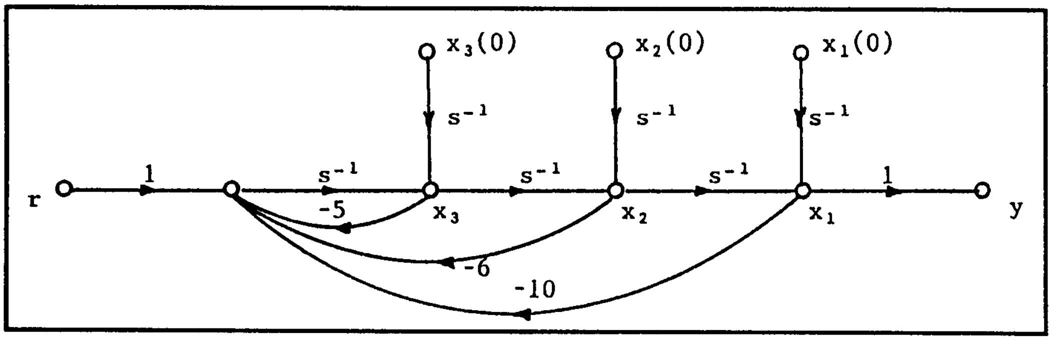
$$\frac{X_1(s)}{R(s)} = \frac{6s^{-2}}{\Delta} = \frac{6}{s^2+7s+25} \qquad \frac{X_2(s)}{R(s)} = \frac{2s^{-1}(1+2s^{-1})}{\Delta} = \frac{2(s+2)}{s^2+7s+25}$$

3-33) MATLAB solutions are in 3-34.

(a) Write the differential equation as

$$\frac{d^3y(t)}{dt^3} = r(t) - 5\frac{d^2y(t)}{dt^2} - 6\frac{dy(t)}{dt} - 10y(t)$$

State diagram:



(b) State equations:

$$\frac{dx_1(t)}{dt} = x_2(t) \qquad \frac{dx_2(t)}{dt} = x_3(t) \qquad \frac{dx_3(t)}{dt} = -10x_1(t) - 6x_2(t) - 5x_3(t) + r(t)$$

(c) Characteristic equation:

$$\Delta = 1 + 5s^{-1} + 6s^{-2} + 10s^{-3} = 0 \qquad s^3 + 5s^2 + 6s + 10 = 0$$

Characteristic equation roots:

-4.1337, $\quad -0.43313 + j1.4938$, $\quad -0.43313 - j1.4938$

(d) Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{s^{-3}}{1 + 5s^{-1} + 6s^{-2} + 10s^{-3}} = \frac{1}{s^3 + 5s^2 + 6s + 10}$$

(e) $R(s) = 1/s$.

$$Y(s) = \frac{1}{s(s^3 + 5s^2 + 6s + 10)} = \frac{0.1}{s} - \frac{0.01519}{s + 4.1337} - \frac{0.08481(s + 0.4331)}{(s + 0.4331)^2 + 2.232} - \frac{0.09953}{(s + 0.4331)^2 + 2.232}$$

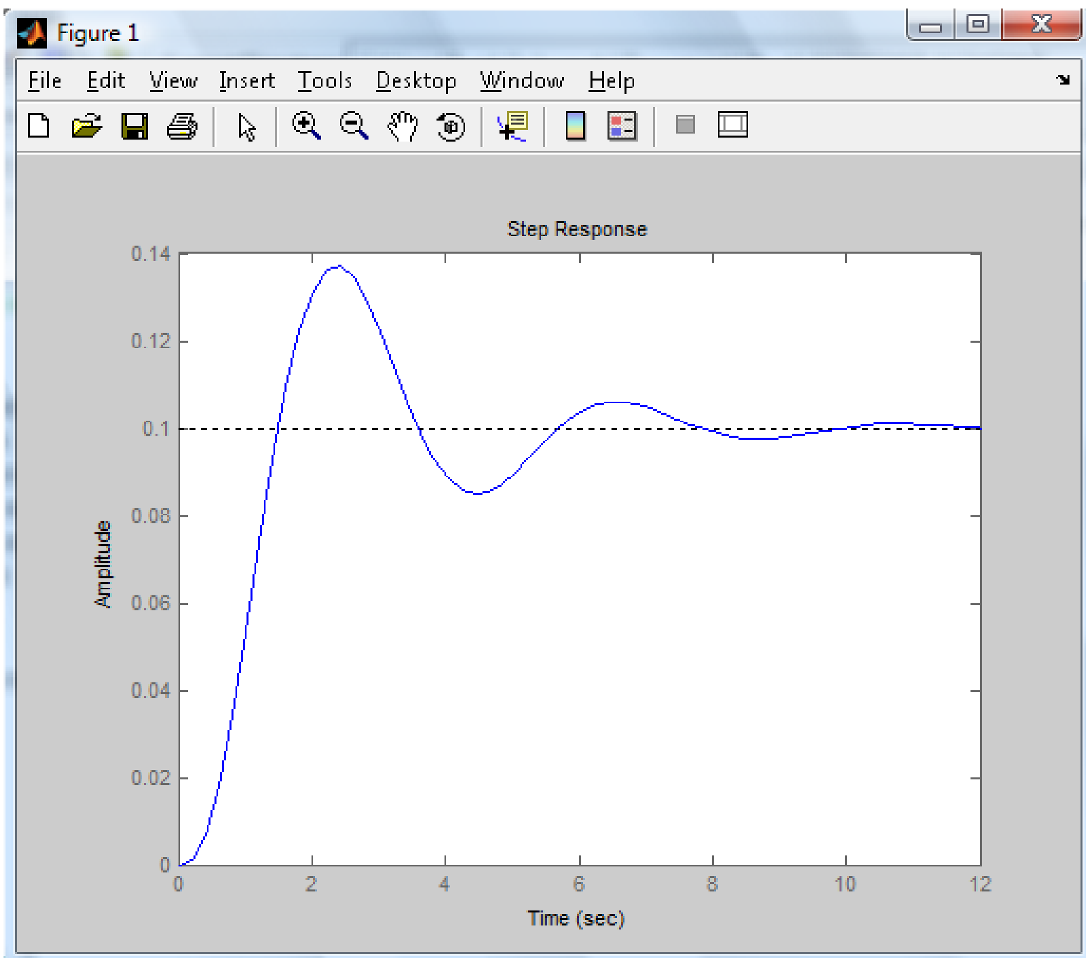
$$y(t) = \left[0.1 - 0.01519e^{-4.1337t} - 0.08481e^{-0.4331t}\cos(1.494t) - 0.06662e^{-0.4331t}\sin(1.494t)\right]u_s(t)$$

3-34) MATLAB

```
clear all
p = [1 5 6 10] % Define polynomial s^3+5*s^2+6*s+10=0
roots(p)
G=tf(1,p)
step(G)
```

p =
   1   5   6 10

ans =
 -4.1337
 -0.4331 + 1.4938i
 -0.4331 - 1.4938i
Transfer function:
      1
----
s^3 + 5 s^2 + 6 s + 10



Alternatively:

```
clear all
syms s
G=1/( s^3 + 5*s^2 + 6*s + 10)
y=ilaplace(G/s)
s=0
yfv=eval(G)

G =
 1/(s^3+5*s^2+6*s+10)

y =
 1/10+1/5660*sum((39*_alpha^2-91+160*_alpha)*exp(_alpha*t),_alpha =
RootOf(_Z^3+5*_Z^2+6*_Z+10))

s =
     0

yfv =
    0.1000
```

Problem finding the inverse Laplace.

Use Toolbox 2-5-1 to find the partial fractions to better find inverse Laplace

```
clear all
B=[1]
A = [1 5 6 10 0] % Define polynomial s*(s^3+5*s^2+6*s+10)=0
[r,p,k]=residue(B,A)

B =
     1
A =
     1     5     6    10     0


r =
  -0.0152
  -0.0424 + 0.0333i
  -0.0424 - 0.0333i
   0.1000
p =
  -4.1337
  -0.4331 + 1.4938i
  -0.4331 - 1.4938i
       0
k =
     []
```

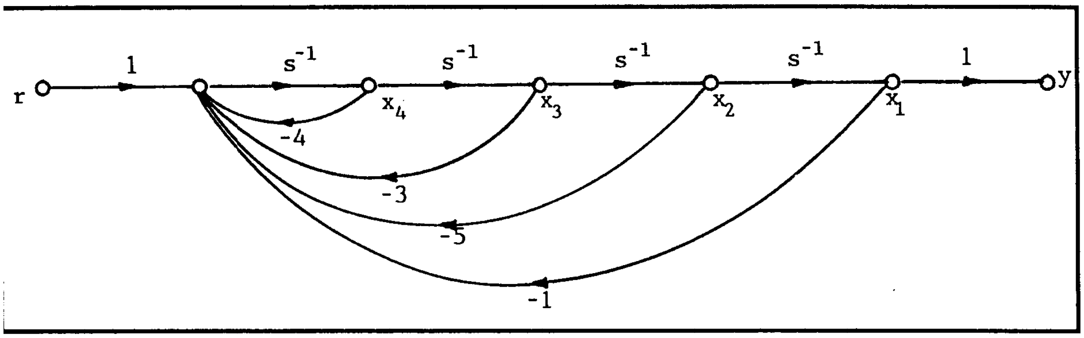
So partial fraction of Y is: $\frac{1}{s}+\frac{-0.0152}{s-4.1337}+\frac{-0.0424\ +\ 0.0333\mathrm{i}}{s-0.4331+1.4938\mathrm{i}}+\frac{-0.0424\ -\ 0.0333\mathrm{i}}{s-0.4331\ -\ 1.4938\mathrm{i}}$

3-35) MATLAB solutions are in 3-36.

(a) Write the differential equation as

$$\frac{d^4y(t)}{dt^4} = r(t) - 4\frac{d^3y(t)}{dt^3} - 3\frac{d^2y(t)}{dt^2} - 5\frac{dy(t)}{dt} - y(t)$$

State diagram:



(b) State equations:

$$\frac{dx_1(t)}{dt} = x_2(t) \qquad \frac{dx_2(t)}{dt} = x_3(t) \qquad \frac{dx_3(t)}{dt} = x_4(t) \qquad \frac{dx_4(t)}{dt} = -x_1(t) - 5x_2(t) - 3x_3(t) - 4x_4(t) + r(t)$$

(c) Characteristic equation:

$$s^4 + 4s^3 + 3s^2 + 5s + 1 = 0$$

Characteristic equation roots:

$-3.5286, \quad -0.2212, \quad -0.1251 + j1.125 \quad -0.1251 - j1.125$

(d) Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{1}{s^4 + 4s^3 + 3s^2 + 5s + 1}$$

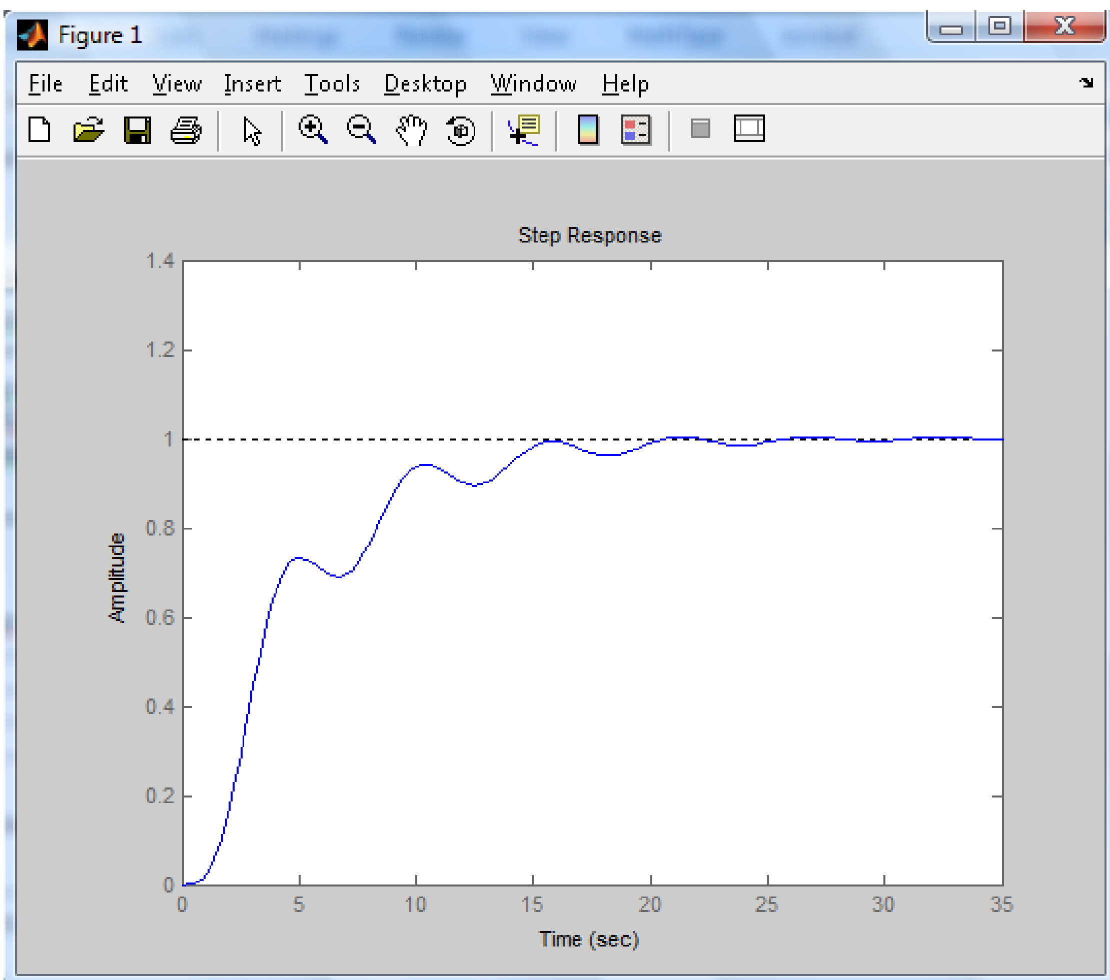
(e) $R(s) = 1/s$.

$$Y(s) = \frac{1}{s(s^4 + 4s^3 + 3s^2 + 5s + 1)} = \frac{1}{s} - \frac{1.072}{s + 0.2212} + \frac{0.006668}{s + 3.5286} + \frac{0.06558(s + 0.1251)}{(s + 0.1251)^2 + 1.2656} - \frac{0.2054}{(s + 0.1251)^2 + 1.2656}$$

$$y(t) = [1 - 1.072e^{-0.2212t} + 0.006668e^{-3.5286t} + 0.06558e^{-0.1251t}\cos(1.125t) - 0.1826e^{-0.1251t}\sin(1.125t)]u_s(t)$$

3-36)

```
clear all
p = [1 4 3 5 1] % Define polynomial s^4+4*s^3+3*s^2+5*s+1=0
roots(p)
G=tf(1,p)
step(G)

p =
     1     4     3     5     1

ans =
  -3.5286
  -0.1251 + 1.1250i
  -0.1251 - 1.1250i
  -0.2212

Transfer function:
              1
-----------------------------
s^4 + 4 s^3 + 3 s^2 + 5 s + 1
```



Alternatively:

```
clear all
syms s t
G=1/(s^4+4*s^3+3*s^2+5*s+1)
y=ilaplace(G/s)
s=0
yfv=eval(G)

G =
1/(s^4+4*s^3+3*s^2+5*s+1)

y =
1-
1/14863*sum((3955*_alpha^3+16873+14656*_alpha^2+7281*_alpha)*exp(_alpha*t),_alp
ha = RootOf(_Z^4+4*_Z^3+3*_Z^2+5*_Z+1))

s =
     0

yfv =
     1
```

Problem finding the inverse Laplace.

Use Toolbox 2-5-1 to find the partial fractions to better find inverse Laplace

```
clear all
B=[1]
A = [1 4 3 5 1] % Define polynomial s^4+4*s^3+3*s^2+5*s+1=0
[r,p,k]=residue(B,A)

B =
     1
A =
     1     4     3     5     1

r =
  -0.0235
  -0.1068 + 0.0255i
  -0.1068 - 0.0255i
   0.2372

p =
  -3.5286
  -0.1251 + 1.1250i
  -0.1251 - 1.1250i
  -0.2212

k =
     []
```

3-37)

(a)

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{G_1G_2G_3}{1+G_1G_2H_1+G_2G_3H_2+G_1G_2G_3} = \frac{100(s+1)}{101s^3+2122s^2+3050s+1010}$$

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{(1+G_1G_2H_1)-G_2G_3G_4}{1+G_1G_2H_1+G_2G_3H_2+G_1G_2G_3} = \frac{(101s^3+2122s^2+2040s)-10(s+1)G_4}{101s^3+2122s^2+3050s+1010}$$

$$\left.\frac{E(s)}{R(s)}\right|_{N=0} = \frac{1+G_2G_3H_2}{1+G_1G_2H_1+G_2G_3H_2+G_1G_2G_3} = \frac{s^3+22s^2+50s+10}{101s^3+2122s^2+3050s+1010}$$

(b)

$$G_4(s) = \frac{1+G_1(s)G_2(s)H_1(s)}{G_2(s)G_3(s)} = \frac{101s^3+2122s^2+2040s}{10(s+1)}$$

(c) Characteristic equation: $101s^3+2122s^2+3050s+1010=0$ $\quad s^3+21.01s^2+30.198s+10=0$

Characteristic equation roots: -0.5029, $\quad -1.0205$, $\quad -19.4867$

(d) $R(s) = 1/s$.

$$E(s) = \frac{s^3+22s^2+50s+10}{s(101s^3+2122s^2+2050s+1010)} \qquad \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0.0099$$

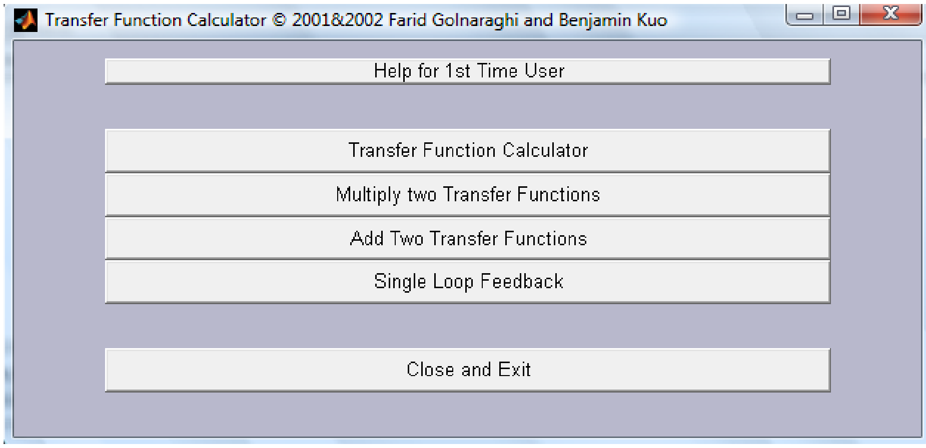
(e) $$Y(s) = \frac{100(s+1)}{s(101s^3+2122s^2+3050s+1010)} = \frac{0.099}{s} + \frac{0.002679}{s+19.49} - \frac{0.002078}{s+1.02} - \frac{0.00996}{s+0.5029}$$

$$y(t) = \left(0.099+0.002679e^{-19.49t} - 0.002078e^{-1.02t} - 0.00996e^{-0.5029t}\right)u_s(t)$$

3-38) MATLAB

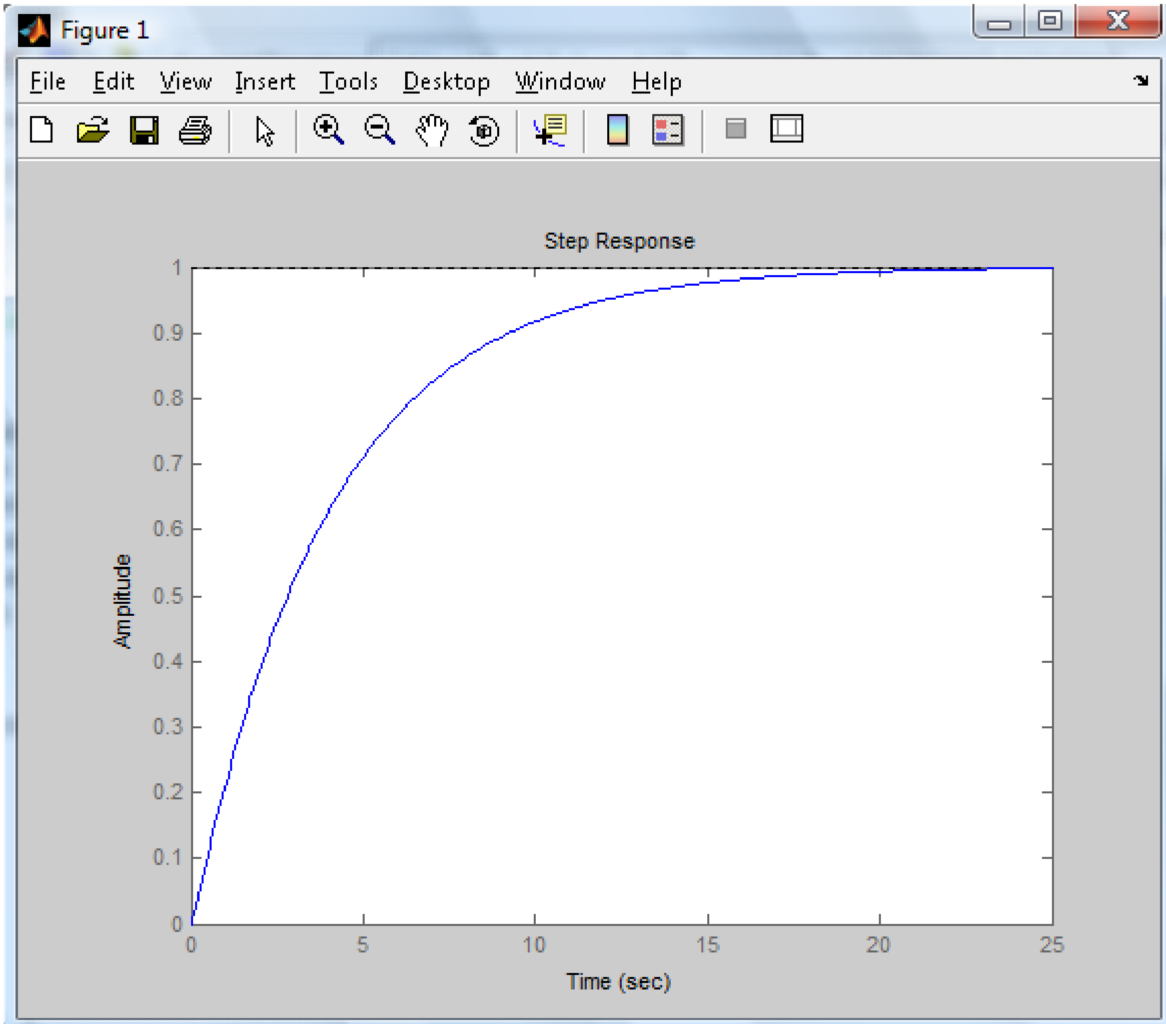
Use TFcal in ACSYS (go to ACSYS folder and type in TFcal in the MATLAB Command Window).

TFcal



Alternatively use toolboxes 3-3-1 and 3-3-2

```
clear all
syms s
G1=100
G2=(s+1)/(s+2)
G3=10/s/(s+20)
G4=(101*s^3+2122*s^2+2040*s)/10/(s+1)
H1=1
H2=1
simplify(G1*G2*G3/(1+G1*G2*H1+G1*G2*H2+G1*G2*G3))

G1 =
   100

G2 =
(s+1)/(s+2)

G3 =
10/s/(s+20)

G4 =
(101/10*s^3+1061/5*s^2+204*s)/(s+1)

H1 =
     1

H2 =
     1

ans =
1000*(s+1)/(201*s^3+4222*s^2+5040*s+1000)

clear all
TF=tf([1000 1000],[201 4222 5040 1000])
step(TF)
```



3-39)

```
clear all
syms s
P1 = 2*s^6+9*s^5+15*s^4+25*s^3+25*s^2+14*s+6 % Define polynomial
P2 = s^6+8*s^5+23*s^4+36*s^3+38*s^2+28*s+16 % Define polynomial
solve(P1, s)
solve(P2, s)
collect(P2-P1)
collect(P2+P1)
collect((P1-P2)*P1)
P1 =
2*s^6+9*s^5+15*s^4+25*s^3+25*s^2+14*s+6

P2 =
s^6+8*s^5+23*s^4+36*s^3+38*s^2+28*s+16

ans =
```

```
                 -1
                 -3
          i*2^(1/2)
         -i*2^(1/2)
 -1/4+1/4*i*7^(1/2)
 -1/4-1/4*i*7^(1/2)

ans =
   -2
   -4
    i
   -i
 -1+i
 -1-i

ans =

-s^6-s^5+8*s^4+11*s^3+13*s^2+14*s+10


ans =

3*s^6+17*s^5+38*s^4+61*s^3+63*s^2+42*s+22

ans =

-60+2*s^12+11*s^11+8*s^10-54*s^9-195*s^8-471*s^7-796*s^6-1006*s^5-1027*s^4-
848*s^3-524*s^2-224*s


Alternative:
clear all
P1 = [2 9 15 25 25 14 6] % Define polynomial
roots(P1)
P2 = [1 8 23 36 38 28 16] % Define polynomial
roots(P2)

 P1 =
     2     9    15    25    25    14     6

ans =
  -3.0000
  -0.0000 + 1.4142i
  -0.0000 - 1.4142i
  -1.0000
  -0.2500 + 0.6614i
  -0.2500 - 0.6614i
```

```
P2 =
     1     8    23    36    38    28    16

ans =

  -4.0000
  -2.0000
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
   0.0000 + 1.0000i
   0.0000 - 1.0000i
```

3-40)

```
clear all
syms s
P6 = (s+1)*(s^2+2)*(s+3)*(2*s^2+s+1) % Define polynomial
P7 = (s^2+1)*(s+2)*(s+4)*(s^2+s+1) % Define polynomial
digits(2)
vpa(solve(P6, s))
vpa(solve(P7, s))
collect(P6)
collect(P7)

P6 =
(s+3)*(s+1)*(2*s^2+s+1)*(s^2+2)

P7 =
 (s^2+1)*(s+2)*(s+4)*(s^2+s+1)

ans =   -1.
        -3.
      1.4*i
     -1.4*i
 -.25+.65*i
 -.25-.65*i

ans =   -2.
        -4.
          i
      -1.*i
 -.50+.85*i
 -.50-.85*i

ans =
 2*s^6+9*s^5+15*s^4+25*s^3+25*s^2+14*s+6

 ans =
 8+s^6+7*s^5+16*s^4+21*s^3+23*s^2+14*s
```

3-41)

Use Toolbox 2-5-1 to find the partial fractions

```
clear all
B= conv(conv(conv([1 1],[1 0 2]),[1 4]),[1 10])
A= conv(conv(conv([1 0],[1 2]),[1 2 5]),[2 1 4])
[r,p,k]=residue(B,A)

B =
     1    15    56    70   108    80

A =
     2     9    26    45    46    40     0

r =
  -1.0600 - 1.7467i
  -1.0600 + 1.7467i
   0.9600
  -0.1700 + 0.7262i
  -0.1700 - 0.7262i
   2.0000

p =
  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
  -2.0000
  -0.2500 + 1.3919i
  -0.2500 - 1.3919i
        0

k =
     []
```

3-42) Use toolbox 3-3-2

```
clear all
B= conv(conv(conv([1 1],[1 0 2]),[1 4]),[1 10])
A= conv(conv(conv([1 0],[1 2]),[1 2 5]),[2 1 4])
G1=tf(B,A)
YR1=G1/(1+G1)
pole(YR1)
```

```
B =
   1   15   56   70  108   80
A =
   2    9   26   45   46   40    0
Transfer function:
 s^5 + 15 s^4 + 56 s^3 + 70 s^2 + 108 s + 80
-----------------------------------------------
2 s^6 + 9 s^5 + 26 s^4 + 45 s^3 + 46 s^2 + 40 s
Transfer function:
2 s^11 + 39 s^10 + 273 s^9 + 1079 s^8 + 3023 s^7 + 6202 s^6 + 9854 s^5 + 12400 s^4
                                        + 11368 s^3 + 8000 s^2 + 3200 s
-------------------------------------------------------------------------------------
4 s^12 + 38 s^11 + 224 s^10 + 921 s^9 + 2749 s^8 + 6351 s^7 + 11339 s^6 + 16074 s^5
                               + 18116 s^4 + 15048 s^3 + 9600 s^2 + 3200 s
ans =
      0
 -0.7852 + 3.2346i
 -0.7852 - 3.2346i
 -2.5822
 -1.0000 + 2.0000i
 -1.0000 - 2.0000i
 -2.0000
 -0.0340 + 1.3390i
 -0.0340 - 1.3390i
 -0.2500 + 1.3919i
 -0.2500 - 1.3919i
 -0.7794
```

```
C= [1 12 47 60]
D= [4 28 83 135 126 62 12]
G2=tf(D,C)
YR2=G2/(1+G2)
pole(YR2)
C =
     1    12    47    60
```

```
D =
     4    28    83   135   126    62    12

Transfer function:
4 s^6 + 28 s^5 + 83 s^4 + 135 s^3 + 126 s^2 + 62 s + 12
-------------------------------------------------------
               s^3 + 12 s^2 + 47 s + 60

Transfer function:

4 s^9+76 s^8+607 s^7+2687 s^6+7327 s^5+12899 s^4+14778 s^3+10618 s^2
                                                              + 4284 s + 720
------------------------------------------------------------------------------
4 s^9+76 s^8+607 s^7+2688 s^6+7351 s^5+13137 s^4+16026 s^3+14267 s^2+9924
s+4320


ans =

  -5.0000
  -4.0000
   0.0716 + 0.9974i
   0.0716 - 0.9974i
  -1.4265 + 1.3355i
  -1.4265 - 1.3355i
  -3.0000
  -2.1451 + 0.3366i
  -2.1451 - 0.3366i
```

3-43) Use Toolbox 3-3-1

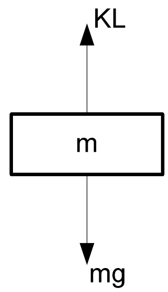
```
G3=G1+G2
G4=G1-G2
G5=G4/G3
G6=G4/(G1*G2)

G3=G1+G2
G4=G1-G2
G5=G4/G3
G6=G4/(G1*G2)

Transfer function:

8 s^12 + 92 s^11 + 522 s^10 + 1925 s^9 + 5070 s^8 + 9978 s^7 + 15154 s^6 +
18427 s^5

                                + 18778 s^4 + 16458 s^3 + 13268 s^2 + 10720 s +
4800
```

```
--------------------------------------------------------------------------------
2 s^9 + 33 s^8 + 228 s^7 + 900 s^6 + 2348 s^5 + 4267 s^4 + 5342 s^3 + 4640 s^2

                                                                             +
2400 s


Transfer function:

-8 s^12 - 92 s^11 - 522 s^10 - 1925 s^9 - 5068 s^8 - 9924 s^7 - 14588 s^6

                         - 15413 s^5 - 9818 s^4 - 406 s^3 + 7204 s^2 + 9760 s +
4800

--------------------------------------------------------------------------------
2 s^9 + 33 s^8 + 228 s^7 + 900 s^6 + 2348 s^5 + 4267 s^4 + 5342 s^3 + 4640 s^2

                                                                             +
2400 s


Transfer function:

-16 s^21 - 448 s^20 - 5904 s^19 - 49252 s^18 - 294261 s^17 - 1.346e006 s^16

        - 4.906e006 s^15 - 1.461e007 s^14 - 3.613e007 s^13 - 7.482e007 s^12

        - 1.3e008 s^11 - 1.883e008 s^10 - 2.234e008 s^9 - 2.078e008 s^8 -
1.339e008 s^7

        - 2.674e007 s^6 + 6.595e007 s^5 + 1.051e008 s^4 + 8.822e007 s^3 +
4.57e007 s^2

                                                                        +
1.152e007 s

--------------------------------------------------------------------------------
16 s^21 + 448 s^20 + 5904 s^19 + 49252 s^18 + 294265 s^17 + 1.346e006 s^16

        + 4.909e006 s^15 + 1.465e007 s^14 + 3.643e007 s^13 + 7.648e007 s^12

        + 1.369e008 s^11 + 2.105e008 s^10 + 2.803e008 s^9 + 3.26e008 s^8 +
3.343e008 s^7

        + 3.054e008 s^6 + 2.493e008 s^5 + 1.788e008 s^4 + 1.072e008 s^3 +
4.8e007 s^2
```

```
                                                                                          +
1.152e007 s


Transfer function:

-16 s^21 - 448 s^20 - 5904 s^19 - 49252 s^18 - 294261 s^17 - 1.346e006 s^16

        - 4.906e006 s^15 - 1.461e007 s^14 - 3.613e007 s^13 - 7.482e007 s^12

        - 1.3e008 s^11 - 1.883e008 s^10 - 2.234e008 s^9 - 2.078e008 s^8 -
1.339e008 s^7

        - 2.674e007 s^6 + 6.595e007 s^5 + 1.051e008 s^4 + 8.822e007 s^3 +
4.57e007 s^2

                                                                                          +
1.152e007 s

-------------------------------------------------------------------------------
8 s^20 + 308 s^19 + 5270 s^18 + 54111 s^17 + 379254 s^16 + 1.955e006 s^15

        + 7.778e006 s^14 + 2.471e007 s^13 + 6.416e007 s^12 + 1.383e008 s^11

        + 2.504e008 s^10 + 3.822e008 s^9 + 4.919e008 s^8 + 5.305e008 s^7 +
4.73e008 s^6

        + 3.404e008 s^5 + 1.899e008 s^4 + 7.643e007 s^3 + 1.947e007 s^2 +
2.304e006 s
```

Chapter 4

4-1) When the mass is added to spring, then the spring will stretch from position O to position L.



The total potential energy is:

$$U_s = \frac{1}{2}K(L+y)^2$$

where y is a displacement from equilibrium position L.

The gravitational energy is:

$$U_g = -mgy$$

The kinetic energy of the mass-spring system is calculated by:

$$T = \frac{1}{2}m\dot{y}^2$$

As we know that $T + U_g + U_s = constant$, then $\frac{1}{2}m\dot{y}^2 - mgy + \frac{1}{2}K(L+y)^2 = constant$

By differentiating from above equation, we have:

$$m\dot{y}\ddot{y} - mg\dot{y} + \dot{y}K(L+y) = 0$$

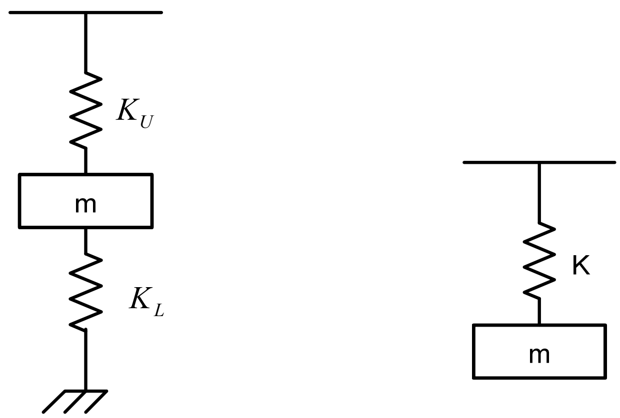
$$m\dot{y}\ddot{y} + Ky\dot{y} + (KL - mg)\dot{y} = 0$$

since KL = mg, therefore:

$$\dot{y}(m\ddot{y} + K\dot{y}) = 0$$

As ẏ cannot be zero because of vibration, then $m\ddot{y} + Ky = 0$

4-2)



$$K_u = K_1 + K_2$$
$$K_L = \frac{K_3K_4}{K_3 + K_4} + K_5$$

$$\begin{aligned} K &= K_u + K_L \\ &= K_1 + K_2 + K_5 + \frac{K_3K_4}{K_3 + K_4} \\ &= \frac{K_1K_3 + K_2K_3 + K_5K_3 + K_1K_4 + K_2K_4 + K_4K_5 + K_3K_4}{K_3 + K_4} \end{aligned}$$

$$\Rightarrow \omega_n = \sqrt{\frac{K}{m}}$$

4-3) a) Rotational kinetic energy: $T_{rot} = \frac{1}{2}J\dot{\theta}^2$

Translational kinetic energy: $T_T = \frac{1}{2}m\dot{y}^2$

Relation between translational displacement and rotational displacement:

$$y = r\theta$$

$$\dot{y} = r\dot{\theta}$$

$$T_{Rot} = \frac{1}{2}\frac{J}{r^2}\dot{y}^2$$

Potential energy: $U = \frac{1}{2}Ky^2$

As we know $T_{Rot} + T_T + U = constant$, then:

$$\frac{1}{2}\frac{J}{r^2}\dot{y}^2 + \frac{1}{2}\, m\dot{y}^2 + \frac{1}{2}\, Ky^2 = constant$$

By differentiating, we have:

$$\frac{J}{r^2}\dot{y}\ddot{y} + m\dot{y}\ddot{y} + Ky\dot{y} = 0$$

$$\dot{y}\left(\frac{J}{r^2}\ddot{y} + m\ddot{y} + Ky\right) = 0$$

Since $\dot{y}$ cannot be zero, then $J\frac{\ddot{y}}{r^2} + m\ddot{y} + Ky = 0$

b)

$\ddot{y} = r\ddot{\theta}$

$J\ddot{\theta}^2 + m\ddot{y} + Ky = 0$

$\frac{Y(s)}{\theta(s)} = -\frac{J}{ms^2 + K}$

c)

$$T_{max} = \frac{1}{2}\, m\, \dot{y}_{max}^2 + \frac{1}{2}\frac{J}{r^2}\dot{y}_{max}^2 = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\dot{y}_{max}^2$$

$$\dot{y}_{max}^2 = \omega_n$$

where $\dot{y} = A$ at the maximum energy.

$$U_{max} = \frac{1}{2}Ky_{max}^2 = \frac{1}{2}KA^2$$

Then:

$$\frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 A^2 = \frac{1}{2}KA^2$$

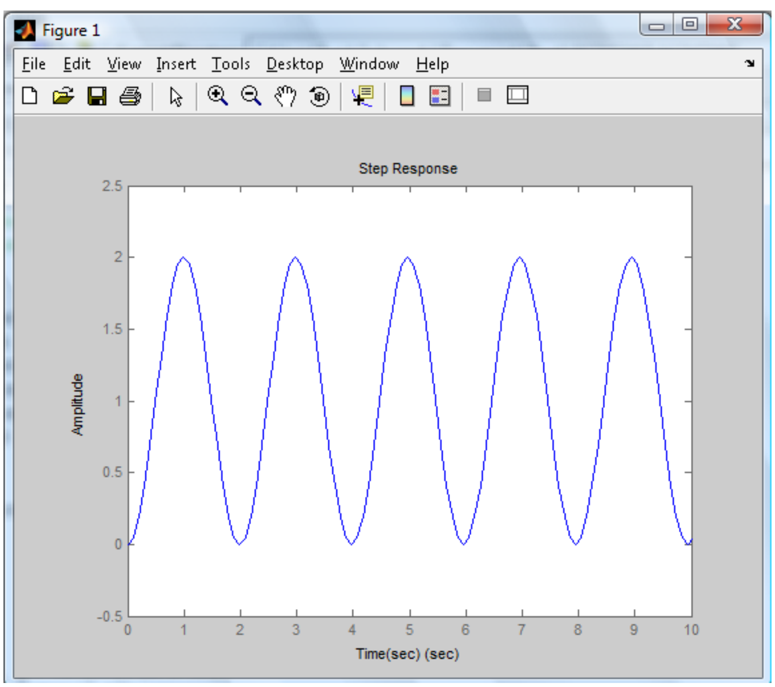
Or:

$$\omega_n = \sqrt{\frac{K}{m + \frac{J}{r^2}}} = r\sqrt{\frac{K}{r^m + J}}$$

d) $G(s) = \dfrac{J}{(ms^2 + K)}$

```
% select values of m, J and K
%Step input
K=10;
J=10;
M=1;
G=tf([10],[1 0 10])
step(G,10)
xlabel( 'Time(sec)');
ylabel('Amplitude');
```

Transfer function:
 10
----
s^2 + 10



4-4)

(a) Force equations:

$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{dy_1}{dt} + B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K\left(y_1 - y_2 \right)$$

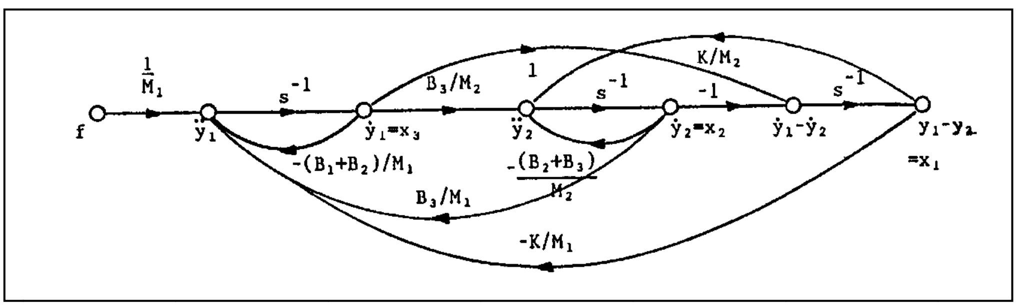
$$B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K\left(y_1 - y_2 \right) + M_2 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt}$$

Rearrange the equations as follows:

$$\frac{d^2 y_1}{dt^2} = -\frac{\left(B_1 + B_3 \right)}{M_1} \frac{dy_1}{dt} + \frac{B_3}{M_1} \frac{dy_2}{dt} - \frac{K}{M_1}\left(y_1 - y_2 \right) + \frac{f}{M_1}$$

$$\frac{d^2 y_2}{dt^2} = \frac{B_3}{M_2} \frac{dy_1}{dt} - \frac{\left(B_2 + B_3 \right)}{M_2} \frac{dy_2}{dt} + \frac{K}{M_2}\left(y_1 - y_2 \right)$$

(i) State diagram: Since $y_1 - y_2$ appears as one unit, the minimum number of integrators is three.



State equations: Define the state variables as $x_1 = y_1 - y_2, \quad x_2 = \frac{dy_2}{dt}, \quad x_3 = \frac{dy_1}{dt}.$

$$\frac{dx_1}{dt} = -x_2 + x_3 \quad \frac{dx_2}{dt} = \frac{K}{M_2}x_1 - \frac{(B_2+B_3)}{M_2}x_2 + \frac{B_3}{M_2}x_3 \quad \frac{dx_3}{dt} = -\frac{K}{M_1}x_1 + \frac{B_3}{M_1}x_2 - \frac{(B_1+B_3)}{M_1}x_3 + \frac{1}{M}f$$

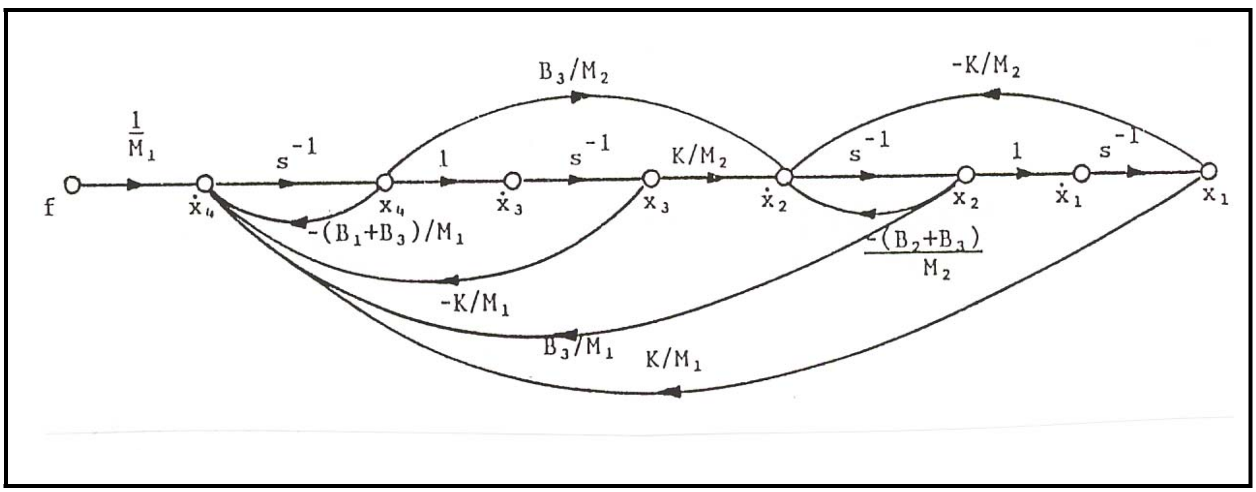
(ii) **State variables:** $x_1 = y_2, \quad x_2 = \frac{dy_2}{dt}, \quad x_3 = y_1, \quad x_4 = \frac{dy_1}{dt}.$

State equations:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{M_2}x_1 - \frac{B_2+B_3}{M_2}x_2 + \frac{K}{M_2}x_3 + \frac{B_3}{M_2}x_4$$

$$\frac{dx_3}{dt} = x_4 \qquad \frac{dx_4}{dt} = \frac{K}{M_1}x_1 + \frac{B_3}{M_1}x_2 - \frac{K}{M_1}x_3 - \frac{B_1+B_3}{M_1}x_4 + \frac{1}{M_1}f$$

State diagram:



Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{M_2 s^2 + (B_2+B_3)s + K}{s\left\{M_1 M_2 s^3 + \left[(B_1+B_3)M_2 + (B_2+B_3)M_1\right]s^2 + \left[K(M_1+M_2) + B_1B_2 + B_2B_3 + B_1B_3\right]s + (B_1+B_2)K\right\}}$$

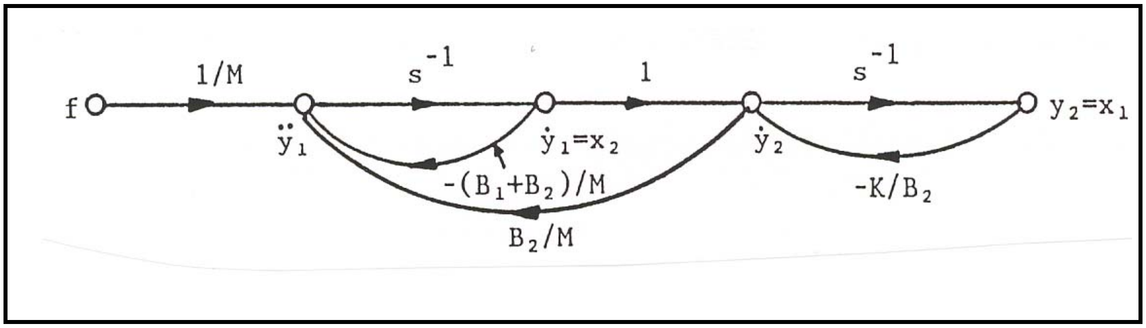
$$\frac{Y_2(s)}{F(s)} = \frac{B_3 s + K}{s\left\{M_1 M_2 s^3 + \left[(B_1+B_3)M_2 + (B_2+B_3)M_1\right]s^2 + \left[K(M_1+M_2) + B_1B_2 + B_2B_3 + B_1B_3\right]s + (B_1+B_2)K\right\}}$$

(b) **Force equations:**

$$\frac{d^2y_1}{dt^2} = -\frac{(B_1+B_2)}{M}\frac{dy_1}{dt} + \frac{B_2}{M}\frac{dy_2}{dt} + \frac{1}{M}f \qquad \frac{dy_2}{dt} = \frac{dy_1}{dt} - \frac{K}{B_2}y_2$$

(i) State diagram:

Define the outputs of the integrators as state variables, $x_1 = y_2, \quad x_2 = \frac{dy_1}{dt}$.



State equations:

$$\frac{dx_1}{dt} = -\frac{K}{B_2}x_1 + x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{M}x_1 - \frac{B_1}{M}x_2 + \frac{1}{M}f$$

(ii) State equations: State variables: $x_1 = y_2, \quad x_2 = y_1, \quad x_3 = \frac{dy_1}{dt}$.

$$\frac{dx_1}{dt} = -\frac{K}{B_2}x_1 + x_3 \qquad \frac{dx_2}{dt} = x_3 \qquad \frac{dx_3}{dt} = -\frac{K}{M}x_1 - \frac{B_1}{M}x_3 + \frac{1}{M}f$$

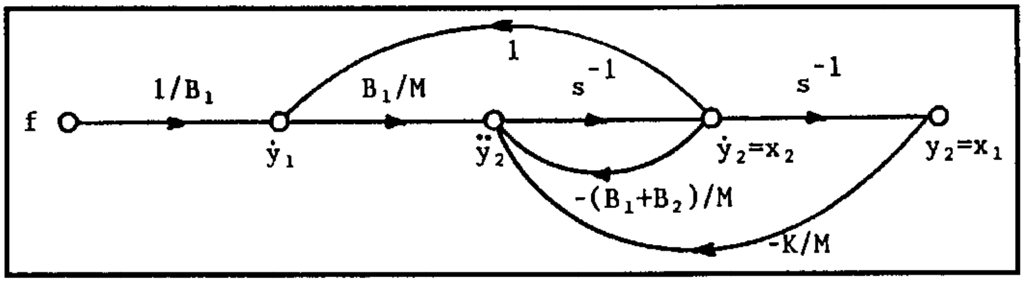
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{B_2 s + K}{s\left[MB_2 s^2 + (B_1B_2 + KM)s + (B_1+B_2)K\right]} \qquad \frac{Y_2(s)}{F(s)} = \frac{B_2}{MB_2 s^2 + (B_1B_2 + KM)s + (B_1+B_2)K}$$

(c) Force equations:

$$\frac{dy_1}{dt} = \frac{dy_2}{dt} + \frac{1}{B_1}f \qquad \frac{d^2y_2}{dt^2} = -\frac{(B_1+B_2)}{M}\frac{dy_2}{dt} + \frac{B_1}{M}\frac{dy_2}{dt} + \frac{B_1}{M}\frac{dy_1}{dt} - \frac{K}{M}y_2$$

(i) State diagram:



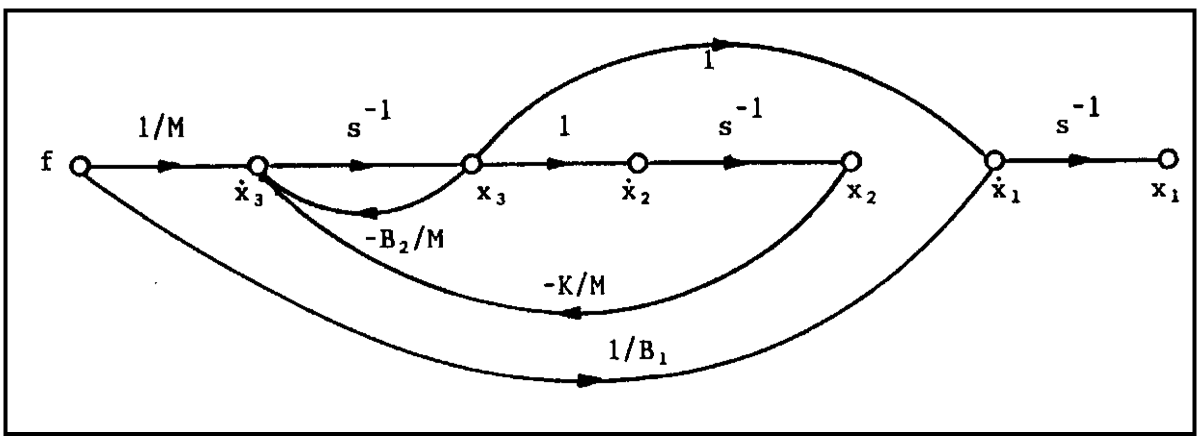
State equations: Define the outputs of integrators as state variables.

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{M}x_1 - \frac{B_2}{M}x_2 + \frac{1}{M}f$$

(ii) State equations: state variables: $x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \dfrac{dy_2}{dt}.$

$$\frac{dx_1}{dt} = x_3 + \frac{1}{B_1}f \qquad \frac{dx_2}{dt} = x_3 \qquad \frac{dx_3}{dt} = -\frac{K}{M}x_2 - \frac{B_2}{M}x_3 + \frac{1}{M}f$$

State diagram:



Transfer functions:

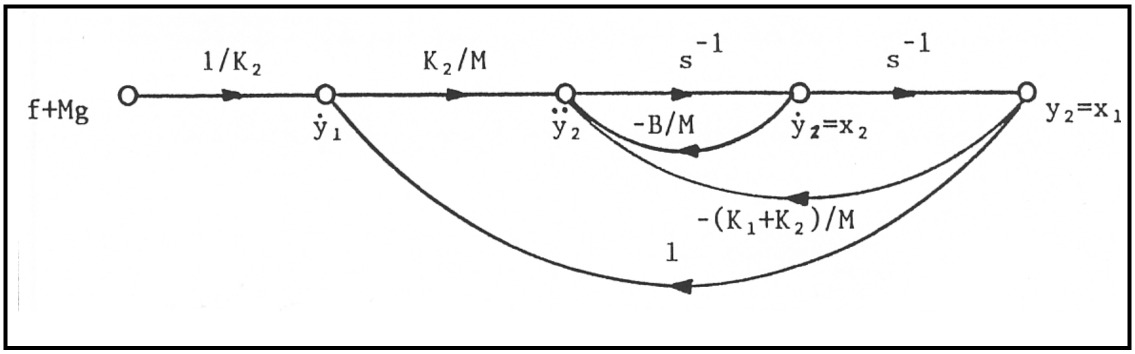
$$\frac{Y_1(s)}{F(s)} = \frac{Ms^2 + (B_1 + B_2)s + K}{B_1 s\left(Ms^2 + B_2 s + K\right)} \qquad \frac{Y_2(s)}{F(s)} = \frac{1}{Ms^2 + B_2 s + K}$$

4-5)

(a) Force equations:

$$y_1 = \frac{1}{K_2}(f + Mg) + y_2 \qquad \frac{d^2 y_2}{dt^2} = -\frac{B}{M}\frac{dy_2}{dt} - \frac{K_1 + K_2}{M} y_2 + \frac{K_2}{M} y_1$$

State diagram:



State equations:
Define the state variables as:

$x_1 = y_2, \quad x_2 = \frac{dy_2}{dt}$

.

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K_1}{M} x_1 - \frac{B}{M} x_2 + \frac{1}{M}(f + Mg)$$

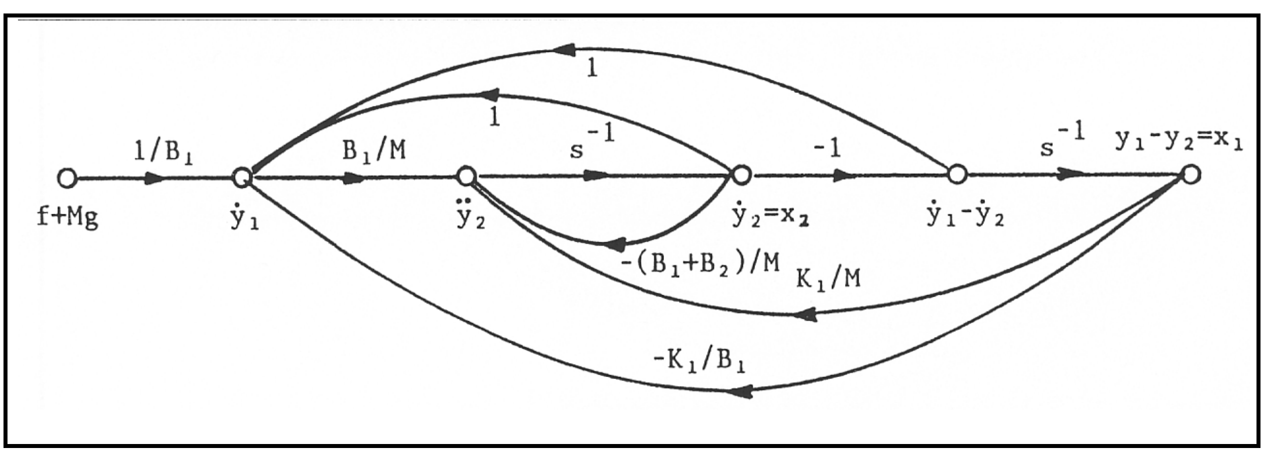
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{s^2 + Bs + K_1 + K_2}{K_2(Ms^2 + Bs + K_1)} \qquad \frac{Y_2(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K_1}$$

(b) Force equations:

$$\frac{dy_1}{dt} = \frac{1}{B_1}\left[f(t) + Mg\right] + \frac{dy_2}{dt} - \frac{K_1}{B_1}(y_1 - y_2) \qquad \frac{d^2 y_2}{dt^2} = \frac{B_1}{M}\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + \frac{K_1}{M}(y_1 - y_2) - \frac{B_2}{M}(y_1 - y_2) - \frac{B_2}{M}\frac{dy_2}{dt}$$

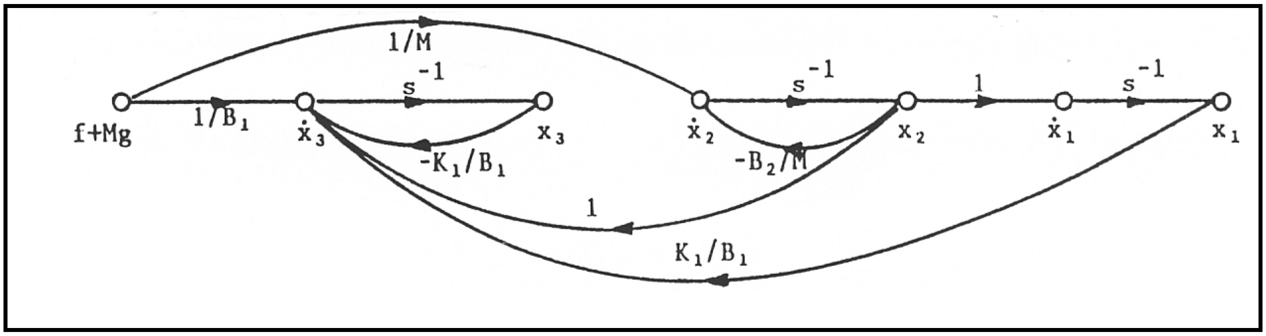
State diagram: (With minimum number of integrators)



To obtain the transfer functions $Y_1(s)/F(s)$ and $Y_2(s)/F(s)$, we need to redefine the state variables as:

$x_1 = y_2$, $x_2 = dy_2/dt$, and $x_3 = y_1$.

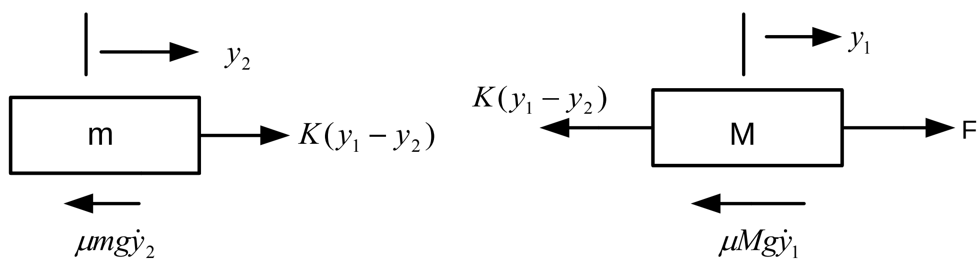
State diagram:



Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{Ms^2 + (B_1 + B_2)s + K_1}{s^2\left[MB_1 s + (B_1 B_2 + MK_1)\right]} \qquad \frac{Y_2(s)}{F(s)} = \frac{Bs + K_1}{s^2\left[MB_1 s + (B_1 B_2 + MK_1)\right]}$$

4-6) a)



b) From Newton's Law:

$$M\ddot{y}_1 = F - K(y_1 - y_2) - \mu Mg\dot{y}_1$$

$$m\ddot{y}_2 = K(y_1 - y_2) - \mu mg\dot{y}_2$$

If y_1 and y_2 are considered as a position and v_1 and v_2 as velocity variables

$$\text{Then: } \begin{cases} \dot{y}_1 = v_1 \\ \dot{y}_2 = v_2 \\ M\dot{v}_1 = F - K(y_1 - y_2) - \mu Mgv_1 \\ m\dot{v}_2 = F - K(y_1 - y_2) - \mu mgv_2 \end{cases}$$

The output equation can be the velocity of the engine, which means $z = v_2$

c)

$$\begin{cases} Ms^2Y_1(s) = F - K\big(Y_1(s) - Y_2(s)\big) - \mu MgsY_1(s) \\ ms^2Y_2(s) = K\big(Y_1(s) - Y_2(s)\big) - \mu mgsY_2(s) \\ Z(s) = V_2(s) = sY(s) \end{cases}$$

Obtaining $\frac{Z(s)}{F(s)}$ requires solving above equation with respect to $Y_2(s)$

From the first equation:

$$(Ms^2 + K + \mu Mgs)Y_1(s) = F + KY_2(s)$$

$$Y_1(s) = \frac{F + KY_2(s)}{Ms^2 + \mu Mgs + K}$$

Substituting into the second equation:

$$ms^2Y_2(s) = \frac{KF + K^2Y_2(s)}{Ms^2 + \mu Mgs + K} - KY_2(s) - \mu mgsY_2(s)$$

By solving above equation:

$$\frac{Z(s)}{F(s)} = \frac{sY_2(s)}{F(s)} = \frac{ms^2 + m\,\mu gs + 1}{Mms^3 + (2Mm\,\mu g)s^2 + (Mk + Mm(\mu g)^2 + mk)s + K\mu g(M + m)}$$

d)

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K}{m} & \frac{K}{M} & -\mu g & 0 \\ \frac{K}{m} & -\frac{K}{M} & 0 & -\mu g \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} F$$

$$Z = [0\ 0\ 0\ 1] \begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} + [0]F$$

4-7) (a) Force equations:

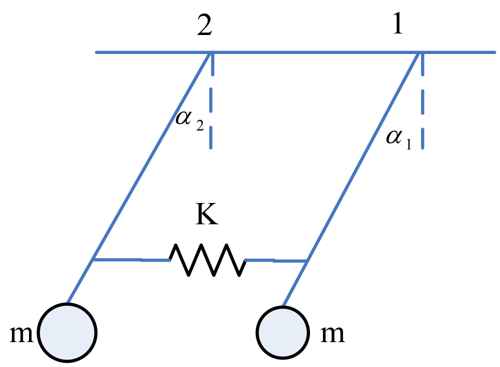
$$f(t) = K_h\left(y_1 - y_2\right) + B_h\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) \qquad K_h\left(y_1 - y_2\right) + B_h\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) = M\frac{d^2y_2}{dt^2} + B_t\frac{dy_2}{dt}$$

(b) State variables: $x_1 = y_1 - y_2, \quad x_2 = \frac{dy_2}{dt}$

State equations:

$$\frac{dx_1}{dt} = -\frac{K_h}{B_h}x_1 + \frac{1}{B_h}f(t) \qquad \frac{dx_2}{dt} = -\frac{B_t}{M}x_2 + \frac{1}{M}f(t)$$

4-8)



For the left pendulum:

$$T_{rot} = ml^2\ddot{\alpha}_2$$

$$U_g = -mgl\sin\alpha_2$$

$$T = K\left(\frac{7}{8}l\right)(\sin\alpha_2 - \sin\alpha_1)\cos\alpha_2\left(\frac{7}{8}l\right) = K\frac{49}{64}l^2(\sin\alpha_2 - \sin\alpha_1)\cos\alpha_2$$

⇨ $T_{rot} + U_g + T = 0$

⇨ $ml^2\ddot{\alpha}_2 + mgl\sin\alpha_2 + \frac{49}{64}Kl^2(\sin\alpha_2 - \sin\alpha_1)\cos\alpha_2$

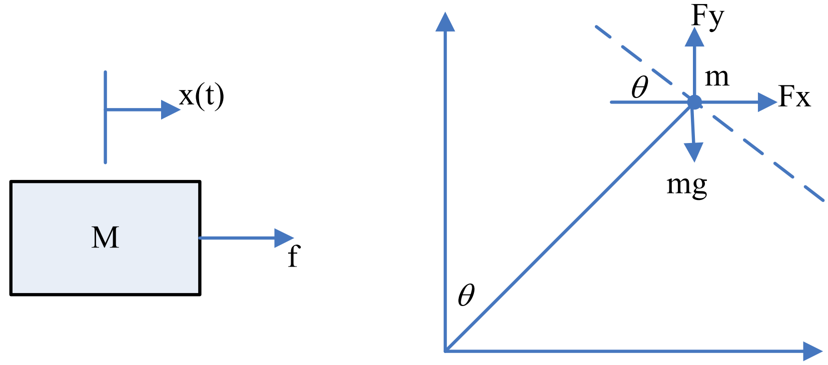
For the right pendulum, we can write the same equation:

$$ml^2\ddot{\alpha}_1 + mgl\sin\alpha_1 + \frac{49}{64}Kl^2(\sin\alpha_1 - \sin\alpha_2)\cos\alpha_1$$

since the angles are small:

$$\Rightarrow \begin{cases}\sin\alpha_2 \cong \alpha_2 \\ \cos\alpha_2 \cong 1 \\ \sin\alpha_1 = \alpha_1 \\ \cos\alpha_1 = 1\end{cases} \qquad \Rightarrow \begin{cases}ml\,\ddot{\alpha}_1 + mg\alpha_1 + \frac{49}{64}Kl(\alpha_1 - \alpha_2) = 0 \\ ml\ddot{\alpha}_2 + mg\alpha_2 + \frac{49}{64}Kl(\alpha_2 - \alpha_1) = 0\end{cases}$$

4-9) a)



b) If we consider the coordinate of centre of gravity of mass m as (x_g, y_g),

Then $x_g = x + l\sin\theta$ and $y_g = l\cos\theta$

From force balance, we have:

$$M\ddot{x} + m\ddot{x}_g = f$$

$$\Rightarrow \; M\ddot{x} + m\ddot{x} + m\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = f$$

From the torque balance, we have:

$$(F_x\cos\theta)l - \left(F_y\sin\theta\right)l = (mg\sin\theta)l$$

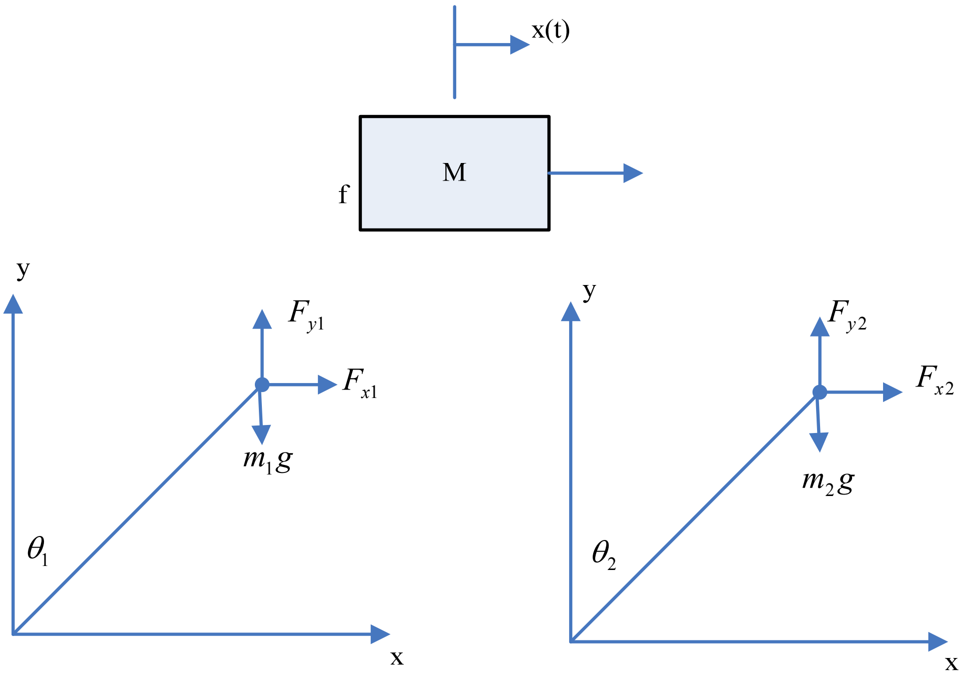
Where:

$$\begin{cases} F_x = m\ddot{x}_g = m\left(\ddot{x} - l\dot{\theta}^2 sin\theta + l\ddot{\theta}\cos\theta\right) \\ F_y = m\ddot{y}_g = -m\left(l\dot{\theta}^2\cos\theta + l\ddot{\theta}\,\sin\theta\right) \end{cases}$$

Substituting these equation:

$$m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta$$

4-10) a)



b) Kinetic energy

(i) For lower pendulum:

$$T_1 = \frac{1}{2} J_1\dot{\theta}_1^2 + \frac{1}{2} m_1 \left\{ \left[\frac{d}{dt}(l_1 \sin\theta_1)\right]^2 + \left[\frac{d}{dt}(l_1 \cos\theta_1)\right]^2 \right\}$$

For upper pendulum:

$$T_2 = \frac{1}{2} J_2\dot{\theta}_2^2 + \frac{1}{2} m_2 \left\{ \left[\frac{d}{dt}(l_2 \sin\theta_2)\right]^2 + \left[\frac{d}{dt}(l_2 \cos\theta_2)\right]^2 \right\}$$

For the cart: $T_3 = \frac{1}{2} M\dot{x}^2$

(ii) Potential energy:

For lower pendulum: $U_1 = m_1 g l_1 \cos\theta_2$

For upper pendulum: $U_2 = m_2 g l_2 \cos\theta_2$

For the cart: $U_3 = 0$

(iii) Total kinetic energy: $T_1 = T_1 + T_2 + T_3$

Total potential energy: $U = U_1 + U_2 + U_3$

The Lagrangian equation of motion is:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial T}{\partial x} + \dfrac{\partial U}{\partial x} = f \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}_1}\right) - \dfrac{\partial T}{\partial \theta_1} + \dfrac{\partial U}{\partial \theta_1} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}_2}\right) - \dfrac{\partial T}{\partial \theta_2} + \dfrac{\partial U}{\partial \theta_2} = 0 \end{cases}$$

Substituting T and U into the Lagrangian equation of motion gives:

$$\begin{cases} (m_1 + m_2 + M)\ddot{x} + m_1 l_1 \,\theta_1 \cos\theta_1 + m_2 l_2 \,\theta_2 \cos\theta_2 = m_1 l_1 \dot{\theta}_1^{\,2} \sin\theta_1 + m_2 l_2 \dot{\theta}_2^{\,2} \sin\theta_2 + f \\ m_1 l_1 \ddot{x} \cos\theta_1 + (J_1 + m_1 l_1^2)\ddot{\theta}_1 = m_1 l_1 g \sin\theta_1 \\ m_2 l_2 \ddot{x} \cos\theta_2 + (J_2 + m_2 l_2^2)\ddot{\theta}_2 = m_2 l_2 g \sin\theta_2 \end{cases}$$

4-11) a) From the Lagrangian equation of motion:

$$\left(\frac{J}{r^2} + m\right)\ddot{p} + \; mg\sin\alpha - mp\dot{\alpha}^2 = 0$$

b) As:

$$\alpha = \frac{d}{L}\,\theta$$

Then

$$\left(\frac{J}{r^2} + m\right)\ddot{p} + mg\sin\left(\frac{d\dot{\theta}}{L}\right) - mp\frac{d}{L}\,\dot{\theta}^2 = 0$$

If we linearize the equation about beam angle α = 0, then sinα ≈ α and sinθ ≈ θ

Then:

$$\left(\frac{J}{r^2} + m\right)\ddot{p} = -mg\frac{d}{\mathcal{L}}\,\theta$$

$$\left(\frac{J}{r^2} + m\right)s^2 P(s) = -\frac{mgd}{L}\,\theta(s)$$

$$\frac{P(s)}{\theta(s)} = \frac{mgd}{s^2 L\left(\frac{J}{r^2} + m\right)}$$

c) Considering

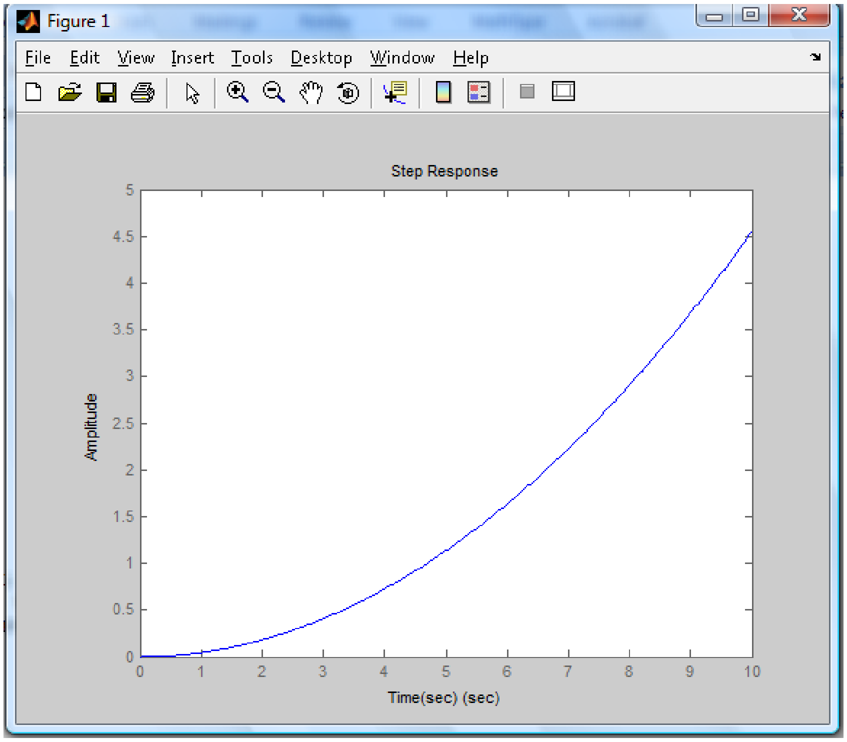
$$\begin{cases} \dot{p} = q \\ \ddot{p} = \dot{q} \end{cases}$$

Then the state-space equation is described as:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{mgd}{L\left(\dfrac{J}{r^2} + m\right)} \end{bmatrix} \theta$$

d) $G(s) = \dfrac{mgd}{(s^2 L(J/r^2 + m))}$

```
clear all
% select values of m, d, r, and J
%Step input
g=10;
J=10;
M=1;
D=0.5;
R=1;
L=5;
G=tf([M*g*D],[L*(J/R^2+M) 0 0])
step(G,10)
xlabel( 'Time(sec)');
ylabel('Amplitude');

Transfer function:
  5
------
55 s^2
```



4-12) If the aircraft is at a constant altitude and velocity, and also the change in pitch angle does not change the speed, then from longitudinal equation, the motion in vertical plane can be written as:

$$\begin{cases} \dot{u} = \dfrac{x}{m} - g\sin\theta - q\omega \\ \dot{\omega} = \dfrac{z}{m} - g\cos\theta + qu \\ \dot{q} = \dfrac{M}{I_{yy}} \\ \dot{\theta} = q \end{cases}$$

Where u is axial velocity, ω is vertical velocity, q is pitch rate, and θ is pitch angle.

Converting the Cartesian components with polar inertial components and replace x, y, z by T, D, and L. Then we have:

$$\begin{cases} \dot{V} = \dfrac{1}{m}[T\cos\alpha - D - mg\sin\gamma] \\ \dot{\gamma} = \dfrac{1}{mV}[T\sin\alpha + L - mg\cos\gamma] \\ \dot{q} = \dfrac{M}{I_{yy}} \\ \dot{\theta} = q \end{cases}$$

Where $\alpha = \theta - \gamma$ is an attack angle, V is velocity, and γ is flight path angle.

It should be mentioned that T, D, L and M are function of variables α and V.

Refer to the aircraft dynamics textbooks, the state equations can be written as:

$$\begin{cases} \dot{\alpha} = A_1\alpha + B_1 q + C_1\gamma \\ \dot{q} = A_2\alpha + B_2 q + C_2\gamma \\ \dot{\theta} = A_3 q \end{cases}$$

b) The Laplace transform of the system is:

$$G(s) = \frac{\theta(s)}{\gamma(s)}$$

By using Laplace transform, we have:

$$\begin{aligned} s\alpha(s) &= A_1\alpha(s) + B_1 q(s) + C_1\gamma(s) \quad (1) \\ sq(s) &= A_2\alpha(s) + B_2 q(s) + C_2\gamma(s) \quad (2) \\ s\theta(s) &= A_3 q(s) \quad (3) \end{aligned}$$

From equation (1):

$$\alpha(s) = \frac{B_1}{s - A_1} q(s) + \frac{C_1}{s - A_1}\gamma(s)$$

Substituting in equation (2) and solving for q(s):

$$q(s) = \frac{C_3(s - A_1) + A_2 C_1}{s(s - A_1) - B_2(s - A_1) - A_2 B_1}\gamma(s)$$

Substituting above expression in equation (3) gives:

$$\frac{\theta(s)}{\gamma(s)} = \frac{(C_2 s + A_2 C_1 - C_2 A_1)A_3}{s[s^2 - (A_1 + B_2)s - (B_2 A_1 + A_2 B_1)]}$$

If we consider $u = \omega^2 \sin\omega t$, then

$$M\ddot{y} + B\dot{y} + Ky = mlu$$

By using Laplace transform:

$$(Ms^2 + Bs + K)Y(s) = mlU(s) \quad (4)$$

Which gives:

$$\frac{Y(s)}{U(s)} = \frac{ml}{Ms^2 + Bs + K}$$

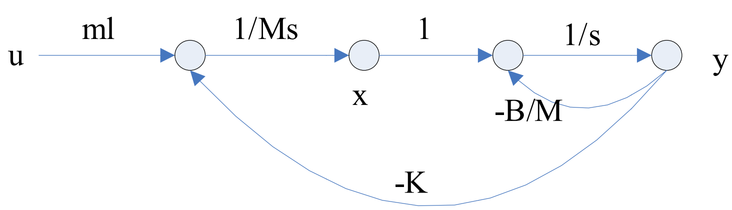
For plotting state flow diagram, equation (4) must be rewritten as:

$$\left(s+\frac{B}{M}+\frac{K}{MS}\right)Y(s)=\frac{ml}{MS}U(s)$$

or

$$\begin{cases} sY(s) = -\frac{B}{M}Y(s)+X(s) \rightarrow \ Y(s) = -\frac{B}{M}Y(s)+\frac{X(s)}{s} \\ X(s) = -\frac{K}{MS}Y(s)+\frac{ml}{MS}U(s) \end{cases}$$

So, the state flow diagram will plotted as:



Also look at section 4-11

4-13) (a) **Torque equation:**

$$\frac{d^2\theta}{dt^2}=-\frac{B}{J}\frac{d\theta}{dt}+\frac{1}{J}T(t)$$

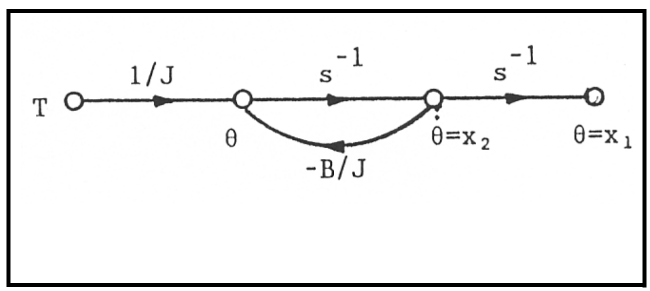
State equations:

$$\frac{dx_1}{dt}=x_2 \qquad \frac{dx_2}{dt}=-\frac{B}{J}x_2+\frac{1}{J}T$$

Transfer function:

$$\frac{\Theta(s)}{T(s)}=\frac{1}{s(Js+B)}$$

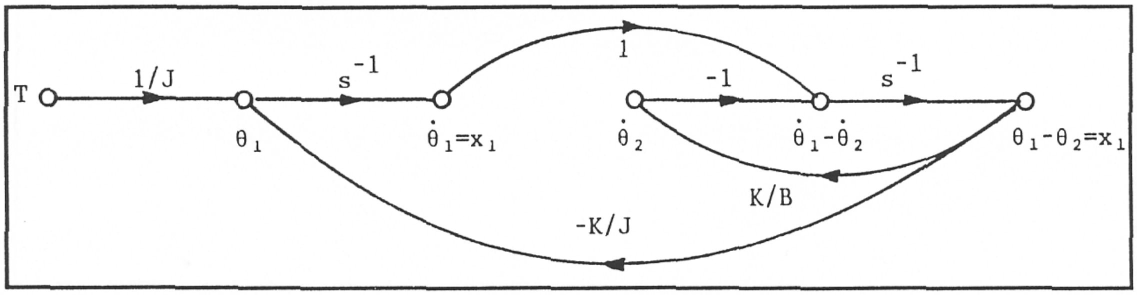
State diagram:



(b) **Torque equations:**

$$\frac{d^2\theta_1}{dt^2}=-\frac{K}{J}(\theta_1-\theta_2)+\frac{1}{J}T \qquad K(\theta_1-\theta_2)=B\frac{d\theta_2}{dt}$$

State diagram: (minimum number of integrators)



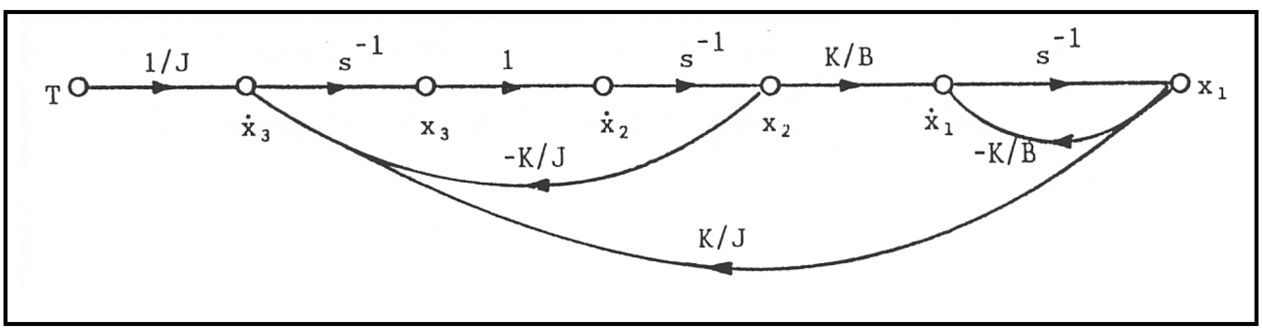
State equations:

$$\frac{dx_1}{dt} = -\frac{K}{B}x_1 + x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{J}x_1 + \frac{1}{J}T$$

State equations: Let $x_1 = \theta_2$, $x_2 = \theta_1$, and $x_3 = \frac{d\theta_1}{dt}$.

$$\frac{dx_1}{dt} = -\frac{K}{B}x_1 + \frac{K}{B}x_2 \qquad \frac{dx_2}{dt} = x_3 \qquad \frac{dx_3}{dt} = \frac{K}{J}x_1 - \frac{K}{J}x_2 + \frac{1}{J}T$$

State diagram:



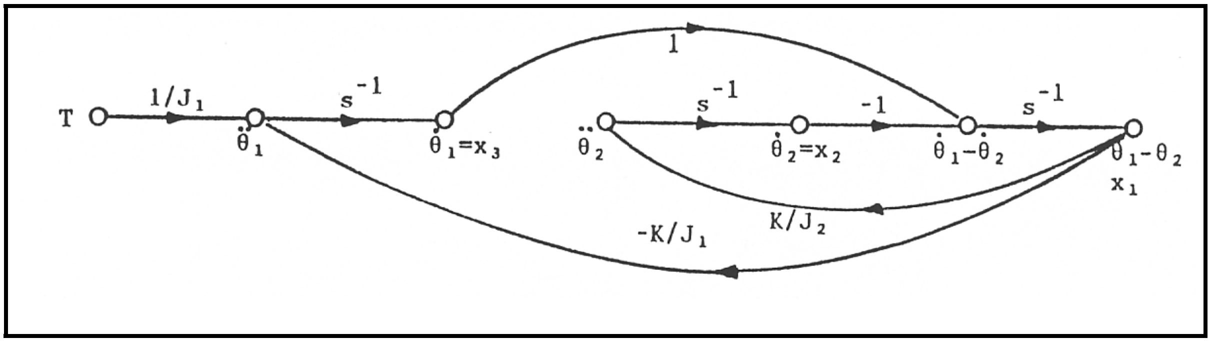
Transfer functions:

$$\frac{\Theta_1(s)}{T(s)} = \frac{Bs + K}{s\left(BJs^2 + JKs + BK\right)} \qquad \frac{\Theta_2(s)}{T(s)} = \frac{K}{s\left(BJs^2 + JKs + BK\right)}$$

(c) **Torque equations:**

$$T(t) = J_1\frac{d^2\theta_1}{dt^2} + K\left(\theta_1 - \theta_2\right) \qquad K\left(\theta_1 - \theta_2\right) = J_2\frac{d^2\theta_2}{dt^2}$$

State diagram:



State equations: state variables: $x_1 = \theta_2, \quad x_2 = \frac{d\theta_2}{dt}, \quad x_3 = \theta_1, \quad x_4 = \frac{d\theta_1}{dt}$.

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K}{J_2}x_1 + \frac{K}{J_2}x_3 \qquad \frac{dx_3}{dt} = x_4 \qquad \frac{dx_4}{dt} = \frac{K}{J_1}x_1 - \frac{K}{J_1}x_3 + \frac{1}{J_1}T$$

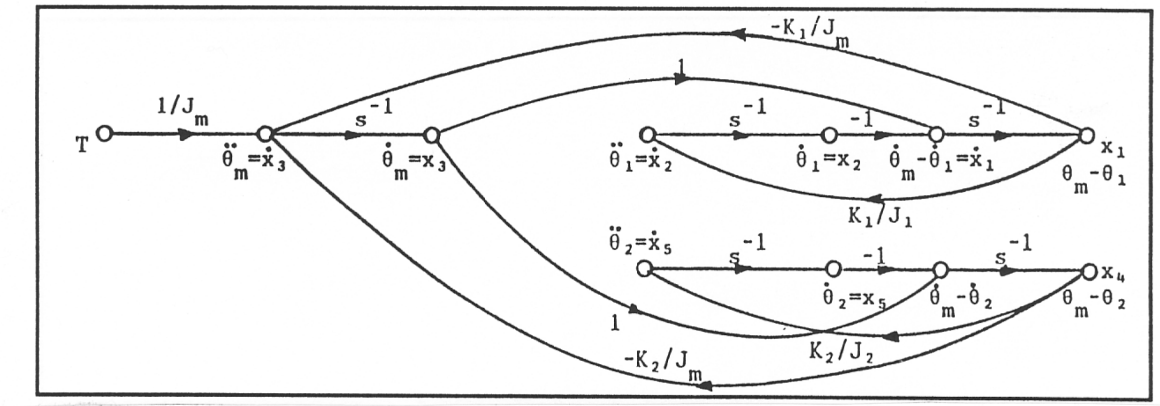
Transfer functions:

$$\frac{\Theta_1(s)}{T(s)} = \frac{J_2 s^2 + K}{s^2\left[J_1 J_2 s^2 + K\left(J_1 + J_2\right)\right]} \qquad \frac{\Theta_2(s)}{T(s)} = \frac{K}{s^2\left[J_1 J_2 s^2 + K\left(J_1 + J_2\right)\right]}$$

(d) Torque equations:

$$T(t) = J_m \frac{d^2\theta_m}{dt^2} + K_1\left(\theta_m - \theta_1\right) + K_2\left(\theta_m - \theta_2\right) \qquad K_1\left(\theta_m - \theta_1\right) = J_1 \frac{d^2\theta_1}{dt^2} \qquad K_2\left(\theta_m - \theta_2\right) = J_2 \frac{d^2\theta_2}{dt^2}$$

State diagram:



State equations: $x_1 = \theta_m - \theta_1, \quad x_2 = \frac{d\theta_1}{dt}, \quad x_3 = \frac{d\theta_m}{dt}, \quad x_4 = \theta_m - \theta_2, \quad x_5 = \frac{d\theta_2}{dt}$.

$$\frac{dx_1}{dt} = -x_2 + x_3 \quad \frac{dx_2}{dt} = \frac{K_1}{J_1}x_1 \quad \frac{dx_3}{dt} = -\frac{K_1}{J_m}x_1 - \frac{K_2}{J_m}x_4 + \frac{1}{J_m}T \quad \frac{dx_4}{dt} = x_3 - x_5 \quad \frac{dx_5}{dt} = \frac{K_2}{J_2}x_4$$

Transfer functions:

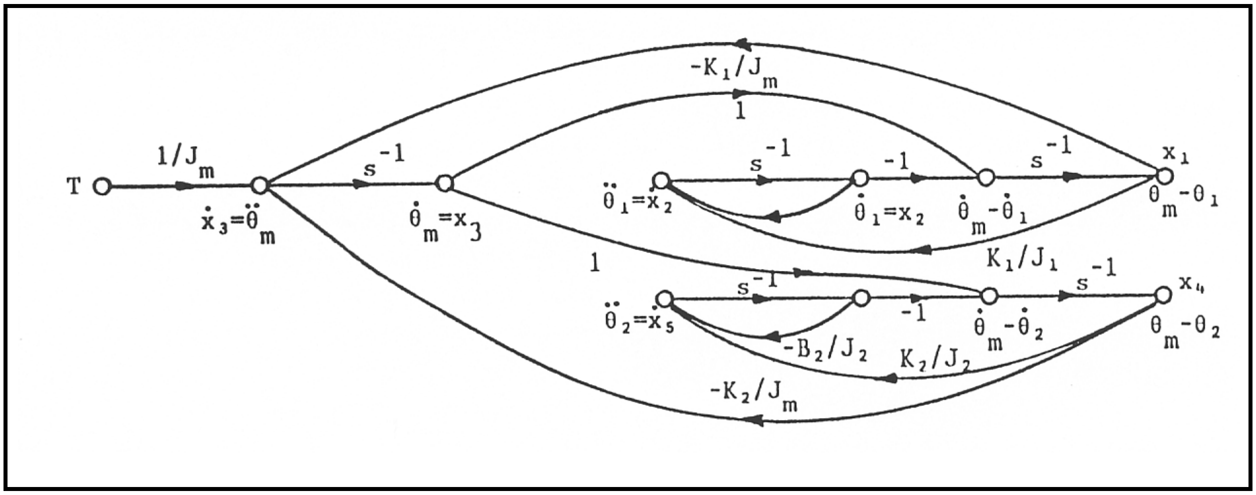
$$\frac{\Theta_1(s)}{T(s)} = \frac{K_1(J_2 s^2 + K_2)}{s^2\left[s^4 + \left(K_1 J_2 J_m + K_2 J_1 J_m + K_1 J_1 J_2 + K_2 J_1 J_2\right)s^2 + K_1 K_2\left(J_m + J_1 + J_2\right)\right]}$$

$$\frac{\Theta_2(s)}{T(s)} = \frac{K_2(J_1 s^2 + K_1)}{s^2\left[s^4 + \left(K_1 J_2 J_m + K_2 J_1 J_m + K_1 J_1 J_2 + K_2 J_1 J_2\right)s^2 + K_1 K_2\left(J_m + J_1 + J_2\right)\right]}$$

(e) Torque equations:

$$\frac{d^2\theta_m}{dt^2} = -\frac{K_1}{J_m}(\theta_m - \theta_1) - \frac{K_2}{J_m}(\theta_m - \theta_2) + \frac{1}{J_m}T \qquad \frac{d^2\theta_1}{dt^2} = \frac{K_1}{J_1}(\theta_m - \theta_1) - \frac{B_1}{J_1}\frac{d\theta_1}{dt} \qquad \frac{d^2\theta_2}{dt^2} = \frac{K_2}{J_2}(\theta_m - \theta_1) - \frac{B_2}{J_2}\frac{d\theta_2}{dt}$$

State diagram:



State variables: $x_1 = \theta_m - \theta_1, \quad x_2 = \frac{d\theta_1}{dt}, \quad x_3 = \frac{d\theta_m}{dt}, \quad x_4 = \theta_m - \theta_2, \quad x_5 = \frac{d\theta_2}{dt}.$

State equations:

$$\frac{dx_1}{dt} = -x_2 + x_3 \quad \frac{dx_2}{dt} = \frac{K_1}{J_1}x_1 - \frac{B_1}{J_1}x_2 \quad \frac{dx_3}{dt} = -\frac{K_1}{J_m}x_1 - \frac{K_2}{J_m}x_4 + \frac{1}{J_m}T \quad \frac{dx_4}{dt} = x_3 - x_5 \quad \frac{dx_5}{dt} = \frac{K_2}{J_2}x_4 - \frac{B_2}{J_2}x_5$$

Transfer functions:

$$\frac{\Theta_1(s)}{T(s)} = \frac{K_1\left(J_2 s^2 + B_2 s + K_2\right)}{\Delta(s)} \qquad \frac{\Theta_2(s)}{T(s)} = \frac{K_2\left(J_1 s^2 + B_1 s + K_1\right)}{\Delta(s)}$$

$$\Delta(s) = s^2\{J_1J_2J_m s^4 + J_m\left(B_1 + B_2\right)s^3 + \left[\left(K_1J_2 + K_2J_1\right)J_m + \left(K_1 + K_2\right)J_1J_2 + B_1B_2J_m\right]s^2$$
$$+\left[\left(B_1K_2 + B_2K_1\right)J_m + B_1K_2J_2 + B_2K_1J_1\right]s + K_1K_2\left(J_m + J_1 + J_2\right)\}$$

4-14)

(a)

$$T_m(t) = J_m\frac{d^2\theta_1}{dt^2} + T_1 \qquad T_1 = \frac{N_1}{N_2}T_2 \qquad T_3 = \frac{N_3}{N_4}T_4 \qquad T_4 = J_L\frac{d^2\theta_3}{dt^2} \qquad T_2 = T_3 \qquad \theta_2 = \frac{N_1}{N_2}\theta_1$$

$$\theta_3 = \frac{N_1N_3}{N_2N_4}\theta_1 \qquad T_2 = \frac{N_3}{N_4}T_4 = \frac{N_3}{N_4}J_L\frac{d^2\theta_3}{dt^2} \qquad T_m = J_m\frac{d^2\theta_1}{dt^2} + \frac{N_1N_3}{N_2N_4}T_4 = \left[J_m + \left[\frac{N_1N_3}{N_2N_4}\right]^2 J_L\right]\frac{d^2\theta_1}{dt^2}$$

(b)

$$T_m = J_m \frac{d^2\theta_1}{dt^2} + T_1 \qquad T_2 = J_2 \frac{d^2\theta_2}{dt^2} + T_3 \qquad T_4 = \left(J_3 + J_L\right)\frac{d^2\theta_3}{dt^2} \qquad T_1 = \frac{N_1}{N_2}T_2 \qquad T_3 = \frac{N_3}{N_4}T_4$$

$$\theta_2 = \frac{N_1}{N_2}\theta_1 \qquad \theta_3 = \frac{N_1 N_3}{N_2 N_4}\theta_1 \qquad T_2 = J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_3}{N_4}T_4 = J_2\frac{d^2\theta_2}{dt^2} + \frac{N_3}{N_4}\left(J_3 + J_L\right)\frac{d^2\theta_3}{dt^2}$$

$$T_m(t) = J_m \frac{d^2\theta_1}{dt^2} + \frac{N_1}{N_2}\left(J_2\frac{d^2\theta_2}{dt^2} + \frac{N_3}{N_4}\left(J_3 + J_4\right)\frac{d^2\theta_3}{dt^2}\right) = \left[J_m + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1N_3}{N_2N_4}\right)^2\left(J_3 + J_L\right)\right]\frac{d^2\theta_1}{dt^2}$$

4-15) (a)

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + T_1 \qquad T_2 = J_L\frac{d^2\theta_L}{dt^2} + T_L \qquad T_1 = \frac{N_1}{N_2}T_2 = nT_2 \qquad \theta_m N_1 = \theta_L N_2$$

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + nJ_L\frac{d^2\theta_L}{dt^2} + nT_L = \left(\frac{J_m}{n} + nJ_L\right)\alpha_L + nT_L \qquad \text{Thus, } \alpha_L = \frac{nT_m - n^2T_L}{J_m + n^2J_L}$$

Set $\frac{\partial\alpha_L}{\partial n} = 0.$ $\left(T_m - 2nT_L\right)\left(J_m + n^2J_L\right) - 2nJ_L\left(nT_m - n^2J_L\right) = 0$ Or, $n^2 + \frac{J_mT_L}{J_LT_m}n - \frac{J_m}{J_L} = 0$

Optimal gear ratio: $n^* = -\frac{J_mT_L}{2J_LT_m} + \frac{\sqrt{J_m^2T_L^2 + 4J_mJ_LT_m^2}}{2J_LT_m}$ where the + sign has been chosen.

(b) When $T_L = 0$, the optimal gear ratio is

$$n^* = \sqrt{J_m / J_L}$$

4-16) (a) Torque equation about the motor shaft:

$$T_m = J_m\frac{d^2\theta_m}{dt^2} + Mr^2\frac{d^2\theta_m}{dt^2} + B_m\frac{d\theta_m}{dt}$$

Relation between linear and rotational displacements:

$$y = r\theta_m$$

(b) Taking the Laplace transform of the equations in part (a), with zero initial conditions, we have

$$T_m(s) = \left(J_m + Mr^2\right)s^2\Theta_m(s) + B_m s\Theta_m(s) \qquad Y(s) = r\Theta_m(s)$$

Transfer function:

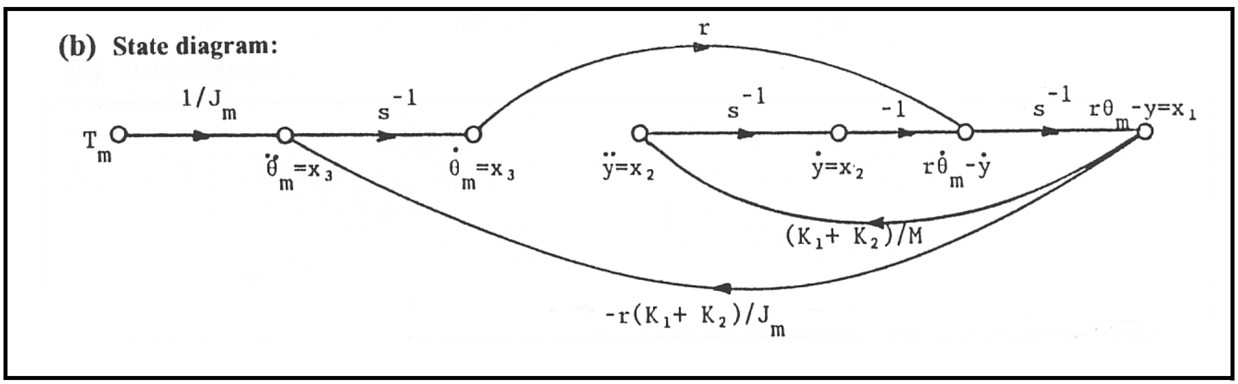
$$\frac{Y(s)}{T_m(s)} = \frac{r}{s\left[\left(J_m + Mr^r\right)s + B_m\right]}$$

4-17) (a)

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + r\left(T_1 - T_2\right) \qquad T_1 = K_2\left(r\theta_m - r\theta_p\right) = K_2\left(r\theta_m - y\right) \qquad T_2 = K_1\left(y - r\theta_m\right)$$

$$T_1 - T_2 = M\frac{d^2 y}{dt^2} \qquad \text{Thus,}\ T_m = J_m \frac{d^2\theta_m}{dt^2} + r\left(K_1 + K_2\right)\left(r\theta_m - y\right) \qquad M\frac{d^2 y}{dt^2} = \left(K_1 + K_2\right)\left(r\theta_m - y\right)$$

(b) State diagram:



(c) State equations:

$$\frac{dx_1}{dt} = rx_3 - x_2 \qquad \frac{dx_2}{dt} = \frac{K_1 + K_2}{M}x_1 \qquad \frac{dx_3}{dt} = \frac{-r\left(K_1 + K_2\right)}{J_m}x_1 + \frac{1}{J_m}T_m$$

(d) Transfer function:

$$\frac{Y(s)}{T_m(s)} = \frac{r\left(K_1 + K_2\right)}{s^2\left[J_m Ms^2 + \left(K_1 + K_2\right)\left(J_m + rM\right)\right]}$$

(e) Characteristic equation:

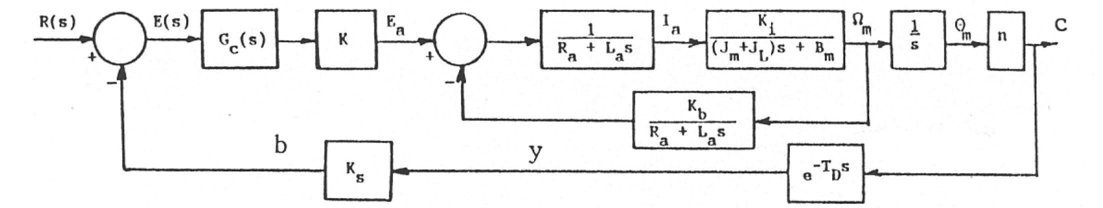
$$s^2\left[J_m Ms^2 + \left(K_1 + K_2\right)\left(J_m + rM\right)\right] = 0$$

4-18) (a) System equations:

$$T_m = K_i i_a = \left(J_m + J_L\right)\frac{d\omega_m}{dt} + B_m\omega_m \qquad e_a = R_a i_a + L_a\frac{di_a}{dt} + K_b\omega_m \qquad y = n\theta_m \qquad y = y\left(t - T_D\right)$$

$$T_D = \frac{d}{V}\ \text{(sec)} \qquad e = r - b \qquad b = K_s y \qquad E_a(s) = KG_c(s)E(s)$$

Block diagram:



(b) Forward-path transfer function:

$$\frac{Y(s)}{E(s)} = \frac{KK_i nG_c(s)e^{-T_D s}}{s\left\{\left(R_a + L_a s\right)\left[\left(J_m + J_L\right)s + B_m\right] + K_b K_i\right\}}$$

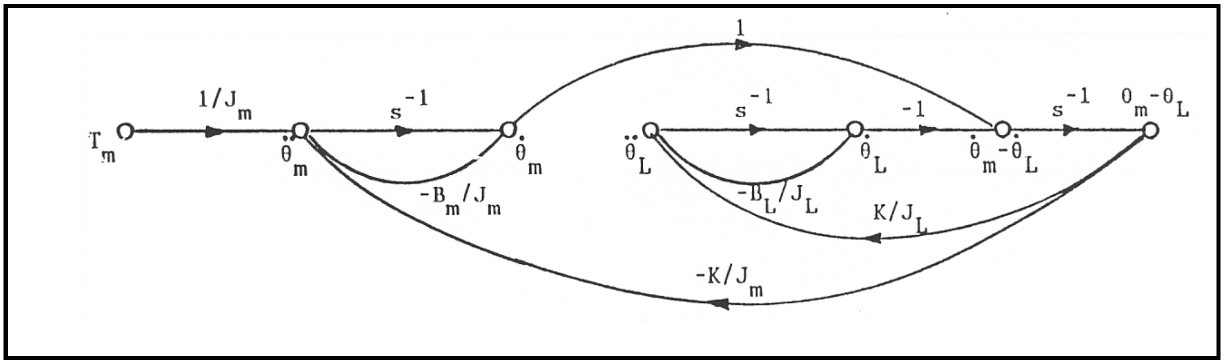
Closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{KK_i nG_c(s)e^{-T_D s}}{s\left(R_a + L_a s\right)\left[\left(J_m + J_L\right)s + B_m\right] + K_b K_i s + KG_c(s)K_i n e^{-T_D s}}$$

4-19) (a) Torque equations:

$$T_m(t) = J_m\frac{d^2\theta_m}{dt^2} + B_m\frac{d\theta_m}{dt} + K\left(\theta_m - \theta_L\right) \qquad K\left(\theta_m - \theta_L\right) = J_L\frac{d^2\theta_L}{dt^2} + B_L\frac{d\theta_L}{dt}$$

State diagram:



(b) Transfer functions:

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K}{\Delta(s)} \qquad \frac{\Theta_m(s)}{T_m(s)} = \frac{J_L s^2 + B_L s + K}{\Delta(s)} \qquad \Delta(s) = s\left[J_m J_L s^3 + \left(B_m J_L + B_L J_m\right)s^2 + \left(KJ_m + KJ_L + B_m B_L\right)s + B_m K\right]$$

(c) Characteristic equation: $\Delta(s) = 0$

(d) Steady-state performance: $T_m(t) = T_m = \text{constant.} \quad T_m(s) = \frac{T_m}{s}.$

$$\lim_{t\to\infty}\omega_m(t) = \lim_{s\to 0} s\Omega_m(s) = \lim_{s\to 0}\frac{J_L s^2 + B_L s + K}{J_m J_L s^3 + \left(B_m J_L + B_L J_m\right)s^2 + \left(KJ_m + KJ_L + B_m B_L\right)s + B_m K} = \frac{1}{B_m}$$

Thus, in the steady state, $\omega_m = \omega_L$.

(e) The steady-state values of ω_m and ω_L do not depend on J_m and J_L.

4-20) (a) Torque equation: (About the center of gravity C)

$$J\frac{d^2\theta}{dt^2} = T_s d_2 \sin\delta + F_d d_1 \qquad F_a d_1 = J_\alpha \alpha_1 = K_F d_1 \theta \qquad \sin\delta \cong \delta$$

Thus,

$$J\frac{d^2\theta}{dt^2} = T_s d_2 \delta + K_F d_1 \theta \qquad J\frac{d^2\theta}{dt^2} - K_F d_1 \theta = T_s d_2 \delta$$

(b) $Js^2\Theta(s) - K_F d_1 \Theta(s) = T_s d_2 \Delta(s)$

(c) With C and P interchanged, the torque equation about C is:

$$T_s\left(d_1 + d_2\right)\delta + F_\alpha d_2 = J\frac{d^2\theta}{dt^2} \qquad T_s\left(d_1 + d_2\right)\delta + K_F d_2 \theta = J\frac{d^2\theta}{dt^2}$$

$$Js^2\Theta(s) - K_F d_2 \Theta(s) = T_s\left(d_1 + d_2\right)\Delta(s) \qquad \frac{\Theta(s)}{\Delta(s)} = \frac{T_s\left(d_1 + d_2\right)}{Js^2 - K_F d_2}$$

4-21) (a) Nonlinear differential equations:

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -k(v) - g(x) + f(t) = -Bv(t) + f(t)$$

With $R_a = 0$, $\phi(t) = \dfrac{e(t)}{K_b v(t)} = K_f i_f(t) = K_f i_f(t) = K_f i_a(t)$ Then, $i_a(t) = \dfrac{e(t0}{K_b K_f v(t)}$

$$f(t) = K_i \phi(t) i_a(t) = \frac{K_i e^2(t)}{K_b^2 K_f v^2(t)}. \qquad \text{Thus,} \quad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$$

(b) State equations: $i_a(t)$ as input.

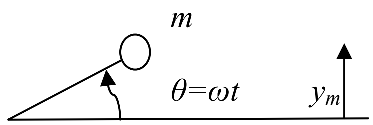
$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -Bv(t) + K_i K_f i_a^2(t)$$

(c) State equations: $\phi(t)$ as input.

$$f(t) = K_i K_f i_a^2(t) \qquad i_a(t) = i_f(t) = \frac{\phi(t)}{K_f}$$

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_f}\phi^2(t)$$

4-22) Define as the angle between mass *m* and the horizontal axis (positive in c..c.w. direction):



Use Newton's second law:

$$m(\ddot{y}+\ddot{y}_m) = -F_m$$
$$(M-m)\ddot{y} = F_m - B\dot{y} - Ky$$
$$\ddot{y}_m = -e\omega^2 \sin\omega t$$
$$\Rightarrow$$
$$M\ddot{y} + B\dot{y} + Ky = me\omega^2 \sin\omega t$$

Where *M* is the Mass of the overall block system. *M-m* is the mass of the block alone.

$$G(s) = \frac{Y}{R} = \frac{me\omega^2}{Ms^2 + Bs + K}$$

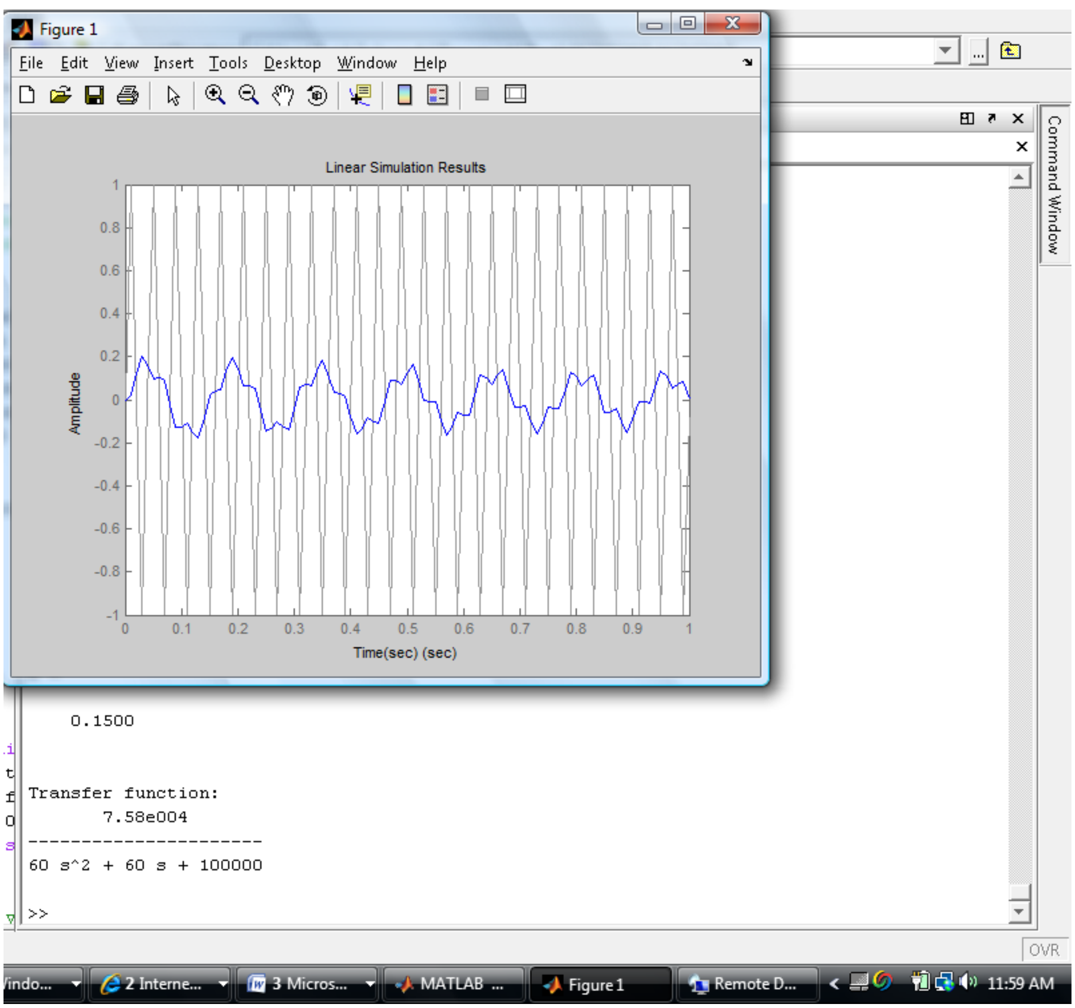
Zero i.c. and input $r(t) = \sin\omega t$

Note $\theta = \omega t$**.** So in case of a step response as asked in the question, ω is a step input and angle θ increases with time – i.e. it is a ramp function. Hence, y_m is a sinusoidal function, where the Laplace transform of a sine function is $\sin(\omega t) = \frac{\omega}{s^2+\omega^2}$

Pick values of the parameters and run MATLAB. See toolbox 5-8-2

```
clear all
m=20.5 %kg
M=60 %kg
K=100000 %N/m
Om=157 %rad/s
B=60 %N-m/s
e=0.15 %m
G=tf([m*e*Om^2],[M B K])
t=0:0.01:1;
u=1*sin(Om*t);
lsim(G,u,t)
xlabel( 'Time(sec)');
ylabel('Amplitude');
```

```
m =
   20.5000

M =
    60

K =
      100000

Om =
   157

B =
    60

e =
    0.1500

Transfer function:
       7.58e004
----------------------
60 s^2 + 60 s + 100000
```



4-23) a) summation of vertical forces gives:

$$\begin{cases} M\ddot{y} + (2K + k)y - kx = F & (1) \\ m\ddot{x} - Ky + kx = 0 & (2) \end{cases}$$

If we consider $\dot{y} = q$ and $\dot{x} = r$, then:

$$\begin{cases} M\dot{q} + (2K + k)y - kx = F \\ m\dot{r} - Ky + kx = 0 \end{cases}$$

The state-space model is:

$$\begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-2K - k}{M} & k & 0 & 0 \\ \dfrac{K}{m} & k & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \\ q \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{M} \\ 0 \end{bmatrix} F$$

b) $G(s) = \frac{Y(s)}{X(s)}$

By applying Laplace transform for equations (1) and (2), we obtain:

$$\begin{cases} [Ms^2 + (2K + k)]Y(s) - kX(s) = F(s) \\ (ms^2 + k)X(s) = kY(s) \end{cases}$$

Which gives:

$$X(s) = \frac{k}{ms^s + k} Y(s)$$

and

$$\left[Ms^2 + (2K + k) - \frac{k^2}{ms^2 + k} \right] Y(s) = F(s)$$

Therefore:

$$\frac{Y(s)}{F(s)} = \frac{ms^2 + k}{Mms^4 + (Mk + 2Km + mk)s^2 + 2Kk}$$

4-24) a) Summation of vertical forces gives:

$$\begin{cases} M\ddot{y} + (B + b)\dot{y} - b\dot{x} + (K + k)y - kx = F \\ m\ddot{x} - b\dot{y} + b\dot{x} - ky - kx = 0 \end{cases}$$

Consider $\dot{y} = q$ and $\dot{x} = r$, then

$$\begin{cases} M\dot{q} + (B+b)q - br + (K+k)y - kx = F \\ m\dot{r} - bq + br - ky - kx = 0 \end{cases}$$

So, the state-space model of the system is:

$$\begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-(K+k)}{M} & \dfrac{Kk}{M} & -\dfrac{B+b}{M} & \dfrac{b}{M} \\ \dfrac{k}{M} & \dfrac{k}{m} & \dfrac{b}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} y \\ x \\ q \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{M} \\ 0 \end{bmatrix} F$$

b) The Laplace transform of the system is defined by:

$$G(s) = \frac{Y(s)}{X(s)}$$

where

$$\begin{cases} \left(Ms^2 + (B+b)s + (K+k)\right)Y(s) - (bs+K)X(s) = F(s) \\ (ms^2 + bs - k)X(s) = (bs+k)Y(s) \end{cases}$$

as a result:

$$X(s) = \frac{bs+k}{ms^2 + bs - k} Y(s)$$

Substituting into above equation:

$$\left[\left(Ms^2 + (B+b)s + (K+k)\right)(ms^2 + bs + k) - (bs+k)^2\right]Y(s) = (ms^2 + bs - k)F(s)$$

$$\frac{Y(s)}{X(s)} = \frac{ms^2 + bs - k}{[Ms^2 + (B+b)s + (K+k)][ms^2 + bs - k] - (bs+k)^2}$$

4-25) a) According to the circuit:

$$\begin{cases} \dfrac{v_{in} - v_1}{2R} + C\dfrac{d}{dt}v_1 + \dfrac{v_{out} - v_1}{2R} = 0 \\ \dfrac{C}{2}\dfrac{d}{dt}(v_{in} - v_2) - \dfrac{v_2}{R} + \dfrac{C}{2}\dfrac{d}{dt}(v_{out} - v_2) = 0 \\ \dfrac{C}{2}\dfrac{d}{dt}(v_2 - v_{out}) + \dfrac{v_1 - v_{out}}{2R} = 0 \end{cases}$$

By using Laplace transform we have:

$$\begin{cases} \dfrac{V_{in}(s) - V_1(s)}{2R} + CsV_1(s) + \dfrac{V_{out}(s) - V_1(s)}{2R} = 0 \\ \dfrac{Cs}{2}\left(V_{in}(s) - V_2(s)\right) - \dfrac{V_2(s)}{R} + \dfrac{Cs}{2}\left(V_{out}(s) - V_2(s)\right) = 0 \\ \dfrac{Cs}{2}\left(V_2(s) - V_{out}(s)\right) + \dfrac{V_1(s) - V_{out}(s)}{2R} = 0 \end{cases}$$

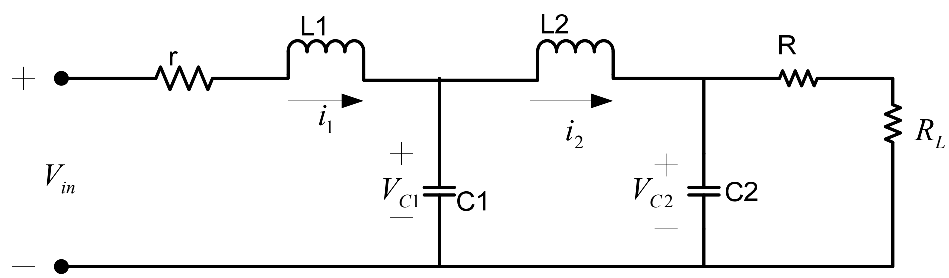
From above equations:

$$\begin{cases} V_1(s) = \dfrac{1}{2(RCS+1)}\left(V_{in}(s) + V_{out}(s)\right) \\ V_2(s) = \dfrac{RCS}{2(RCS+1)}\left(V_{in}(s) + V_{out}(s)\right) \end{cases}$$

Substituting $V_1(s)$ and $V_2(s)$ into preceding equations, we obtain:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R^2C^2s^2 + 1}{R^2C^2s^2 + 4RCs + 1}$$

b) Measuring V_{out} requires a load resistor, which means:



Then we have:

$$\begin{cases} L_1 \dfrac{d}{dt} i_1 = v_{in} - r i_1 - v_{C1} \\ C_1 \dfrac{d}{dt} v_{C1} = i_1 - i_2 \\ L_2 \dfrac{d}{dt} i_2 = v_{C1} - v_{C2} \\ C_2 \dfrac{d}{dt} v_{C2} = i_2 - \dfrac{v_{C2}}{R + R_L} \end{cases}$$

When

$$v_{out} = \frac{R_L}{R + R_L} v_{C2}$$

If $R_L \gg R$, then $v_{out} = v_{C2}$

By using Laplace transform we have:

$$\begin{cases} L_1 sI_1(s) = V_{in}(s) - rI_1(s) - V_{C1}(s) \\ C_1 sV_{C1}(s) = I_1(s) - I_2(s) \\ L_2 sI_2(s) = V_{C1}(s) - V_{C2}(s) \\ C_2 sV_{C2}(s) = I_2(s) - \dfrac{V_{C2}(s)}{R+R_L} \end{cases}$$

Therefore:

$$I_2(s) = \frac{C_2(R+R_L)+1}{R+R_L} V_{C2}(s)$$

$$V_{C1}(s) = \frac{L_2C_2s(R+R_L)+s+(R+R_L)}{R+R_L} V_{C2}(s)$$

$$I_1(s) = \frac{L_2C_2C_1s^2(R+R_L)+C_1s^2+C_1s(R+R_L)+C_2(R+R_L)+1}{R+R_L} V_{C2}$$

$\frac{V_{C2}(s)}{V_{in}(s)}$ can be obtained by substituting above expressions into the first equation of the state variables of the system.

4-26) a) The charge q is related to the voltage across the plate: $q = C(d)v_C$

The force f_v produced by electric field is:

$$f_v = \frac{q^2}{2\varepsilon A}$$

Since the electric force is opposes the motion of the plates, then the equation of the motion is written as:

$$M\ddot{d} + B\dot{d} + Kd + [sgn(\dot{d})]f_v = f(t)$$

The equations for the electric circuit are:

$$\begin{cases} v = iR + L\dfrac{d}{dt}i + v_C \\ C\dfrac{dv_C}{dt} = i \end{cases}$$

As we know, $i = \frac{d}{dt}q = \dot{q}$ and $q = Cv_C$, then:

$$\begin{cases} v = R\dot{q} + L\ddot{q} + \dfrac{q}{C} \\ C\dfrac{dv_C}{dt} = i \end{cases}$$

Since $C(d) = \frac{\varepsilon A}{d}$, then :

$$\begin{cases} M\ddot{d} + B\dot{d} + Kd + \left[sgn(\dot{d})\right]\dfrac{q^2}{2\varepsilon A} = f(t) \\ v_C = R\dot{q} + L\ddot{q} + \dfrac{dq}{\varepsilon A} \\ C\dfrac{dv_C}{dt} = \dot{q} \end{cases}$$

b) As $q = Cv_C$ then $q^2 = Cqv_C$

If $sgn(\dot{d}) = 1$

$$\begin{cases} M\ddot{d} + B\dot{d} + Kd + \dfrac{Cqv_C}{2\varepsilon A} = f(t) \\ v_C = R\dot{q} + L\ddot{q} + \dfrac{dq}{\varepsilon A} \\ C\dfrac{dv_C}{dt} = \dot{q} \end{cases}$$

Then the transfer function is:

$$\begin{cases} (Ms^2 + Bs + K)D(s) + \dfrac{CQ(s) * V_C(s)}{2\varepsilon A} = F(s) \\ V_C(s) = (Ls^2 + Rs)Q(s) + \dfrac{D(s) * Q(s)}{\varepsilon A} \\ CV_C(s) = Q(s) \end{cases}$$

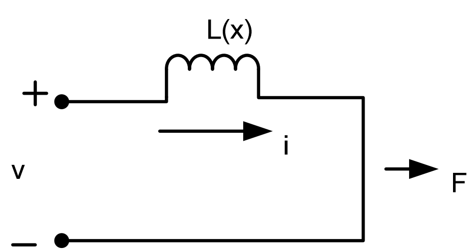
If $sgn(\dot{d}) = -1$

$$\begin{cases} M\ddot{d} + B\dot{d} + Kd - \dfrac{Cqv_C}{2\varepsilon A} = f(t) \\ v_C = R\dot{q} + L\ddot{q} + \dfrac{dq}{\varepsilon A} \\ C\dfrac{dv_C}{dt} = \dot{q} \end{cases}$$

Then the transfer function is:

$$\begin{cases} (Ms^2 + Bs + K)D(s) - \dfrac{CQ(s) * V_C(s)}{2\varepsilon A} = F(s) \\ V_C(s) = (Ls^2 + Rs)Q(s) + \dfrac{D(s) * Q(s)}{\varepsilon A} \\ CV_C(s) = Q(s) \end{cases}$$

4-27) a) The free body diagram is:



where F is required force for holding the core in the equilibrium point against magnetic field

b) The current of inductor, i, and the force, F, are function of flux, Φ, and displacement, x.

Also, we know that

$$i = \frac{\Phi}{L(x)}$$

The total magnetic field is:

$$W(\Phi, x) = \int_0^{\Phi} \frac{\Phi}{L(x)} d\Phi = \frac{\Phi^2}{2L(x)}$$

where W is a function of electrical and mechanical power exerted to the inductor, so:

$$\begin{cases} \mathrm{i} = \dfrac{\partial \mathrm{W}}{\partial \Phi} = \dfrac{\Phi}{\mathrm{L(x)}} \\ \mathrm{F} = \dfrac{\partial \mathrm{W}}{\partial \mathrm{x}} = -\dfrac{\Phi^2}{2\mathrm{L}^2(\mathrm{x})} \dfrac{dL(x)}{dx} \end{cases}$$

As $v = \dot{\Phi}$, then:

$$\begin{cases} v = L(x)\dfrac{di}{dt} + \dfrac{dL(x)}{dt} i\dot{x} \\ F = -\dfrac{1}{2}\dfrac{dL(x)}{dx} i^2 \end{cases}$$

c) Changing the flux requires a sinusoidal movement, and then we can conclude that:

$$x = A \sin \omega t$$

if the inductance is changing relatively, then L(x) = Lx, where L is constant.

Also, the current is changing with the rate of changes in displacement. It means:

$$\mathrm{i} = -\mathrm{B}\dot{\mathrm{x}}$$

So:

$$i = -AB\omega \cos \omega t$$
$$L(x) = LA \sin \omega t$$

Substituting these equations into the state-space equations gives:

$$\Rightarrow v = LA \sin \omega t \, (AB\, \omega^2 \sin \omega t) + L(-AB\omega \cos\omega\, t)(A\omega \cos \omega t)$$

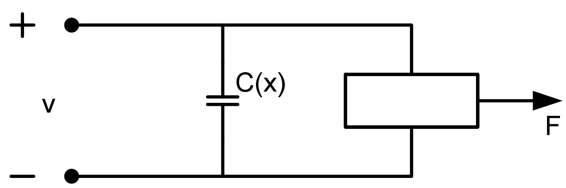
$$= LBA^2\omega^2(\sin^2 \omega t - \cos^2 \omega t)$$

$$F = -\frac{1}{2} L(AB\omega \cos \omega t)^2$$

Therefore:

$$\frac{\mathrm{V(s)}}{\mathrm{F(s)}} = \mathcal{L}\left\{-\frac{2}{\mathrm{B}}[\tan^2 \omega\, t - 1]\right\}$$

4-28) a) The free body diagram is:



where F is the external force required for holding the plate in the equilibrium point against the electrical field.

b) The voltage of capacitors, v, and the force, F, are function of charge, q, and displacement, x.

Also, we know

$$v = \frac{q}{C(x)}$$

The total electrical force between plates is:

$$W(q,x) = \int_0^q \frac{q}{C(x)} dq = \frac{q^2}{2C(x)}$$

Where W is a function of electrical and mechanical power exerted to the capacitor, so:

$$\begin{cases} v = \dfrac{\partial W}{\partial q} = \dfrac{q}{C(x)} \\ F = \dfrac{\partial W}{\partial x} = -\dfrac{q^2}{2C^2(x)}\dfrac{dC(x)}{dx} \end{cases}$$

As $i = \frac{dq}{dt}$, then:

$$\begin{cases} i = C(x)\dfrac{dv}{dt} + \dfrac{dC(x)}{dt} v\dot{x} \\ F = -\dfrac{1}{2}\dfrac{dC(x)}{dx} v^2 \end{cases}$$

c) The same as Problem 4.28,

Consider: $\begin{cases} x = A \sin\omega\, t \\ C(x) = C\, x \\ v = -B\dot{x} \end{cases}$

Then solve the equations.

4-29) According to the circuit:

$$\frac{v_{in} - v_-}{r} = \frac{v_- - v_{out}}{R}$$

or

$$v_- = \frac{R}{R+r} v_{in} + \frac{r}{R+r} v_{out}$$

As an op-amp is modeled with the following equation:

$$v_{out} = \frac{A}{s+1}[v_+ - v_-]$$

Then:

$$\begin{aligned} v_{out} &= \frac{A}{s+1}\left[v_{in} - \frac{R}{R+r} v_{in} - \frac{Rr}{R+r} v_{out}\right] \\ &= \frac{A}{s+1}\left[\frac{r}{R+r} v_{in} - \frac{r}{R+r} v_{out}\right] \\ &= \frac{Ar}{(s+1)(R+r)}(v_{in} - v_{out}) \end{aligned}$$

$$\Longrightarrow \qquad \frac{v_{out}}{v_{in}} = \frac{\frac{Ar}{(s+1)(R+r)}}{1 + \frac{Ar}{(s+1)(R+r)}} = \frac{Ar}{(s+1)(R+r) + Ar}$$

4-30) a) Positive feedback ratio:

$$F_p = \frac{r}{r + R_L}$$

b) Negative feedback ratio:

$$F_N = \frac{R_{in}}{R_{in} + R_f}$$

c) According to the circuit:

$$\begin{cases} \dfrac{v_{in} - v_-}{R_{in}} = \dfrac{v_- - v_{out}}{R_f} \\ \dfrac{v_{out} - v_+}{R_L} = \dfrac{v_+}{r} \end{cases}$$

Therefore:

$$\begin{cases} v_- = \dfrac{R_f}{R_{in}+R_f} v_{in} + \dfrac{R_{in}}{R_{in}+R_f} v_{out} = (1-F_N)v_{in} + F_N v_{out} \\ v_+ = \dfrac{r}{r+R_L} v_{out} = F_p v_{out} \end{cases}$$

As

$$v_{out} = \frac{10^7}{s+1}[v_+ - v_-]$$

then:

$$v_{out} = \frac{10^7}{s+1}\left[F_p v_{out} - (1-F_N)v_{in} - F_N v_{out}\right]$$

which gives:

$$\frac{v_{out}}{v_{in}} = \frac{1-F_N}{10^7\left[s+1-10^7\left(F_p+F_N\right)\right]}$$

It is stable when $1 - 10^7(F_P + F_N) > 0$ which means: $F_p > F_N + 10^{-7}$

4-31) a) If the drop voltage of R_{in} is called v_1

Then:

$$\frac{(v_{in}-v)}{R_{in}} - \frac{v_1}{R} - C\frac{d}{dt}v_1 = 0$$

Also:

$$\frac{v_{in}-v_1}{R_{in}} + \frac{v_{out}}{R_f} = 0$$

Then:

$$v_1 = v_{in} + \frac{R_{in}}{R_f} v_{out}$$

Substituting this expression into the above equation gives:

$$\frac{v_{in}}{R} + C\frac{dv_{in}}{dt} + \frac{1}{R_f}\left(\left(1+\frac{R}{R_{in}}\right)v_{out}\right) + \frac{RC}{R_f}\frac{dv_{out}}{dt} = 0$$

As a result:

$$\left(\frac{1}{R}+Cs\right)v_{in}(s) = -\frac{1}{R_f}\left[\left(1+\frac{R}{R_{in}}\right)+RCs\right]v_{out}(s)$$

Or

$$\frac{v_{out}}{v_{in}} = -\frac{R_f(R_{in}Cs+1)}{RR_{in}Cs+(R_{in}+R)}$$

b) If the dropped voltage across resistor R_f is called v_f, then

$$\begin{cases} \dfrac{v_{in}}{R_{in}}+\dfrac{v_f}{R_f}=0 \\ \dfrac{v_f-v_{out}}{R}+C\dfrac{d}{dt}(v_f-v_{out})-\dfrac{v_{in}}{R_{in}}=0 \end{cases}$$

As a result:

$$v_f = -\frac{R_f}{R_{in}}v_{in}$$

Substituting into the second equation gives:

$$-\frac{R_f}{RR_{in}}v_{in}-\frac{v_{out}}{R}-C\frac{R_f}{R_{in}}\frac{dv_{in}}{dt}-C\frac{d}{dt}v_{out}-\frac{v_{in}}{R_{in}}=0$$

or

$$-\left(\frac{v_{out}}{R}+C\frac{dv_{out}}{dt}\right)=\frac{R_f+R}{RR_{in}}v_{in}+\frac{CR_f}{R_{in}}\frac{dv_{in}}{dt}$$

As a result:

$$\frac{v_{out}}{v_{in}} = -\frac{1}{R_{in}}\frac{RR_fCs+R_f+R}{RCs+1}$$

4-32) The heat flow-in changes with respect to the electric power as:

$$\dot{q}_{in} = K\frac{v^2}{R}$$

where R is the resistor of the heater.

The heat flow-out can be defined as:

$$\dot{q}_{out} = \frac{T_1-T_2}{K_f}$$

where K_f is the heat flow coefficient between actuator and air, T_1 and T_2 are temperature of actuator and ambient.

Since the temperature changes with the differences in heat flows:

$$\frac{dT_1}{dt} = \frac{1}{C}(\dot{q}_{in} - \dot{q}_{out}) = \frac{1}{C}\left(K\frac{v^2}{R} - \frac{1}{K_f}(T_1 - T_2)\right)$$

where C is the thermal capacitor.

The displacement of actuator is changing proportionally with the temperature differences:

$$x = A(T_1 - T_2)$$

If we consider the T_2 is a constant for using inside a room, then

$$T_1 = \frac{x}{A} + T_2$$

Therefore:

$$\frac{dT_1}{dt} = \frac{1}{A}\frac{dx}{dt}$$

$$\frac{1}{A}\left(\frac{dx}{dt} - \frac{x}{K_f}\right) = \frac{K}{CT}v^2$$

By linearizing the right hand side of the equation around point $v = v_o$

$$\frac{1}{A}\left(\frac{dx}{dt} - \frac{x}{R_f}\right) = \frac{Kv_o}{CR}\ (2v_2 - 1)$$

Or

$$\frac{CR}{KAv_o}\left(\frac{dx}{dt} - \frac{x}{K_f}\right) = 2v - 1$$

If we consider the right hand side of the above equation as two inputs to the system as: $u_1(t) = 2v$ and $u_2(t) = 1$ or $u_2(t) = u_s(t)$, then:

$$\left[\frac{X(s)}{V(s)}\right]_{u_2(t)=0} = \frac{2KAv_o}{CR(s-1)}$$

4-33) Due to insulation, there is no heat flow through the walls. The heat flow through the sides is:

$$\begin{cases} q_{1,2} = \dfrac{2\pi K_v H}{\ln\left(\frac{r_2}{r_1}\right)}(T_1 - T_2) & (1) \\ q_{2,a} = \dfrac{2\pi K_i H}{\ln\left(\frac{r_3}{r_2}\right)}(T_2 - T_a) & (2) \end{cases}$$

Where T_1 and T_2 are the temperature at the surface of each cylinder.

As $q_{1,2} = q_{2,a}$, then from equation (1) and (2), we obtain:

$$T_2 = \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\,\pi K_i H} q_{1,2} + T_a \qquad (3)$$

The conduction or convection at:

$$\begin{cases} \textit{the surface of the oil}: \quad q_o = C_h(\pi r_1^2)(T_1 - T_a) & (4) \\ \textit{the face of forgoing}: \quad q_f = C_h A\left(T_f - T_1\right) & (5) \\ \textit{the bottom at the vat}: \quad q_v = \dfrac{K_v}{h}(\pi r_1^2)(T_1 - T_a) & (6) \end{cases}$$

The thermal capacitance dynamics gives:

$$\begin{cases} m_o C_o \dfrac{d}{dt} T_1 = q_f - q_{1,2} - q_v - q_o & (7) \\ mC\dfrac{d}{dt} T_f = -q_f & (8) \end{cases}$$

Where $m_o = \pi r_1^2 H d_0$

According to the equation (7) and (8), T_1 and T_f are state variables.

Substituting equation (3), (4), (5) and (6) into equation (7) and (8) gives the model of the system.

4-34) As heat transfer from power supply to enclosure by radiation and conduction, then:

$$C_p \frac{d}{dt} T_p = q_p - q_r - q_c \qquad (1)$$

$$q_r = \frac{\sigma\left(T_p^4 - T_e^4\right)}{\left[\dfrac{1-\varepsilon_1}{\varepsilon_1 A_p} + \dfrac{1}{A_p F} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_e}\right]} = \frac{\sigma\left(T_p^4 - T_e^4\right)}{R_p + \dfrac{1}{A_p F} + R_e} \qquad (2)$$

$$q_c = \left(\frac{K_1 A_1}{\Delta x}\right)\left(T_p - T_e\right) = \frac{T_p - T_s}{R_E} \qquad (3)$$

Also the enclosure loses heat to the air through its top. So:

$$C_e \frac{d}{dt} T_e = q_r + q_c - q_e - C_t A_t (T_e - T_a) \qquad (4)$$

Where

$$q_e = \left(\frac{K_2 A_2}{\Delta x}\right)(T_e - T_s) = \frac{T_e - T_s}{R_s} \qquad (5)$$

And C_t is the convective heat transfer coefficient and A_t is the surface area of the enclosure.

The changes if the temperature of heat sink is supposed to be zero, then:

$$C \frac{d}{dt} T_{sink} = q_e - q_s = 0$$

Therefore $q_e = q_s$ where $q_s = C_s A_s (T_s - T_a)$, as a result:

$$\frac{T_e - T_s}{R_s} = C_s A_s (T_s - T_a) \qquad (6)$$

According to the equations (1) and (4), T_p and T_e are state variables. The state model of the system is given by substituting equations (2), (3), and (6) into these equations give.

4-35) If the temperature of fluid B and A at the entrance and exit are supposed to be T_{BN} and T_{BX}, and T_{AN} and T_{AX}, respectively. Then:

$$\begin{cases} q_B = \dot{m}_B C_B (T_{BX} - T_{BN}) & (1) \\ q_A = \dot{m}_A C_A (T_{AX} - T_{AN}) & (2) \end{cases}$$

The thermal fluid capacitance gives:

$$\begin{cases} C_B \frac{d}{dT} T_{Bx} = -q_B - q_{B-A} & (3) \\ C_A \frac{d}{dT} T_{Ax} = -q_A + q_{B-A} & (4) \end{cases}$$

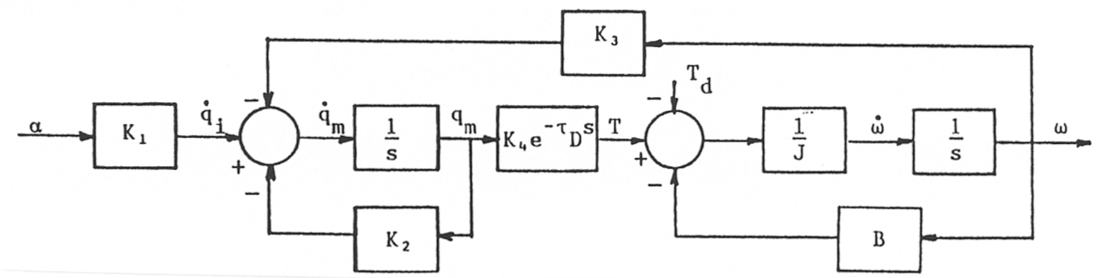
From thermal conductivity:

$$q_{B-A} = \frac{T_{Bx} - T_{Ax}}{\frac{1}{C_i A_i} + \frac{\ln\left(\frac{R_o}{R_i}\right)}{2\,\pi K L} + \frac{1}{C_o A_o}} \qquad (5)$$

Where C_i and C_o are convective heat transfer coefficient of the inner and outer tube; A_i and A_o are the surface of inner and outer tube; R_i and R_o are the radius of inner and outer tube.

Substituting equations (1), (2), and (5) into equations (3) and (4) gives the state model of the system.

4-36) (a) Block diagram:



(b) Transfer function:

$$\frac{\Omega(s)}{\alpha(s)} = \frac{K_1 K_4 e^{-\tau_D s}}{Js^2 + (JK_L + B)s + K_2 B + K_3 K_4 e^{-\tau_D s}}$$

(c) Characteristic equation:

$$Js^2 + (JK_L + B)s + K_2 B + K_3 K_4 e^{-\tau_D s} = 0$$

(d) Transfer function:

$$\frac{\Omega(s)}{\alpha(s)} \cong \frac{K_1 K_4 (2 - \tau_D s)}{\Delta(s)}$$

Characteristic equation:

$$\Delta(s) \cong J\tau_D s^3 + (2J + JK_2\tau_D + B\tau_D)s^2 + (2JK_2 + 2B - \tau_D K_2 B - \tau_D K_3 K_4)s + 2(K_2 B + K_3 K_4) = 0$$

4-37) The total potential energy is:

$$U = \frac{1}{2}\mu A y^2 - \left(-\frac{1}{2}\mu A y^2\right) = \mu A y^2$$

The total kinetic energy is:

$$T = \frac{AL\mu}{2g}\dot{y}^2$$

Therefore:

$$\frac{AL\mu}{2g}\dot{y}^2 = \mu A y^2$$

$$\frac{L}{2g}\dot{y}^2 = y^2$$

As a result:

$$\dot{y} = \sqrt{\frac{2g}{L}}y$$

So, the natural frequency of the system is calculated by: $\omega = \sqrt{\frac{2g}{L}}$

Also, by assuming $y(t) = Y\sin(\omega t + \theta)$ and substituting into $\frac{L}{2y}\dot{y}^2 = y^2$ yields the same result when calculated for maximum displacement.

4-38) If the height of the reservoir, the surge tank and the storage tank are assumed to be H, h_1 and h_2, then potential energy of reservoir and storage tank are:

$$\begin{cases} P_1 = \rho g H \\ P_t = \rho g h_2 \end{cases}$$

For the pipeline we have:

$$Pl\frac{d}{dt}Q = A(P_1 - P_2) + \rho A g(z_1 - z_2) - F_f$$

The surge tank dynamics can be written as:

$$P_s = \rho g h_1$$
$$A_s\frac{d}{dt}h_1 = Q_{2-s}(between\ pipe\ at\ point\ 2\ and\ surge\ tank)$$

At the turbine generator, we have:

$$\left(P_{tg} - P_t\right)Q_{2-v} = I$$

where I is a known input and Q_{2-v} is the fluid flow transfer between point 2 and valve. The behaviour of the valve in this system can be written as:

$$\begin{cases} Q_{2-s} = C_s sgn(P_2 - P_s)(|P_2 - P_s|)^{\frac{1}{\alpha_s}} \\ Q_{2-v} = C_v sgn(P_v - P_{tg})(|P_v - P_{tg}|)^{\frac{1}{\alpha_v}} \end{cases}$$

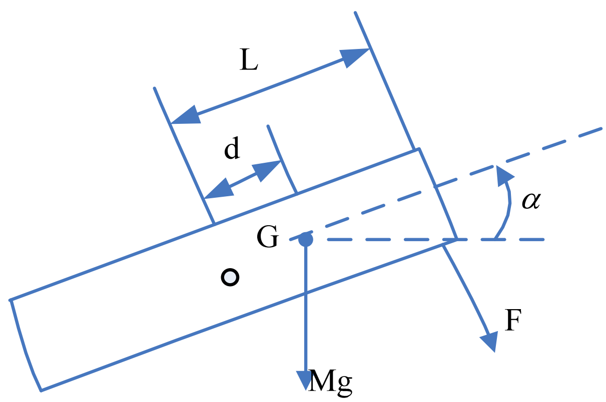
Regarding Newton's Law:

$$\begin{cases} P_2 = P_v \\ Q = Q_{2-v} + Q_{2-s} \end{cases}$$

According to above equations, it is concluded that Q and h_1 are state variables of the system.

The state equations can be rewritten by substituting P_2, P_v, P_s and $Q_{2\text{-}v}$ from other equations.

4-39)



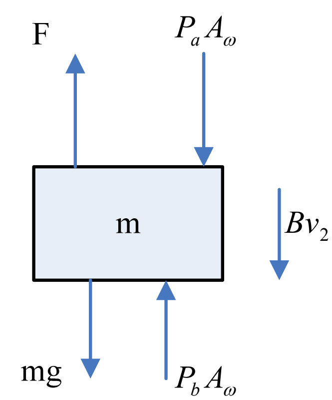
If the beam rotate around small angle of α ($\cos\alpha \cong 1$), then

$$\begin{cases} J\dfrac{d}{dt}\omega = T_{in} - Mgd - FL \\ F = \dfrac{AE(L\alpha - y)}{H - y} \end{cases}$$

where A and E are cross sectional area and elasticity of the cable; H is the distance between point O and the bottom of well, and y is the displacement.

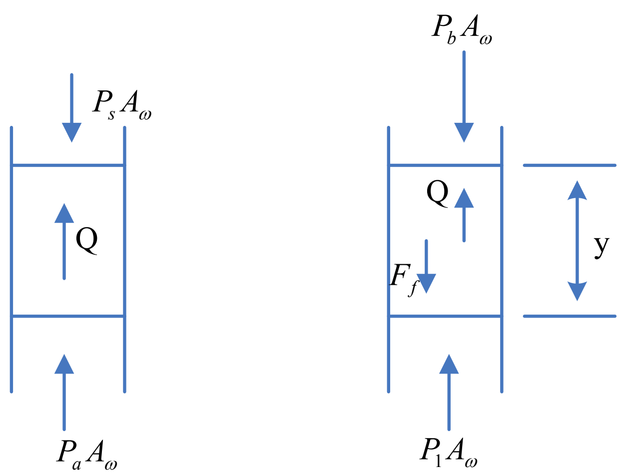
On the other hand, Newton's Law gives:

$$m\frac{d}{dt}v = P_b A_w + F - P_a A_w - Bv_2 - mg$$



where B is the viscous friction coefficient, A_w is the cross sectional area of the well; P_1 and P_2 are pressures above and below the mass m.

The dynamic for the well can be written as two pipes separating by mass m:



a)   b)

$$\begin{cases} \dfrac{dQ}{dt} = \dfrac{A_w}{\rho y}(P_1 - P_b) + \dfrac{A_w g}{y}(0 - y) - \dfrac{F_f}{\rho y} \\ \dfrac{dQ_1}{dt} = \dfrac{A_w}{\rho[H - D - y]}(P_a - P_s) + \dfrac{8A_w g}{y}(0 - y) - \dfrac{F_{f1}}{\rho y} \end{cases}$$

Where D is the distance between point O and ground, P_s is the pressure at the surface and known. If the diameter of the well is assumed to be r, the F_f for the laminar flow is

$$F_f = 32\frac{\mu y Q}{r^2}$$

Therefore:

$$\begin{cases} \dfrac{dQ}{dt} = \dfrac{A_w}{\rho y}(P_1 - P_b) - A_w g - 32\dfrac{\mu Q}{\rho r^2} \\ \dfrac{dQ_1}{dt} = \dfrac{A_w}{\rho[H - D - y]}(P_a - P_s) - A_w g - 32\dfrac{\mu Q_1}{\rho r^2} \end{cases}$$

The state variables of the system are ω, v, y, Q, Q_1.

4-40) For the hydraulic amplifier, we have:

$$\begin{cases} Q = N y_v \\ Q = A\dfrac{dy_p}{dt} \end{cases}$$

As a result

$$\frac{d}{dt}y_p = \frac{N}{A}y_v$$

where N is a constant and A is the cross sectional area.

For the walking beam:

$$y_v = \frac{l_1 y_2 - l_2 y_p}{l_1 + l_2}$$

For the spring: $F = K(y_1 - y_2)$

The angular velocity of the lever is assumed as:

$$\begin{cases} \omega_x = 0 \rightarrow \ \Omega_x = 0 \\ \omega_y = \omega \ \rightarrow \ \Omega_y = \omega \\ \omega_z = \dot{\alpha} \ \rightarrow \ \Omega_z = 0 \end{cases}$$

The moments of inertia of the lever are calculated as:

$$\begin{cases} J_{xx} = mL^2 \sin^2\alpha \\ J_{xy} = m(L\cos\alpha + r)(L\sin\alpha) \\ J_{yy} = m(L\cos\alpha + r)^2 \\ J_{yz} = J_{zx} = 0 \\ J_{zz} = m[(L\cos\alpha + r)^2 + L^2\sin^2\alpha] \end{cases}$$

where L is the length of lever and r is the offset from the center of rotation.

According to the equation of angular motion:

$$\begin{cases} T_x = -\dot{J}_{xy}\omega - J_{xy}\dot{\omega} + J_{zz}\omega_z\omega \\ T_y = \dot{J}_{yy}\omega + J_{yy}\dot{\omega} \\ T_z = \dot{J}_{zz}\omega_z + J_{zz}\dot{\omega}_z + J_{xy}\omega^2 \end{cases}$$

Also:

$$T_z = f_y d + \frac{F}{2}(r - d\sin\alpha) - mg(L\cos\alpha + r\ )$$

Due to force balance, we can write:

$$f_y - \frac{F}{2} - mg = m(L\sin\alpha\ \ \dot{\omega}_2 - L\sin\alpha\ \ \dot{\omega}_2^2)$$

Therefore $\dot{\omega}_2$ can be calculated form above equations.

On the other hand, $-y_2 = r\sin\alpha$, and $\frac{dy_2}{dt} = v_2$ and $\frac{d}{dt}\alpha = \omega_2$, the dynamic of the system is:

$$\begin{cases} J\dfrac{d}{dt}\omega = \dfrac{n_1 Q}{\omega} - B\ \omega - T_Z \\ Q = n_2\dot{y}_p \end{cases}$$

where B is the viscous friction coefficient, and n_1 and n_2 are constant.

The state variables of the systems are α ,y_p, ω and ω_2.

4-41) If the capacitances of the tanks are assumed to be C_1 and C_2 respectively, then

$$\begin{cases} C_1 \dfrac{dh_1}{dt} = (q_{i1} - q1) \\ C_2 \dfrac{dh_2}{dt} = (q_1 + q_{i2} - q_o) \\ q_1 = \dfrac{h_1 - h_2}{R_1} \\ q_o = \dfrac{h_2}{R_2} \end{cases}$$

Therefore:

$$\begin{cases} \dfrac{dh_1}{dt} = \dfrac{1}{C_1}\left(q_{i1} - \dfrac{h_1 - h_2}{R_1}\right) \\ \dfrac{dh_2}{dt} = \dfrac{1}{C_2}\left(\dfrac{h_1 - h_2}{R_1} + q_{i2} - \dfrac{h_2}{R_2}\right) \end{cases}$$

Asa result:

$$\begin{bmatrix} \dfrac{dh_1}{dt} \\ \dfrac{dh_2}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1C_1} & \dfrac{1}{R_1C_1} \\ \dfrac{1}{R_1C_2} & -\dfrac{R_1 + R_2}{R_1R_2C_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{C_1} & 0 \\ 0 & \dfrac{1}{C_2} \end{bmatrix} \begin{bmatrix} q_{i1} \\ q_{i2} \end{bmatrix}$$

4-42) The equation of motion is:

$$M\ddot{x} + B(\dot{x} - \dot{y}) + K(x - y) = 0$$

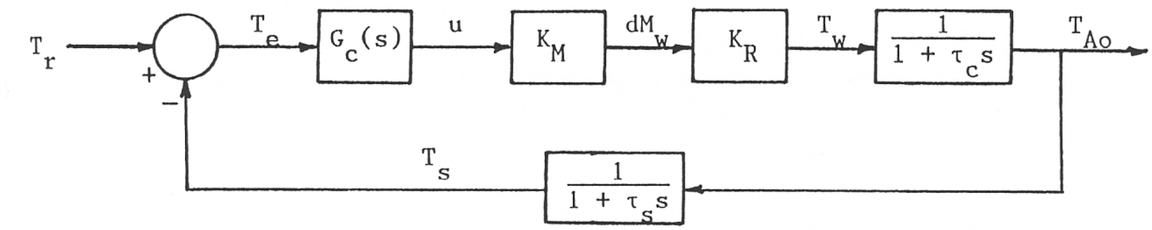
Considering $z = x - y$ gives:

$$M(\dot{z} - \ddot{y}) + B\dot{z} + Kz = 0$$

or

$$\ddot{z} + \frac{B}{M}\dot{z} + \frac{K}{M}z = \ddot{y}$$

4-43) (a) Block diagram:



(b) Transfer function:

$$\frac{T_{AO}(s)}{T_r(s)} = \frac{K_M K_R}{(1+\tau_c s)(1+\tau_s s) + K_m K_R} = \frac{3.51}{20s^2 + 12s + 4.51}$$

4-44)

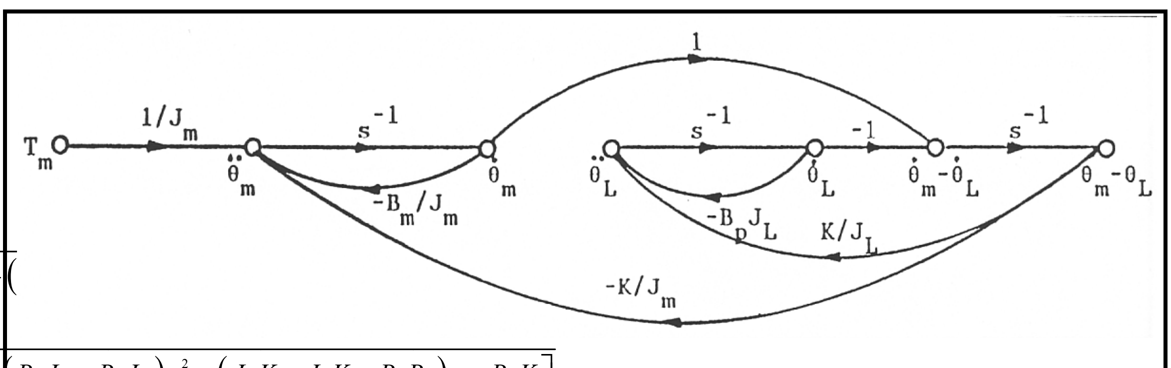
System equations:

$$T_m(t) = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + K(\theta_m - \theta_L) \qquad K(\theta_m - \theta_L) = J_L \frac{d^2\theta_L}{dt^2} + B_p \frac{d\theta_L}{dt}$$

Output equation: $e_o = \dfrac{E\theta_L}{20\pi}$

State diagram:

Transfer function:



$$\frac{\Theta_L(s)}{T_m(s)} = \frac{}{s\left[J_m J_L s^3 + \left(\right.\right.}$$

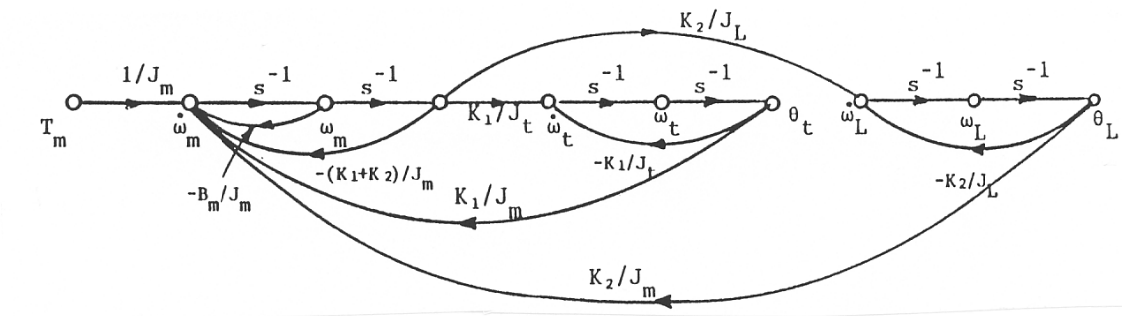
$$\frac{E_o(s)}{T_m(s)} = \frac{}{s\left[J_m J_L s^3 + (B_m J_L + B_p J_m)s^2 + (J_m K + J_L K + B_m B_p)s + B_m K\right]}$$

4-45) (a) State equations:

$$\frac{d\theta_L}{dt} = \omega_L \qquad \frac{d\omega_L}{dt} = \frac{K_2}{J_L}\theta_m - \frac{K_2}{J_L}\theta_L \qquad \frac{d\theta_t}{dt} = \omega_t \qquad \frac{d\omega_t}{dt} = \frac{K_1}{J_t}\theta_m - \frac{K_1}{J_t}\theta_t$$

$$\frac{d\theta_m}{dt} = \omega_m \qquad \frac{d\omega_m}{dt} = -\frac{B_m}{J_m}\omega_m - \frac{(K_1 + K_2)}{J_m}\theta_m + \frac{K_1}{J_m}\theta_t + \frac{K_2}{J_m}\theta_L + \frac{1}{J_m}T_m$$

(b) State diagram:



(c) Transfer functions:

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K_2\left(J_t s^2 + K_1\right)}{\Delta(s)} \qquad \frac{\Theta_t(s)}{T_m(s)} = \frac{K_1\left(J_L s^2 + K_2\right)}{\Delta(s)} \qquad \frac{\Theta_m(s)}{T_m(s)} = \frac{J_t J_L s^4 + \left(K_1 J_L + K_2 J_t\right)s^2 + K_1 K_2}{\Delta(s)}$$

$$\Delta(s) = s[J_m J_L s^5 + B_m J_L J_t s^4 + \left(K_1 J_L J_t + K_2 J_L J_t + K_1 J_m J_L + K_2 J_m J_t\right)s^3$$
$$+ B_m J_L\left(K_1 + K_2\right)s^2 + K_1 K_2\left(J_L + J_t + J_m\right)s + B_m K_1 K_2] = 0$$

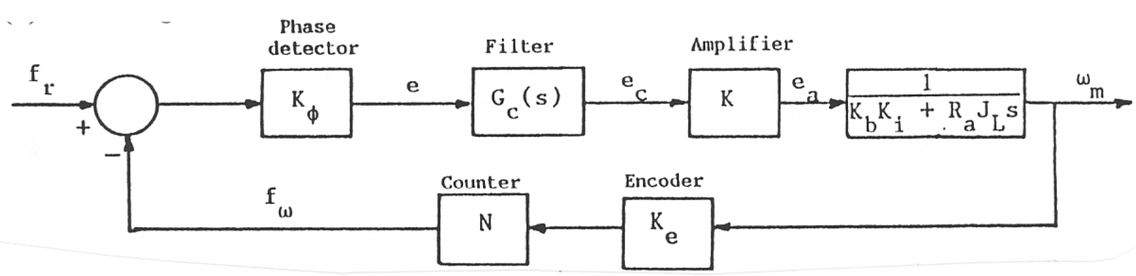
(d) Characteristic equation: $\Delta(s) = 0$.

4-46)

(a) Transfer function:

$$G(s) = \frac{E_c(s)}{E(s)} = \frac{1 + R_2 Cs}{1 + \left(R_1 + R_2\right)Cs}$$

(b) Block diagram:



(c) Forward-path transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{K\left(1+R_2Cs\right)}{\left[1+\left(R_1+R_2\right)Cs\right]\left(K_bK_i+R_aJ_Ls\right)}$$

(d) Closed-loop transfer function:

$$\frac{\Omega_m(s)}{F_r(s)} = \frac{K_\phi K\left(1+R_2Cs\right)}{\left[1+\left(R_1+R_2\right)Cs\right]\left(K_bK_i+R_aJ_Ls\right)+K_\phi KK_eN\left(1+R_2Cs\right)}$$

(e)

$$G_c(s) = \frac{E_c(s)}{E(s)} = \frac{\left(1+R_2Cs\right)}{R_1Cs}$$

Forward-path transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{K\left(1+R_2Cs\right)}{R_1Cs\left(K_bK_i+R_aJ_Ls\right)}$$

Closed-loop transfer function:

$$\frac{\Omega_m(s)}{F_r(s)} = \frac{K_\phi K\left(1+R_2Cs\right)}{R_1Cs\left(K_bK_i+R_aJ_Ls\right)+K_\phi KK_eN\left(1+R_2Cs\right)}$$

$K_e = 36$ pulses / rev $= 36/2\pi$ pulses / rad = 5.73 pulses / rad.

(f) $f_r = 120$ pulses / sec $\omega_{\mathrm{m}} = 200$ RPM $= 200(2\pi/60)$ rad / sec

$f_\omega = NK_e\omega_m = 120$ pulses / sec $= N(36/2\pi)200(2\pi/60) = 120N$ pulses / sec

Thus, *N* = 1. For $\omega_m = 1800$ RPM, $120 = N(36/2\pi)1800(2\pi/60) = 1080N$. Thus, $N = 9$.

4-47) If the incremental encoder provides a pulse at every edge transition in the two signals of channels A and B, then the output frequency is increased to four times of input frequency.

4-48) (a)

$$\left.\frac{\Omega_m(s)}{T_L(s)}\right|_{\omega_r=0} = \frac{\dfrac{-1}{B+Js}\left(1+K_1H_e(s)+\dfrac{K_1H_i(s)}{R_a+L_as}\right)}{\Delta(s)} \cong \frac{\dfrac{-K_1}{B+Js}\left(H_e(s)+\dfrac{H_i(s)}{R_a+L_as}\right)}{\Delta(s)} = 0$$

Thus,

$$H_e(s) = -\frac{H_i(s)}{R_a + L_a s} \qquad \frac{H_i(s)}{H_e(s)} = -\left(R_a + L_a s\right)$$

(b)

$$\left.\frac{\Omega_m(s)}{\Omega_r(s)}\right|_{T_L=0} = \frac{\dfrac{K_1 K_i}{\left(R_a + L_a s\right)\left(B + Js\right)}}{\Delta(s)}$$

$$\Delta(s) = 1 + K_1 H_e(s) + \frac{K_1 K_b}{\left(R_a + L_a s\right)\left(B + Js\right)} + \frac{K_1 H_i(s)}{R_a + L_a s} + \frac{K_1 K_i K_b H_e(s)}{\left(R_a + L_a s\right)\left((B + Js\right)}$$

$$= 1 + \frac{K_1 K_b}{\left(R_a + L_a s\right)\left(B + Js\right)} + \frac{K_1 K_i}{\left(R_a + L_a s\right)\left(B + Js\right)}$$

$$\left.\frac{\Omega_m(s)}{\Omega_r(s)}\right|_{T_L=0} = \frac{K_1 K_i}{\left(R_a + L_a s\right)\left(B + Js\right) + K_i K_b + K_1 K_i K_b H_e(s)} \cong \frac{1}{K_b H_e(s)}$$

4-49) (a) **Cause-and-effect equations:** $\theta_e = \theta_r - \theta_o \qquad e = K_s \theta_e \qquad e_a = Ke$

$$\frac{di_a}{dt} = -\frac{R_a}{L_a} i_a + \frac{1}{L_a}\left(e_a - e_b\right) \qquad T_m = K_i i_a$$

$$\frac{d^2\theta_m}{dt^2} = -\frac{B_m}{J_m}\frac{d\theta_m}{dt} + \frac{1}{J} T_m - \frac{nK_L}{J_m}\left(n\theta_m - \theta_o\right) \qquad T_2 = \frac{T_m}{n} \qquad \theta_2 = n\theta_m$$

$$\frac{d^2\theta_o}{dt^2} = \frac{K_L}{J_L}\left(\theta_2 - \theta_o\right)$$

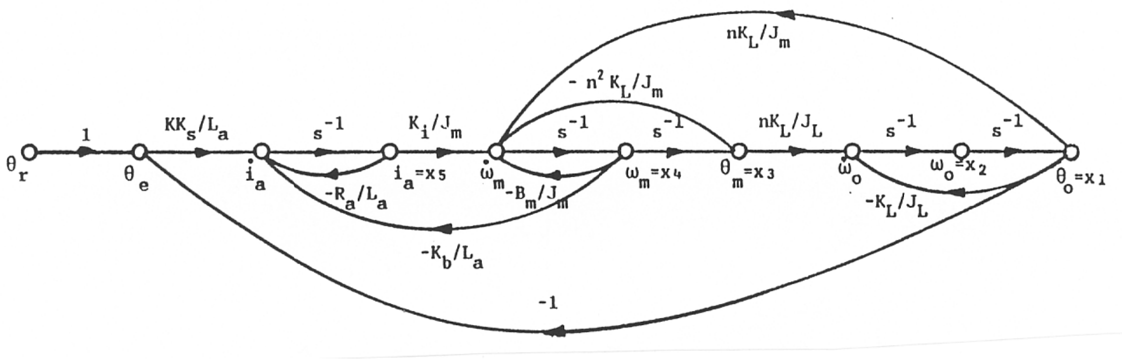
State variables: $x_1 = \theta_o, \quad x_2 = \omega_o, \quad x_3 = \theta_m, \quad x_4 = \omega_m, \quad x_5 = i_a$

State equations:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{K_L}{J_L} x_1 + \frac{nK_L}{J_L} x_3 \qquad \frac{dx_3}{dt} = x_4$$

$$\frac{dx_4}{dt} = -\frac{nK_L}{J_m} x_1 - \frac{n^2 K_L}{J_m} x_3 - \frac{B_m}{J_m} x_4 + \frac{K_i}{J_m} x_5 \qquad \frac{dx_5}{dt} = -\frac{KK_s}{L_a} x_1 - \frac{K_b}{L_a} x_4 - \frac{R_a}{L_a} x_5 + \frac{KK_s}{L_a}\theta_r$$

(b) **State diagram:**



(c) Forward-path transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_sK_inK_L}{s\left[J_mJ_LL_as^4 + J_L\left(R_aJ_m + B_mJ_m + B_mL_a\right)s^3 + \left(n^2K_LL_aJ_L + K_LJ_mL_a + B_mR_aJ_L\right)s^2 + \left(n^2R_aK_LJ_L + R_aK_LJ_m + B_mK_LL_a\right)s + K_iK_bK_L + R_aB_mK_L\right]}$$

Closed-loop transfer function:

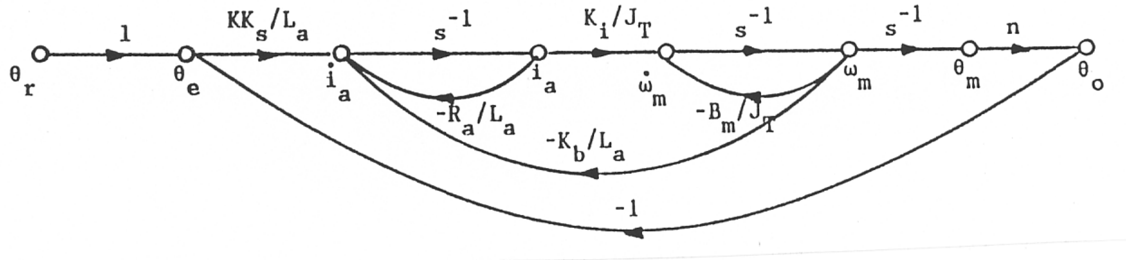
$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_sK_inK_L}{J_mJ_LL_as^5 + J_L\left(R_aJ_m + B_mJ_m + B_mL_a\right)s^4 + \left(n^2K_LL_aJ_L + K_LJ_mL_a + B_mR_aJ_L\right)s^3 + \left(n^2R_aK_LJ_L + R_aK_LJ_m + B_mK_LL_a\right)s^2 + \left(K_iK_bK_L + R_aB_mK_L\right)s + nKK_sK_iK_L}$$

(d) $K_L = \infty, \quad \theta_o = \theta_2 = n\theta_m$. J_L is reflected to motor side so $J_T = J_m + n^2 J_L$.

State equations:

$$\frac{d\omega_m}{dt} = -\frac{B_m}{J_T}\omega_m + \frac{K_i}{J_T}i_a \qquad \frac{d\theta_m}{dt} = \omega_m \qquad \frac{di_a}{dt} = -\frac{R_a}{L_a}i_a + \frac{KK_s}{L_a}\theta_r - \frac{KK_s}{L_a}n\theta_m - \frac{K_b}{L_a}\omega_m$$

State diagram:



Forward-path transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_sK_in}{s\left[J_TL_as^2 + \left(R_aJ_T + B_mL_a\right)s + R_aB_m + K_iK_b\right]}$$

Closed-loop transfer function:

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_sK_in}{J_TL_as^3 + \left(R_aJ_T + B_mL_a\right)s^2 + \left(R_aB_m + K_iK_b\right)s + KK_sK_in}$$

From part (c), when $K_L = \infty$, all the terms without K_L in $\Theta_o(s)/\Theta_e(s)$ and $\Theta_o(s)/\Theta_r(s)$ can be neglected.

The same results as above are obtained.

4-50) (a) **System equations:**

$$f = K_ii_a = M_T\frac{dv}{dt} + B_Tv \qquad e_a = R_ai_a + \left(L_a + L_{as}\right)\frac{di_a}{dt} - L_{as}\frac{di_s}{dt} + e_b \qquad 0 = R_si_s + \left(L_s + L_{as}\right)\frac{di_s}{dt} - L_{as}\frac{di_a}{dt}$$

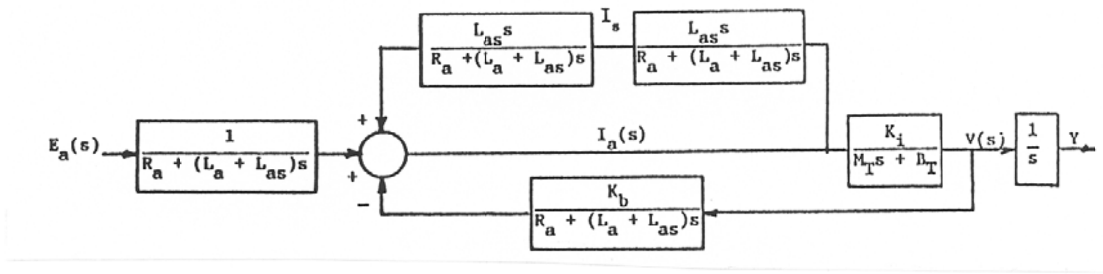
(b) Take the Laplace transform on both sides of the last three equations, with zero initial conditions, we have

$$K_iI_a(s) = \left(M_Ts + B_T\right)V(s) \qquad E_a(s) = \left[R_a + \left(L_a + L_{as}\right)s\right]I_a(s) - L_{as}sI_s(s) + K_bV(s)$$
$$0 = -L_{as}sI_a(s) + \left[R_s + s\left(L_s + L_{as}\right)\right]I_s(s)$$

Rearranging these equations, we get

$$V(s) = \frac{K_i}{M_Ts + B_T}I_a(s) \qquad Y(s) = \frac{V(s)}{s} = \frac{K_i}{s\left(M_Ts + B_T\right)}I_a(s)$$
$$I_a(s) = \frac{1}{R_a + \left(L_a + L_{as}\right)s}\left[E_a(s) + L_{as}sI_s(s) - K_bV(s)\right] \qquad I_s(s) = \frac{L_{as}s}{R_a + \left(L_a + L_{as}\right)s}I_a(s)$$

Block diagram:



(c) **Transfer function:**

$$\frac{Y(s)}{E_a(s)} = \frac{K_i\left[R_s + \left(L_s + L_{as}\right)s\right]}{s\left[R_a + \left(L_a + L_{as}\right)s\right]\left[R_s + \left(L_s + L_{as}\right)s\right]\left(M_T s + B_T\right) + K_i K_b\left[R_s + \left(L_a + L_{as}\right)s\right] - L_{as}^2 s^2\left(M_T s + B_T\right)}$$

4-51) (a) **Cause-and-effect equations:**

$$\theta_e = \theta_r - \theta_L \qquad e = K_s\theta_e \qquad K_s = 1 \text{ V/rad} \qquad e_a = Ke \qquad i_a = \frac{e_a - e_b}{R_a}$$

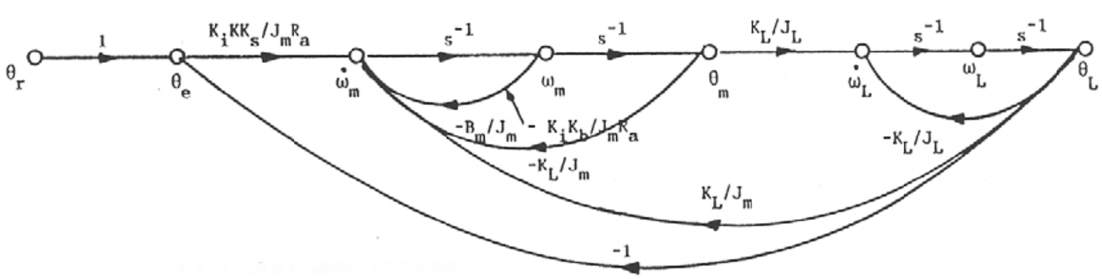
$$T_m = K_i i_a \qquad \frac{d\omega_m}{dt} = \frac{1}{J_m}T_m - \frac{B_m}{J_m}\omega - \frac{K_L}{J_m}\left(\theta_m - \theta_L\right) \qquad \frac{d\omega_L}{dt} = \frac{K_L}{J_L}\left(\theta_m - \theta_L\right) \qquad e_b = K_b\omega_m$$

$$K_b = 15.5 \text{ V / KRPM } = \frac{15.5}{1000 \times 2\pi / 60} = 0.148 \text{ V / rad / sec}$$

State equations:

$$\frac{d\theta_L}{dt} = \omega_L \qquad \frac{d\omega_L}{dt} = \frac{K_L}{J_L}\theta_m - \frac{K_L}{J_L}\theta_L \qquad \frac{d\theta_m}{dt} = \omega_m \qquad \frac{d\omega_m}{dt} = -\frac{B_m}{J_m}\omega_m - \frac{K_L}{J_m}\theta_L + \frac{1}{J_m}\frac{K_i}{R_a}\left(KK_s\theta_e - K_b\omega_m\right)$$

(b) **State diagram:**



(c) **Forward-path transfer function:**

$$G(s) = \frac{K_i K K_s K_L}{s\left[J_m J_L R_a s^3 + \left(B_m R_a + K_i K_b\right)J_L s^2 + R_a K_L\left(J_L + J_m\right)s + K_L\left(B_m R_a + K_i K_b\right)\right]}$$

$J_m R_a J_L = 0.03 \times 1.15 \times 0.05 = 0.001725 \qquad B_m R_a J_L = 10 \times 1.15 \times 0.05 = 0.575 \qquad K_i K_b J_L = 21 \times 0.148 \times 0.05 = 0.1554$

$R_a K_L J_L = 1.15 \times 50000 \times 0.05 = 2875 \qquad R_a K_L J_m = 1.15 \times 50000 \times 0.03 = 1725 \qquad K_i K K_s K_L = 21 \times 1 \times 50000 K = 1050000 K$

$K_L\left(B_m R_a + K_i K_b\right) = 50000(10 \times 1.15 + 21 \times 0.148) = 730400$

$$G(s) = \frac{608.7 \times 10^6 K}{s\left(s^3 + 423.42 s^2 + 2.6667 \times 10^6 s + 4.2342 \times 10^8\right)}$$

(d) Closed-loop transfer function:

$$M(s) = \frac{\Theta_L(s)}{\Theta_r(s)} = \frac{G(s)}{1+G(s)} = \frac{K_i K K_s K_L}{J_m J_L R_a s^4 + \left(B_m R_a + K_i K_b\right) J_L s^3 + R_a K_L \left(J_L + J_m\right) s^2 + K_L\left(B_m R_a + K_i K_b\right) s + K_i K K_s K_L}$$

$$M(s) = \frac{6.087 \times 10^8 K}{s^4 + 423.42 s^3 + 2.6667 \times 10^6 s^2 + 4.2342 \times 10^8 s + 6.087 \times 10^8 K}$$

Characteristic equation roots:

$K = 1$	$K = 2738$	$K = 5476$
$s = -1.45$	$s = \pm j1000$	$s = 405 \pm j1223.4$
$s = -159.88$	$s = -211.7 \pm j1273.5$	$s = -617.22 \pm j1275$
$s = -131.05 \pm j1614.6$		

4-52) (a) Nonlinear differential equations:

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -k(v) - g(x) + f(t) = -Bv(t) + f(t)$$

With $R_a = 0$, $\phi(t) = \dfrac{e(t)}{K_b v(t)} = K_f i_f(t) = K_f i_f(t) = K_f i_a(t)$ Then, $i_a(t) = \dfrac{e(t0}{K_b K_f v(t)}$

$$f(t) = K_i \phi(t) i_a(t) = \frac{K_i e^2(t)}{K_b^2 K_f v^2(t)}. \qquad \text{Thus,} \quad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$$

(b) State equations: $i_a(t)$ as input.

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -Bv(t) + K_i K_f i_a^2(t)$$

(c) State equations: $\phi(t)$ as input.

$$f(t) = K_i K_f i_a^2(t) \qquad i_a(t) = i_f(t) = \frac{\phi(t)}{K_f}$$

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_f}\phi^2(t)$$

4-53) (a) Differential equations:

$$K_i i_a = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + K\left(\theta_m - \theta_L\right) + B\left(\frac{d\theta_m}{dt} - \frac{d\theta_L}{dt}\right)$$

$$K\left(\theta_m - \theta_L\right) + B\left(\frac{d\theta_m}{dt} - \frac{d\theta_L}{dt}\right) = \left(J_L \frac{d^2\theta_L}{dt^2} + B_L \frac{d\theta_L}{dt}\right) + T_L$$

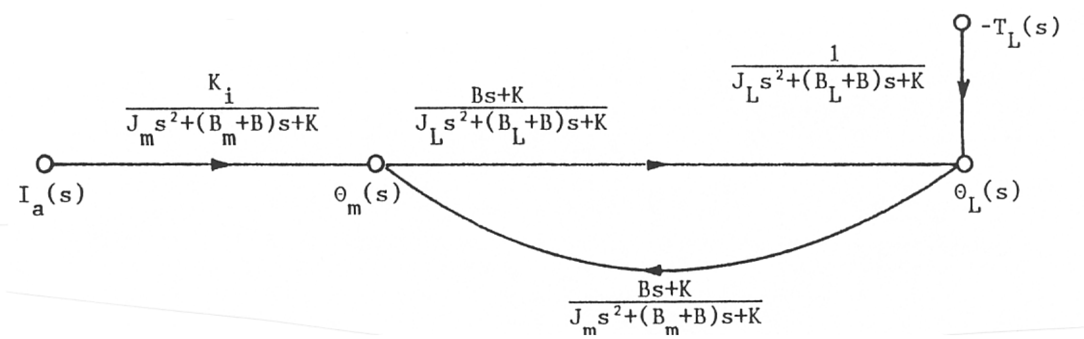
(b) Take the Laplace transform of the differential equations with zero initial conditions, we get

$$K_i I_a(s) = \left(J_m s^2 + B_m s + Bs + K\right)\Theta_m(s) + \left(Bs + K\right)\Theta_L(s)$$
$$\left(Bs + K\right)\Theta_m(s) - \left(Bs + K\right)\Theta_L(s) = \left(J_L s^2 + B_L\right)s\Theta_L(s) + T_L(s)$$

Solving for $\Theta_m(s)$ and $\Theta_L(s)$ from the last two equations, we have

$$\Theta_m(s) = \frac{K_i}{J_m s^2 + \left(B_m + B\right)s + K} I_a(s) + \frac{Bs + K}{J_m s^2 + \left(B_m + B\right)s + K}\Theta_L(s)$$
$$\Theta_L(s) = \frac{Bs + K}{J_L s^2 + \left(B_L + B\right)s + K}\Theta_m(s) - \frac{T_L(s)}{J_L s^2 + \left(B_L + B\right)s + K}$$

Signal flow graph:



(c) Transfer matrix:

$$\begin{bmatrix}\Theta_m(s)\\ \Theta_L(s)\end{bmatrix}=\frac{1}{\Delta_o(s)}\begin{bmatrix}K_i\left[J_Ls^2+(B_L+B)s+K\right] & Bs+K\\ K_i(Bs+K) & J_ms^2+(B_m+B)s+K\end{bmatrix}\begin{bmatrix}I_a(s)\\ -T_L(s)\end{bmatrix}$$

$$\Delta_o(s)=J_LJ_ms^3+\left[J_L(B_m+B)+J_m(B_L+B)\right]s^3+\left[B_LB_m+(B_L+B_M)B+(J_m+J_L)K\right]s^2+K(B_L+B)s$$

4-54) As e^{-T_ds} can be estimated by:

$$e^{-T_ds}\cong\frac{1-\left(\frac{T_ds}{2}\right)}{1+\left(\frac{T_ds}{2}\right)}\cong\frac{2-T_ds}{2+T_ds}$$

Therefore:

$$G(s)=\frac{K(2-T_ds)}{(2+T_ds)(\tau_1s+1)(\tau_2s+1)}$$

As a result:

Poles: $-\frac{1}{\tau_1},-\frac{1}{\tau_2},-\frac{2}{T_d}$

zeros: $\frac{2}{T_d}$

4-55) By approximatinge^{-sT}:

$$e^{-Ts}=\frac{1-\frac{Ts}{2}}{1+\frac{Ts}{2}}$$

a)

$$G(s)=\frac{1-\frac{Ls}{2}}{(Ts+1)\left(1+\frac{Ls}{2}\right)}$$

Therefore:

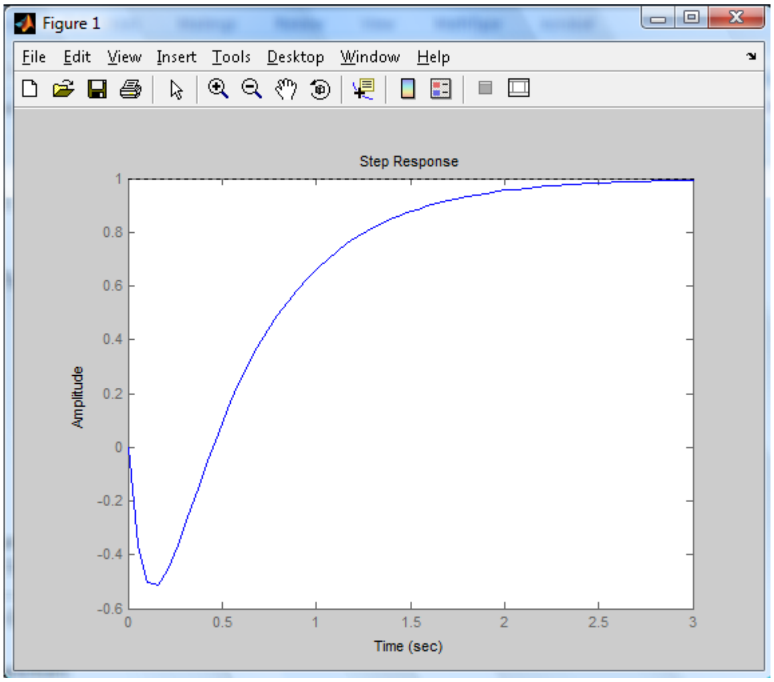
$$G(j\omega)=\frac{1-\frac{jL\omega}{2}}{(jT\omega+1)\left(1+\frac{j\omega L}{2}\right)}$$

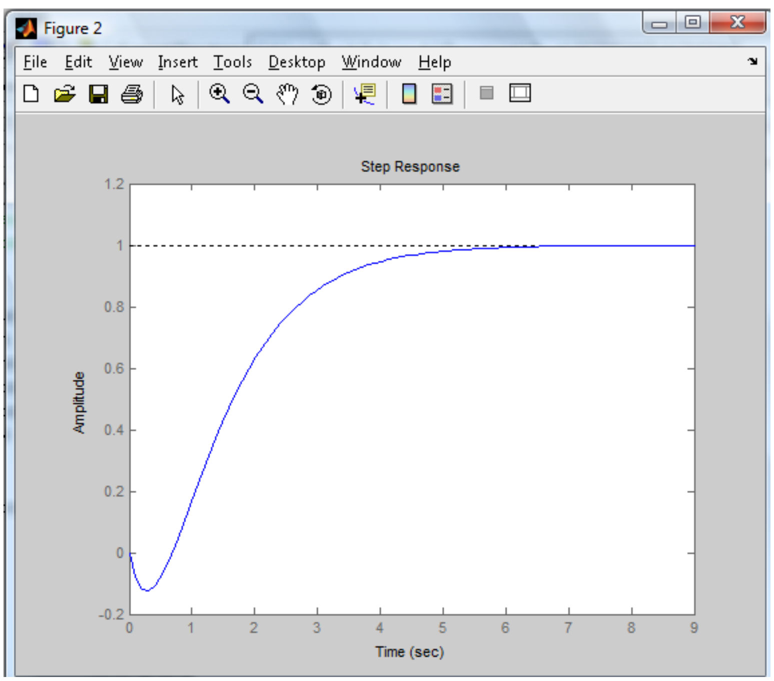
b)

$$G(s) = \frac{2 + 2s\dfrac{1-\frac{s}{2}}{1+\frac{s}{2}} + 4\dfrac{1-s}{1+s}}{s^2 + 3s + 2}$$
$$= \frac{2(2+s)(1+s) + s(2-s)(1+s) + 4(1-s)(2+s)}{(s+2)^2(s+1)^2}$$
$$= \frac{-s^3 - s^2 + 4s + 4}{(s+2)^2(s+1)^2}$$

4-56) MATLAB

```
clear all
L=1
T=0.1
G1=tf([-1/2 1],conv([0.1 1],[1/2 1]))
figure(1)
step(G1)
G2=tf([-1 -1 4 4], conv (conv ([1 2],[1 2]),conv([1 1],[1 1])))
figure(2)
step(G2)

L =
    1

T =
   0.1000

Transfer function:
    -0.5 s + 1
--------------------
0.05 s^2 + 0.6 s + 1

Transfer function:
    -s^3 - s^2 + 4 s + 4
-------------------------------
s^4 + 6 s^3 + 13 s^2 + 12 s + 4
```





4-57) 4-23 (a) Differential equations: $\left[L(y) = \frac{L}{y} \right]$

$$e(t) = Ri(t) + \frac{d\left[L(y)i(t)\right]}{dt} = Ri(t) + i(t)\frac{dL(y)}{dy}\frac{dy(t)}{dt} + \frac{L}{y}\frac{di(t)}{dt} = Ri(t) - \frac{L}{y^2}i(t)\frac{dy(t)}{dt} + \frac{L}{y}\frac{di(t)}{dt}$$

$$My(t) = Mg - \frac{Ki^2(t)}{y^2(t)} \qquad \text{At equilibrium, } \frac{di(t)}{dt} = 0, \quad \frac{dy(t)}{dt} = 0, \quad \frac{d^2y(t)}{dt^2} = 0$$

Thus, $i_{eq} = \dfrac{E_{eq}}{R}$ $\qquad \dfrac{dy_{eq}}{dt} = 0$ $\qquad y_{eq} = \dfrac{E_{eq}}{R}\sqrt{\dfrac{K}{Mg}}$

(b) Define the state variables as $x_1 = i, x_2 = y,$ and $x_3 = \dfrac{dy}{dt}.$

Then, $x_{1eq} = \dfrac{E_{eq}}{R}$ $\qquad x_{2eq} = \dfrac{E_{eq}}{R}\sqrt{\dfrac{K}{Mg}}$ $\qquad x_{3eq} = 0$

The differential equations are written in state equation form:

$$\frac{dx_1}{dt} = -\frac{R}{L}x_1x_2 + \frac{x_1x_3}{x_2} + \frac{x_2}{L}e = f_1 \qquad \frac{dx_2}{dt} = x_3 = f_2 \qquad \frac{dx_3}{dt} = g - \frac{K}{M}\frac{x_1^2}{x_2^2} = f_3$$

(c) Linearization:

$$\frac{\partial f_1}{\partial x_1} = -\frac{R}{L}x_{2eq} + \frac{x_{3eq}}{x_{2eq}} = -\frac{E_{eq}}{L}\sqrt{\frac{K}{Mg}} \qquad \frac{\partial f_1}{\partial x_2} = -\frac{R}{L}x_{1eq} - \frac{x_1x_3}{x_2^2} + \frac{E_{eq}}{L} = 0 \qquad \frac{\partial f_1}{\partial x_3} = \frac{x_{1eq}}{x_{2eq}} = \sqrt{\frac{Mg}{K}}$$

$$\frac{\partial f_1}{\partial e} = \frac{x_{2eq}}{L} = \frac{1}{L}\sqrt{\frac{K}{Mg}}\frac{E_{eq}}{R} \qquad \frac{\partial f_2}{\partial x_1} = 0 \qquad \frac{\partial f_2}{\partial x_2} = 0 \qquad \frac{\partial f_2}{\partial x_3} = 1 \qquad \frac{\partial f_2}{\partial e} = 0$$

$$\frac{\partial f_3}{\partial x_1} = -\frac{2K}{M}\frac{x_{1eq}}{x_{2eq}^2} = -\frac{2Rg}{E_{eq}} \qquad \frac{\partial f_3}{\partial x_2} = \frac{2K}{M}\frac{x_{1eq}^2}{x_{2eq}^3} = \frac{2Rg}{E_{eq}}\sqrt{\frac{Mg}{K}} \qquad \frac{\partial f_3}{\partial e} = 0$$

The linearized state equations about the equilibrium point are written as: $\Delta\dot{\mathbf{x}} = \mathbf{A}^*\Delta\mathbf{x} + \mathbf{B}^*\Delta e$

$$\mathbf{A}^* = \begin{bmatrix} -\dfrac{E_{eq}}{L}\sqrt{\dfrac{K}{Mg}} & 0 & \sqrt{\dfrac{Mg}{K}} \\ 0 & 0 & 0 \\ -\dfrac{2Rg}{E_{eq}} & \dfrac{2Rg}{E_{eq}}\sqrt{\dfrac{Mg}{K}} & 0 \end{bmatrix} \qquad \mathbf{B}^* = \begin{bmatrix} \dfrac{E_{eq}}{RL}\sqrt{\dfrac{K}{Mg}} \\ 0 \\ 0 \end{bmatrix}$$

4-58)

(a) Differential equations:

$$M_1\frac{d^2y_1(t)}{dt^2} = M_1g - B\frac{dy_1(t)}{dt} - \frac{Ki^2(t)}{y_1^2(t)} + Ki^2(t)\frac{1}{\left[y_2(t) - y_1(t)\right]^2}$$

$$M_2\frac{d^2y_2(t)}{dt^2} = M_2g^2 - B\frac{dy_2(t)}{dt} - \frac{Ki^2(t)}{\left[y_2(t)-y_1(t)\right]^2}$$

Define the state variables as $x_1 = y_1, \quad x_2 = \frac{dy_1}{dt}, \quad x_3 = y_2, \quad x_4 = \frac{dy_2}{dt}.$

The state equations are:

$$\frac{dx_1}{dt} = x_2 \qquad M_1\frac{dx_2}{dt} = M_1g - Bx_2 - \frac{Ki^2}{x_1^2} + \frac{Ki^2}{\left(x_3-x_1\right)^2} \qquad \frac{dx_3}{dt} = x_4 \qquad M_2\frac{dx_4}{dt} = M_2g - Bx_4 - \frac{Ki^2}{\left(x_3-x_1\right)^2}$$

At equilibrium, $\frac{dx_1}{dt} = 0, \ \frac{dx_2}{dt} = 0, \ \frac{dx_3}{dt} = 0, \ \frac{dx_4}{dt} = 0.$ Thus, $x_{2eq} = 0$ and $x_{4eq} = 0.$

$$M_1g - \frac{KI^2}{X_1^2} + \frac{KI^2}{\left(X_3-X_1\right)^2} = 0 \qquad M_2g - \frac{KI^2}{\left(X_3-X_1\right)^2} = 0$$

Solving for *I*, with $X_1 = 1$, we have

$$Y_2 = X_3 = 1 + \left(\frac{M_1+M_2}{M_2}\right)^{1/2} \qquad I = \left(\frac{\left(M_1+M_2\right)g}{K}\right)^{1/2}$$

(b) Nonlinear state equations:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = g - \frac{B}{M_1}x_2 - \frac{K}{M_1x_1^2}i^2 + \frac{Ki^2}{M_1\left(x_3-x_1\right)^2} \qquad \frac{dx_3}{dt} = x_4 \qquad \frac{dx_4}{dt} = g - \frac{B}{M_2}x_4 - \frac{Ki^2}{M_2\left(x_3-x_1\right)^2}$$

(c) Linearization:

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = 0 \qquad \frac{\partial f_1}{\partial x_3} = 0 \qquad \frac{\partial f_1}{\partial x_4} = 0 \qquad \frac{\partial f_1}{\partial i} = 0$$

$$\frac{\partial f_2}{\partial x_1} = \frac{2KI^2}{M_1x_1^3} + \frac{2KI^2}{M_1(X_3-X_1)^3} \qquad \frac{\partial f_2}{\partial x_2} = -\frac{B}{M_1} \qquad \frac{\partial f_2}{\partial x_3} = \frac{-2KI^2}{M_1\left(X_3-X_1\right)^3} \qquad \frac{\partial f_2}{\partial x_4} = 0$$

$$\frac{\partial f_2}{\partial i} = \frac{2KI}{M_1}\left(\frac{-1}{X_1^2} + \frac{1}{\left(X_3-X_1\right)^2}\right) \qquad \frac{\partial f_3}{\partial x_1} = 0 \qquad \frac{\partial f_3}{\partial x_2} = 0 \qquad \frac{\partial f_3}{\partial x_3} = 0 \qquad \frac{\partial f_3}{\partial x_4} = 1 \qquad \frac{\partial f_3}{\partial i} = 0$$

$$\frac{\partial f_4}{\partial x_1} = \frac{-2KI^2}{M_2\left(X_3-X_1\right)^3} \qquad \frac{\partial f_4}{\partial x_2} = 0 \qquad \frac{\partial f_4}{\partial x_3} = \frac{2KI^2}{M_2\left(X_3-X_1\right)^3} \qquad \frac{\partial f_4}{\partial x_4} = -\frac{B}{M_2} \qquad \frac{\partial f_4}{\partial i} = \frac{-2KI}{M_2\left(X_3-X_1\right)^2}$$

Linearized state equations: $M_1 = 2, \ M_2 = 1, \ g = 32.2, \ B = 0.1, \ K = 1.$

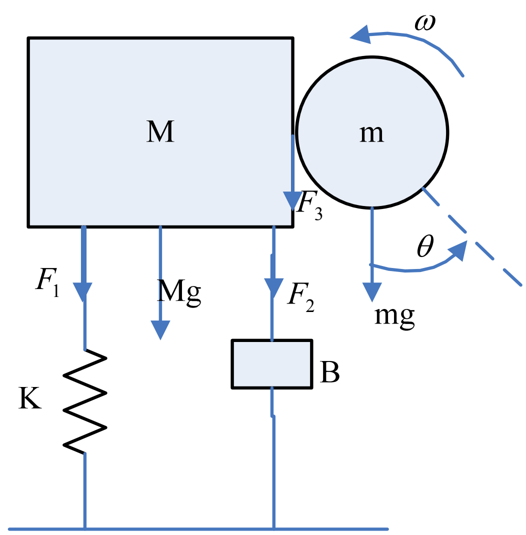
$$I = \left(\frac{32.2(1+2)}{1}\right)^{1/2} X_1 = \sqrt{96.6}X_1 = 9.8285 X_1 \qquad X_1 = \frac{1}{9.8285} = 1$$

$$X_3 = \left(1+\sqrt{1+2}\right)X_1 = 2.732 X_1 = Y_2 = 2.732 \qquad X_3 - X_1 = 1.732$$

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2KI^2}{M_1}\left(\frac{1}{X_1^3} + \frac{1}{(X_3 - X_1)^3}\right) & \frac{-B}{M_1} & \frac{-2KI^2}{M_1(X_3-X_1)^3} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2KI^2}{M_2(X_3-X_1)^3} & 0 & \frac{2KI^2}{M_2(X_3-X_1)^3} & \frac{-B}{M_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 115.2 & -0.05 & -18.59 & 0 \\ 0 & 0 & 0 & 1 \\ -37.18 & 0 & 37.18 & -0.1 \end{bmatrix}$$

$$\mathbf{B}^* = \begin{bmatrix} 0 \\ \frac{2KI}{M_1}\left(\frac{-1}{X_1^2} + \frac{1}{(X_3 - X_1)^2}\right) \\ 0 \\ \frac{-2KI}{M_2(X_3-X_1)^2} \end{bmatrix} = \begin{bmatrix} 0 \\ -6.552 \\ 0 \\ -6.552 \end{bmatrix}$$

4-59) a)



b) The equation of the translational motion is:

$$\begin{cases} \dfrac{Mdv}{dt} = Mg - F_1 + F_2 - F_3 \quad (1) \\ F_1 = Ky \\ \dfrac{dy}{dt} = v \\ F_2 = -Bv \end{cases}$$

The equation of rotational motion is:

$$\begin{cases} J\dfrac{d\omega}{dt} = F_3 r \\ \dfrac{d\theta}{dt} = \omega \end{cases}$$

where $J = \frac{1}{2}mr^2$

Also, the relation between rotational and translational motion defines:

$$\begin{cases} v = r\,\omega \\ y = r\,\theta \end{cases}$$

Therefore, substituting above expression into the first equation gives:

$$F_3 = \left(\frac{m}{2M + m}\right)(Mg - Ky - Bv)$$

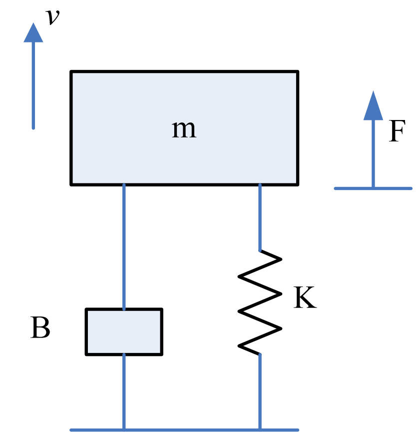
The resulted state space equations are:

$$\begin{cases} \dfrac{d}{dt}\omega = \left(\dfrac{2}{r}\right)\left(\dfrac{Mg - Kr\theta - Br\omega}{2\mu + m}\right) \\ \dfrac{d}{dt}\theta = \omega \end{cases}$$

c) According to generalized elements:

1) Viscous friction can be replaced by a resistor where R = B
2) Spring can be replaced by a capacitor where $C = \frac{1}{k}$
3) Mass M and m can be replaced by two inductors where $L_1 = M$ and $L_2 = m$. Then the angular velocity is measured as a voltage of the inductor L_2
4) The gear will be replaced by a transformer with the ratio of $N = \frac{1}{r}$
5) The term Mg is also replaced by an input voltage of $V_e = Mg$

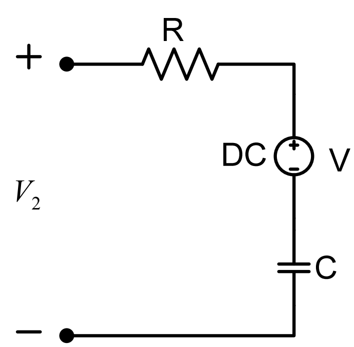
4-60) As the base is not moving then the model can be reduced to:



Therefore:

1) As $m\frac{dv}{dt} = F$, they can be replaced by a inductor with L = m
2) Friction B can be replaced by a resistor where R = B
3) Spring can be replaced by a capacitor where $C = \frac{1}{k}$
4) The force F is replaced by a current source where $I_s = F$

4-61)



$$R = f_R(Q) = C(|P - P_2|)^{\left(\frac{1}{\alpha}\right)}$$

$$C = \frac{A}{\rho g}$$

$$V = \rho g h$$

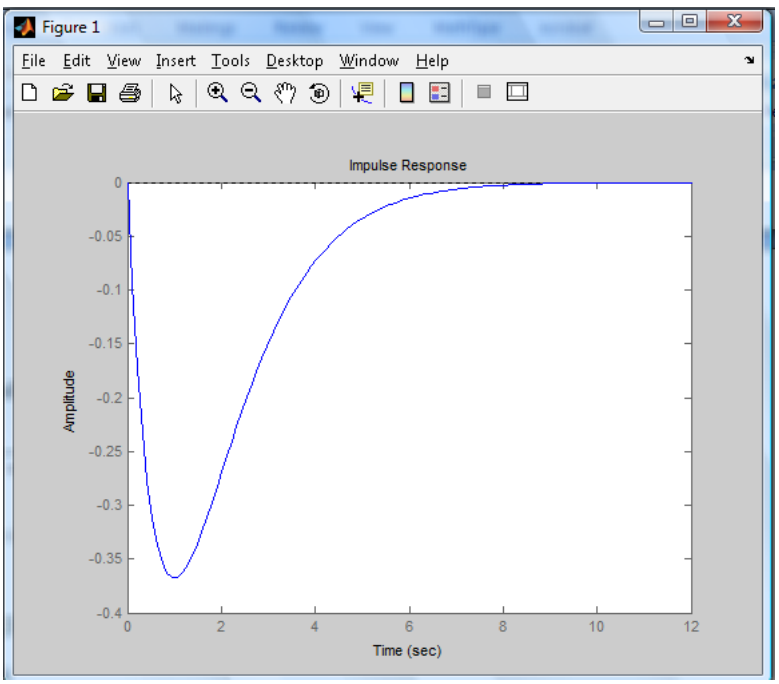
4-62) Recall Eq. (4-324)

$$\frac{Z(s)}{\ddot{Y}(s)} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

Set $\ddot{Y}(s) = impulse$ **, pick** $\begin{matrix}\omega_n = 1, \\ \varsigma = 1\end{matrix}$ **for simplicity.**

```
clear all
G=tf([-1],[1 2 1])
figure(1)
impulse(G)
```

```
Transfer function:
     -1
-------------
s^2 + 2 s + 1
```



4-63)

Use Eq. (4-329).

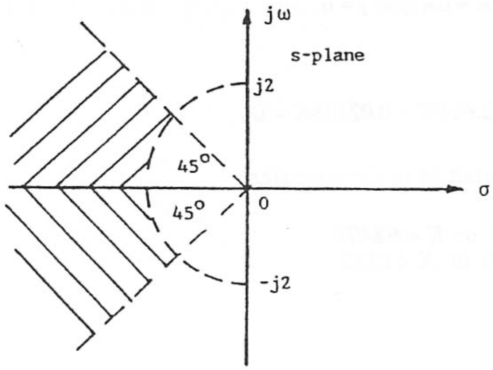
$$Z(s) = \frac{\frac{K_m r}{R_a}}{\left(\frac{L_a}{R_a}s+1\right)(Js^2+Bs+K)+\frac{K_m K_b}{R_a}s}V_a(s)$$

$$-\frac{\left(\frac{L_a}{R_a}s+1\right)r}{\left(\frac{L_a}{R_a}s+1\right)(Js^2+Bs+K)+\frac{K_m K_b}{R_a}s}mr\ddot{Y}(s)$$

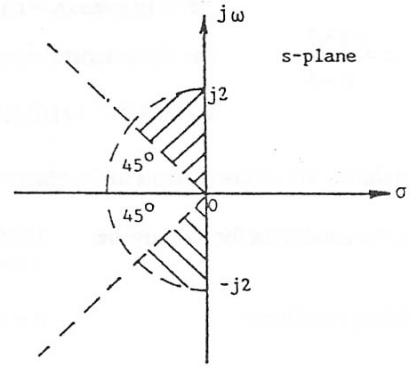
For $\frac{L_a}{R_a}$=0 (very small) the format of the equation is similar to Eq. (4-324), and we expect the same response for the disturbance input. Except, $Z(s) = \frac{\frac{K_m r}{R_a}}{(Js^2+Bs+K)+\frac{K_m K_b}{R_a}s}V_a(s)$ can be used to reduce the effects of disturbance. See Chapter 6.

Chapter 5

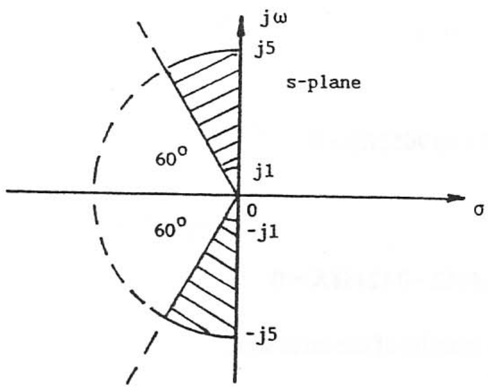
5-1 (a) $\zeta \geq 0.707 \quad \omega_n \geq 2 \text{ rad/sec}$



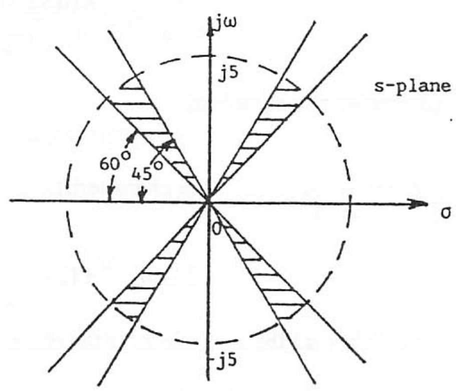
(b) $0 \leq \zeta \leq 0.707 \quad \omega_n \leq 2 \text{ rad/sec}$



(c) $\zeta \leq 0.5 \quad 1 \leq \omega_n \leq 5 \text{ rad/sec}$



(d) $0.5 \leq \zeta \leq 0.707 \quad \omega_n \leq 0.5 \text{ rad/sec}$



5-2 (a) Type 0 **(b)** Type 0 **(c)** Type 1 **(d)** Type 2 **(e)** Type 3 **(f)** Type 3

(g) type 2 **(h)** type 1

5-3 (a) $K_p = \lim_{s \to 0} G(s) = 1000 \qquad K_v = \lim_{s \to 0} sG(s) = 0 \qquad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(b) $K_p = \lim_{s \to 0} G(s) = \infty \qquad K_v = \lim_{s \to 0} sG(s) = 1 \qquad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(c) $K_p = \lim_{s \to 0} G(s) = \infty$ $\quad K_v = \lim_{s \to 0} sG(s) = K$ $\quad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(d) $K_p = \lim_{s \to 0} G(s) = \infty$ $\quad K_v = \lim_{s \to 0} sG(s) = \infty$ $\quad K_a = \lim_{s \to 0} s^2 G(s) = 1$

(e) $K_p = \lim_{s \to 0} G(s) = \infty$ $\quad K_v = \lim_{s \to 0} sG(s) = 1$ $\quad K_a = \lim_{s \to 0} s^2 G(s) = 0$

(f) $K_p = \lim_{s \to 0} G(s) = \infty$ $\quad K_v = \lim_{s \to 0} sG(s) = \infty$ $\quad K_a = \lim_{s \to 0} s^2 G(s) = K$

5-4 (a)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = 1000$	$1/1001$
$tu_s(t)$	$K_v = 0$	$\propto$
$t^2 u_s(t)/2$	$K_a = 0$	$\propto$

(b)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = 1$	1
$t^2 u_s(t)/2$	$K_a = 0$	$\propto$

(c)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0

$tu_s(t)$	$K_v = K$	$1/K$
$t^2u_s(t)/2$	$K_a = 0$	$\propto$

The above results are valid if the value of *K* corresponds to a stable closed-loop system.

(d) The closed-loop system is unstable. It is meaningless to conduct a steady-state error analysis.

(e)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = 1$	1
$t^2u_s(t)/2$	$K_a = 0$	$\propto$

(f)

Input	Error Constants	Steady-state Error
$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = \infty$	0
$t^2u_s(t)/2$	$K_a = K$	$1/K$

The closed-loop system is stable for all positive values of *K*. Thus the above results are valid.

5-5 (a) $K_H = H(0) = 1$ $$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{s+1}{s^3+2s^2+3s+3}$$

$$a_0 = 3, \quad a_1 = 3, \quad a_2 = 2, \quad b_0 = 1, \quad b_1 = 1.$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \frac{2}{3}$$

Unit-ramp input:

$$a_0 - b_0 K_H = 3 - 1 = 2 \neq 0. \text{ Thus } e_{ss} = \infty.$$

Unit-parabolic Input:

$$a_0 - b_0 K_H = 2 \neq 0 \text{ and } a_1 - b_1 K_H = 1 \neq 0. \text{ Thus } e_{ss} = \infty.$$

(b) $K_H = H(0) = 5$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{s^2 + 5s + 5} \qquad a_0 = 5, \quad a_1 = 5, \quad b_0 = 1, \quad b_1 = 0.$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \frac{1}{5}\left(1 - \frac{5}{5}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5}{25} = \frac{1}{5}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(c) $K_H = H(0) = 1/5$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{s + 5}{s^4 + 15s^3 + 50s^2 + s + 1}$$ The system is stable.

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 50, \quad a_3 = 15, \quad b_0 = 5, \quad b_1 = 1$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = 5\left(1 - \frac{5/5}{1}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 4/5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{1 - 1/5}{1/5} = 4$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(d) $K_H = H(0) = 10$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{s^3 + 12s^2 + 5s + 10}$$ The system is stable.

$$a_0 = 10, \quad a_1 = 5, \quad a_2 = 12, \quad b_0 = 1, \quad b_1 = 0, \quad b_2 = 0$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \frac{1}{10}\left(1 - \frac{10}{10}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \quad i = 1: \quad a_1 - b_1 K_H = 5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5}{100} = 0.05$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

5-6 (a) $M(s) = \dfrac{s+4}{s^4 + 16s^3 + 48s^2 + 4s + 4} \quad K_H = 1$ The system is stable.

$$a_0 = 4, \quad a_1 = 4, \quad a_2 = 48, \quad a_3 = 16, \quad b_0 = 4, \quad b_1 = 1, \quad b_2 = 0, \quad b_3 = 0$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \left(1 - \frac{4}{4}\right) = 0$$

Unit-ramp input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 4 - 1 = 3 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{4-1}{4} = \frac{3}{4}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(b) $M(s) = \dfrac{K(s+3)}{s^3 + 3s^2 + (K+2)s + 3K}$ $\quad K_H = 1 \quad$ The system is stable for $K > 0$.

$$a_0 = 3K, \quad a_1 = K+2, \quad a_2 = 3, \quad b_0 = 3K, \quad b_1 = K$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \left(1 - \frac{3K}{3K}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = K + 2 - K = 2 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{K+2-K}{3K} = \frac{2}{3K}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

The above results are valid for $K > 0$.

(c) $M(s) = \dfrac{s+5}{s^4 + 15s^3 + 50s^2 + 10s}$ $\quad H(s) = \dfrac{10s}{s+5} \quad K_H = \lim_{s \to 0} \dfrac{H(s)}{s} = 2$

$$a_0 = 0, \quad a_1 = 10, \quad a_2 = 50, \quad a_3 = 15, \quad b_0 = 5, \quad b_1 = 1$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(\frac{a_2 - b_1 K_H}{a_1}\right) = \frac{1}{2}\left(\frac{50 - 1 \times 2}{10}\right) = 2.4$$

Unit-ramp Input:

$$e_{ss} = \infty$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(d) $M(s) = \dfrac{K(s+5)}{s^4 + 17s^3 + 60s^2 + 5Ks + 5K}$ $\quad K_H = 1$ The system is stable for 0 < *K* < 204.

$$a_0 = 5K, \quad a_1 = 5K, \quad a_2 = 60, \quad a_3 = 17, \quad b_0 = 5K, \quad b_1 = K$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) = \left(1 - \frac{5K}{5K}\right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \qquad i = 1: \quad a_1 - b_1 K_H = 5K - K = 4K \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5K - K}{5K} = \frac{4}{5}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

The results are valid for 0 < *K* < 204.

5-7)

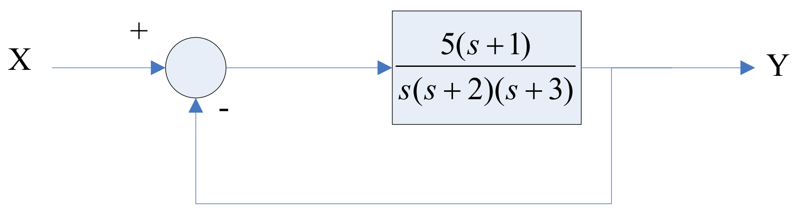
$$\frac{Y(s)}{X(s)} = \frac{\left(\left(\frac{s+1}{s+3}\right)\frac{s}{s(s+2)}\right)}{1+\frac{5(s+1)}{s(s+2)(s+3)}} = \frac{5(s+1)}{s^3+5s^2+11s+5}$$

⇨ Type of the system is zero

Pole: s = -2.2013 + 1.8773i, s = -2.2013 - 1.8773i, and s = -0.5974

Zero: s = -1

5-8)



$$G(s) = \frac{5(s+1)}{s(s+2)(s+3)}$$

a) Position error: $K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0}\frac{5(s+1)}{s(s+2)(s+3)} = \infty$

b) Velocity error: $K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0}\frac{5(s+1)}{(s+2)(s+3)} = \frac{5}{6}$

c) Acceleration error: $K_a = \lim_{s\to\infty} s^2G(s) = \lim_{s\to\infty}\frac{5s(s+1)}{(s+2)(s+3)} = 0$

5-9) a) Steady state error for unit step input:

$e_{ss} = \frac{1}{1+K_p}$

Referring to the result of problem 5-8, $K_p = \infty$ ➔ $e_{ss} = 0$

b) Steady state error for ramp input:

$e_{ss} = \frac{1}{K_v}$

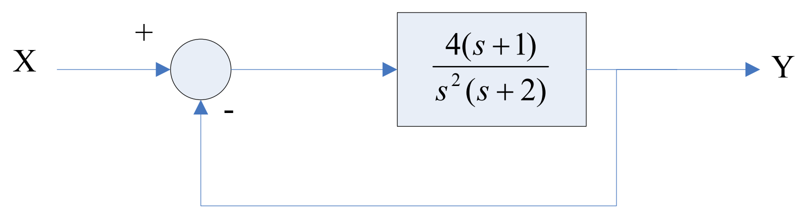
Regarding the result of problem 5-8, $K_v = \frac{5}{6}$ ➔ $e(\infty) = \frac{6}{5}$

c) Steady state error for parabolic input:

$$e_{ss} = \frac{1}{K_a}$$

Regarding the result of problem 5-8, $K_a = 0$ ➔ $e(\infty) = \infty$

5-10)



a) Step error constant: $K_p = \lim_{s\to 0} \frac{4(s+1)}{s^2(s+2)} = \infty$

b) Ramp error constant: $K_v = \lim_{s\to 0} \frac{4(s+1)}{s(s+2)} = \infty$

c) Parabolic error constant: $K_a = \lim_{s\to 0} \frac{4(s+1)}{s+2} = 2$

5-11) $X = \frac{5}{2s} - \frac{3}{s^2} + \frac{4}{s^3} = \frac{5}{2}X_1(s) - 3X_2(s) + 4X_3(s)$

where x_1 is a unit step input, x_2 is a ramp input, and x_3 is a unit parabola input. Since the system is linear, then the effect of X(s) is the summation of effect of each individual input.

That is: $e(\infty) = \frac{5}{2}e_1(\infty) - 3e_2(\infty) + 4e_3(\infty)$

So:

$$\begin{cases} e_{step} = \frac{1}{1+K_p} = 0 \\ e_{ramp} = \frac{1}{Kv} = 0 \\ e_{parabolic} = \frac{1}{K_a} = \frac{1}{2} \end{cases}$$

⇨ $e_{ss} = 4\left(\frac{1}{2}\right) = 2$

5-12) The step input response of the system is:

$$Y(s) = G(s)U(s) = \frac{1-k}{s(s-k)} = \frac{1}{1+k}\left[\frac{1}{s} - \frac{1}{s-k}\right]$$

Therefore:

$$y(t) = \frac{1}{1+k}\left[e^{Kt} + 1\right]u(t)$$

The rise time is the time that unit step response value reaches from 0.1 to 0.9. Then:

$$t_r = \frac{1}{1+k}\left[e^{0.9k} - e^{0.1k}\right]$$

It is obvious that $t_r > 0$, then:

$$\frac{1}{1+k}\left[e^{0.9k} - e^{0.1k}\right] > 0$$

As $|k| < 1$, then $\frac{1}{1+k} > 0$

Therefore $e^{0.9k} - e^{0.1k} > 0$ or $e^{0.9k} > e^{0.1k}$

which yields: $k > 0$

5-13)

$$G(s) = \frac{Y(s)}{E(s)} = \frac{KG_p(s)/20s}{1+K_tG_p(s)} = \frac{100K}{20s(1+0.2s+100K_t)}$$ Type-1 system.

Error constants: $K_p = \infty, \quad K_v = \frac{5K}{1+100K_t}, \quad K_a = 0$

(a) $r(t) = u_s(t)$: $e_{ss} = \frac{1}{1+K_p} = 0$

(b) $r(t) = tu_s(t)$: $e_{ss} = \frac{1}{K_v} = \frac{1+100K_t}{5K}$

(c) $r(t) = t^2u_s(t)/2$: $e_{ss} = \frac{1}{K_a} = \infty$

5-14

$$G_p(s) = \frac{100}{(1+0.1s)(1+0.5s)} \qquad G(s) = \frac{Y(s)}{E(s)} = \frac{KG_p(s)}{20s\left[1+K_tG_p(s)\right]}$$

$$G(s) = \frac{100K}{20s\left[(1+0.1s)(1+0.5s)+100K_t\right]}$$

Error constants: $K_p = \infty, \quad K_v = \dfrac{5K}{1+100K_t}, \quad K_a = 0$

(a) $r(t) = u_s(t)$: $e_{ss} = \dfrac{1}{1+K_p} = 0$

(b) $r(t) = tu_s(t)$: $e_{ss} = \dfrac{1}{K_v} = \dfrac{1+100K_t}{5K}$

(c) $r(t) = t^2u_s(t)/2$: $e_{ss} = \dfrac{1}{K_a} = \infty$

Since the system is of the third order, the values of *K* and K_t must be constrained so that the system is stable. The characteristic equation is

$$s^3 + 12s^2 + \left(20 + 2000K_t\right)s + 100K = 0$$

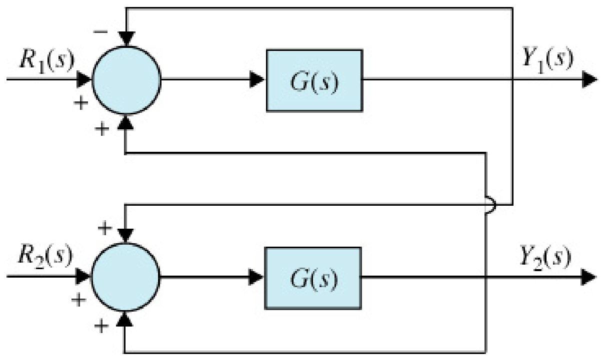
Routh Tabulation:

s^3	1	$20+2000K_t$
s^2	12	$100K$
s^1	$\dfrac{240+24000K_t-100K}{12}$	
s^0	$100K$	

Stability Conditions: $K > 0$ $\qquad 12\left(1+100K_t\right) - 5K > 0 \quad \text{or} \quad \dfrac{1+100K_t}{5K} > \dfrac{1}{12}$

Thus, the minimum steady-state error that can be obtained with a unit-ramp input is 1/12.

5-15 (a) From Figure 3P-29,



$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{1 + \dfrac{K_1K_2}{R_a + L_a s} + \dfrac{K_iK_b + KK_1K_iK_t}{(R_a + L_a s)(B_t + J_t s)}}{1 + \dfrac{K_1K_2}{R_a + L_a s} + \dfrac{K_iK_b + KK_1K_iK_t}{(R_a + L_a s)(B_t + J_t s)} + \dfrac{KK_sK_1K_iN}{s(R_a + L_a s)(B_t + J_t s)}}$$

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{s\left[(R_a + L_a s)(B_t + J_t s) + K_1K_2(B_t + J_t s) + K_iK_b + KK_1K_iK_t\right]}{L_aJ_ts^3 + (L_aB_t + R_aJ_t + K_1K_2J_t)s^2 + (R_aB_t + K_iK_b + KK_iK_1K_t + K_1K_2B_t)s + KK_sK_1K_iN}$$

$$\theta_r(t) = u_s(t), \quad \Theta_r(s) = \frac{1}{s} \qquad \lim_{s\to 0} s\Theta_e(s) = 0$$

Provided that all the poles of $s\Theta_e(s)$ are all in the left-half *s*-plane.

(b) For a unit-ramp input, $\Theta_r(s) = 1/s^2$.

$$e_{ss} = \lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} s\Theta_e(s) = \frac{R_aB_t + K_1K_2B_t + K_iK_b + KK_1K_iK_t}{KK_sK_1K_iN}$$

if the limit is valid.

5-16 (a) Forward-path transfer function: [$n(t) = 0$]:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{\dfrac{K(1+0.02s)}{s^2(s+25)}}{1+\dfrac{KK_t s}{s^2(s+25)}} = \frac{K(1+0.02s)}{s\left(s^2+25s+KK_t\right)}$$

Type-1 system.

Error Constants: $K_p = \infty, \quad K_v = \dfrac{1}{K_t}, \quad K_a = 0$

For a unit-ramp input, $r(t) = tu_s(t), \quad R(s) = \dfrac{1}{s^2}, \quad e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \dfrac{1}{K_v} = K_t$

Routh Tabulation:

s^3	1	$KK_t + 0.02K$
s^2	25	K
s^1	$\dfrac{25K\left(K_t+0.02\right)-K}{25}$	
s^0	K	

Stability Conditions: $K > 0 \qquad 25\left(K_t + 0.02\right) - K > 0 \quad \text{or} \quad K_t > 0.02$

(b) With $r(t) = 0$, $n(t) = u_s(t), \quad N(s) = 1/s.$

System Transfer Function with *N*(*s*) as Input:

$$\frac{Y(s)}{N(s)} = \frac{\dfrac{K}{s^2(s+25)}}{1+\dfrac{K(1+0.02s)}{s^2(s+25)}+\dfrac{KK_t s}{s^2(s+25)}} = \frac{K}{s^3+25s^2+K\left(K_t+0.02\right)s+K}$$

Steady-State Output due to *n*(*t*):

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 1$$

if the limit is valid.

5-17 You may use MATLAB in all Routh Hurwitz calculations.

1. Activate MATLAB

2. Go to the directory containing the ACSYS software.

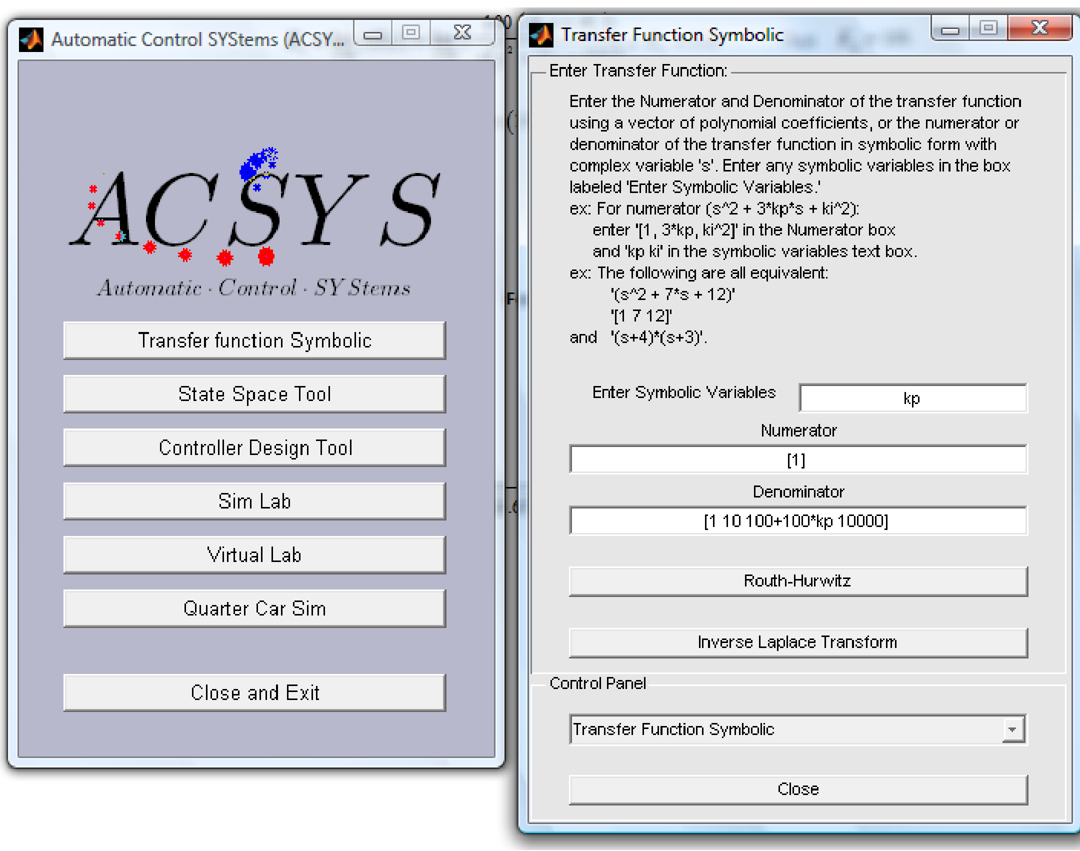
3. Type in

Acsys

4. Then press the “transfer function Symbolic” and enter the Characteristic equation

5. Then press the “Routh Hurwitz” button

6. For example look at below Figures



(a) $n(t) = 0, \quad r(t) = tu_s(t).$

Forward-path Transfer function:

$$G(s) = \left.\frac{Y(s)}{E(s)}\right|_{n=0} = \frac{K(s+\alpha)(s+3)}{s\left(s^2-1\right)}$$ Type-1 system.

Ramp-error constant: $K_v = \lim_{s\to 0} sG(s) = -3K\alpha$

Steady-state error: $e_{ss} = \frac{1}{K_v} = -\frac{1}{3K_v}$

Characteristic equation: $s^3 + Ks^2 + [K(3+\alpha)-1]s + 3\alpha K = 0$

Routh Tabulation:

s^3	1	$3K + \alpha K - 1$
s^2	K	$3\alpha K$
s^1	$\frac{K(3K+\alpha K-1)-3\alpha K}{K}$	
s^0	$3\alpha K$	

Stability Conditions: $3K + \alpha K - 1 - 3\alpha > 0$ or $K > \frac{1+3K}{3+\alpha}$

$\alpha K > 0$

(b) When $r(t) = 0$, $n(t) = u_s(t), \quad N(s) = 1/s.$

Transfer Function between $n(t)$ and $y(t)$:

$$\left.\frac{Y(s)}{N(s)}\right|_{r=0} = \frac{\frac{K(s+3)}{s^2-1}}{1+\frac{K(s+\alpha)(s+3)}{s\left(s^2-1\right)}} = \frac{Ks(s+3)}{s^3+Ks^2+[K(s+\alpha)-1]s+3\alpha K}$$

Steady-State Output due to $n(t)$:

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0$$ if the limit is valid.

5-18

$$\text{Percent maximum overshoot} = 0.25 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Thus

$$\pi\zeta\sqrt{1-\zeta^2} = -\ln 0.25 = 1.386 \qquad \pi^2\zeta^2 = 1.922\left(1-\zeta^2\right)$$

Solving for ζ from the last equation, we have ζ = 0.404.

Peak Time $t_{\max} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.01$ sec. Thus, $\omega_n = \dfrac{\pi}{0.01\sqrt{1-(0.404)^2}} = 343.4$ rad / sec

Transfer Function of the Second-order Prototype System:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{117916}{s^2 + 277.3s + 117916}$$

Extended MATLAB solutions of problems similar to 5-19-5-27 appear later on – e.g. 5-58

5-19 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + \left(5 + 500K_t\right)s + 25K}$$

Characteristic equation:

$$s^2 + \left(5 + 500K_t\right)s + 25K = 0$$

For a second-order prototype system, when the maximum overshoot is 4.3%, $\zeta = 0.707$.

$$\omega_n = \sqrt{25K}, \qquad 2\zeta\omega_n = 5 + 500K_t = 1.414\sqrt{25K}$$

Rise Time:

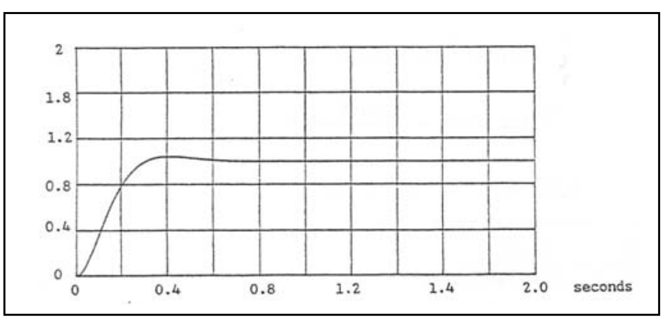
$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = \frac{2.164}{\omega_n} = 0.2 \quad \text{sec} \qquad \text{Thus} \quad \omega_n = 10.82 \quad \text{rad / sec}$$

Thus, $K = \frac{\omega_n^2}{25} = \frac{(10.82)^2}{25} = 4.68 \qquad 5 + 500K_t = 1.414\omega_n = 15.3$ Thus $K_t = \frac{10.3}{500} = 0.0206$

With $K = 4.68$ and $K_t = 0.0206$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{117}{s^2 + 15.3s + 117}$$

Unit-step Response:



y = 0.1 at *t* = 0.047 sec.

y = 0.9 at *t* = 0.244 sec.

$t_r = 0.244 - 0.047 = 0.197$ sec.

$$y_{\max} = 0.0432 \quad \text{(4.32\% max. overshoot)}$$

5-20 **Closed-loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

When Maximum overshoot = 10%, $\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -\ln 0.1 = 2.3 \qquad \pi^2\zeta^2 = 5.3(1-\zeta^2)$

Solving for ζ , we get ζ = 0.59.

The Natural undamped frequency is $\omega_n = \sqrt{25K} \qquad$ Thus, $5 + 500K_t = 2\zeta\omega_n = 1.18\omega_n$

Rise Time:

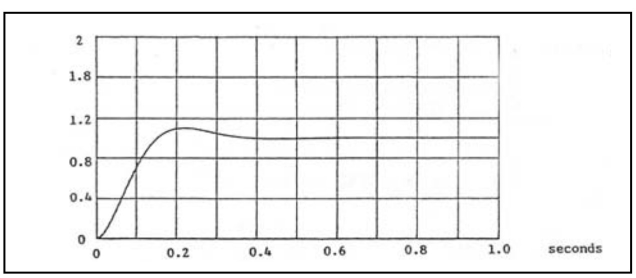
$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.1 = \frac{1.7696}{\omega_n} \text{ sec.} \quad \text{Thus } \omega_n = 17.7 \text{ rad / sec}$$

$$K = \frac{\omega_n^2}{25} = 12.58 \qquad \text{Thus } K_t = \frac{15.88}{500} = 0.0318$$

With *K* = 12.58 and $K_t = 0.0318$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{313}{s^2 + 20.88s + 314.5}$$

Unit-step Response:



y = 0.1 when *t* = 0.028 sec.

y = 0.9 when *t* = 0.131 sec.

$t_r = 0.131 - 0.028 = 0.103$ sec.

$y_{\max} = 1.1 \quad (10\% \quad \text{max. overshoot})$

5-21 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)}=\frac{25K}{s^2+(5+500K_t)s+25K}$$

Characteristic Equation:

$$s^2+\left(5+500K_t\right)s+25K=0$$

When Maximum overshoot = 20%, $\frac{\pi\zeta}{\sqrt{1-\zeta^2}}=-\ln 0.2=1.61$ $\quad \pi^2\zeta^2=2.59\left(1-\zeta^2\right)$

Solving for ζ , we get $\zeta=0.456.$

The Natural undamped frequency $\omega_n=\sqrt{25K}$ $\quad 5+500K_t=2\zeta\omega_n=0.912\omega_n$

Rise Time:

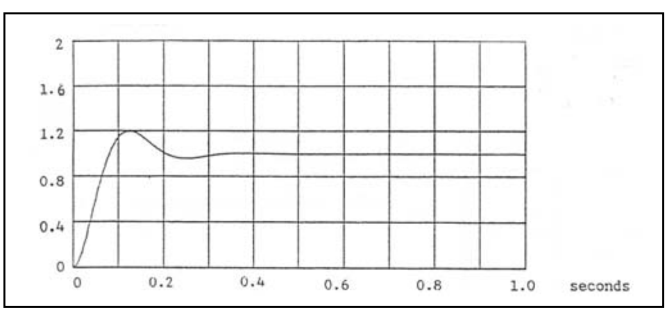
$$t_r=\frac{1-0.4167\zeta+2.917\zeta^2}{\omega_n}=0.05=\frac{1.4165}{\omega_n}\ \text{sec.}\quad \text{Thus,}\ \ \omega_n=\frac{1.4165}{0.05}=28.33$$

$$K=\frac{\omega_n^2}{25}=32.1 \qquad 5+500K_t=0.912\omega_n=25.84 \qquad \text{Thus,}\ \ K_t=0.0417$$

With *K* = 32.1 and $K_t=0.0417$, the system transfer function is

$$\frac{Y(s)}{R(s)}=\frac{802.59}{s^2+25.84s+802.59}$$

Unit-step Response:



y = 0.1 when *t* = 0.0178 sec.

y = 0.9 when *t* = 0.072 sec.

$t_r=0.072-0.0178=0.0542$ sec.

$y_{max}=1.2$ (20% max. overshoot)

5-22 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

Delay time $t_d \cong \frac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = 0.1$ sec.

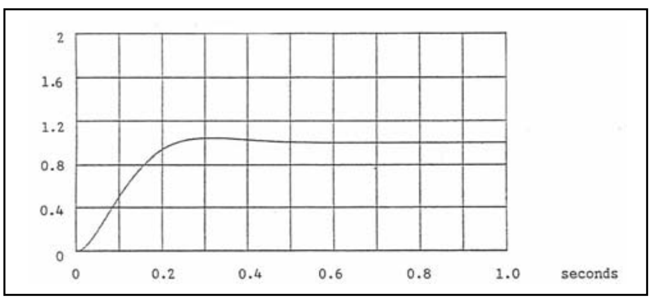
When Maximum overshoot = 4.3%, $\zeta = 0.707$. $t_d = \frac{1.423}{\omega_n} = 0.1$ sec. Thus $\omega_n = 14.23$ rad/sec.

$K = \left(\frac{\omega_n}{5}\right)^2 = \left(\frac{14.23}{5}\right)^2 = 8.1$ $5 + 500K_t = 2\zeta\omega_n = 1.414\omega_n = 20.12$ Thus $K_t = \frac{15.12}{500} = 0.0302$

With *K* = 20.12 and $K_t = 0.0302$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{202.5}{s^2 + 20.1s + 202.5}$$

Unit-Step Response:



When *y* = 0.5, *t* = 0.1005 sec.

Thus, $t_d = 0.1005$ sec.

$y_{\max} = 1.043$ (4.3% max. overshoot)

5-23 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + \left(5 + 500K_t\right)s + 25K}$$

Characteristic Equation:

$$s^2 + \left(5 + 500K_t\right)s + 25K = 0$$

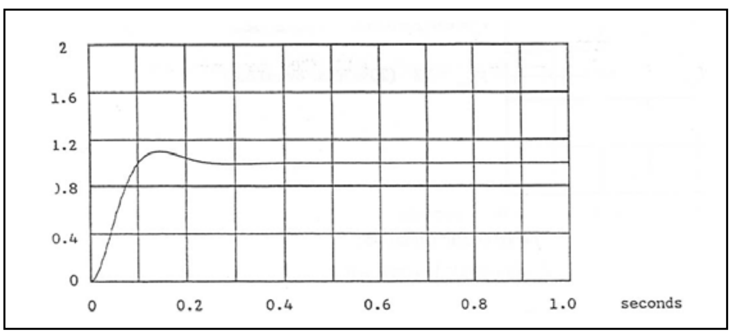
Delay time $t_d \cong \dfrac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = 0.05 = \dfrac{1.337}{\omega_n}$ Thus, $\omega_n = \dfrac{1.337}{0.05} = 26.74$

$$K = \left(\frac{\omega_n}{5}\right)^2 = \left(\frac{26.74}{5}\right)^2 = 28.6 \qquad 5 + 500K_t = 2\zeta\omega_n = 2 \times 0.59 \times 26.74 = 31.55 \quad \text{Thus } K_t = 0.0531$$

With K = 28.6 and $K_t = 0.0531$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{715}{s^2 + 31.55s + 715}$$

Unit-Step Response:



y = 0.5 when t = 0.0505 sec.

Thus, $t_d = 0.0505$ sec.

$y_{max} = 1.1007 \quad (10.07\% \text{ max. overshoot})$

5-24 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

For Maximum overshoot = 0.2, $\zeta = 0.456$.

Delay time $t_d = \dfrac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = \dfrac{1.2545}{\omega_n} = 0.01$ sec.

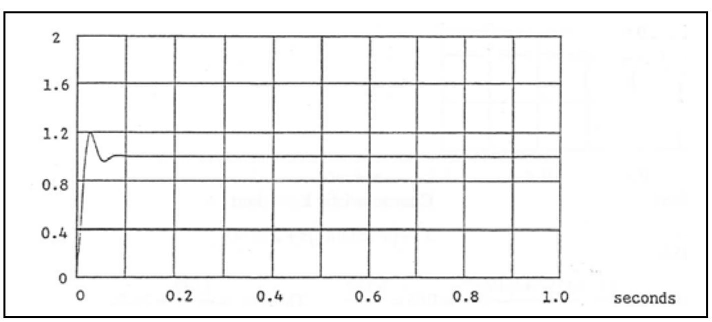
Natural Undamped Frequency $\omega_n = \dfrac{1.2545}{0.01} = 125.45$ rad/sec. Thus, $K = \left(\dfrac{\omega_n}{5}\right)^2 = \dfrac{15737.7}{25} = 629.5$

$5 + 500K_t = 2\zeta\omega_n = 2 \times 0.456 \times 125.45 = 114.41$ Thus, $K_t = 0.2188$

With *K* = 629.5 and $K_t = 0.2188$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{15737.7}{s^2 + 114.41s + 15737.7}$$

Unit-step Response:



y = 0.5 when *t* = 0.0101 sec.

Thus, $t_d = 0.0101$ sec.

$y_{\max} = 1.2$ (20% max. overshoot)

5-25 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + \left(5 + 500K_t\right)s + 25K}$$

Characteristic Equation:

$$s^2 + \left(5 + 5000K_t\right)s + 25K = 0$$

$$\zeta = 0.6 \qquad 2\zeta\omega_n = 5 + 500K_t = 1.2\omega_n$$

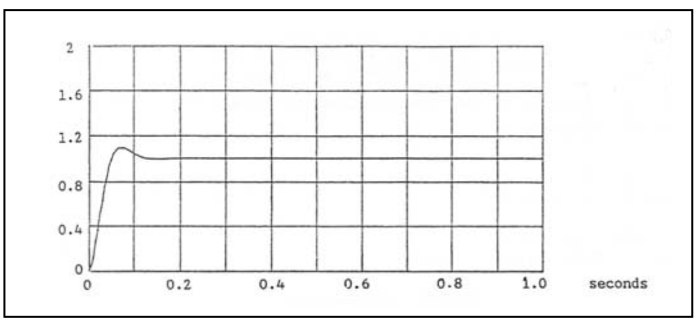
Settling time $t_s \cong \frac{3.2}{\zeta\omega_n} = \frac{3.2}{0.6\omega_n} = 0.1$ sec. Thus, $\omega_n = \frac{3.2}{0.06} = 53.33 \text{ rad / sec}$

$$K_t = \frac{1.2\omega_n - 5}{500} = 0.118 \qquad K = \frac{\omega_n^2}{25} = 113.76$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{2844}{s^2 + 64s + 2844}$$

Unit-step Response:



$y(t)$ reaches 1.00 and never exceeds this

value at t = 0.098 sec.

Thus, $t_s = 0.098$ sec.

5-26 (a) **Closed-Loop Transfer Function:** **Characteristic Equation:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

$$s^2 + (5 + 500K_t)s + 25K = 0$$

For maximum overshoot = 0.1, $\zeta = 0.59$. $5 + 500K_t = 2\zeta\omega_n = 2 \times 0.59\omega_n = 1.18\omega_n$

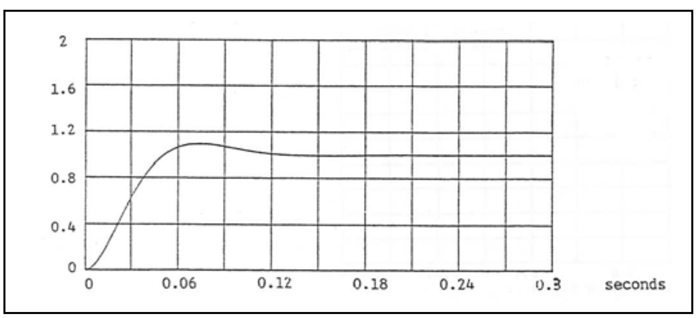
Settling time: $t_s = \frac{3.2}{\zeta\omega_n} = \frac{3.2}{0.59\omega_n} = 0.05$ sec. $\omega_n = \frac{3.2}{0.05 \times 0.59} = 108.47$

$$K_t = \frac{1.18\omega_n - 5}{500} = 0.246 \qquad K = \frac{\omega_n^2}{25} = 470.63$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{11765.74}{s^2 + 128s + 11765.74}$$

Unit-Step Response:



$y(t)$ reaches 1.05 and never exceeds this value at t = 0.048 sec.

Thus, $t_s = 0.048$ sec.

(b) For maximum overshoot = 0.2, $\zeta = 0.456$. $5 + 500K_t = 2\zeta\omega_n = 0.912\omega_n$

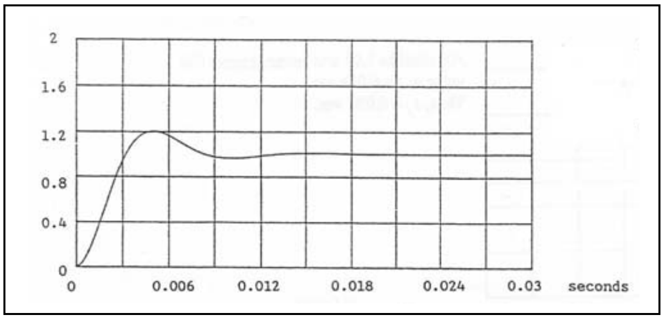
Settling time $t_s = \frac{3.2}{\zeta\omega_n} = \frac{3.2}{0.456\omega_n} = 0.01$ sec. $\omega_n = \frac{3.2}{0.456 \times 0.01} = 701.75$ rad / sec

$$K_t = \frac{0.912\omega_n - 5}{500} = 1.27$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{492453}{s^2 + 640s + 492453}$$

Unit-Step Response:



$y(t)$ reaches 1.05 and never exceeds this value at t = 0.0074 sec.

Thus, $t_s = 0.0074 \quad \text{sec.}$ This is less than the calculated value of 0.01 sec.

5-27 **Closed-Loop Transfer Function:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + \left(5 + 500K_t\right)s + 25K}$$

Characteristic Equation:

$$s^2 + \left(5 + 500K_t\right)s + 25K = 0$$

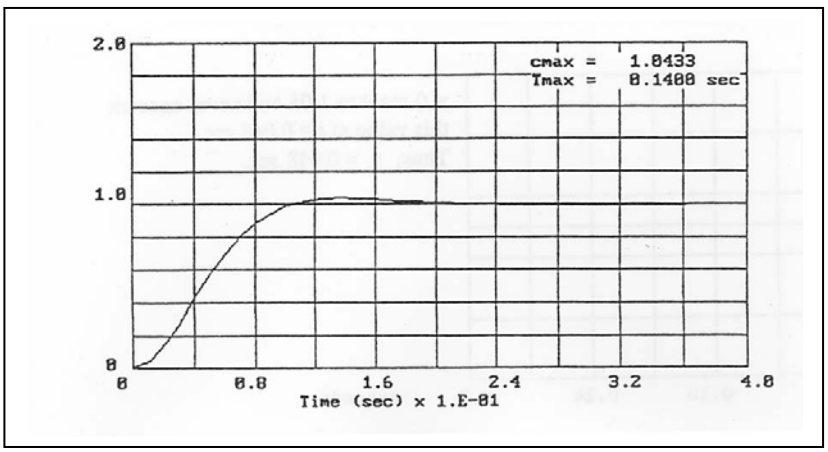
Damping ratio $\zeta = 0.707$. Settling time $t_s = \frac{4.5\zeta}{\omega_n} = \frac{3.1815}{\omega_n} = 0.1$ sec. Thus, $\omega_n = 31.815$ rad/sec.

$$5 + 500K_t = 2\zeta\omega_n = 44.986 \quad \text{Thus,} \quad K_t = 0.08 \qquad K = \frac{\omega_n^2}{2\zeta} = 40.488$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{1012.2}{s^2 + 44.986s + 1012.2}$$

Unit-Step Response: The unit-step response reaches 0.95 at t = 0.092 sec. which is the measured t_s.



5-28 (a) When $\zeta = 0.5$, the rise time is

$$t_r \cong \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = \frac{1.521}{\omega_n} = 1 \text{ sec.} \quad \text{Thus } \omega_n = 1.521 \text{ rad/sec.}$$

The second-order term of the characteristic equation is written

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 1.521s + 2.313 = 0$$

The characteristic equation of the system is $s^3 + (a+30)s^2 + 30as + K = 0$

Dividing the characteristic equation by $s^2 + 1.521s + 2.313$, we have

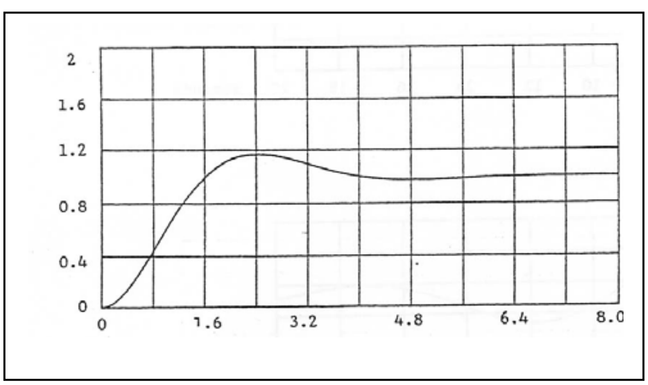
$$\begin{array}{r|l} & s + (28.48 + a) \\ \hline s^2 + 1.521s + 2.313 & s^3 + (a+30)s^2 + 30as + K \\ & s^3 + 1.521s^2 + 2.313s \\ \hline & (28.48 + a)s^2 + (30a - 2.323)s + K \\ & (28.48 + a)s^2 + (1.521a + 43.32)s + 65.874 + 2.313a \\ \hline & (28.48a - 45.63)s + K - 0.744 - 2.313a \end{array}$$

For zero remainders, $28.48a = 45.63$ Thus, $a = 1.6$ $K = 65.874 + 2.313a = 69.58$

Forward-Path Transfer Function:

$$G(s) = \frac{69.58}{s(s+1.6)(s+30)}$$

Unit-Step Response:



y = 0.1 when t = 0.355 sec.

y = 0.9 when t = 1.43 sec.

Rise Time:

$t_r = 1.43 - 0.355 = 1.075$ sec.

(b) The system is type 1.

(i) For a unit-step input, $e_{ss} = 0.$

(ii) For a unit-ramp input, $K_v = \lim_{s\to 0} sG(s) = \dfrac{K}{30a} = \dfrac{60.58}{30 \times 1.6} = 1.45 \qquad e_{ss} = \dfrac{1}{K_v} = 0.69$

5-29 (a) Characteristic Equation:

$$s^3 + 3s^2 + (2+K)s - K = 0$$

Apply the Routh-Hurwitz criterion to find the range of *K* for stability.

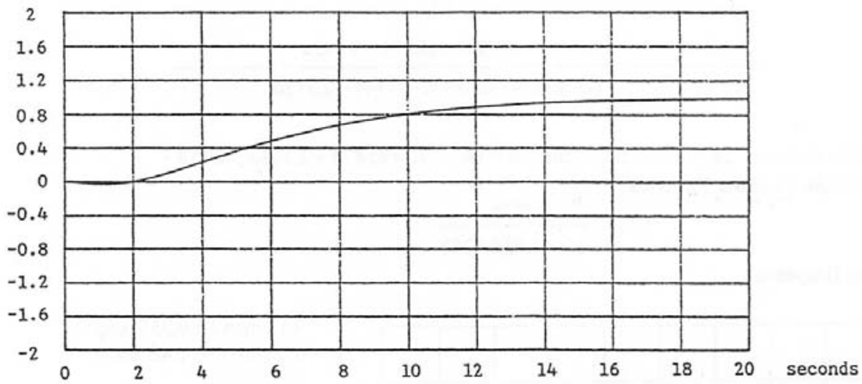
Routh Tabulation:

s^3	1	$2+K$
s^2	3	$-K$
s^1	$\dfrac{6+4K}{3}$	
s^0	$-K$	

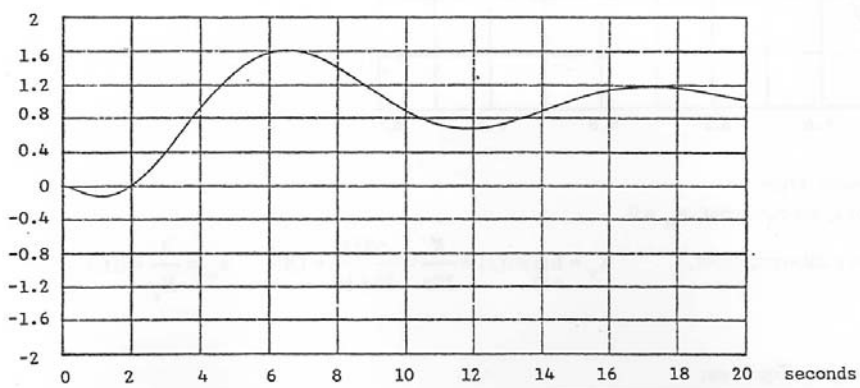
Stability Condition: -1.5 < *K* < 0 This simplifies the search for *K* for two equal roots.

When *K* = –0.27806, the characteristic equation roots are: –0.347, –0.347, and –2.3054.

(b) Unit-Step Response: (*K* = –0.27806)



(c) Unit-Step Response $(K = -1)$



The step responses in (a) and (b) all have a negative undershoot for small values of t. This is due to the zero of $G(s)$ that lies in the right-half s-plane.

5-30 (a) The state equations of the closed-loop system are:

$$\frac{dx_1}{dt} = -x_1 + 5x_2 \qquad \frac{dx_2}{dt} = -6x_1 - k_1 x_1 - k_2 x_2 + r$$

The characteristic equation of the closed-loop system is

$$\Delta = \begin{vmatrix} s+1 & -5 \\ 6+k_1 & s+k_2 \end{vmatrix} = s^2 + (1+k_2)s + (30 + 5k_1 + k_2) = 0$$

For $\omega_n = 10 \text{ rad/sec}, \quad 30 + 5k_1 + k_2 = \omega_n^2 = 100.$ Thus $5k_1 + k_2 = 70$

(b) For $\zeta = 0.707,\ 2\zeta\omega_n = 1 + k_2.$ Thus $\omega_n = 1 + \frac{k_2}{1.414}.$

$$\omega_n^2 = \frac{(1+k_2)^2}{2} = 30 + 5k_1 + k_2 \qquad \text{Thus} \quad k_2^2 = 59 + 10k_1$$

(c) For $\omega_n = 10 \text{ rad/sec}$ and $\zeta = 0.707,$

$$5k_1 + k_2 = 100 \quad \text{and} \quad 1 + k_2 = 2\zeta\omega_n = 14.14 \quad \text{Thus} \quad k_2 = 13.14$$

Solving for k_1, we have $k_1 = 11.37.$

(d) The closed-loop transfer function is

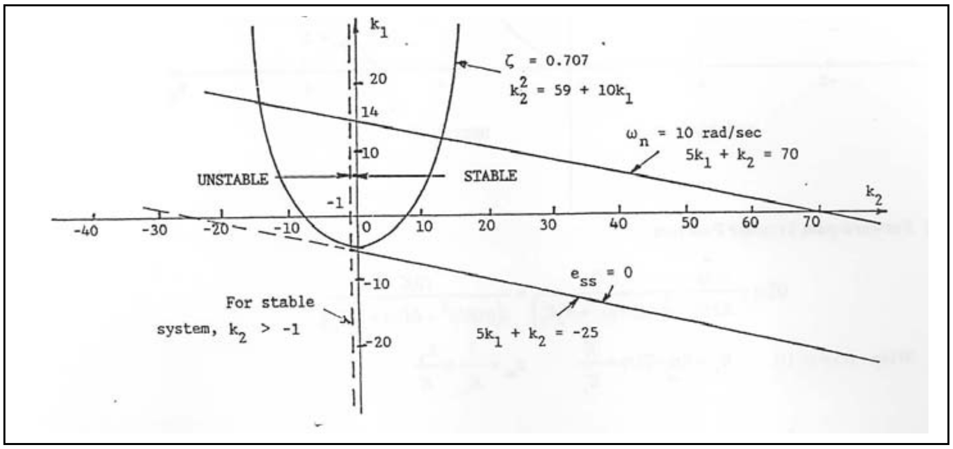
$$\frac{Y(s)}{R(s)} = \frac{5}{s^2 + (k_2+1)s + (30 + 5k_1 + k_2)} = \frac{5}{s^2 + 14.14s + 100}$$

For a unit-step input, $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{5}{100} = 0.05$

(e) For zero steady-state error due to a unit-step input,

$$30 + 5k_1 + k_2 = 5 \qquad \text{Thus} \qquad 5k_1 + k_2 = -25$$

Parameter Plane k_1 versus k_2:



5-31 (a) Closed-Loop Transfer Function

$$\frac{Y(s)}{R(s)} = \frac{100\left(K_P + K_D s\right)}{s^2 + 100K_D s + 100K_P}$$

The system is stable for $K_P > 0$ and $K_D > 0$.

(b) Characteristic Equation:

$$s^2 + 100K_D s + 100K_P = 0$$

(b) For $\zeta = 1, \quad 2\zeta\omega_n = 100K_D$.

$$\omega_n = 10\sqrt{K_P} \quad \text{Thus} \quad 2\omega_n = 100K_D = 20\sqrt{K_P} \qquad K_D = 0.2\sqrt{K_P}$$

(c) See parameter plane in part **(g)**.

(d) See parameter plane in part **(g)**.

(e) Parabolic error constant $K_a = 1000 \ \sec^{-2}$

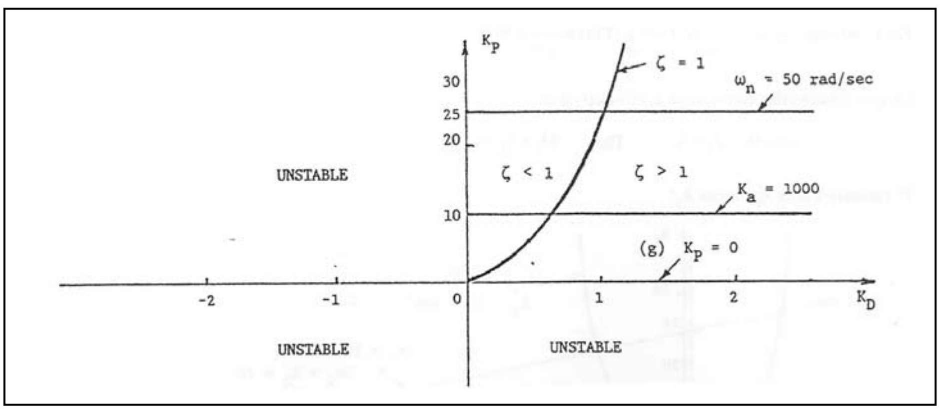
$$K_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} 100\left(K_P + K_D s\right) = 100K_P = 1000 \qquad \text{Thus} \quad K_P = 10$$

(f) Natural undamped frequency $\omega_n = 50$ rad/sec.

$$\omega_n = 10\sqrt{K_P} = 50 \qquad \text{Thus} \qquad K_P = 25$$

(g) When $K_P = 0$,

$$G(s) = \frac{100K_D s}{s^2} = \frac{100K_D}{s} \qquad \text{(pole-zero cancellation)}$$



5-32 (a) Forward-path Transfer Function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{KK_i}{s\left[Js(1+Ts)+K_iK_t\right]} = \frac{10K}{s\left(0.001s^2+0.01s+10K_t\right)}$$

When $r(t) = tu_s(t)$, $\quad K_v = \lim_{s\to 0} sG(s) = \frac{K}{K_t} \qquad e_{ss} = \frac{1}{K_v} = \frac{K_t}{K}$

(b) When $r(t) = 0$

$$\frac{Y(s)}{T_d(s)} = \frac{1+Ts}{s\left[Js(1+Ts)+K_iK_t\right]+KK_i} = \frac{1+0.1s}{s\left(0.001s^2+0.01s+10K_t\right)+10K}$$

For $T_d(s) = \frac{1}{s} \qquad \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{1}{10K}$ if the system is stable.

(c) The characteristic equation of the closed-loop system is

$$0.001s^3 + 0.01s^2 + 0.1s + 10K = 0$$

The system is unstable for $K > 0.1$. So we can set K to just less than 0.1. Then, the minimum value of the steady-state value of $y(t)$ is

$$\left.\frac{1}{10K}\right|_{K=0.1^-} = 1^+$$

However, with this value of K, the system response will be very oscillatory. The maximum overshoot will be nearly 100%.

(d) For $K = 0.1$, the characteristic equation is

$$0.001s^3 + 0.01s^2 + 10K_ts + 1 = 0 \quad \text{or} \quad s^3 + 10s^2 + 10^4K_ts + 1000 = 0$$

For the two complex roots to have real parts of –2/5. we let the characteristic equation be written as

$$(s+a)(s^2+5s+b)=0 \qquad \text{or} \qquad s^3+(s+5)s^2+(5a+b)s+ab=0$$

Then, $a+5=10 \qquad a=5 \qquad ab=1000 \qquad b=200 \qquad 5a+b=10^4 K_t \qquad K_t=0.0225$

The three roots are: $s=-a=-5 \qquad s=-a=-5 \qquad s=-2.5\pm j13.92$

5-33) Rise time: $t_r \cong \frac{0.8+2.5\,\xi}{\omega_n} = \frac{0.8+2.5*0.6}{5} = 0.56\ sec$

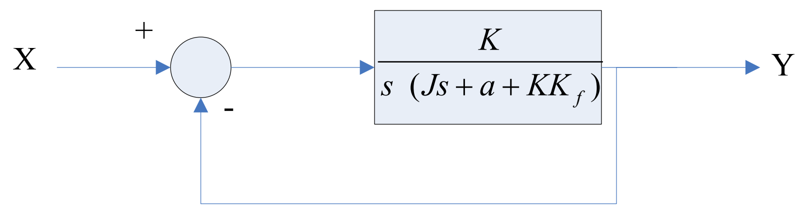
Peak time: $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{3.14}{5\sqrt{0.64}} = 0.785\ sec$

Maximum overshoot: $M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} = e^{-\frac{0.6\pi}{0.8}} = 0.095$

Settling time: $t_s \cong \frac{3.2}{\xi\omega_n} \qquad 0<\xi<0.69$

⇨ $t_s \cong \frac{3.2}{0.615} \cong 1.067\ sec$

5-34)



⇨ $\frac{Y(s)}{X(s)} = \frac{K}{Js^2+(a+KK_f)s+K}$

⇨ $M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.2 \rightarrow \xi = 0.456$

⇨ $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.1 \rightarrow \omega_n = 0.353$

⇨ $G(s) = \frac{\omega_n^2}{s(s+2\,\xi\omega_n)} = \frac{\frac{K}{J}}{s\left(s+\frac{a+KK_f}{J}\right)}$

⇨ $$\begin{cases} \omega_n = \sqrt{\frac{K}{J}} & \rightarrow \quad K = 0.125 \\ 2\,\xi\omega_n = \frac{a+KK_f}{J} & \rightarrow \quad K_f = \frac{2J\,\xi\omega_n - a}{K} \cong -5.42 \end{cases}$$

⇨ $t_r = \frac{0.8+2.5f}{\omega_n} \cong 5.49\ sec$

⇨ $t_s = \frac{3.2}{\xi\omega_n} \cong 19.88\ sec$

5-35) a)

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 + u_1 + u_2 \\ \dot{x}_2 = 6.5x_1 + u_1 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

$$\begin{cases} sX_1(s) = -X_1(s) - X_2(s) + U_1(s) + U_2(s) & (1) \\ sX_2(s) = 6.5X_1(s) + U_1(s) & (2) \\ Y_1(s) = X_1(s) \\ Y_2(s) = X_2(s) \end{cases}$$

⇨ $(s+1)x_1(s) = -\frac{6.5}{s}X(s) + \frac{U_1(s)}{s} + U_1(s) + U_2(s)$

⇨ $(s^2+s+6.5)X_1(s) = (s-1)U_1(s) + U_2(s)$

⇨ $Y_1(s) = X_1(s) = \frac{s-1}{s^2+s+6.5}U_1(s) + \frac{5}{s^2+s+6.5}U_2(s)$

Substituting into equation (2) gives:

$$Y_2(s) = X_2(s) = \frac{s+7.5}{s^2+s+6.5}U_1(s) + \frac{6.5}{s^2+s+6.5}U_2(s)$$

Since the system is multi input and multi output, there are 4 transfer functions as:

$$\left[\frac{Y_1(s)}{U_1(s)}\right]_{U_2=0}, \left[\frac{Y_1(s)}{U_2(s)}\right]_{U_1=0}, \left[\frac{Y_2(s)}{U_1(s)}\right]_{U_2=0}, \left[\frac{Y_2(s)}{U_2(s)}\right]_{U_1=0}$$

To find the unit step response of the system, let's consider

$$\frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2+s+6.5} = \frac{\omega_n^2}{s(s^2+2\,\xi\omega_n+\,\omega_n^2)}$$

$$where \begin{cases} \omega_n^2 = 6.5 \;\rightarrow\; \omega_n = \sqrt{6.5} \\ 2\,\xi\omega_n = 1 \;\rightarrow \xi = \frac{1}{2\,\omega_n} = \frac{1}{2\sqrt{6.5}} \end{cases}$$

By looking at the Laplace transform function table:

$$y(t) = 1 - \frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}\,t+\theta\right)$$

where $\theta = \cos^{-1}\xi$

b)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + u \\ y = x_1 \end{cases}$$

Therefore:

$$\begin{cases} sX_1(s) = X_2(s) \\ sX_2(s) = -X_1(s) - X_2(s) + U(s) \\ Y(s) = X_1(s) \end{cases}$$

As a result:

$$s^2X_1(s) = -X_1(s) - sX_1(s) + U(s)$$

which means:

$$X_1(s) = \frac{1}{s^2 + s + 1}U(s)$$

The unit step response is:

$$Y(s) = X_1(s) = \frac{1}{s(s^2 + s + 1)}$$

Therefore as a result:

$$y(t) = 1 - \frac{1}{\sqrt{1 - \xi^2}} e^{-\xi\omega_n t} \sin \omega_n \sqrt{1 - \xi^2}\, t$$

where $\omega_n = 1$ and $\xi \omega_n = 1/2$

c)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + 4 \\ \dot{x}_3 = x_1 \\ y = x_3 \end{cases}$$

Therefore:

$$\begin{cases} sX_1(s) = X_2(s) \\ (s + 1)X_2(s) = -X_1(s) + U(s) \rightarrow s(s + 1)X_1(s) = -x_1(s) + U(s) \\ sX_3(s) = X_1(s) \\ Y(s) = X_3(s) \quad \rightarrow \quad Y(s) = \dfrac{X_1(s)}{s} \end{cases}$$

As a result, the step response of the system is:

$$Y(s) = \frac{1}{s^2(s^2 + s + 1)}$$

By looking up at the Laplace transfer function table:

$Y(s) = \frac{\omega_n^2}{s^2(s^2+2\,\xi\omega_n+\omega_n^2)}$

where $\omega_n = 1$, and $2\xi\,\omega_n = 1$ ➔ $\xi = 1/2$

$y(t) = t - \frac{2\xi}{\omega_n} + \frac{1}{\omega_n\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2} + \theta\right)$

where $\theta = \cos^{-1}(2\xi^2 - 1) = \cos^{-1}(0.5)$, therefore,

$$y(t) = t - 1 + \frac{2}{\sqrt{3}} e^{-\xi\omega_n t}\sin\left(\frac{\sqrt{3}}{2}t + \theta\right)$$

5-36) MATLAB CODE

```
(a)

clear all
Amat=[-1 -1;6.5 0]
Bmat=[1 1;1 0]
Cmat=[1 0;0 1]
Dmat=[0 0;0 0]
disp(' State-Space Model is:')
Statemodel=ss(Amat,Bmat,Cmat,Dmat)
[mA,nA]=size(Amat);
rankA=rank(Amat);
disp(' Characteristic Polynomial:')
chareq=poly(Amat);

% p = poly(A) where A is an n-by-n matrix returns an n+1 element
%row vector whose elements are the coefficients of the characteristic
%polynomialdet(sI-A). The coefficients are ordered in descending powers.

[mchareq,nchareq]=size(chareq);
syms 's';

poly2sym(chareq,s)
disp(' Equivalent Transfer Function Model is:')

Hmat=Cmat*inv(s*eye(2)-Amat)*Bmat+Dmat
```

Since the system is multi input and multi output, there are 4 transfer functions as:

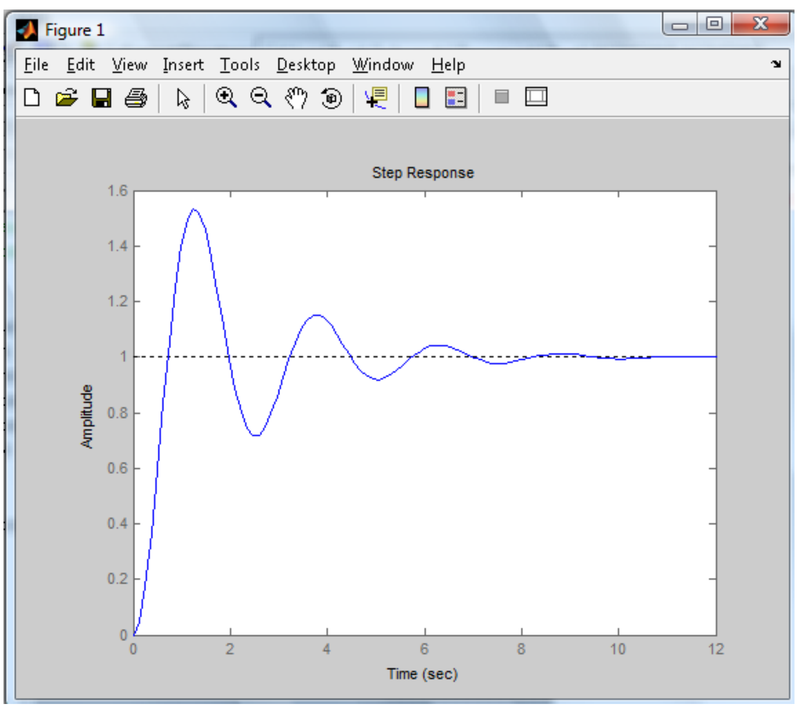
$$\left[\frac{Y_1(s)}{U_1(s)}\right]_{U_2=0}, \left[\frac{Y_1(s)}{U_2(s)}\right]_{U_1=0}, \left[\frac{Y_2(s)}{U_1(s)}\right]_{U_2=0}, \left[\frac{Y_2(s)}{U_2(s)}\right]_{U_1=0}$$

To find the unit step response of the system, let's consider

$$\frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2 + s + 6.5}$$

Let's obtain this term and find $Y_2(s)$ time response for a step input.

```
H22=Hmat(2,2)
ilaplace(H22/s)
Pretty(H22)
H22poly=tf([13/2],chareq)
step(H22poly)

H22 =

13/(2*s^2+2*s+13)

ans =

1-1/5*exp(-1/2*t)*(5*cos(5/2*t)+sin(5/2*t))

                                         13
                                   ---------------
                                      2
                                   2 s  + 2 s + 13

Transfer function:
     6.5
-------------
s^2 + s + 6.5
```



`To find the step response H11, H12, and H21 follow the same procedure.`

Other parts are the same.

5-37) Impulse response:

a) $Y(s) = \frac{6.5}{s^2+s+6.5}$ and $U(s) = 1$, therefore,

$$y(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}\, t\right)$$

b) $Y(s) = \frac{1}{s^2+s+1}$

$$y(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}\, t\right)$$

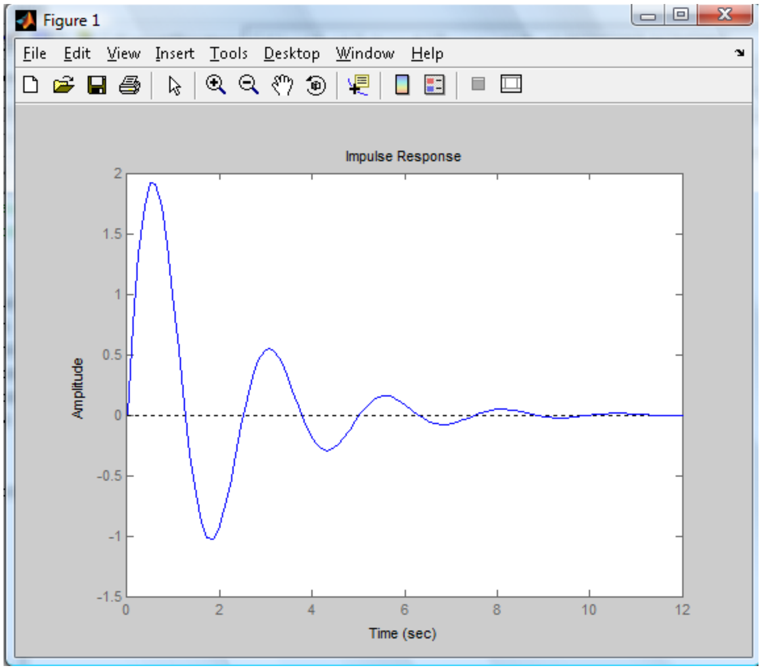
c) $Y(s) = \frac{1}{s(s^2+s+1)}$

$$y(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}\, t + \alpha\right)$$

where $\alpha = \cos^{-1}\xi$

5-38) Use the approach in 5-36 except:

```
H22=Hmat(2,2)
ilaplace(H22)
Pretty(H22)
H22poly=tf([13/2],chareq)
impulse(H22poly)

H22 =

13/(2*s^2+2*s+13)

ans =

13/5*exp(-1/2*t)*sin(5/2*t)

                 13
          ---------------
              2
          2 s  + 2 s + 13

Transfer function:
     6.5
-------------
s^2 + s + 6.5
```



Other parts are the same.

5-39) a) The displacement of the bar is:

$$x = L\sin\theta$$

Then the equation of motion is:

$$B\left(\frac{dy}{dt} - \frac{dx}{dt}\right) - K\tau = 0$$

As x is a function of θ and changing with time, then

$$\frac{dx}{dt} = L\frac{d\theta}{dt}\cos\theta$$

If θ is small enough, then sin θ ≈ θ and cos θ ≈ 1. Therefore, the equation of motion is rewritten as:

$$B\left(\dot{y} - L\dot{\theta}\right) - KL\,\theta = 0$$

$$L\left(B\dot{\theta} + K\,\theta\right) = B\dot{y}$$

$$L(Bs + K)\theta(s) = BsY(s)$$

$$\frac{\theta(s)}{Y(s)} = \frac{Bs}{L(Bs+K)}$$

(b) To find the unit step response, you can use the symbolic approach shown in Toolbox 2-1-1:

```
clear all
%s=tf('s');
syms s B L K
Theta=B*s/s/L/(B*s+K)
ilaplace(Theta)

Theta =
B/L/(B*s+K)

ans =
1/L*exp(-K*t/B)
```

Alternatively, assign values to B L K and find the step response – see solution to problem 5-36.

5-40 (a) $K_t = 10000$ oz-in/rad

The Forward-Path Transfer Function:

$$G(s) = \frac{9\times10^{12}K}{s\left(s^4 + 5000s^3 + 1.067\times10^7 s^2 + 50.5\times10^9 s + 5.724\times10^{12}\right)}$$
$$= \frac{9\times10^{12}K}{s(s+116)(s+4883)(s+41.68+j3178.3)(s+41.68-j3178.3)}$$

Routh Tabulation:

s^5	1	1.067×10^7	5.724×10^{12}
s^4	5000	50.5×10^9	$9\times10^{12}K$
s^3	5.7×10^5	$5.72\times10^{12} - 1.8\times10^9 K$	0
s^2	$2.895\times10^8 + 1.579\times10^7 K$	$9\times10^{12}K$	

$$s^1 \quad \frac{16.6\times10^{13} + 8.473\times10^{12}K - 2.8422\times10^9 K^2}{29 + 1.579K}$$

$$s^0 \quad 9\times10^{12}K$$

From the s^1 row, the condition of stability is $165710 + 8473K - 2.8422K^2 > 0$

or $K^2 - 2981.14K - 58303.427 < 0$ or $(K + 19.43)(K - 3000.57) < 0$

Stability Condition: 0 < *K* < 3000.56

The critical value of *K* for stability is 3000.56. With this value of *K*, the roots of the characteristic equation are: –4916.9, –41.57 + *j*3113.3, –41.57 + *j*3113.3, –*j*752.68, and *j*752.68

(b) $K_L = 1000$ oz-in/rad. The forward-path transfer function is

$$G(s) = \frac{9 \times 10^{11} K}{s\left(s^4 + 5000s^3 + 1.582 \times 10^6 s^2 + 5.05 \times 10^9 s + 5.724 \times 10^{11}\right)}$$
$$= \frac{9 \times 10^{11} K}{s(1 + 116.06)(s + 4882.8)(s + 56.248 + j1005)(s + 56.248 - j1005)}$$

(c) Characteristic Equation of the Closed-Loop System:

$$s^5 + 5000s^4 + 1.582 \times 10^6 s^3 + 5.05 \times 10^9 s^2 + 5.724 \times 10^{11} s + 9 \times 10^{11} K = 0$$

Routh Tabulation:

s^5	1	1.582×10^6	5.724×10^{11}
s^4	5000	5.05×10^9	$9 \times 10^{11} K$
s^3	5.72×10^5	$5.724 \times 10^{11} - 1.8 \times 10^8 K$	0
s^2	$4.6503 \times 10^7 + 1.5734 \times 10^6 K$	$9 \times 10^{11} K$	
s^1	$\dfrac{26.618 \times 10^{18} + 377.43 \times 10^{15} K - 2.832 \times 10^{14} K^2}{4.6503 \times 10^7 + 1.5734 \times 10^6 K}$		
s^0	$9 \times 10^{11} K$		

From the s^1 row, the condition of stability is $26.618 \times 10^4 + 3774.3K - 2.832K^2 > 0$

Or, $K^2 - 1332.73K - 93990 < 0$ or $(K - 1400)(K + 67.14) < 0$

Stability Condition: $0 < K < 1400$

The critical value of *K* for stability is 1400. With this value of K, the characteristic equation root are:

–4885.1, –57.465 + *j*676, –57.465 – *j*676, *j*748.44, and –*j*748.44

(c) $K_L = \infty$.

Forward-Path Transfer Function:

$$G(s) = \frac{nK_s K_i K}{s\left[L_a J_T s^2 + \left(R_a J_T + R_m L_a\right)s + R_a B_m + K_i K_b\right]} \qquad J_T = J_m + n^2 J_L$$

$$= \frac{891100K}{s\left(s^2 + 5000s + 566700\right)} = \frac{891100K}{s(s+116)(s+4884)}$$

Characteristic Equation of the Closed-Loop System:

$$s^3 + 5000s^2 + 566700s + 891100K = 0$$

Routh Tabulation:

s^3	1	566700
s^2	5000	$891100K$
s^1	$566700 - 178.22K$	
s^0	$8991100K$	

From the s^1 row, the condition of *K* for stability is 566700 – 178.22*K* > 0.

Stability Condition: $0 < K < 3179.78$

The critical value of K for stability is 3179.78. With K = 3179.78, the characteristic equation roots are

–5000, j752.79, and –j752.79.

When the motor shaft is flexible, K_L is finite, two of the open-loop poles are complex. As the shaft becomes stiffer, K_L increases, and the imaginary parts of the open-loop poles also increase. When $K_L = \infty$, the shaft is rigid, the poles of the forward-path transfer function are all real. Similar effects are observed for the roots of the characteristic equation with respect to the value of K_L.

5-41 (a)

$$G_c(s) = 1 \qquad G(s) = \frac{100(s+2)}{s^2 - 1} \qquad K_p = \lim_{s \to 0} G(s) = -200$$

When $d(t) = 0$, the steady-state error due to a unit-step input is

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1-200} = -\frac{1}{199} = -0.005025$$

(b)

$$G_c(s) = \frac{s+\alpha}{s} \qquad G(s) = \frac{100(s+2)(s+\alpha)}{s\left(s^2 - 1\right)} \qquad K_p = \infty \qquad e_{ss} = 0$$

(c)

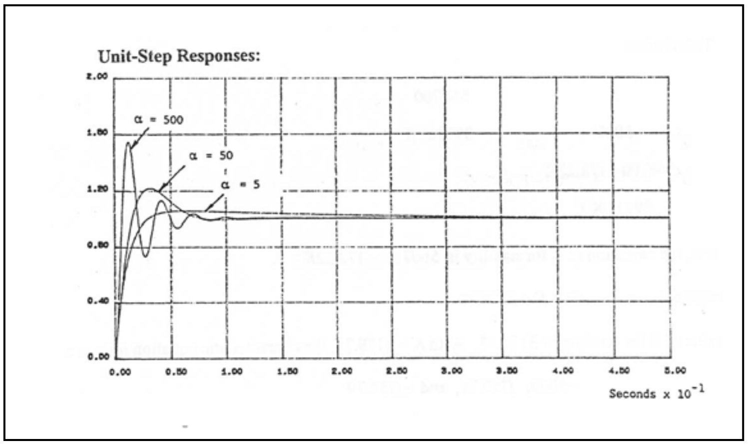
$\alpha = 5$ maximum overshoot = 5.6%

$\alpha = 50$ maximum overshoot = 22%

$\alpha = 500$ maximum overshoot = 54.6%

As the value of α increases, the maximum overshoot increases because the damping effect of the zero at $s=-\alpha$ becomes less effective.

Unit-Step Responses:



(d) $r(t)=0$ and $G_c(s)=1$. $d(t)=u_s(t)$ $\qquad D(s)=\dfrac{1}{s}$

System Transfer Function: (*r* = 0)

$$\left.\frac{Y(s)}{D(s)}\right|_{r=0}=\frac{100(s+2)}{s^3+100s^2+(199+100\alpha)s+200\alpha}$$

Output Due to Unit-Step Input:

$$Y(s)=\frac{100(s+2)}{s\left[s^3+100s^2+(199+100\alpha)s+200\alpha\right]}$$

$$y_{ss}=\lim_{t\to\infty}y(t)=\lim_{s\to 0}sY(s)=\frac{200}{200\alpha}=\frac{1}{\alpha}$$

(e) $r(t)=0,\quad d(t)=u_s(t)$

$$G_c(s)=\frac{s+\alpha}{s}$$

System Transfer Function [*r*(*t*) = 0]

$$\left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100s(s+20}{s^3+100s^2+(199+100\alpha)s+200\alpha} \qquad D(s)=\frac{1}{s}$$

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0$$

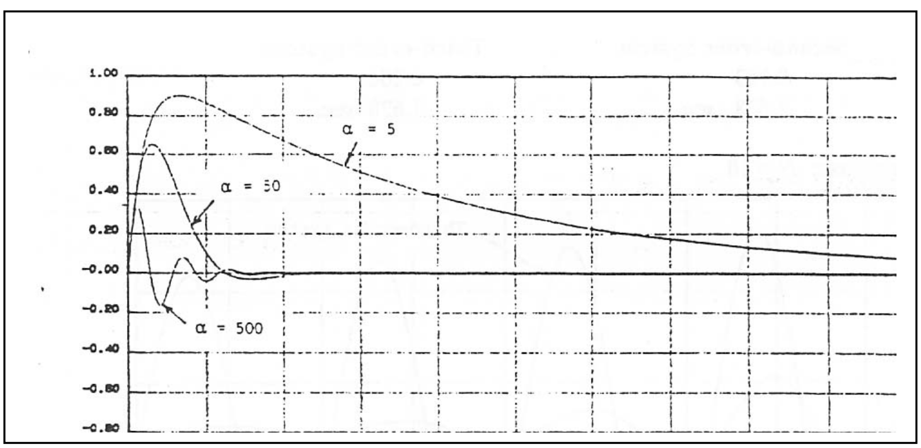
(f)

$$\alpha = 5 \qquad \left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100s(s+2)}{s^3+100s^2+699s+1000}$$

$$\alpha = 50 \qquad \left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100s(s+2)}{s^3+100s^2+5199s+10000}$$

$$\alpha = 5000 \qquad \left.\frac{Y(s)}{D(s)}\right|_{r=0} = \frac{100s(s+2)}{s^3+100s^2+50199s+100000}$$

Unit-Step Responses:



(g) As the value of α increases, the output response *y*(*t*) due to *r*(*t*) becomes more oscillatory, and the overshoot is larger. As the value of α increases, the amplitude of the output response *y*(*t*) due to *d*(*t*) becomes smaller and more oscillatory.

5-42 (a) Forward-Path Transfer function: **Characteristic Equation:**

$$G(s) = \frac{H(s)}{E(s)} = \frac{10N}{s(s+1)(s+10)} \cong \frac{N}{s(s+1)} \qquad s^2 + s + N = 0$$

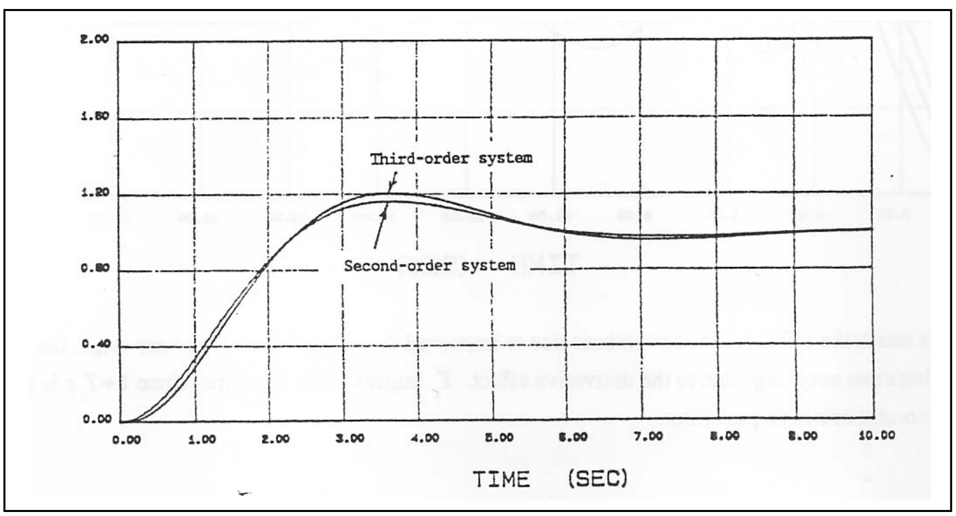
N=1: Characteristic Equation: $s^2 + s + 1 = 0$ $\zeta = 0.5$ $\omega_n = 1$ rad/sec.

Maximum overshoot $= e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.163$ (16.3%) Peak time $t_{max} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 3.628$ sec.

N=10: Characteristic Equation: $s^2 + s = 10 = 0$ $\zeta = 0.158$ $\omega_n = 10$ rad/sec.

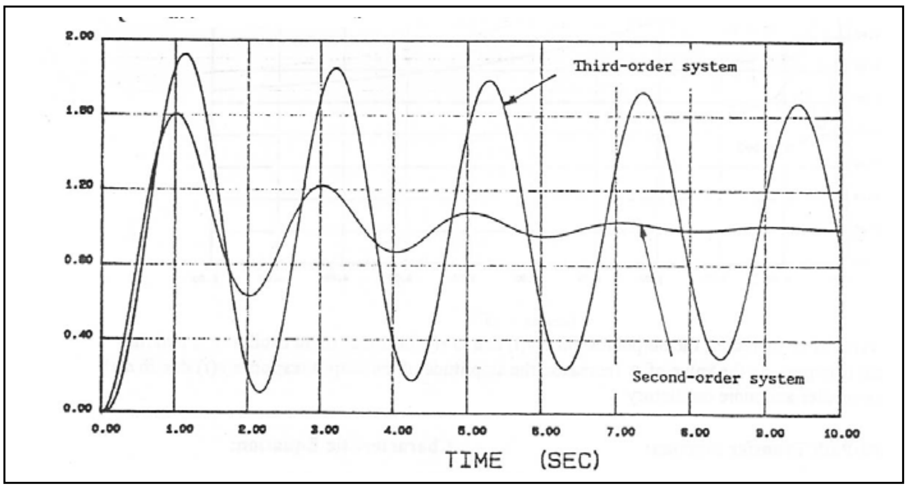
Maximum overshoot $= e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.605$ (60.5%) Peak time $t_{max} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.006$ sec.

(b) Unit-Step Response: *N* = 1



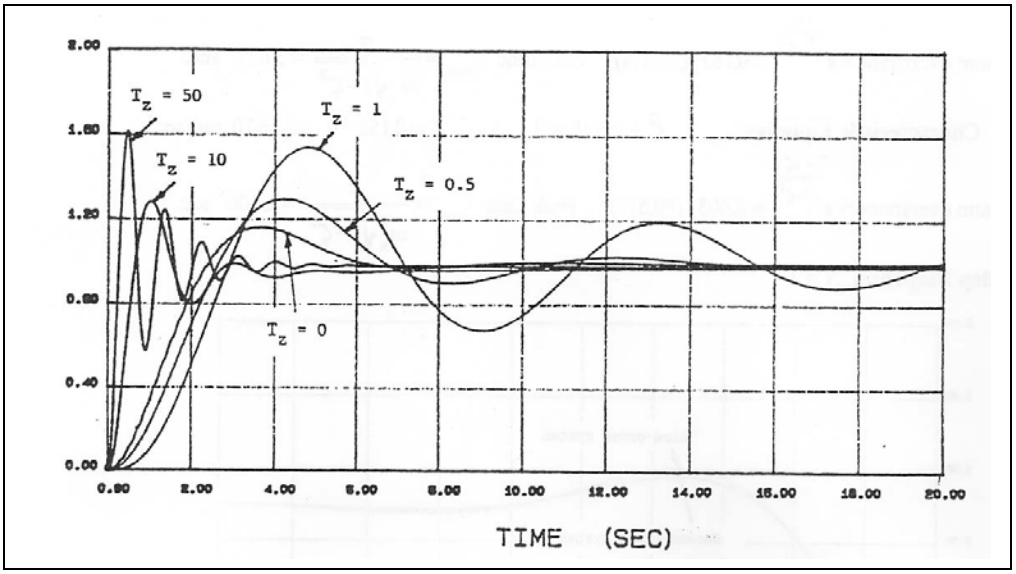
	Second-order System	Third-order System
Maximum overshoot	0.163	0.206
Peak time	3.628 sec.	3.628 sec.

Unit-Step Response: *N* = 10



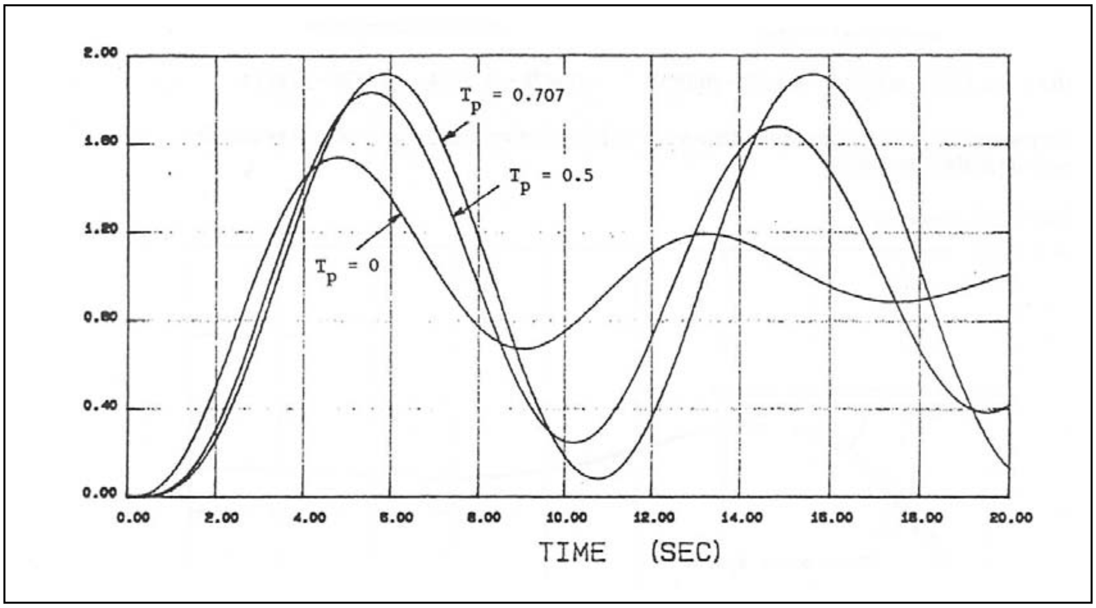
	Second-order System	Third-order System
Maximum overshoot	0.605	0.926
Peak time	1.006 sec.	1.13 sec.

5-43 **Unit-Step Responses:**



When T_z is small, the effect is lower overshoot due to improved damping. When T_z is very large, the overshoot becomes very large due to the derivative effect. T_z improves the rise time, since $1+T_z s$ is a derivative control or a high-pass filter.

5-44 **Unit-Step Responses**



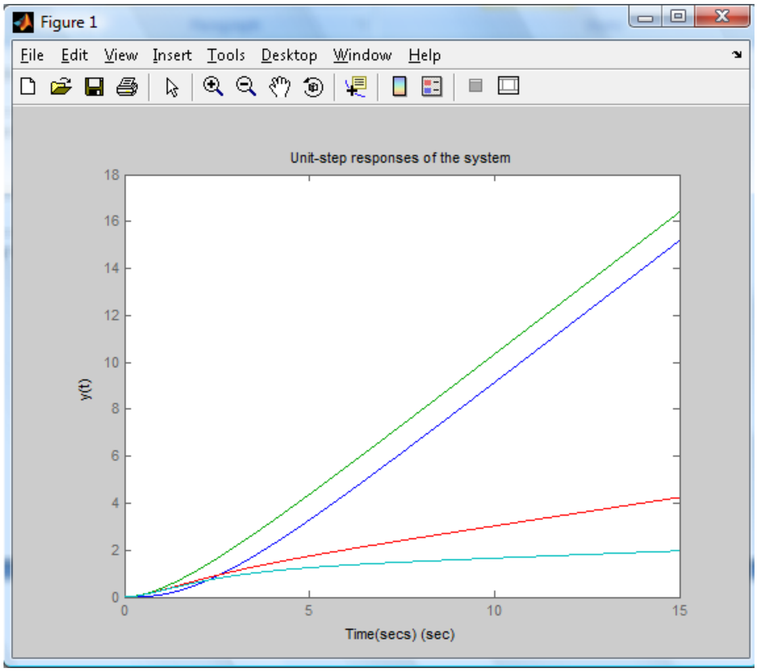
The effect of adding the pole at $s = -\frac{1}{T_p}$ to *G*(*s*) is to increase the rise time and the overshoot. The system is less stable. When $T_p > 0.707$, the closed-loop system is stable.

5-45) You may use the ACSYS software developed for this book. For description refer to Chapter 9. **We use a MATLAB code similar to toolbox 2-2-1 and those in Chapter 5 to solve this problem.**

(a) Using Toolbox 5-9-3

```
clear all
num = [];
den = [0 -0.55 -1.5];
G=zpk(num,den,1)
t=0:0.001:15;
step(G,t);
hold on;
for Tz=[1 5 20];
t=0:0.001:15;
num = [-1/Tz];
den = [0 -0.55 -1.5];
G=zpk(num,den,1)
step(G,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

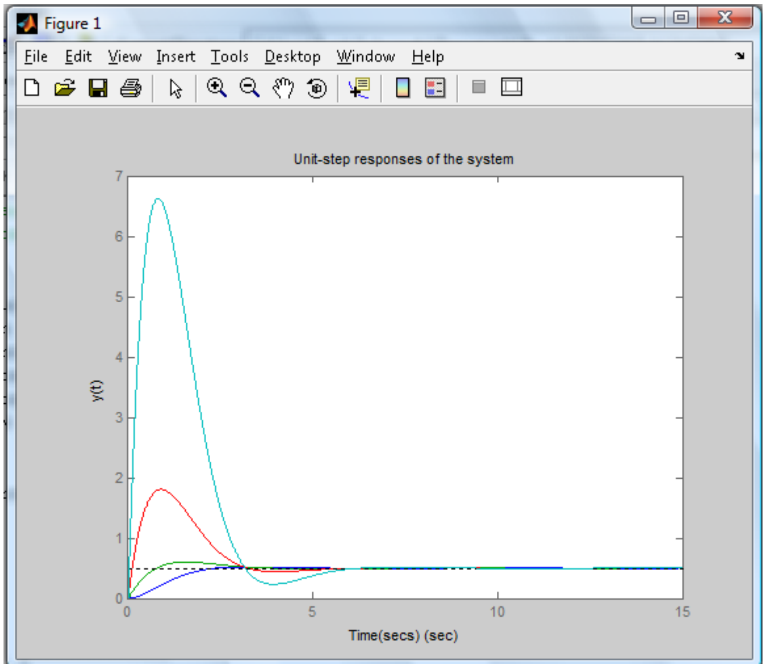
```
Zero/pole/gain:
        1
------------------
s (s+0.55) (s+1.5)


Zero/pole/gain:
      (s+1)
------------------
s (s+0.55) (s+1.5)


Zero/pole/gain:
     (s+0.2)
------------------
s (s+0.55) (s+1.5)


Zero/pole/gain:
     (s+0.05)
------------------
s (s+0.55) (s+1.5)
```



(b)

```
clear all
for Tz=[0 1 5 20];
t=0:0.001:15;
num = [Tz 1];
den = [1 2 2];
G=tf(num,den)
step(G,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')

Transfer function:
      1
-------------
s^2 + 2 s + 2


Transfer function:
    s + 1
-------------
s^2 + 2 s + 2


Transfer function:
   5 s + 1
-------------
s^2 + 2 s + 2
```

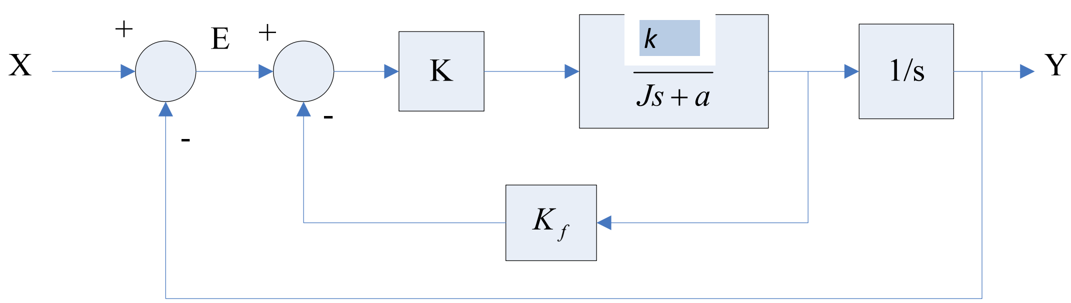
```
Transfer function:
  20 s + 1
-------------
s^2 + 2 s + 2
```



Follow the same procedure for other parts.

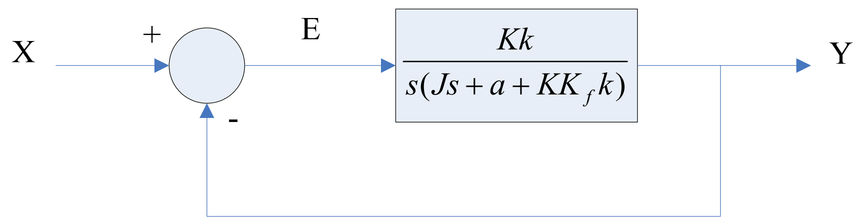
5-46) Since the system is linear we use superposition to find Y, for inputs X and D

First, consider D = 0



Then

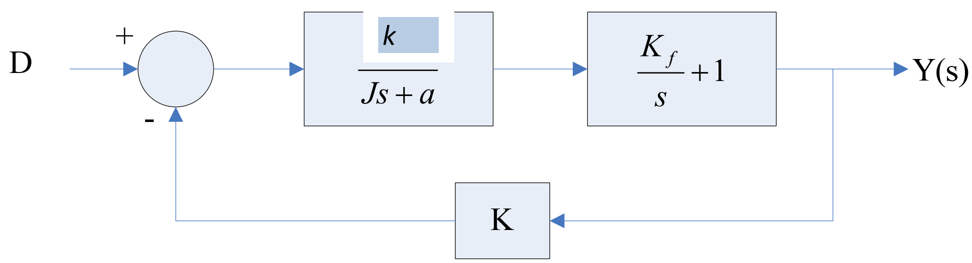
$$\frac{Y}{X} = G(s) = \frac{G_1}{1+G_1} = \frac{Kk}{s(Js+a+KkK_f)+Kk}\ ;\ G_1 = \frac{Kk}{s(Js+a+KkK_f)}$$



According to above block diagram:

$$E(s) = X(s) - Y(s) = X(s) - \frac{X(s)G_1(s)}{1+G_1(s)} = \frac{1}{1+G_1(s)}X(s)$$

Now consider X = 0, then:



Accordingly,

$$G_2(s) = \frac{k(K_f + s)}{s(Js+a)}$$

and

$$Y(s) = \frac{G_2(s)}{1+KG_2(s)}D(s) = \frac{k(K_f + s)}{s(Js+a) + k(K_f + s)K}D(s)$$

In this case, *E(s) = D-KY(s)*

$$E(s) = D(s) - \frac{KG_2(s)}{1+KG_2(s)}D(s) = -\frac{1}{1+KG_2(s)}D(s)$$

Now the steady state error can be easily calculated by:

$$\begin{cases} for\ unit\ step\ input\ X = 1/s:\ \ e_{ss} = \lim_{s\to 0}\dfrac{1}{1+G_1(s)} = \dfrac{Kk}{s\left(Js+a+KkK_f\right)} \\ \lim_{s\to 0}\dfrac{1}{1+\dfrac{Kk}{s\left(Js+a+KkK_f\right)}} = \lim_{s\to 0}\dfrac{s\left(Js+a+KkK_f\right)}{s\left(Js+a+KkK_f\right)+Kk} = 0 \\ for\ ramp\ input\ D = \dfrac{1}{s^2}:\ \ e_{ss} = \lim_{s\to 0} - \dfrac{1}{s\left(1+KG_2(s)\right)} \\ = \lim_{s\to 0}\dfrac{-1}{s\left(1+K\dfrac{k\left(K_f+s\right)}{s(Js+a)}\right)} \\ = \lim_{s\to 0}\dfrac{-s(Js+a)}{s\left(s(Js+a)+Kk\left(K_f+s\right)\right)} = - \dfrac{-a}{KkK_f} \end{cases}$$

(c) The overall response is obtained through superposition

$$Y(s) = Y(s)\big|_{D=0} + Y(s)\big|_{X=0}$$
$$y(t) = y(t)\big|_{d(t)=0} + y(t)\big|_{x(t)=0}$$

$$Y(s) = \frac{Kk}{s\left(Js+a+KkK_f\right)+Kk}X(s) + \frac{k\left(K_f+s\right)}{s(Js+a)+k\left(K_f+s\right)K}D(s)$$

MATLAB

```
clear all
syms s K k J a Kf
X=1/s;
D=1/s^2
Y=K*k*X/(s*(J*s+a+K*k*Kf)+K*k)+k*(Kf+s)*D/(s*(J*s+a)+k*(Kf+s)*K)
ilaplace(Y)
D =
1/s^2
Y =
K*k/s/(s*(J*s+a+K*k*Kf)+K*k)+k*(Kf+s)/s^2/(s*(J*s+a)+k*(Kf+s)*K)
ans =

1+t/K+1/k/K^2/Kf/(a^2+2*a*K*k+K^2*k^2-
4*J*K*k*Kf)^(1/2)*sinh(1/2*t/J*(a^2+2*a*K*k+K^2*k^2-4*J*K*k*Kf)^(1/2))*exp(-
1/2*(a+K*k)/J*t)*(a^2+a*K*k-2*J*K*k*Kf)-cosh(1/2*t/J*(a^2+2*a*K*k*Kf+K^2*k^2*Kf^2-
4*J*K*k)^(1/2))*exp(-1/2*(a+K*k*Kf)/J*t)-(a+K*k*Kf)/(a^2+2*a*K*k*Kf+K^2*k^2*Kf^2-
4*J*K*k)^(1/2)*sinh(1/2*t/J*(a^2+2*a*K*k*Kf+K^2*k^2*Kf^2-4*J*K*k)^(1/2))*exp(-
1/2*(a+K*k*Kf)/J*t)+1/k/K^2/Kf*a*(-1+exp(-
1/2*(a+K*k)/J*t)*cosh(1/2*t/J*(a^2+2*a*K*k+K^2*k^2-4*J*K*k*Kf)^(1/2)))
```

5-47) (a) Find the $\int_0^\infty e(t)dt$ when *e(t)* is the error in the unit step response.

As the system is stable then $\int_0^\infty e(t)dt$ will converge to a constant value:

$$\int_0^\infty e(t)dt = \lim_{s\to 0} s\frac{E(s)}{s} = \lim_{s\to 0} E(s)$$

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)} = \frac{(A_1s+1)(A_2s+1)\ldots(A_ns+1)}{(B_1s+1)(B_2s+1)\ldots(B_ms+1)} \quad n \le m$$

Try to relate this to Equation (5-40).

$$\mathrm{E}(s) = X(s) - Y(s) = X(s) - \frac{G(s)}{1+G(s)}X(s) = \frac{1}{1+G(s)}X(s)$$

$$= \left(1 - \frac{(A_1s+1)(A_2s+1)\ldots(A_ns+1)}{(B_1s+1)(B_2s+1)\ldots(B_ms+1)}\right)X(s) = \left(\frac{(B_1s+1)(B_2s+1)\ldots(B_ms+1) - (A_1s+1)(A_2s+1)\ldots(A_ns+1)}{(B_1s+1)(B_2s+1)\ldots(B_ms+1)}\right)X(s)$$

$$\lim_{s\to 0} E(s) = \lim_{s\to 0}\frac{1}{s}\left(\frac{(B_1s+1)(B_2s+1)\ldots(B_ms+1) - (A_1s+1)(A_2s+1)\ldots(A_ns+1)}{(B_1s+1)(B_2s+1)\ldots(B_ms+1)}\right)$$

$$= \left(B_1 + B_2 + \cdots + B_m\right) - \left(A_1 + A_2 + \cdots + A_n\right)$$

$$G(s) = \left(\frac{(B_1s+1)(B_2s+1)\ldots(B_ms+1)}{(B_1s+1)(B_2s+1)\ldots(B_ms+1) - (A_1s+1)(A_2s+1)\ldots(A_ns+1)}\right) - 1$$

$$G(s) = \left(\frac{(A_1s+1)(A_2s+1)\ldots(A_ns+1)}{(B_1s+1)(B_2s+1)\ldots(B_ms+1) - (A_1s+1)(A_2s+1)\ldots(A_ns+1)}\right)$$

$$e_{ss} = \lim_{s\to 0} s\frac{1}{s}\left(1 - \frac{(A_1s+1)(A_2s+1)\ldots(A_ns+1)}{(B_1s+1)(B_2s+1)\ldots(B_ms+1)}\right) = 0$$

(b) Calculate $\frac{1}{K} = \frac{1}{\lim_{s \to 0} sG(s)}$

Recall

$$\mathrm{E}(\mathrm{s}) = X(s) - Y(s) = X(s) - \frac{G(s)}{1+G(s)} X(s) = \frac{1}{1+G(s)} X(s)$$

Hence

$$\lim_{s \to 0} \mathrm{E}(\mathrm{s}) = \lim_{s \to 0} \frac{1}{1+G(s)} X(s) = \lim_{s \to 0} \frac{1}{s+sG(s)} = \lim_{s \to 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

Ramp Error Constant

5-48)

$$\frac{C(s)}{R(s)} = \frac{10(s+K)}{(s+p)(s+25)+(s+K)10} = \frac{10(s+K)}{s^2+(35+p)s+(25\,p+10K)}$$

Comparing with the second order prototype system and matching denominators:

$$\begin{cases} 25\,p + 10K = \omega_n^2 \\ 35 + p = 2\,\xi\omega_n \end{cases}$$

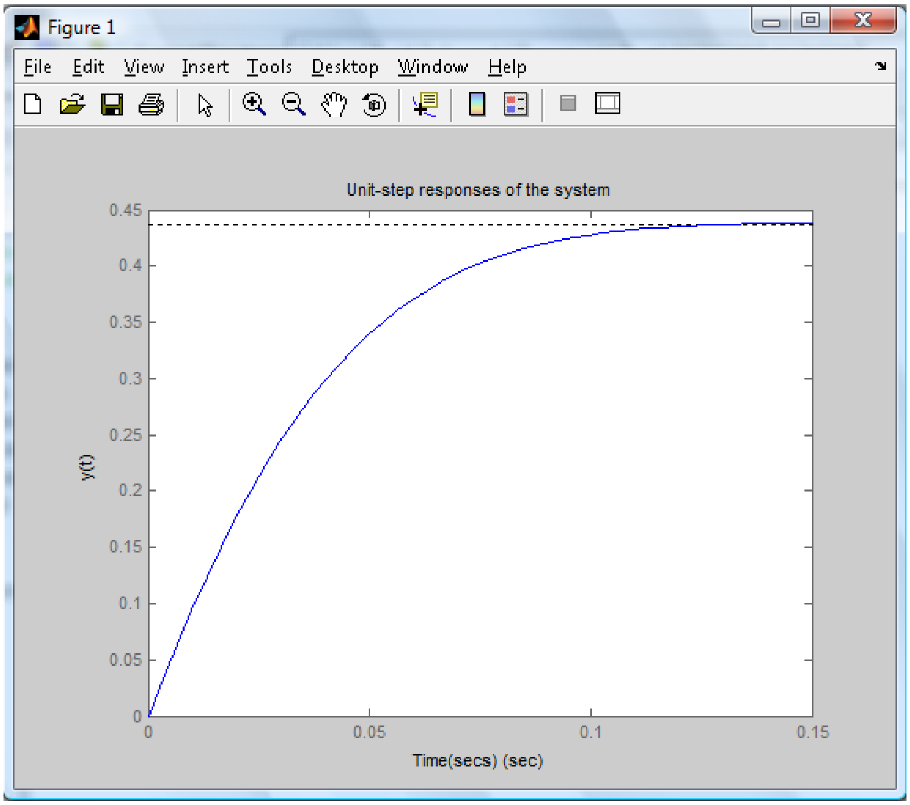
$$\begin{cases} M_\rho = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) \geq 0.25 \;\rightarrow\; \frac{\pi\xi}{\sqrt{1-\xi^2}} \geq 1.386 \rightarrow \xi \geq 0.210 \\ t_s = \frac{3.2}{\xi\omega_n} \leq 0.1 \rightarrow \;\omega_n \leq 80,\;\; when\; 0 < \xi < 0.69 \end{cases}$$

Let $\xi = 0.4$ and $\omega_n = 80$

Then

$$\begin{cases} p = (2\,\xi\omega_n)(35) = 29 \\ K = \frac{\omega_n^2 - 25\,p}{100} = 56.25 \end{cases}$$

```
clear all
p=29;
K=56.25;
num = [10 10*K];
den = [1 35+p 25*p+10*K];
G=tf(num,den)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')

Transfer function:
  10 s + 562.5
-----------------
s^2 + 64 s + 1288
```

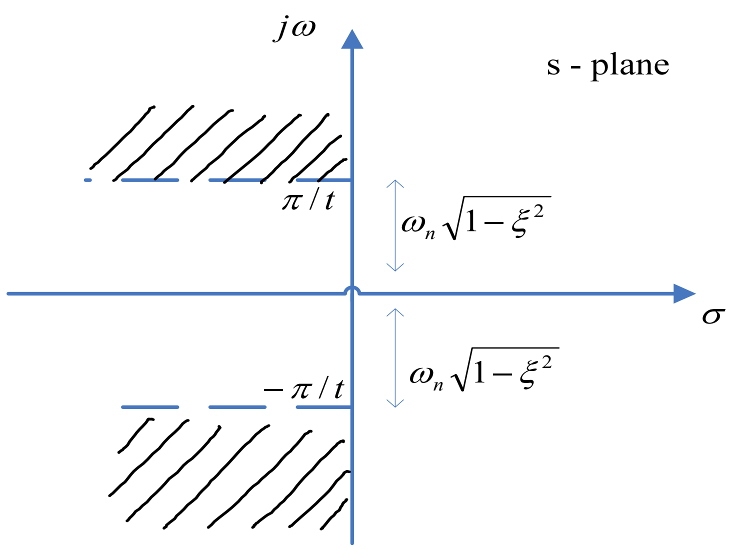


5-49) According to the maximum overshoot:

$$t_{max} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

which should be less than t, then

$$\frac{\pi}{\omega_n\sqrt{1-\xi^2}} < t \text{ or } \omega_n\sqrt{1-\xi^2} > \frac{\pi}{t}$$



5-50) Using a 2nd order prototype system format, from Figure 5-15, ω_n is the radial distance from the complex conjugate roots to the origin of the s-plane, then ω_n with respect to the origin of the shown region is $\omega_n \approx 3.6$.

Therefore the natural frequency range in the region shown is around $2.6 \leq \omega_n \leq 4.6$

On the other hand, the damping ratio ζ at the two dashed radial lines is obtained from:

$$\begin{cases} \zeta_1 = \cos(\pi/2 - \alpha_1) = \sin\alpha_1 \\ \zeta_2 = \cos(\pi/2 - \alpha_2) = \sin\alpha_2 \end{cases}$$

The approximation from the figure gives:

$$\begin{cases} \zeta_1 \approx 0.56 \\ \zeta_2 \approx 0.91 \end{cases}$$

Therefore $0.56 \leq \zeta \leq 0.91$

b)

$$\frac{C(s)}{R(s)} = \frac{KK_p(s+K_I)}{s^2 + (p + KK_p)s + KK_pK_I}$$

As $K_p=2$, then:

$$\frac{C(s)}{R(s)} = \frac{2K(s+K_I)}{s^2 + 2(K+1)s + 2KK_I}$$

If the roots of the characteristic equations are assumed to be lied in the centre of the shown region:

$$\begin{cases} P_1 = s - 3 - 2j \\ P_2 = s - 3 + 2j \end{cases} \Rightarrow s^2 + 6s + 13 = 0$$

Comparing with the characteristic equation:

$$\begin{cases} 2(K+1) = 6 \rightarrow \quad K = 2 \\ 2KK_I = 13 \rightarrow K_I = 3.25 \end{cases}$$

c) The characteristic equation

$$s^2 + 2(p + KK_p)s + KK_pK_I = 0$$

is a second order polynomial with two roots. These two roots can be determined by two terms $2(p+KK_p)$ and KK_pK_I which includes four parameters. Regardless of the p and K_p values, we can always choose K and K_I so that to place the roots in a desired location.

5-51) a) $J_m s^2\theta_m(s) + \left(B + \frac{K_1K_2}{R}\right)s\theta(s) = \frac{K_1}{R}V(s)$

$$\frac{\theta_m(s)}{V(s)} = \frac{\frac{K_1}{R}}{s\left(s + \left(B + \frac{K_1K_2}{R}\right)\right)}$$

By substituting the values:

$$\frac{\theta_m(s)}{V(s)} = \frac{0.2}{s(s + 0.109)}$$

b) Speed of the motor is $\frac{d\theta}{dt} = \omega$

$$\frac{\omega(s)}{V(s)} = \frac{s\theta(s)}{V(s)} = \frac{0.2}{s + 0.109}$$

$e_{ss} = V \lim_{s\to 0} G(s) = 10\frac{0.2}{0.109} = 19.23$

(c)

$$\frac{\theta_m(s)}{V(s)} = \frac{0.2}{s(s + 0.109)}$$

d)

$$\theta_m(s) = \frac{0.2}{s(s + 0.109)}V(s)$$
$$= \frac{0.2}{s(s + 0.109)}K\left(\theta_p(s) - \theta_m(s)\right)$$

As a result:

$$\frac{\theta_m(s)}{\theta_p(s)} = \frac{0.2K}{s^2 + 0.109s + 0.2K}$$

e) As $M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.2s$, then, $\xi = 0.404$

According to the transfer function, $2\,\xi\omega_n = 0.109$, then $\omega_n = \frac{0.109}{(2)(0.404)} \approx 0.14 rad/sec$

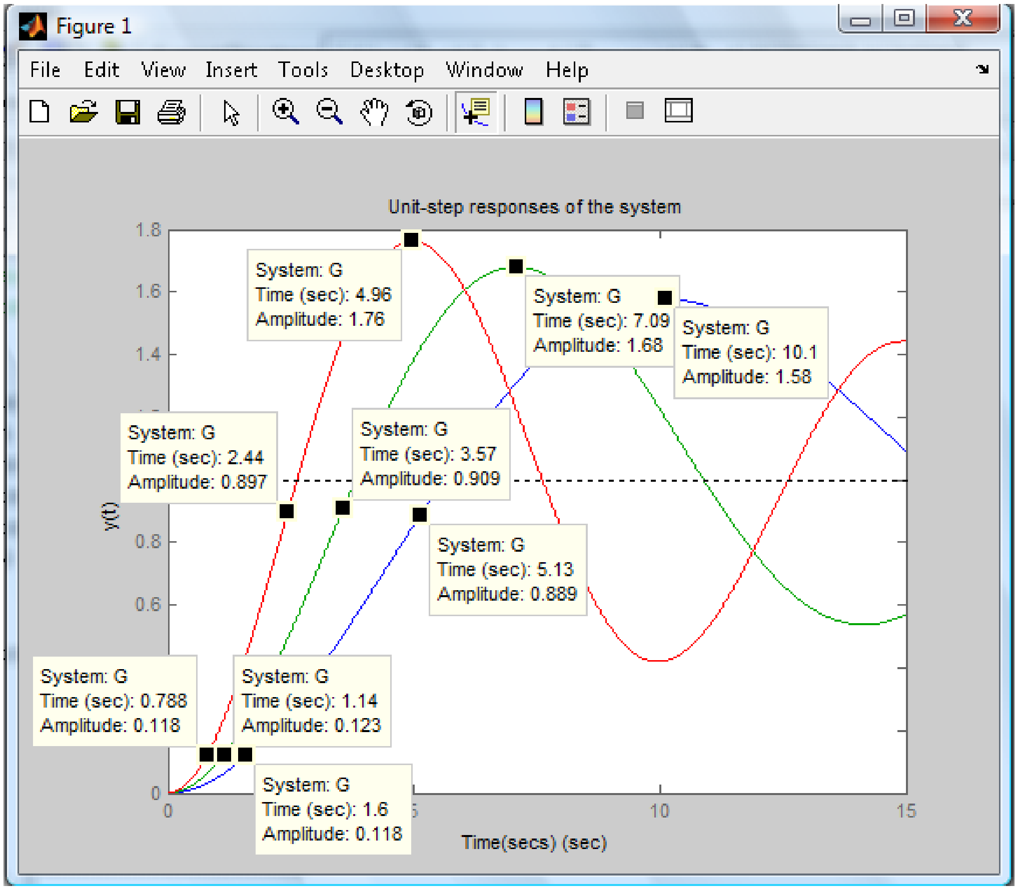
where $\omega_n^2 = 0.2K$ ⇨ $K<0.0845$

f) As $t_r = \frac{0.8+2.5\,\xi}{\omega_n} \approx \frac{1.8}{\omega_n}$, then $\omega_n \geq 0.6$,

as $\omega_n^2 = 0.2K$, therefore; $K \geq 1$

g) MATLAB

```
clear all
for K=[0.5 1 2];
t=0:0.001:15;
num = [0.2*K];
den = [1 0.109 0.2*K];
G=tf(num,den)
step(G,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')


Transfer function:
        0.1
-------------------
s^2 + 0.109 s + 0.1


Transfer function:
        0.2
-------------------
s^2 + 0.109 s + 0.2


Transfer function:
        0.4
-------------------
s^2 + 0.109 s + 0.4
```



Rise time decreases with *K* increasing.

Overshoot increases with *K*.

5-52) $M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.1$, therefore, $\xi = 0.59$

As $t_s = \frac{3.2}{\xi\omega_n}$, then, $\omega_n = \frac{3.2}{(0.59)(1.5)} \approx 3.62$

$$\frac{Y(s)}{X(s)} = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K(s+a)}{s(s+3)(s+b)+K(s+a)}$$
$$= \frac{K(s+a)}{s^3+(3+b)s^2+(3b+K)s+Ka}$$

Therefore:

$$s^3+(3+b)s^2+(3b+K)s+Ka = (s+p)(s^2+2\,\xi\omega_n s+\omega^2)$$

If p is a non-dominant pole; therefore comparing both sides of above equation and:

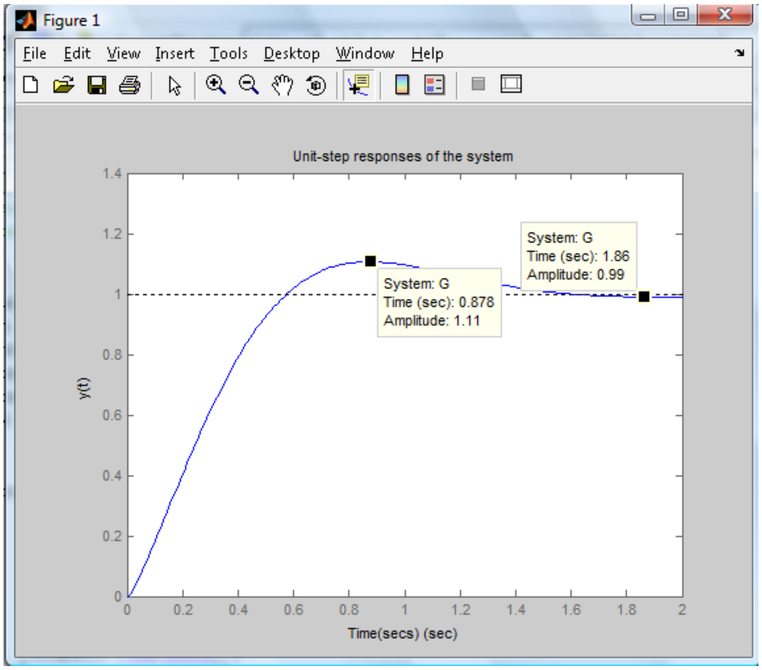
$$\begin{cases} 2\,\xi\omega_n + p = 3+b \\ 2\xi\omega_n p + \omega_n^2 = 3b+k \\ \omega_n^2 p = Ka \end{cases}$$

If we consider p = 10a (non-dominant pole), $\xi = 0.6$ and $\omega_n = 4$, then:

$$\begin{cases} 4.8+10a = 3+b \\ 48a+16 = 3b+K \\ 160a = Ka \;\rightarrow\; K = 160 \end{cases}$$

$$\begin{cases} a = 8.3 \\ b = 89.8 \\ p = 83 \end{cases}$$

```
clear all
K=160;
a=8.3;
b=89.8;
p=83;
num = [K K*a];
den = [1 3+b 3*b+K K*a];
G=tf(num,den)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')

Transfer function:
         160 s + 1328
-------------------------------
s^3 + 92.8 s^2 + 429.4 s + 1328
```



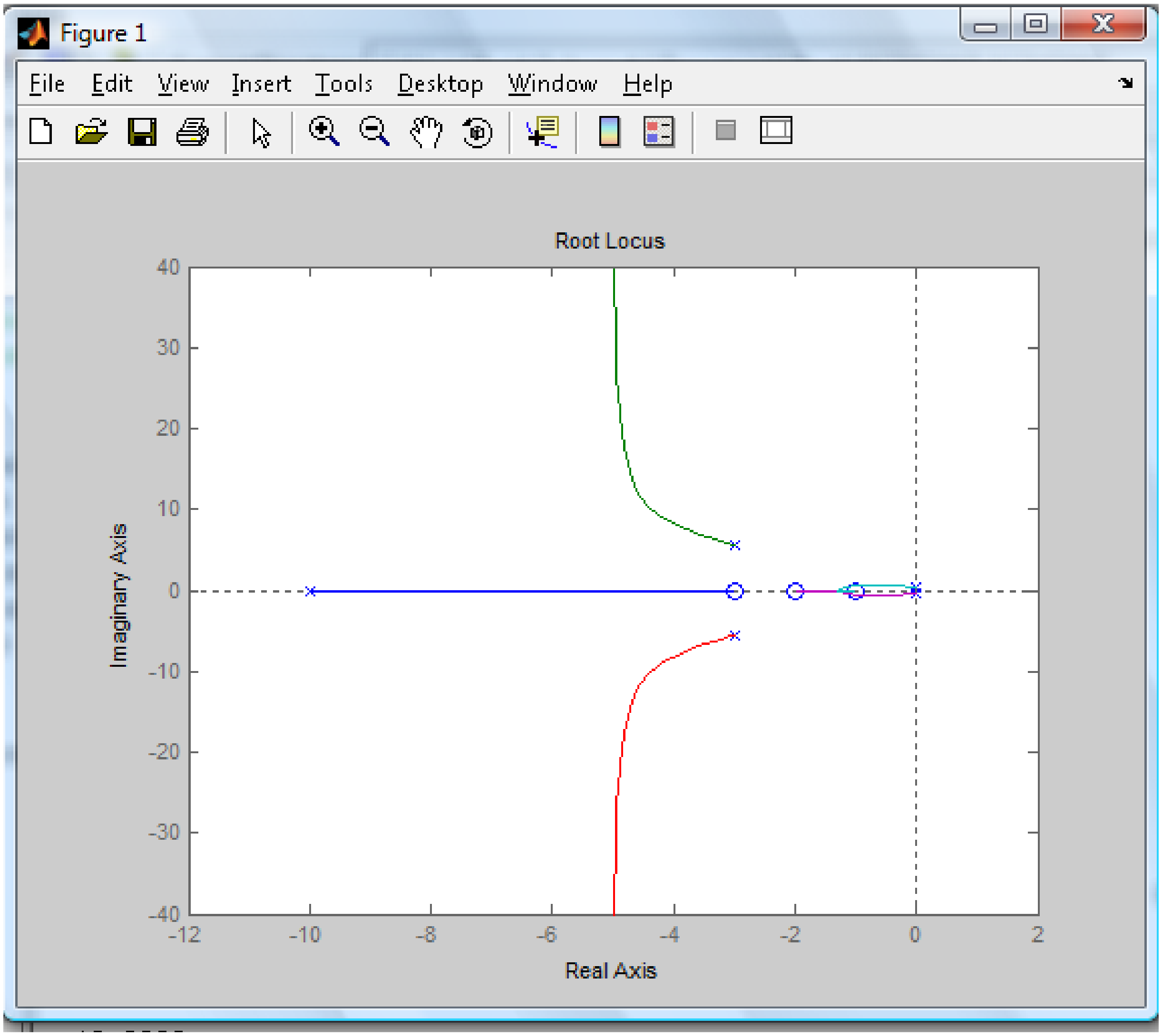
Both Overshoot and settling time values are met. No need to adjust parameters.

5-53) For the controller and the plant to be in series and using a unity feedback loop we have:

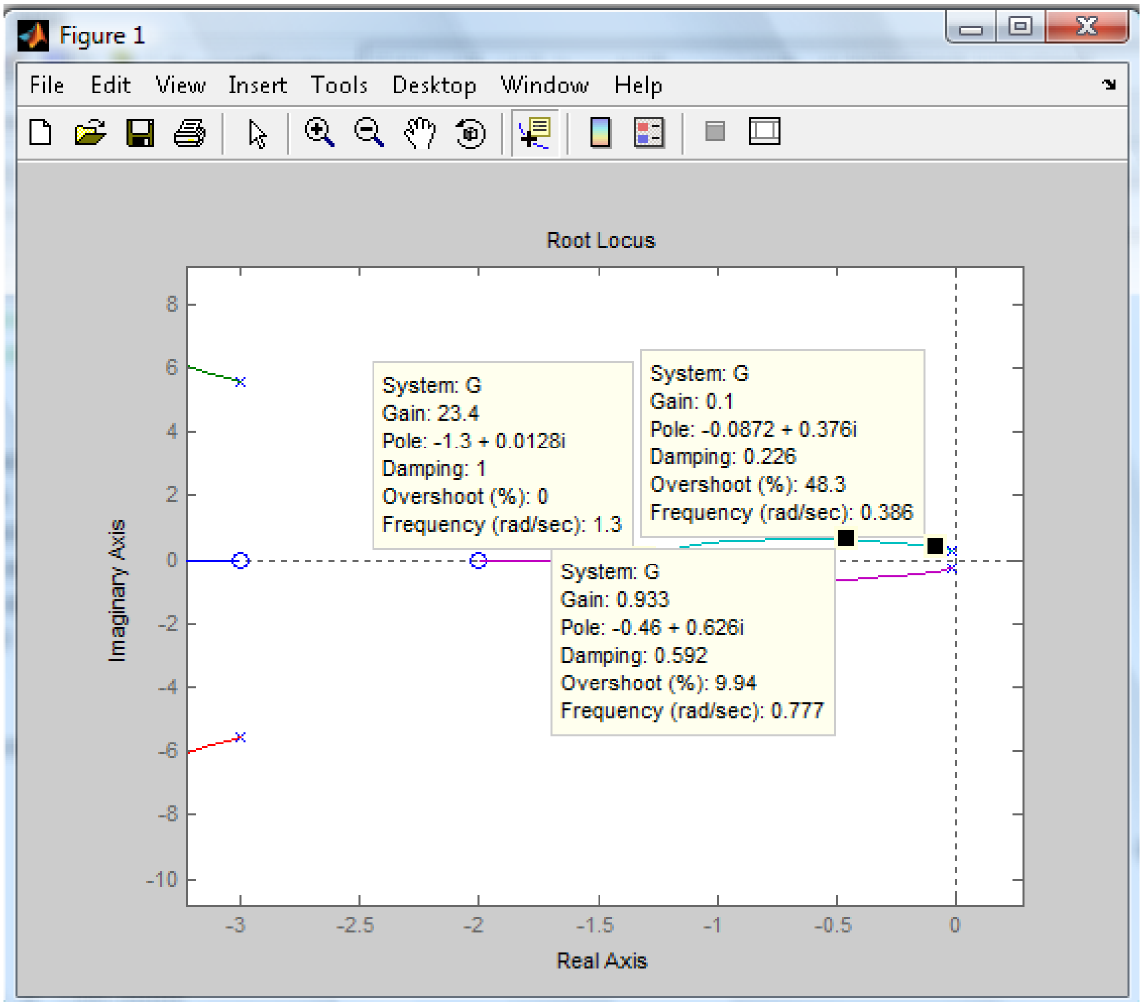
MATLAB

USE toolbox 5-8-3

```
clear all
num=[-1 -2 -3];
denom=[-3+sqrt(9-40) -3-sqrt(9-40) -0.02+sqrt(.004-.07) -0.02-sqrt(.004-.07) -10];
G=zpk(num,denom,60)
rlocus(G)

Zero/pole/gain:
         60 (s+1) (s+2) (s+3)
-----------------------------------------------
(s+10) (s^2  + 0.04s + 0.0664) (s^2  + 6s + 40)
```



Note the system has two dominant complex poles close to the imaginary axis. Lets zoom in the root locus diagram and use the cursor to find the parameter values.

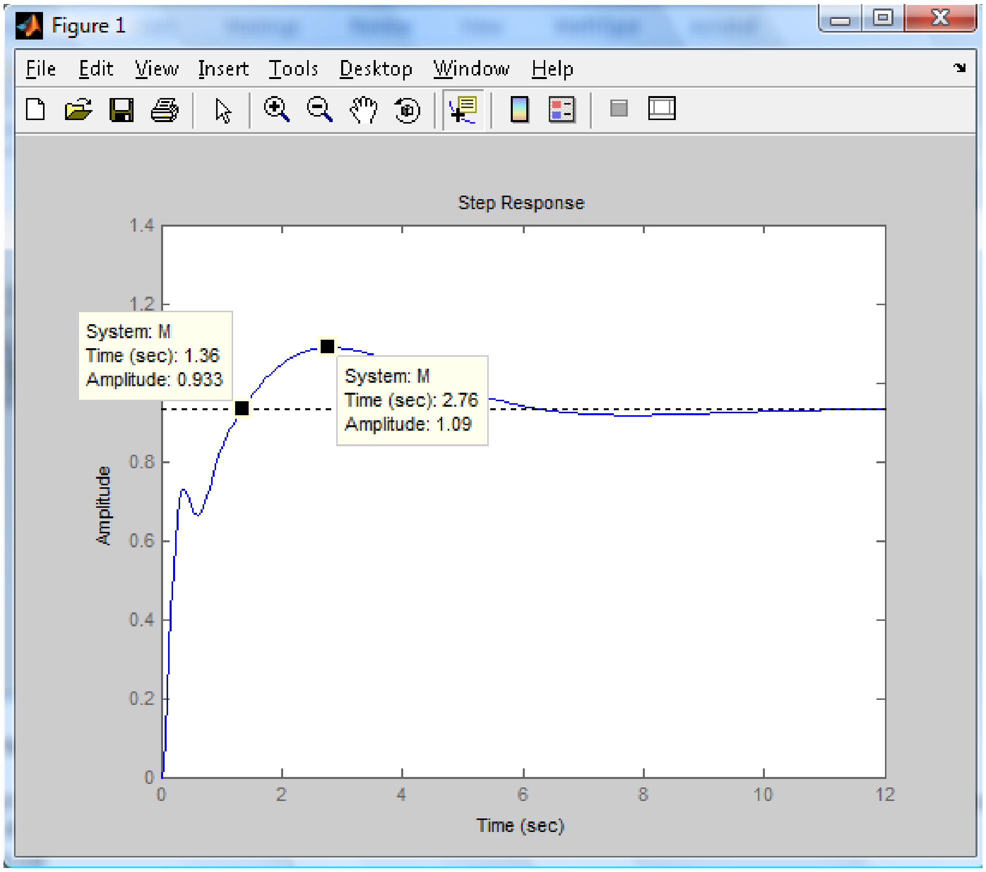


As shown for K=0.933 the dominant closed loop poles are at -0.46±j 0.626 AND OVERSHOOT IS ALMOST 10%.

Increasing K will push the poles closer towards less dominant zeros and poles. As a process the design process becomes less trivial and more difficult.

To confirm use

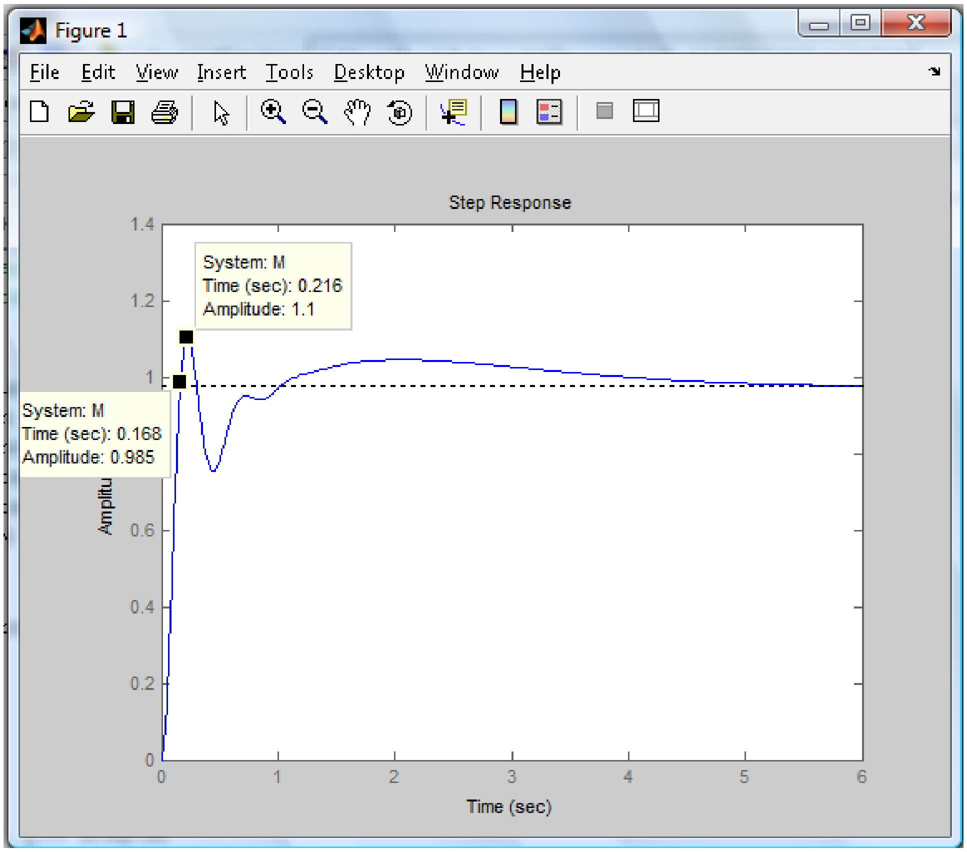
```
M=feedback(G*.933,1) %See toolbox 5-4-2
step(M)
Zero/pole/gain:

          55.98 (s+3) (s+2) (s+1)

-----------------------------------------------------------

(s+7.048) (s^2  + 0.9195s + 0.603) (s^2  + 8.072s + 85.29)
```



To reduce rise time, the poles have to move to left to make the secondary poles more dominant. As a result the little bump in the left hand side of the above graph should rise. Try K=3:

```
>> M=feedback(G*3,1)

Zero/pole/gain:
             180 (s+3) (s+2) (s+1)
---------------------------------------------------------
(s+5.01) (s^2  + 1.655s + 1.058) (s^2  + 9.375s + 208.9)

>> step(M)
```

****Try a higher K value, but looking at the root locus and the time plots, it appears that the overshoot and rise time criteria will never be met simultaneously.**



K=5

M=feedback(G*5,1) %See toolbox 5-4-2
step(M)

Zero/pole/gain:

$$\frac{300\ (s+3)\ (s+2)\ (s+1)}{(s+4.434)\ (s^2 + 1.958s + 1.252)\ (s^2 + 9.648s + 329.1)}$$

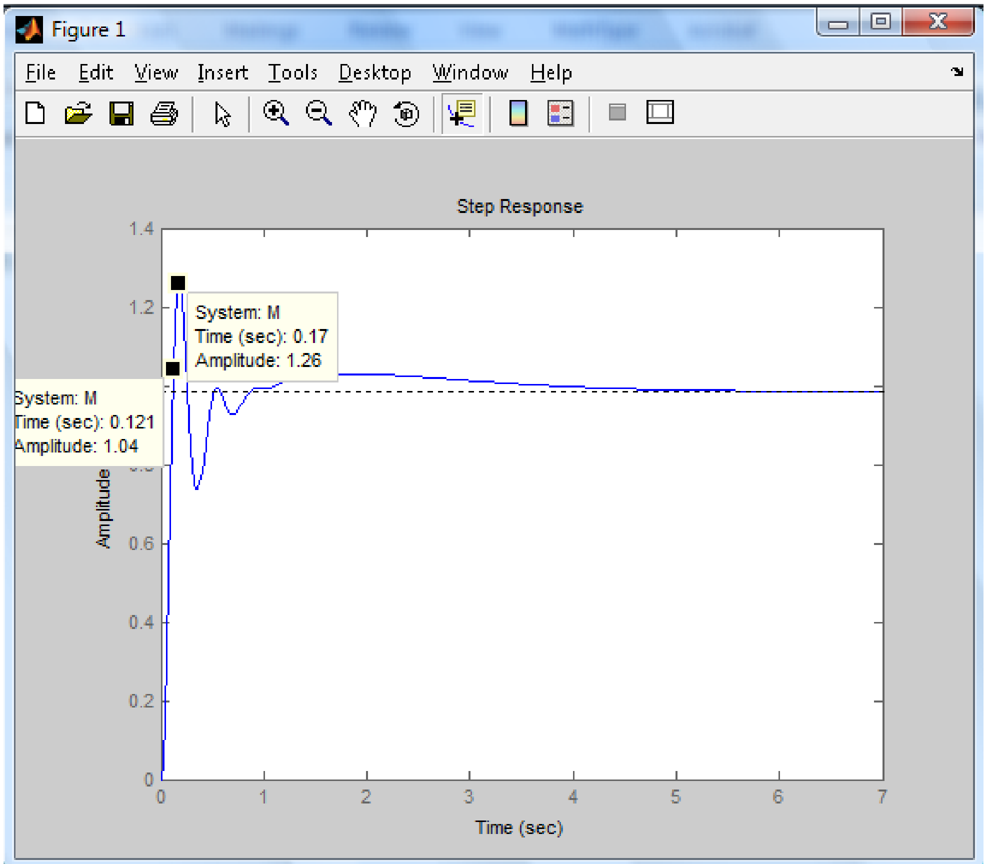


Figure 1
File Edit View Insert Tools Desktop Window Help
Step Response
System: M
Time (sec): 0.17
Amplitude: 1.26
System: M
Time (sec): 0.121
Amplitude: 1.04
Amplitude
Time (sec)
1.4
1.2
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7

5-54) Forward-path Transfer Function:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20+a)s^2 + (200+20a)s + 200a - K}$$

For type 1 system, $200a - K = 0$ Thus $K = 200a$

Ramp-error constant:

$$K_v = \lim_{s \to 0} sG(s) = \frac{K}{200 + 20a} = \frac{200a}{200 + 20a} = 5$$ Thus $a = 10$ $K = 2000$

MATLAB Symbolic tool can be used to solve above. We use it to find the roots for the next part:

```
>> syms s a K

>>solve(5*200+5*20*a-200a)

ans =

10

>> D=(s^2+20*s+200)*s+a))

 D =

(s^2+20*s+200)*(s+a)

>> expand(D)

ans =

s^3+s^2*a+20*s^2+20*s*a+200*s+200*a

>> solve(ans,s)

ans =

      -a

 -10+10*i

 -10-10*i
```

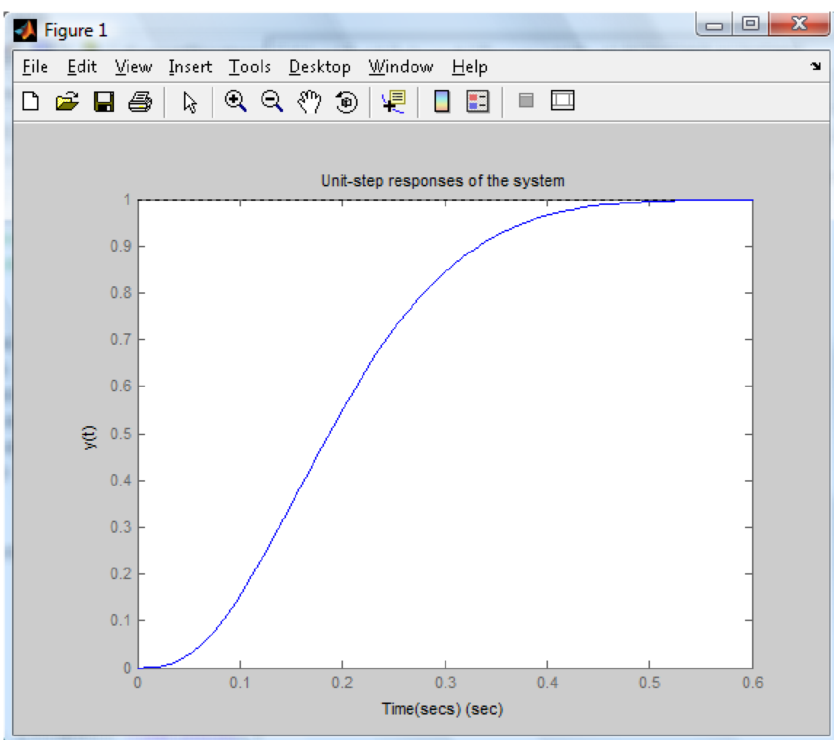
The forward-path transfer function is The controller transfer function is

$$G(s) = \frac{2000}{s\left(s^2 + 30s + 400\right)} \qquad G_c(s) = \frac{G(s)}{G_p(s)} = \frac{20\left(s^2 + 10s + 100\right)}{\left(s^2 + 30s + 400\right)}$$

The maximum overshoot of the unit-step response is 0 percent.

MATLAB

```
clear all
K=2000;
a=10;
num = [];
den = [-10+10i -10-10i -a];
G=zpk(num,den,K)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')


Zero/pole/gain:
          2000
-------------------------

(s+10) (s^2  + 20s + 200)
```



Clearly PO=0.

5-55)

Forward-path Transfer Function:

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{K}{s^3+(20+a)s^2+(200+20a)s+200a-K}$$

For type 1 system, $200a-K=0$ Thus $K = 200a$

Ramp-error constant:

$$K_v=\lim_{s\to 0} sG(s)=\frac{K}{200+20a}=\frac{200a}{200+20a}=9$$ Thus $a = 90$ $K = 18000$

MATLAB Symbolic tool can be used to solve above. We use it to find the roots for the next part:

```
>> syms s a K

solve(9*200+9*20*a-200*a)

 ans =

 90

>>D=(s^2+20*s+200)*s+a))

 D =

(s^2+20*s+200)*(s+a)

>> expand(D)

ans =

s^3+s^2*a+20*s^2+20*s*a+200*s+200*a

>> solve(ans,s)

ans =

      -a

 -10+10*i

 -10-10*i
```

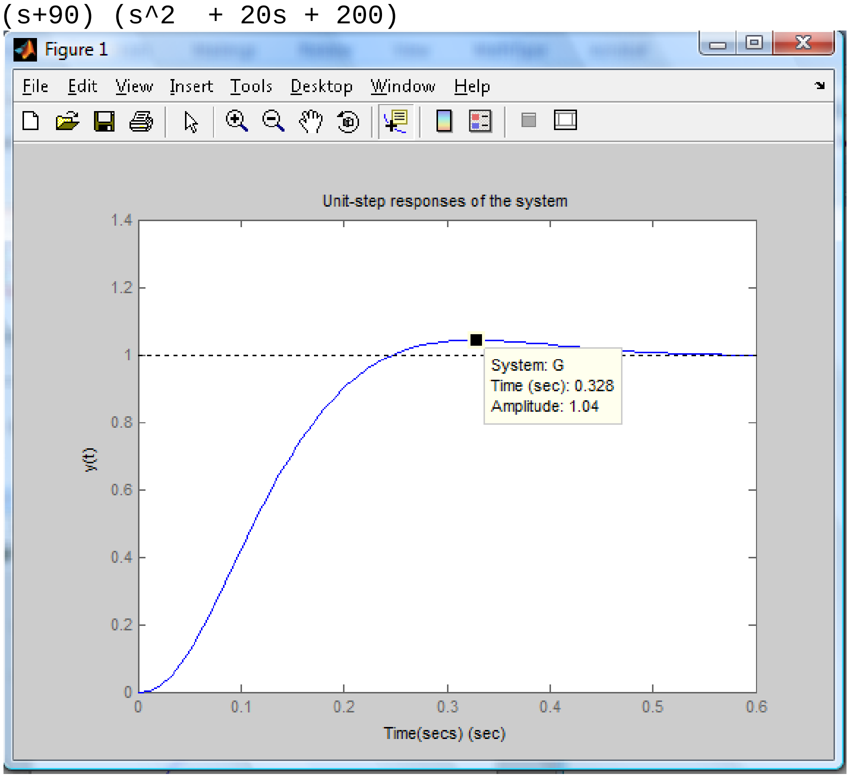
The forward-path transfer function is The controller transfer function is

$$G(s) = \frac{18000}{s\left(s^2 + 110s + 2000\right)} \qquad G_c(s) = \frac{G(s)}{G_p(s)} = \frac{180\left(s^2 + 10s + 100\right)}{\left(s^2 + 110s + 2000\right)}$$

The maximum overshoot of the unit-step response is 4.3 percent.

From the expression for the ramp-error constant, we see that as *a* or *K* goes to infinity, K_v approaches 10. Thus the maximum value of K_v that can be realized is 10. The difficulties with very large values of *K* and *a* are that a high-gain amplifier is needed and unrealistic circuit parameters are needed for the controller.

```
clear all
K=18000;
a=90;
num = [];
den = [-10+10i -10-10i -a];
G=zpk(num,den,K)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')

Zero/pole/gain:
          18000
-------------------------
(s+90) (s^2  + 20s + 200)
```



PO is less than 4.

5-56) (a) Ramp-error Constant:

MATLAB

```
clear all
syms s Kp Kd kv
Gnum=(Kp+Kd*s)*1000
Gden= (s*(s+10))
G=Gnum/Gden
Kv=s*G
s=0
eval(Kv)

Gnum =
 1000*Kp+1000*Kd*s

Gden =
 s*(s+10)

G =
 (1000*Kp+1000*Kd*s)/s/(s+10)

Kv =
 (1000*Kp+1000*Kd*s)/(s+10)

 s =
    0

ans =
 100*Kp
```

$$K_v = \lim_{s\to 0} s\frac{1000\left(K_P + K_D s\right)}{s(s+10)} = \frac{1000K_P}{10} = 100K_P = 1000 \qquad \text{Thus} \qquad K_P = 10$$

```
Kp=10

clear s

syms s

Mnum=(Kp+Kd*s)*1000/s/(s+10)

Mden=1+(Kp+Kd*s)*1000/s/(s+10)
```

Kp =

10

Mnum =

(10000+1000*Kd*s)/s/(s+10)

Mden =

1+(10000+1000*Kd*s)/s/(s+10)

ans =

(s^2+10*s+10000+1000*Kd*s)/s/(s+10)

Characteristic Equation: $s^2 + (10 + 1000K_D)s + 1000K_P = 0$

Match with a 2nd order prototype system

$\omega_n = \sqrt{1000K_P} = \sqrt{10000} = 100$ rad/sec $\quad 2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.5 \times 100 = 100$

solve(10+1000*Kd-100)

ans =

9/100

Thus $K_D = \dfrac{90}{1000} = 0.09$

Use the same procedure for other parts.

(b) For $K_v = 1000$ and $\zeta = 0.707$, and from part (a), $\omega_n = 100$ rad/sec,

$2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.707 \times 100 = 141.4$ Thus $K_D = \dfrac{131.4}{1000} = 0.1314$

(c) For $K_v = 1000$ and $\zeta = 1.0$, and from part (a), $\omega_n = 100$ rad/sec,

$2\zeta\omega_n = 10 + 1000K_D = 2 \times 1 \times 100 = 200$ Thus $K_D = \dfrac{190}{1000} = 0.19$

5-57) **The ramp-error constant:**

$$K_v = \lim_{s\to 0} s\frac{1000\left(K_P + K_D s\right)}{s(s+10)} = 100K_P = 10{,}000 \qquad \text{Thus} \quad K_P = 100$$

The forward-path transfer function is: $G(s) = \dfrac{1000\left(100 + K_D s\right)}{s(s+10)}$

```
clear all
for KD=0.2:0.2:1.0;
num = [-100/KD];
den = [0 -10];
G=zpk(num,den,1000);
M=feedback(G,1)
step(M);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

Zero/pole/gain:

```
    1000 (s+500)

----------------------

(s^2  + 1010s + 5e005)
```

Zero/pole/gain:

```
   1000 (s+250)

-------------------

(s+434.1) (s+575.9)
```

Zero/pole/gain:

```
  1000 (s+166.7)

-------------------

(s+207.7) (s+802.3)
```

Zero/pole/gain:

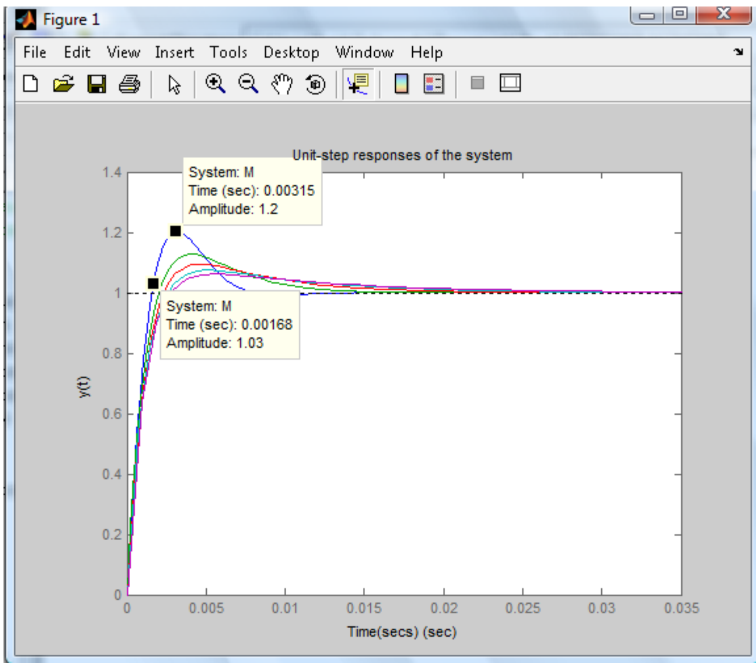
1000 (s+125)

(s+144.4) (s+865.6)

Zero/pole/gain:

1000 (s+100)

(s+111.3) (s+898.7)



Use the cursor to obtain the PO and tr values.

For part b the maximum value of KD results in the minimum overshoot.

5-58) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{4500K\left(K_P + K_D s\right)}{s(s+361.2)}$$

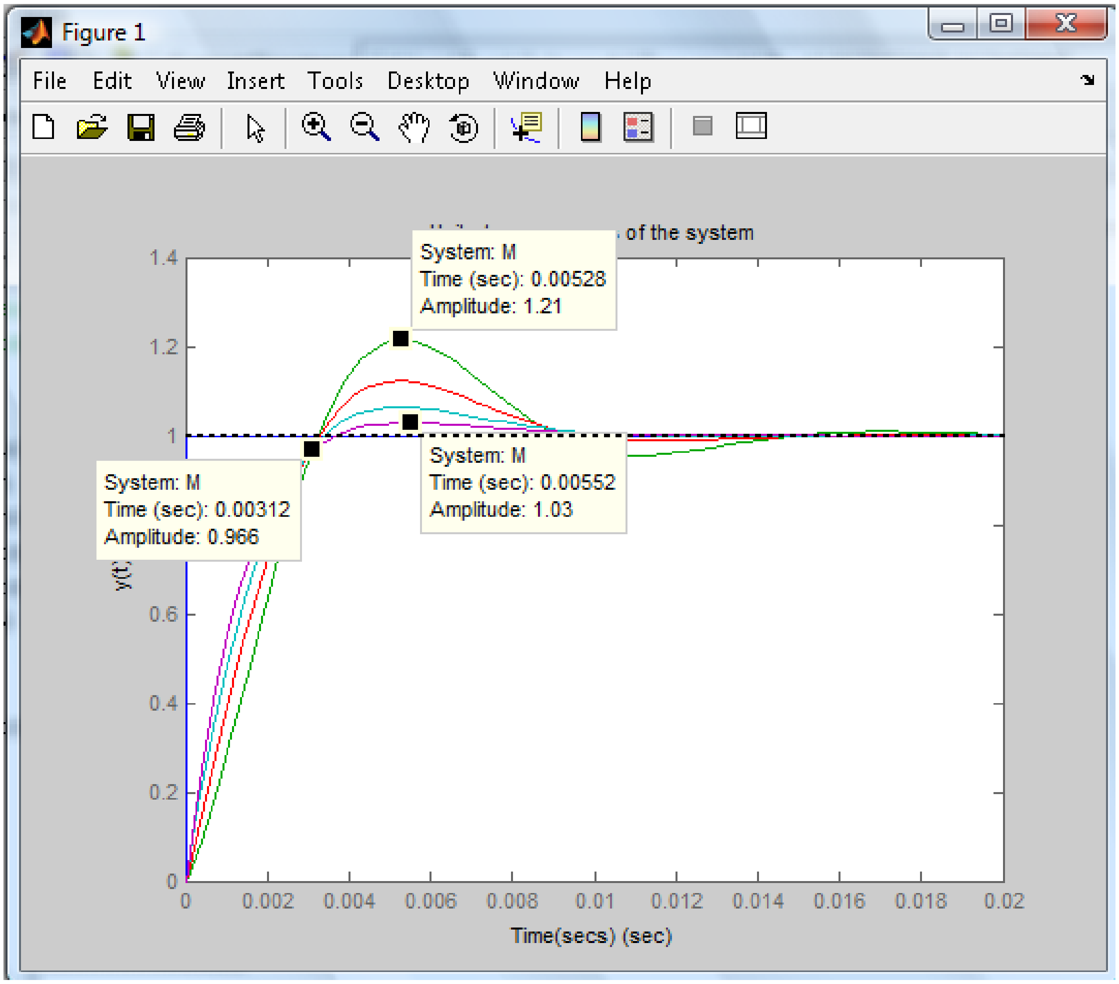
Ramp Error Constant: $K_v = \lim_{s\to 0} sG(s) = \frac{4500KK_P}{361.2} = 12.458KK_P$

$$e_{ss} = \frac{1}{K_v} = \frac{0.0802}{KK_P} \le 0.001$$ Thus $KK_P \ge 80.2$ Let $K_P = 1$ and $K = 80.2$

```
clear all
KP=1;
K=80.2;
figure(1)
num = [-KP];
den = [0 -361.2];
G=zpk(num,den,4500*K)
M=feedback(G,1)
step(M)
hold on;
for KD=00.0005:0.0005:0.002;
num = [-KP/KD];
den = [0 -361.2];
G=zpk(num,den,4500*K*KD)
M=feedback(G,1)
step(M)
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

```
Zero/pole/gain:
360900 (s+1)
------------
s (s+361.2)


Zero/pole/gain:
   360900 (s+1)
----------------------
(s+0.999) (s+3.613e005)


Zero/pole/gain:
180.45 (s+2000)
---------------
 s (s+361.2)


Zero/pole/gain:
    180.45 (s+2000)
```

```
---------------------------
(s^2  + 541.6s + 3.609e005)


Zero/pole/gain:
360.9 (s+1000)
--------------
 s (s+361.2)


Zero/pole/gain:
     360.9 (s+1000)
---------------------------
(s^2  + 722.1s + 3.609e005)


Zero/pole/gain:
541.35 (s+666.7)
----------------
  s (s+361.2)


Zero/pole/gain:
    541.35 (s+666.7)
---------------------------
(s^2  + 902.5s + 3.609e005)


Zero/pole/gain:
721.8 (s+500)
-------------
 s (s+361.2)


Zero/pole/gain:
     721.8 (s+500)
--------------------------
(s^2  + 1083s + 3.609e005)
```



K_D	t_r (sec)	t_s (sec)	Max Overshoot (%)
0	0.00221	0.0166	37.1
0.0005	0.00242	0.00812	21.5
0.0010	0.00245	0.00775	12.2
0.0015	0.0024	0.0065	6.4
0.0016	0.00239	0.00597	5.6
0.0017	0.00238	0.00287	4.8
0.0018	0.00236	0.0029	4.0
0.0020	0.00233	0.00283	2.8

5-59) The forward-path Transfer Function: *N* = 20

$$G(s) = \frac{200\left(K_P + K_D s\right)}{s(s+1)(s+10)}$$

To stabilize the system, we can reduce the forward-path gain. Since the system is type 1, reducing the gain does not affect the steady-state liquid level to a step input. Let $K_P = 0.05$

$$G(s) = \frac{200\left(0.05 + K_D s\right)}{s(s+1)(s+10)}$$

ALSO try other Kp values and compare your results.

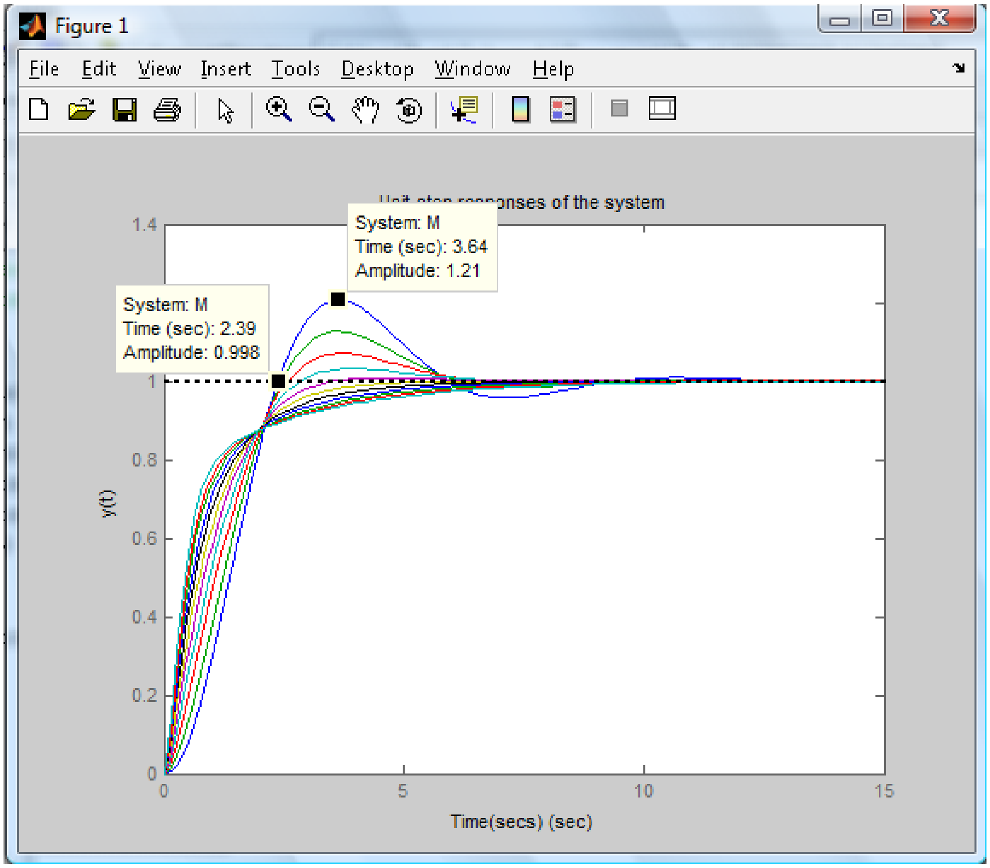
```
clear all
figure(1)
KD=0
num = [];
den = [0 -1 -10];
G=zpk(num,den,200*0.05)
M=feedback(G,1)
step(M)
hold on;
for KD=0.01:0.01:0.1;
KD
num = [-0.05/KD];
G=zpk(num,den,200*KD)
M=feedback(G,1)
step(M)
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

```
KD =
  0

Zero/pole/gain:
   10
--------------
s (s+1) (s+10)

Zero/pole/gain:
          10
-----------------------------------
(s+10.11) (s^2  + 0.8914s + 0.9893)

KD =
  0.0100
```

```
Zero/pole/gain:
  2 (s+5)
--------------
s (s+1) (s+10)

Zero/pole/gain:
         2 (s+5)
---------------------------------
(s+9.889) (s^2  + 1.111s + 1.011)

KD =
   0.0200

Zero/pole/gain:
  4 (s+2.5)
--------------
s (s+1) (s+10)


Zero/pole/gain:
         4 (s+2.5)
---------------------------------
(s+9.658) (s^2  + 1.342s + 1.035)


KD =
   0.0300

Zero/pole/gain:
 6 (s+1.667)
--------------
s (s+1) (s+10)

Zero/pole/gain:
        6 (s+1.667)
---------------------------------
(s+9.413) (s^2  + 1.587s + 1.062)

KD =
   0.0400


Zero/pole/gain:
  8 (s+1.25)
--------------
s (s+1) (s+10)


Zero/pole/gain:
        8 (s+1.25)
---------------------------------
(s+9.153) (s^2  + 1.847s + 1.093)
```

```
KD =
   0.0500

Zero/pole/gain:
  10 (s+1)
--------------
s (s+1) (s+10)

Zero/pole/gain:
      10 (s+1)
-------------------------
(s+8.873) (s+1.127) (s+1)

KD =
   0.0600

Zero/pole/gain:
12 (s+0.8333)
--------------
s (s+1) (s+10)

Zero/pole/gain:
      12 (s+0.8333)
------------------------------
(s+8.569) (s+1.773) (s+0.6582)

KD =
   0.0700

Zero/pole/gain:
14 (s+0.7143)
--------------
s (s+1) (s+10)


Zero/pole/gain:
      14 (s+0.7143)
-----------------------------
(s+8.232) (s+2.221) (s+0.547)

KD =
   0.0800

Zero/pole/gain:
 16 (s+0.625)
--------------
s (s+1) (s+10)

Zero/pole/gain:
      16 (s+0.625)
-----------------------------
(s+7.85) (s+2.673) (s+0.4765)
```

```
KD =
   0.0900

Zero/pole/gain:
18 (s+0.5556)
--------------
s (s+1) (s+10)


Zero/pole/gain:
      18 (s+0.5556)
------------------------------
(s+7.398) (s+3.177) (s+0.4255)

KD =
   0.1000

Zero/pole/gain:
 20 (s+0.5)
--------------
s (s+1) (s+10)

Zero/pole/gain:
        20 (s+0.5)
------------------------------
(s+0.3861) (s+3.803) (s+6.811)
```



Unit-step Response Attributes:

K_D	t_s (sec)	Max Overshoot (%)
0.01	5.159	12.7
0.02	4.57	7.1
0.03	2.35	3.2
0.04	2.526	0.8
0.05	2.721	0
0.06	3.039	0
0.10	4.317	0

When $K_D = 0.05$ the rise time is 2.721 sec, and the step response has no overshoot.

5-60) (a) For $e_{ss} = 1$,

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{200\left(K_P + K_D s\right)}{s(s+1)(s+10)} = 20K_P = 1 \qquad \text{Thus} \quad K_P = 0.05$$

Forward-path Transfer Function:

$$G(s) = \frac{200\left(0.05 + K_D s\right)}{s(s+1)(s+10)}$$

Because of the choice of Kp this is the same as previous part.

5-61)

(a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P + \dfrac{K_I}{s}\right)}{s^2 + 10s + 100} \qquad \text{For } K_v = 10, \qquad K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 10$$

Thus $K_I = 10$.

$$G_{cl}(s) = \frac{100\left(K_P s + K_I\right)}{s^3 + 10s^2 + 100s + 100\left(K_P s + K_I\right)} = \frac{100\left(K_P s + 10\right)}{s^3 + 10s^2 + 100(1+K_P)s + 1000}$$

(b) Let the complex roots of the characteristic equation be written as $s = -\sigma + j15$ and $s = -\sigma - j15$.

The quadratic portion of the characteristic equation is $s^2 + 2\sigma s + \left(\sigma^2 + 225\right) = 0$

The characteristic equation of the system is $s^3 + 10s^2 + \left(100 + 100K_P\right)s + 1000 = 0$

The quadratic equation must satisfy the characteristic equation. Using long division and solve for zero remainder condition.

$$
\begin{array}{r|l}
 & s+(10-2\sigma) \\
\hline
s^2+2\sigma s+\sigma^2+225 & s^3+10s^2+\left(100+100K_P\right)s+1000 \\
 & s^3+2\sigma s^2+\left(\sigma^2+225\right)s \\
\hline
 & (10-2\sigma)s^2+\left(100K_P-\sigma^2-125\right)s+1000 \\
 & (10-2\sigma)s^2+\left(20\sigma-4\sigma^2\right)s+(10-2\sigma)\left(s^2+225\right) \\
\hline
 & \left(100K_P+3\sigma^2-20\sigma-125\right)s+2\sigma^3-10\sigma^2+450\sigma-1250
\end{array}
$$

For zero remainder, $2\sigma^3-10\sigma^2+450\sigma-1250=0$ (1)

and $100K_P+3\sigma^2-20\sigma-125=0$ (2)

The real solution of Eq. (1) is $\sigma = 2.8555$. From Eq. (2),

$$K_P = \frac{125+20\sigma-3\sigma^2}{100} = 1.5765$$

The characteristic equation roots are: $s=-2.8555+j15,\quad -2.8555-j15,\ \text{and}\ \ s=-10+2\sigma=-4.289$

(c) Root Contours:

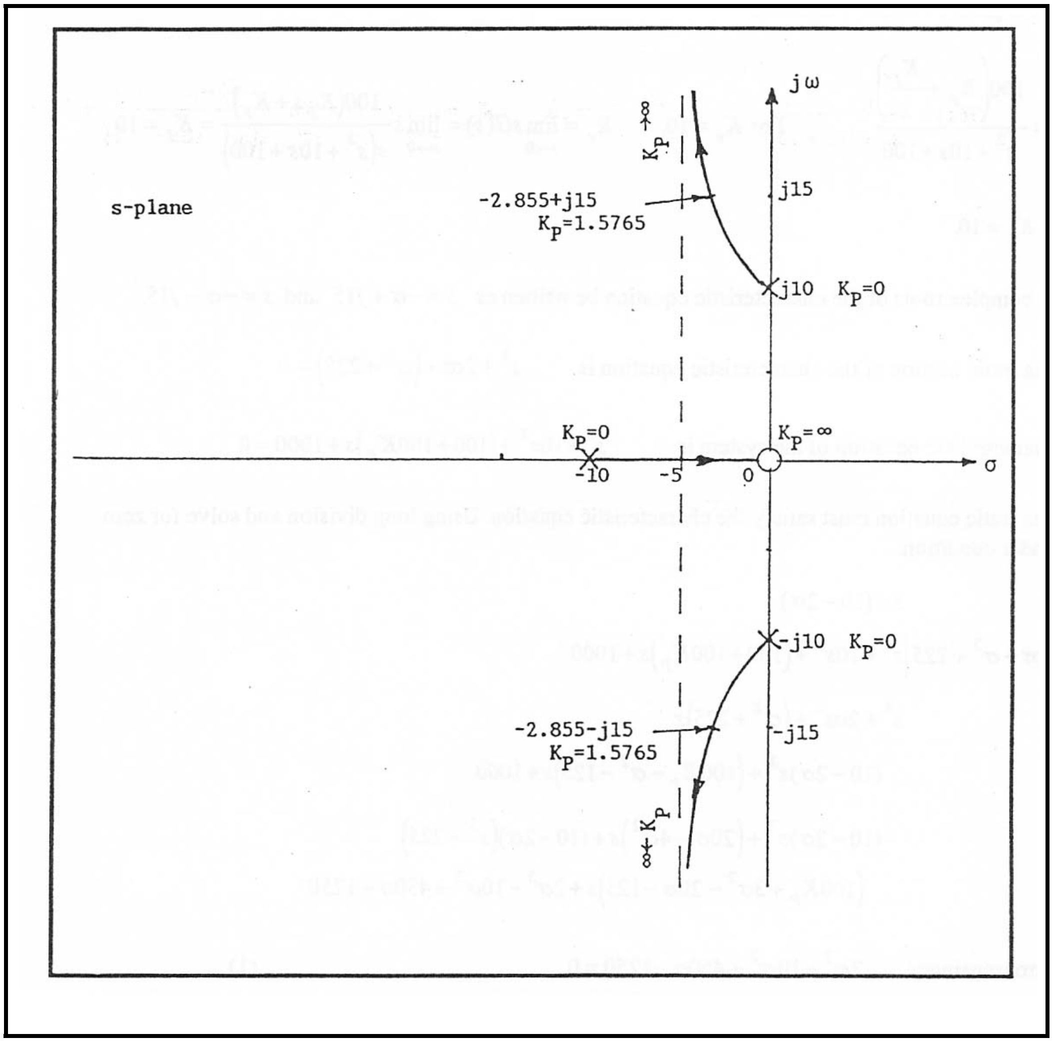
Dividing both sides of $s^3+10s^2+\left(100+100K_P\right)s+1000=0$ by the terms that do not contain K_p **we have:**

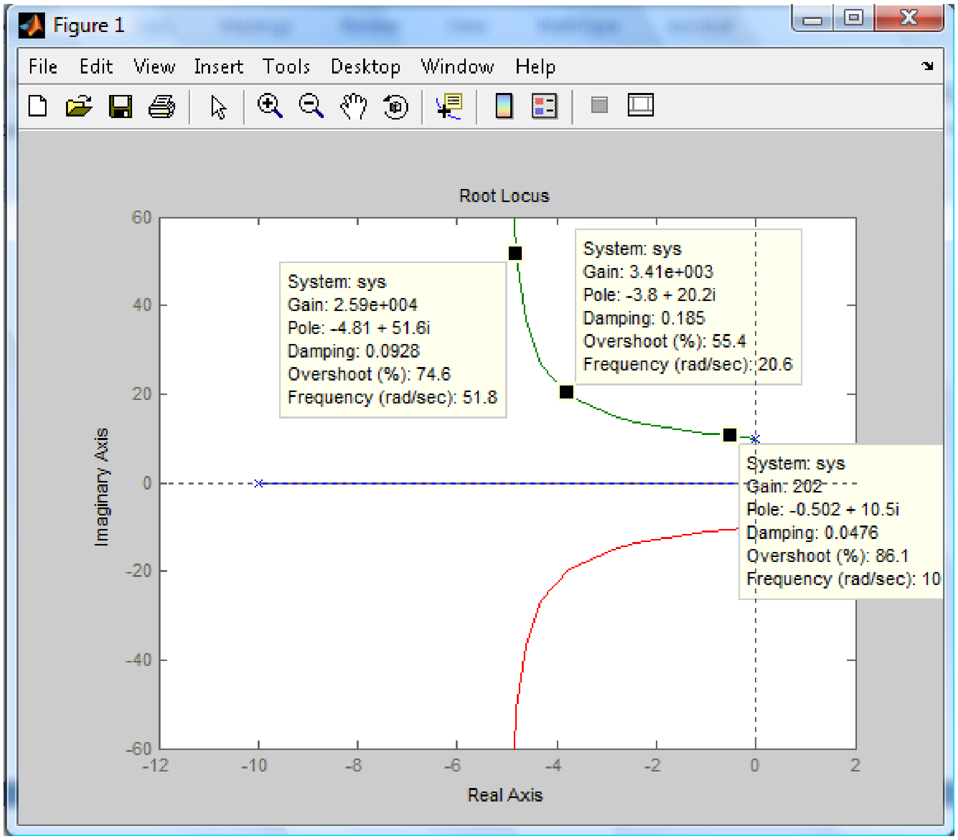
$$1+\frac{100K_P s}{s^3+10s^2+100s+1000}=1+G_{eq}$$

$$G_{eq}(s)=\frac{100K_P s}{s^3+10s^2+100s+1000}=\frac{100K_P s}{(s+10)\left(s^2+100\right)}$$

Root Contours: See Chapter 9 toolbox 9-5-2 for more information

```
clear all
Kp =.001;
num = [100*Kp 0];
den = [1 10 100 1000];
rlocus(num,den)
```


jω
s-plane
Kp→∞
-2.855+j15
Kp=1.5765
j15
j10 Kp=0
Kp=0
Kp=∞
-10
-5
0
σ
-j10 Kp=0
-2.855-j15
Kp=1.5765
-j15
Kp→∞



5-62) (a) Forward-path Transfer Function:

$$G(s)=\frac{100\left(K_P+\frac{K_I}{s}\right)}{s^2+10s+100} \qquad \text{For } K_v=10, \qquad K_v=\lim_{s\to 0} sG(s)=\lim_{s\to 0} s\frac{100\left(K_P s+K_I\right)}{s\left(s^2+10s+100\right)}=K_I=10$$

Thus the forward-path transfer function becomes

$$G(s)=\frac{100\left(10+K_P s\right)}{s\left(s^2+10s+100\right)}$$

$$G_{cl}(s)=\frac{100\left(K_P s+K_I\right)}{s^3+10s^2+100s+100\left(K_P s+K_I\right)}=\frac{100\left(K_P s+10\right)}{s^3+10s^2+100(1+K_P)s+1000}$$

```
clear all
for Kp=.4:0.4:2;
num = [100*Kp 1000];
den =[1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL);
step(GCL)
hold on;
end
```

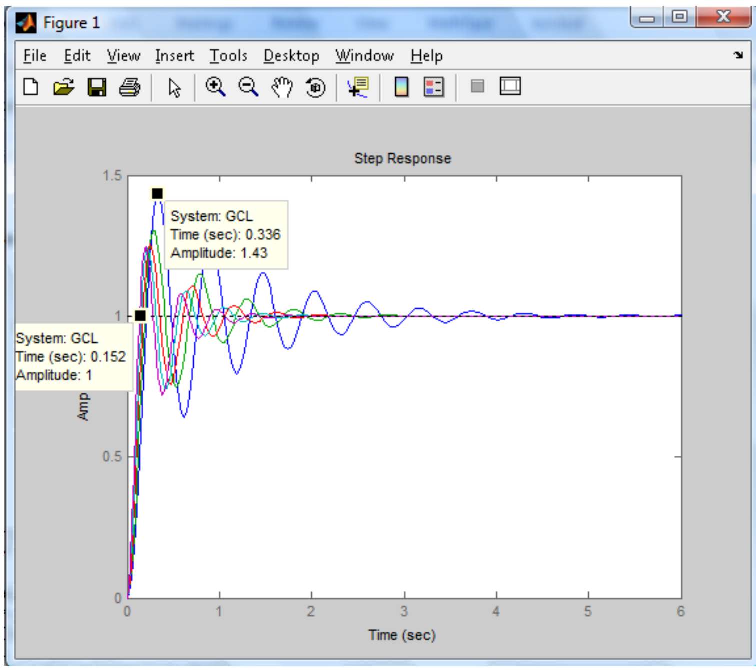
Use the curser to find the maximum overshoot and rise time. For example when $K_p = 2$, PO=43 and tr100%=0.152 sec.

Transfer function:

200 s + 1000

----

s^3 + 10 s^2 + 300 s + 1000



5-63)

(a) **Forward-path Transfer Function:**

$$G(s) = \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)}$$

For $K_v = 100$,

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 100 \qquad \text{Thus} \quad K_I = 100.$$

(b) The characteristic equation is $s^3 + 10s^2 + \left(100 + 100K_P\right)s + 100K_I = 0$

Routh Tabulation:

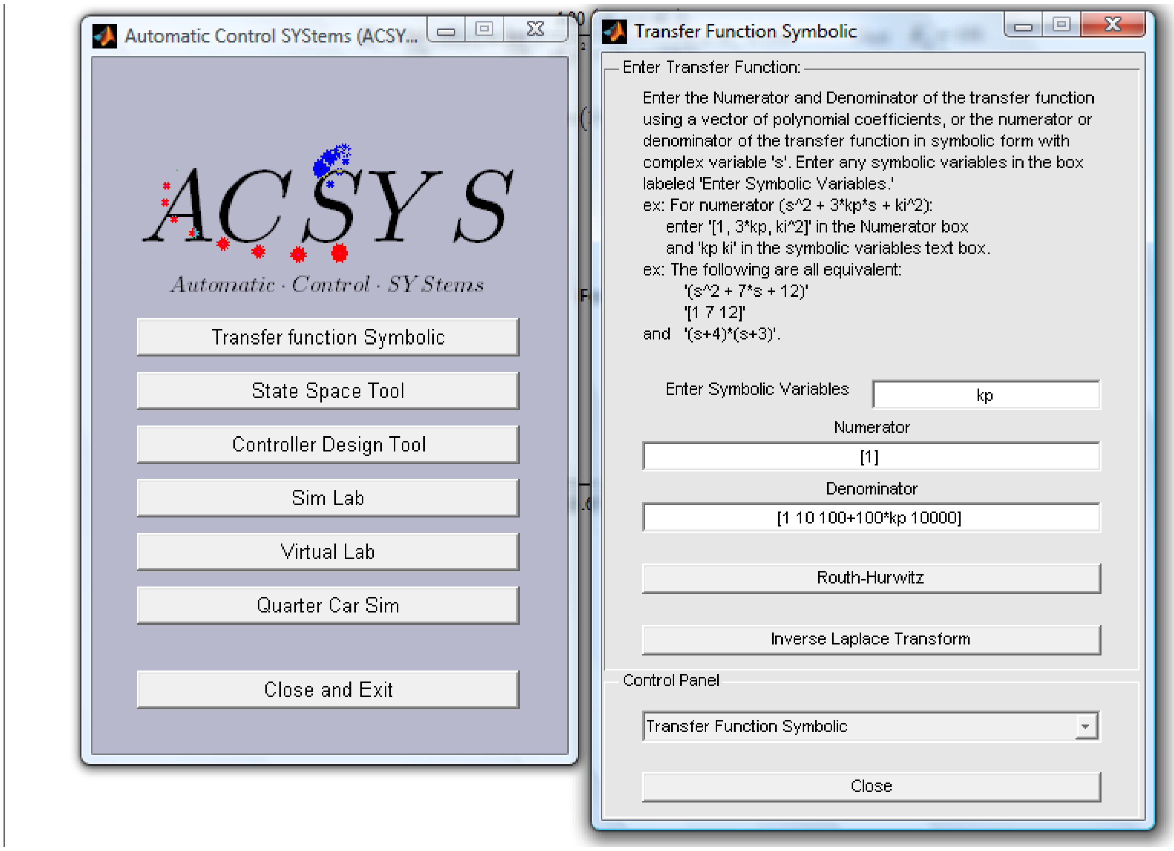
s^3	1	$100+100K_P$
s^2	10	10,000
s^1	$100K_P - 900$	0
s^0	10,000	

For stability, $100K_P - 900 > 0$ Thus $K_P > 9$

7. Activate MATLAB
8. Go to the directory containing the ACSYS software.
9. Type in

Acsys

10. Then press the “transfer function Symbolic” and enter the Characteristic equation
11. Then press the “Routh Hurwitz” button



RH =

[1, 100+100*kp]

[10, 10000]

[-900+100*kp, 0]

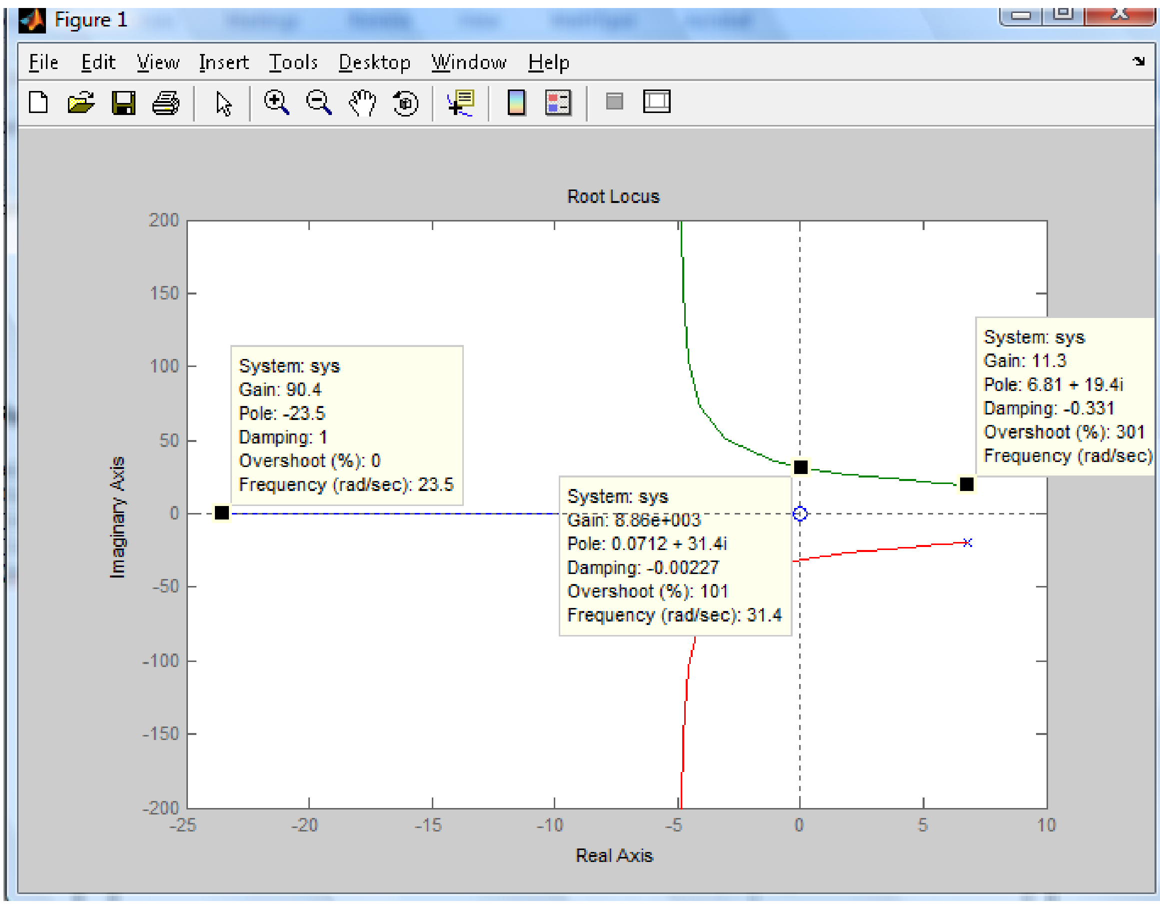
[(-9000000+1000000*kp)/(-900+100*kp), 0]

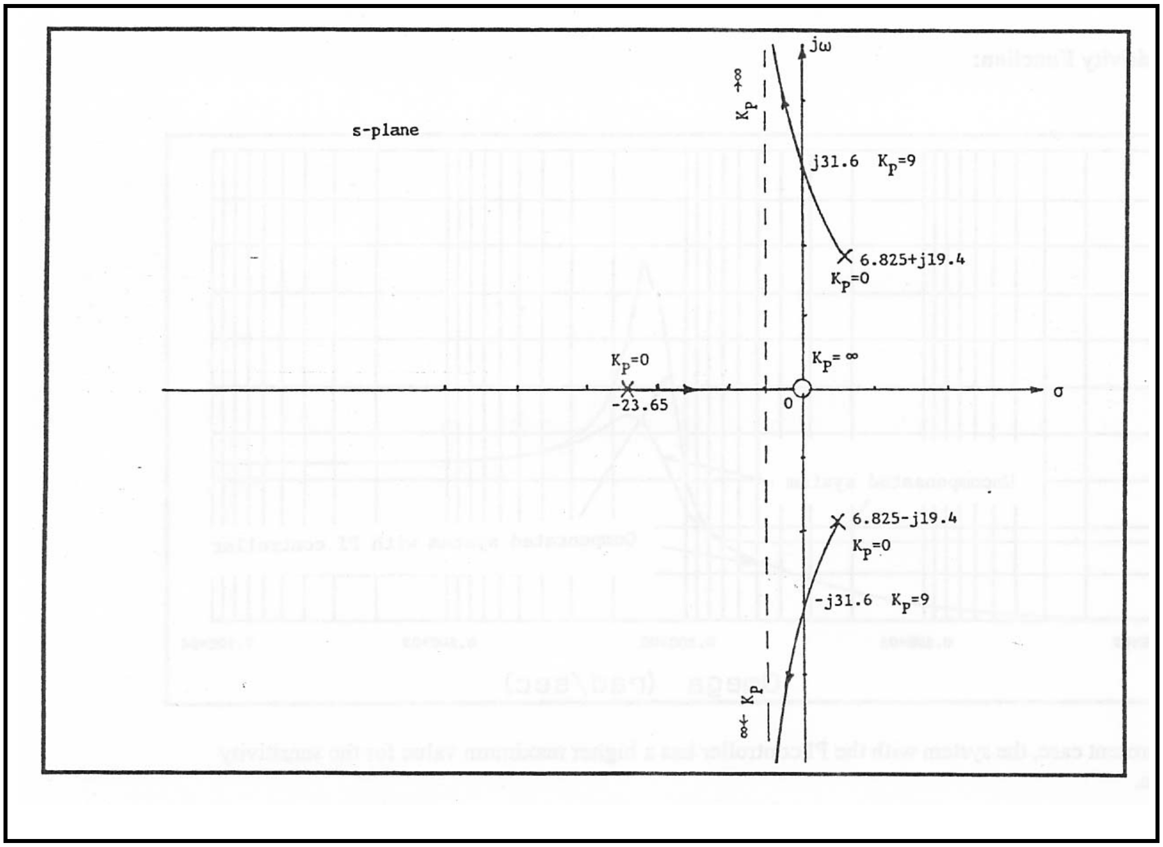
Root Contours:

$$G_{eq}(s) = \frac{100K_P s}{s^3 + 10s^2 + 100s + 10{,}000} = \frac{100K_P s}{(s+23.65)(s-6.825+j19.4)(s-6.825-j19.4)}$$

Root Contours: See Chapter 9 toolbox 9-5-2 for more information

```
clear all
Kp =.001;
num = [100*Kp 0];
den = [1 10 100 10000];
rlocus(num,den)
```





(c) $K_I = 100$

$$G(s) = \frac{100\left(K_P s + 100\right)}{s\left(s^2 + 10s + 100\right)}$$

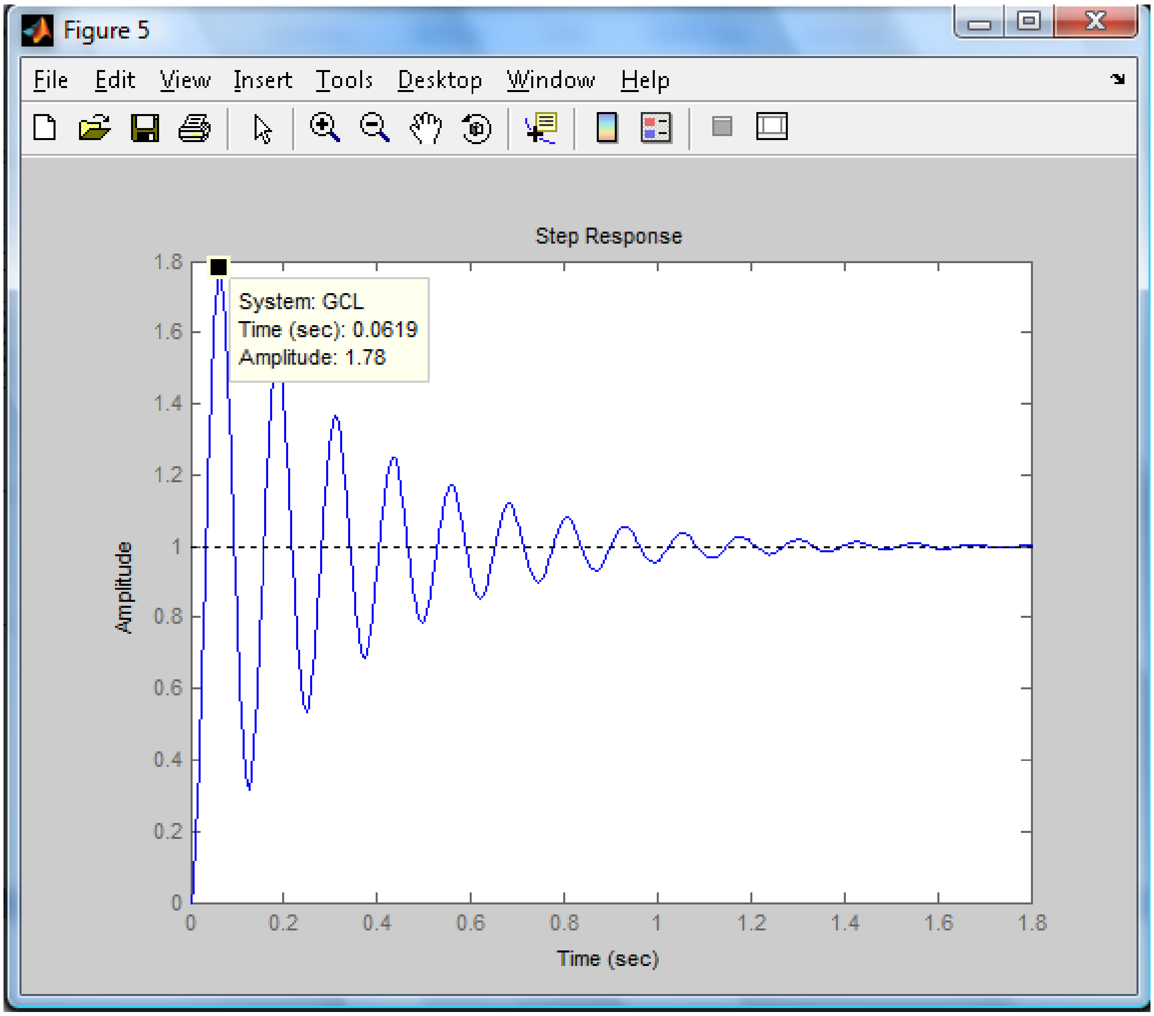
The following maximum overshoots of the system are computed for various values of K_P.

```
clear all
Kp=[15 20 22 24 25 26 30 40 100 1000];
[N,M]=size(Kp);
for i=1:M
num = [100*Kp(i) 10000];
den = [1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL);
figure(i)
step(GCL)
end
```

K_P	15	20	22	24	25	26	30	40	100	1000

y_{max}	1.794	1.779	1.7788	1.7785	1.7756	1.779	1.782	1.795	1.844	1.859

When K_P = 25, minimum y_{max} = 1.7756



Use: **close all** to close all the figure windows.

5-64) MATLAB solution is the same as 5-63.

(a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} \qquad \text{For } K_v = \frac{100K_I}{100} = 10, \qquad K_I = 10$$

(b) Characteristic Equation: $s^3 + 10s^2 + 100\left(K_P + 1\right)s + 1000 = 0$

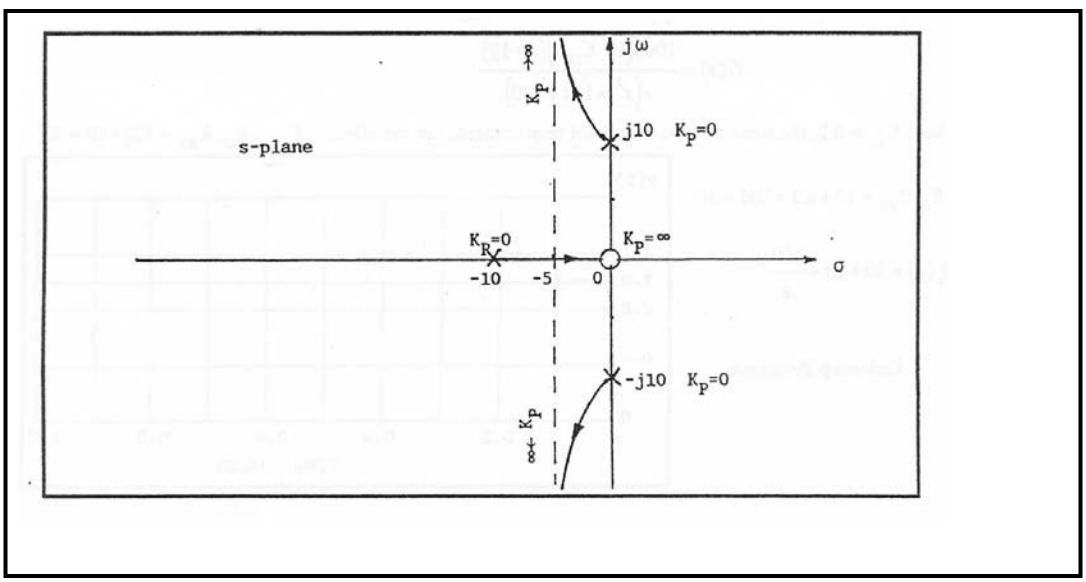
Routh Tabulation:

s^3	1	$100+100K_P$
s^2	10	1000
s^1	$100K_P$	0
s^0	1000	

For stability, $K_P > 0$

Root Contours:

$$G_{eq}(s) = \frac{100K_P s}{s^3 + 10s^2 + 100s + 1000}$$



(c) The maximum overshoots of the system for different values of K_P ranging from 0.5 to 20 are computed and tabulated below.

K_P	0.5	1.0	1.6	1.7	1.8	1.9	2.0	3.0	5.0	10	20
y_{max}	1.393	1.275	1.2317	1.2416	1.2424	1.2441	1.246	1.28	1.372	1.514	1.642

When K_P = 1.7, maximum y_{max} = 1.2416

5-65)

$$G_c(s) = K_P + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_P s + K_I}{s} = (1 + K_{D1}s)\left(K_{P2} + \frac{K_{I2}}{s}\right)$$

where

$$K_P = K_{P2} + K_{D1}K_{I2} \qquad K_D = K_{D1}K_{P2} \qquad K_I = K_{I2}$$

Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{100\left(K_D s^2 + K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)}$$

And rename the ratios: $K_D / K_P = A, \quad K_I / K_P = B$

Thus

$$K_v = \lim_{s \to 0} sG(s) = 100\frac{K_I}{100} = 100$$
$$K_I = 100$$

For K_D being sufficiently small:

Forward-path Transfer Function:

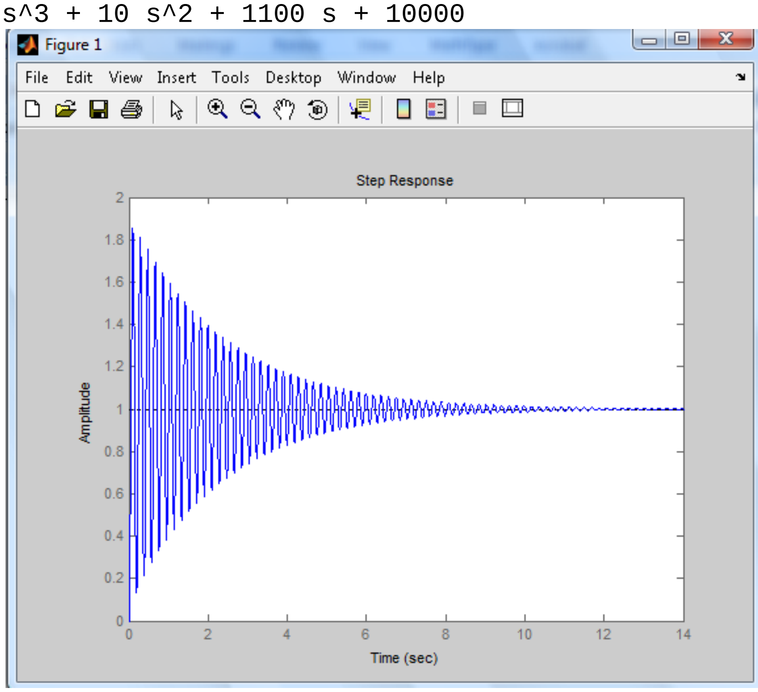
$$G(s)=\frac{100\left(K_p s+100\right)}{s\left(s^2+10s+100\right)}$$

Characteristic Equation:

$$s^3+10s^2+\left(100+100K_p\right)s+10,000=0$$

For stability, $K_p>9$. Select $K_p=10$ and observe the response.

```
clear all
Kp=10;
num = [100*Kp 10000];
den =[1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL)
step(GCL)

Transfer function:
       1000 s + 10000
-----------------------------
s^3 + 10 s^2 + 1100 s + 10000
```

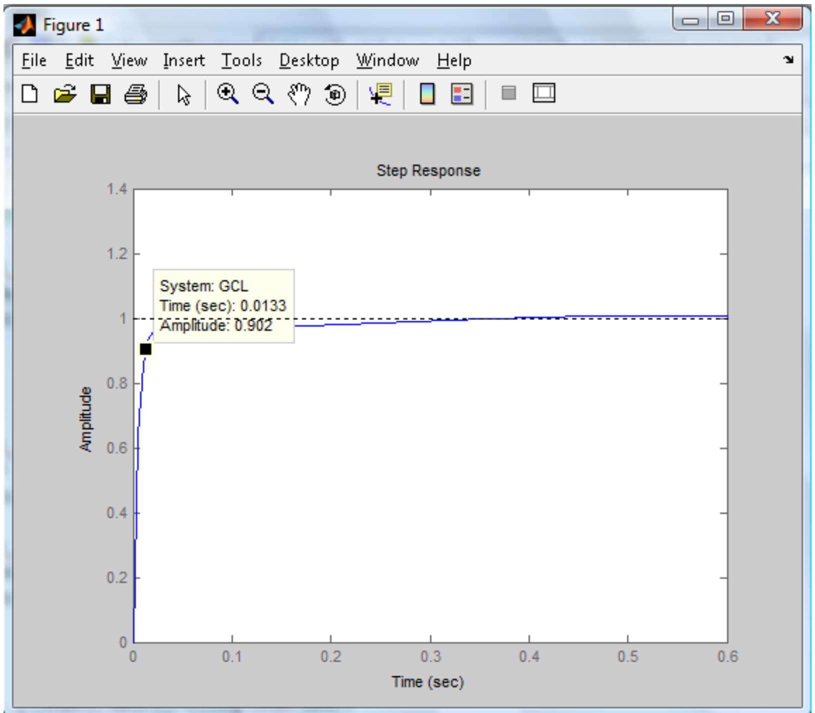


Obviously by increasing Kp more oscillations will occur. Add K_D to reduce oscillations.

```
clear all
Kp=10;
Kd=2;
num = [100*Kd 100*Kp 10000];
den =[1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL)
step(GCL)
```

```
Transfer function:
   200 s^2 + 1000 s + 10000
------------------------------
s^3 + 210 s^2 + 1100 s + 10000
```

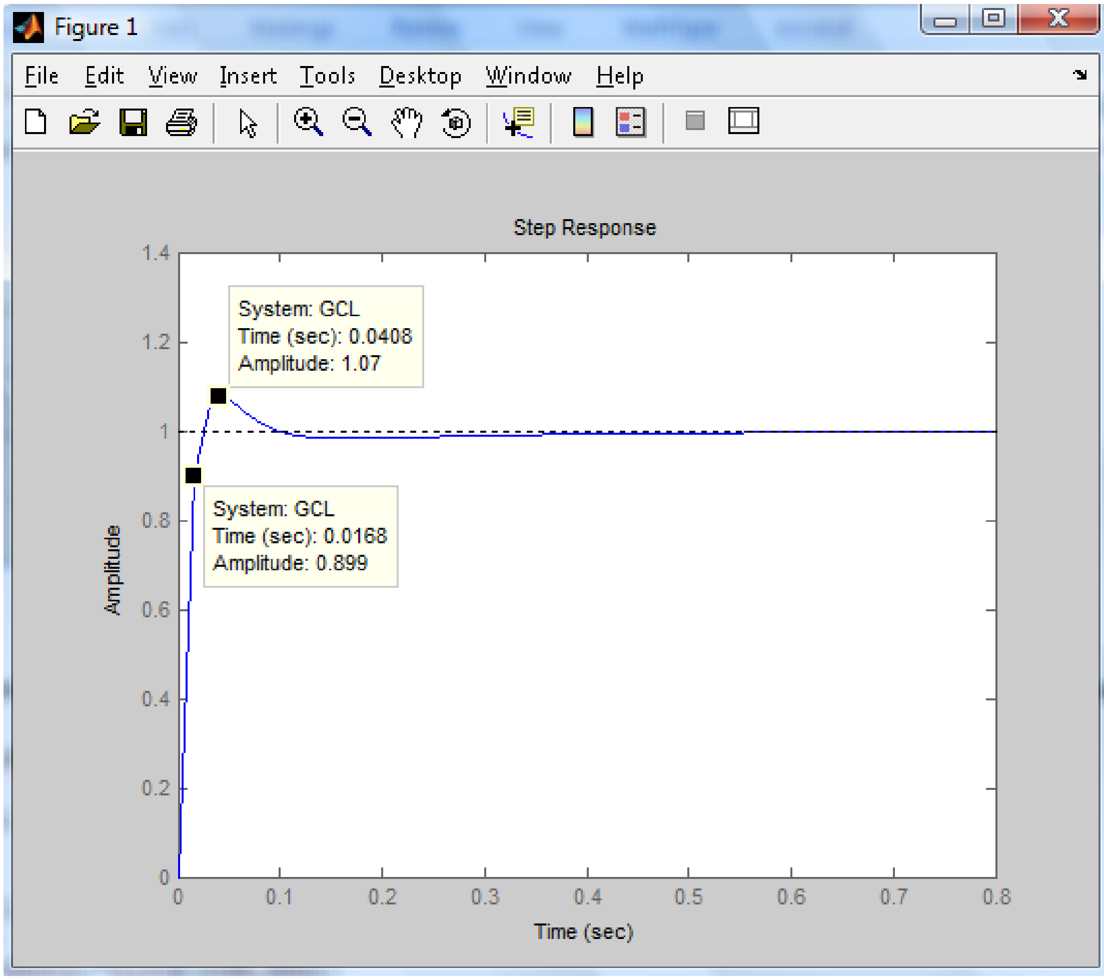
Unit-step Response



The rise time seems reasonable. But we need to increase Kp to improve approach to steady state.

Increase Kp to Kp=30.

```
clear all
Kp=30;
Kd=1;
num = [100*Kd 100*Kp 10000];
den =[1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL)
step(GCL)

Transfer function:
   100 s^2 + 3000 s + 10000
------------------------------
s^3 + 110 s^2 + 3100 s + 10000
```



To obtain a better response continue adjusting KD and KP.

5-66) This problem has received extended treatment in Chapter 6, Control Lab – see Section 6-6.

For the sake simplicity, this problem we assume the control force *f(t)* is applied in parallel to the spring *K* and damper *B*. We will not concern the details of what actuator or sensors are used.

Lets look at Figure 4-84 and equations 4-322 and 4-323.

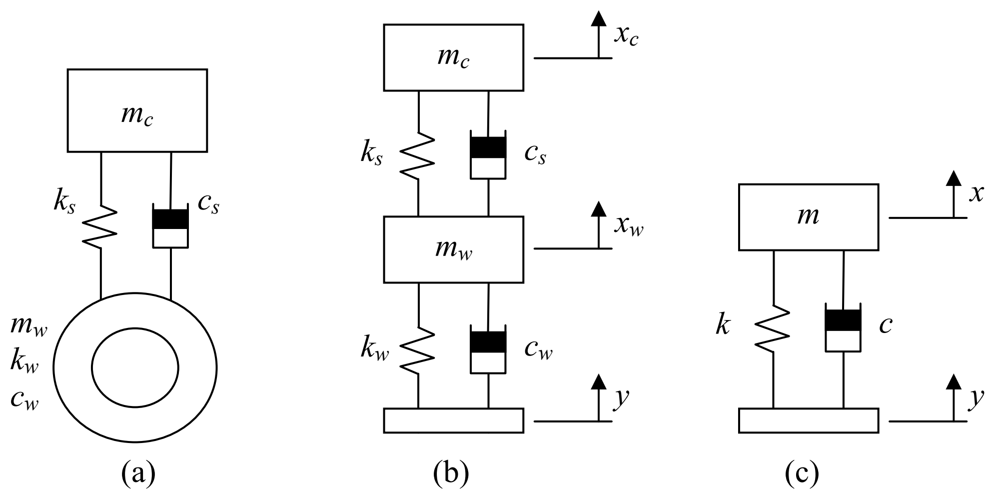


Figure 4-84 **Quarter car model realization: (a) quarter car, (b) 2 degree of freedom, and (c) 1 degree of freedom model.**

The equation of motion of the system is defined as follows:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = c\dot{y}(t) + ky(t) \qquad (4\text{-}322)$$

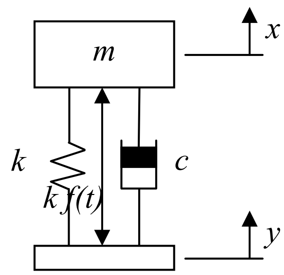
which can be simplified by substituting the relation *z(t) = x(t)-y(t)* and non-dimensionalizing the coefficients to the form

$$\ddot{z}(t) + 2\zeta\omega_n\dot{z}(t) + \omega_n{}^2 z(t) = -\ddot{y}(t) \qquad (4\text{-}323)$$

The Laplace transform of Eq. (4-323) yields the input output relationship

$$\frac{Z(s)}{\ddot{Y}(s)} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} \qquad (4\text{-}324)$$

Now let's apply control – see section 6-6 for more detail.



For simplicity and better presentation, we have scaled the control force as *kf(t)* we rewrite (4-324) as:

$$\begin{aligned} & m\ddot{x}(t) + c\dot{x}(t) + kx(t) = c\dot{y}(t) + ky(t) + kf(t) \\ & \ddot{z}(t) + 2\zeta\omega_n\dot{z}(t) + \omega_n^{\ 2}z(t) = -\ddot{y}(t) + \omega_n^{\ 2}f(t) \\ & s^2 + 2\zeta\omega_n s + \omega_n^{\ 2} = -A(s) + \omega_n^{\ 2}F(s) \\ & A(s) = \ddot{Y}(s) \end{aligned} \qquad (4\text{-}324)$$

Setting the controller structure such that the vehicle bounce $Z(s) = X(s) - Y(s)$ **is minimized:**

$$F(s) = 0 - \left(K_P + K_D s + \frac{K_I}{s}\right)Z(s)$$

$$\frac{Z(s)}{A(s)} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}\left(1 + K_P + K_D s + \dfrac{K_I}{s}\right)}$$

$$\frac{Z(s)}{A(s)} = \frac{-s}{s^3 + 2\zeta\omega_n s^2 + \omega_n^{\ 2}\left(\left(1 + K_P\right)s + K_D s^2 + K_I\right)}$$

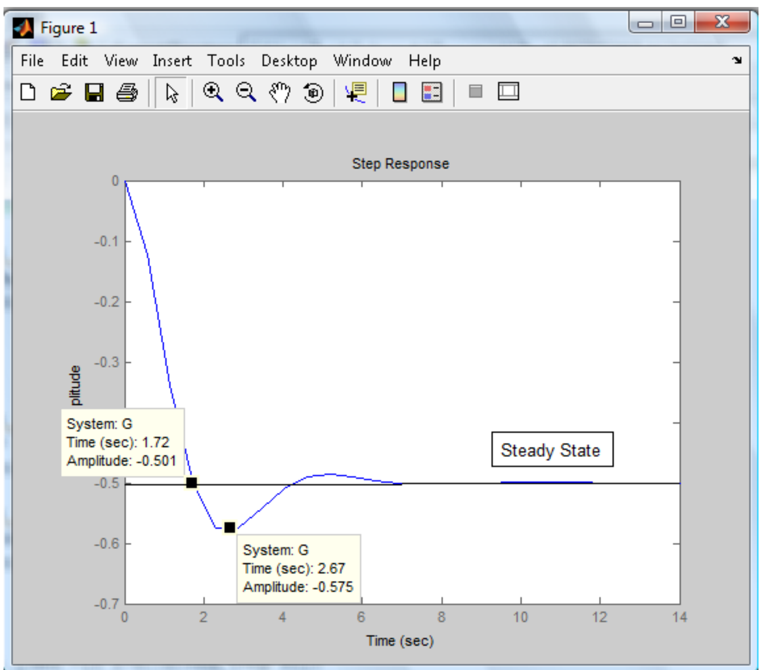
See Equation (6-4).

For proportional control K_D=K_I=0.

Pick $\varsigma = 0.707$ and $\omega_n = 1$ for simplicity. **This is now an underdamped system.**

Use MATLAB to obtain response now.

```
clear all
Kp=1;
Kd=0;
Ki=0;
num = [-1 0];
den =[1 2*0.707+Kd 1+Kp Ki];
G=tf(num,den)
step(G)

Transfer function:
         -s
---------------------
s^3 + 1.414 s^2 + 2 s
```



Adjust parameters to get the desired response if necessary.

The process is the same for parts b, c and d.

5-67) Replace F(s) with

$$F(s) = X_{ref} - \left(K_P + K_D s + \frac{K_I}{s} \right) X(s)$$

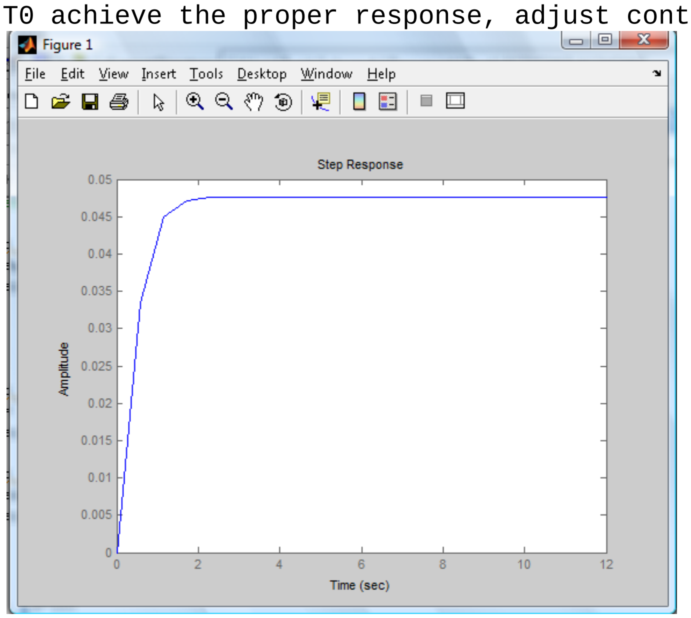
$$2\zeta\omega_n = \frac{B}{M}$$

$$\omega_n^2 = \frac{K}{M}$$

$$\frac{X(s)}{X_{ref}(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \left(\omega_n^2 + K_P + K_D s + \frac{K_I}{s} \right)}$$

Use MATLAB to obtain response now.

```
clear all
Kp=1;
Kd=0;
Ki=0;
B=10;
K=20;
M=1;
omega=sqrt(K/M);
zeta=(B/M)/2/omega;
num = [1 0];
den =[1 2*zeta*omega+Kd omega^2+Kp Ki];
G=tf(num,den)
step(G)
ransfer function:
         s
-------------------
s^3 + 10 s^2 + 21 s

T0 achieve the proper response, adjust controller gains accordingly.
```



5-68) From problem 4-3

a) Rotational kinetic energy: $T_{rot} = \frac{1}{2}J\dot{\theta}^2$

Translational kinetic energy: $T_T = \frac{1}{2}m\dot{y}^2$

Relation between translational displacement and rotational displacement:

$$y = r\theta$$

$$\dot{y} = r\dot{\theta}$$

$$T_{Rot} = \frac{1}{2}\frac{J}{r^2}\dot{y}^2$$

Potential energy: $U = \frac{1}{2}Ky^2$

As we know $T_{Rot} + T_T + U = constant$, then:

$$\frac{1}{2}\frac{J}{r^2}\dot{y}^2 + \frac{1}{2}\,m\dot{y}^2 + \frac{1}{2}\,Ky^2 = constant$$

By differentiating, we have:

$$\frac{J}{r^2}\dot{y}\ddot{y} + m\dot{y}\ddot{y} + Ky\dot{y} = 0$$

$$\dot{y}\left(\frac{J}{r^2}\ddot{y} + m\ddot{y} + Ky\right) = 0$$

Since $\dot{y}$ cannot be zero, then $J\frac{\ddot{y}}{r^2} + m\ddot{y} + Ky = 0$

b)

$\ddot{y} = r\ddot{\theta}$

$J\ddot{\theta}^2 + m\ddot{y} + Ky = 0$

$\frac{Y(s)}{\theta(s)} = -\frac{J}{ms^2 + K}$

c)

$$T_{max} = \frac{1}{2}\, m\, \dot{y}_{max}^2 + \frac{1}{2}\frac{J}{r^2}\dot{y}_{max}^2 = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\dot{y}_{max}^2$$

$$\dot{y}_{max}^2 = \omega_n$$

where $\dot{y} = A$ at the maximum energy.

$$U_{max} = \frac{1}{2}Ky_{max}^2 = \frac{1}{2}KA^2$$

Then:

$$\frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 A^2 = \frac{1}{2}KA^2$$

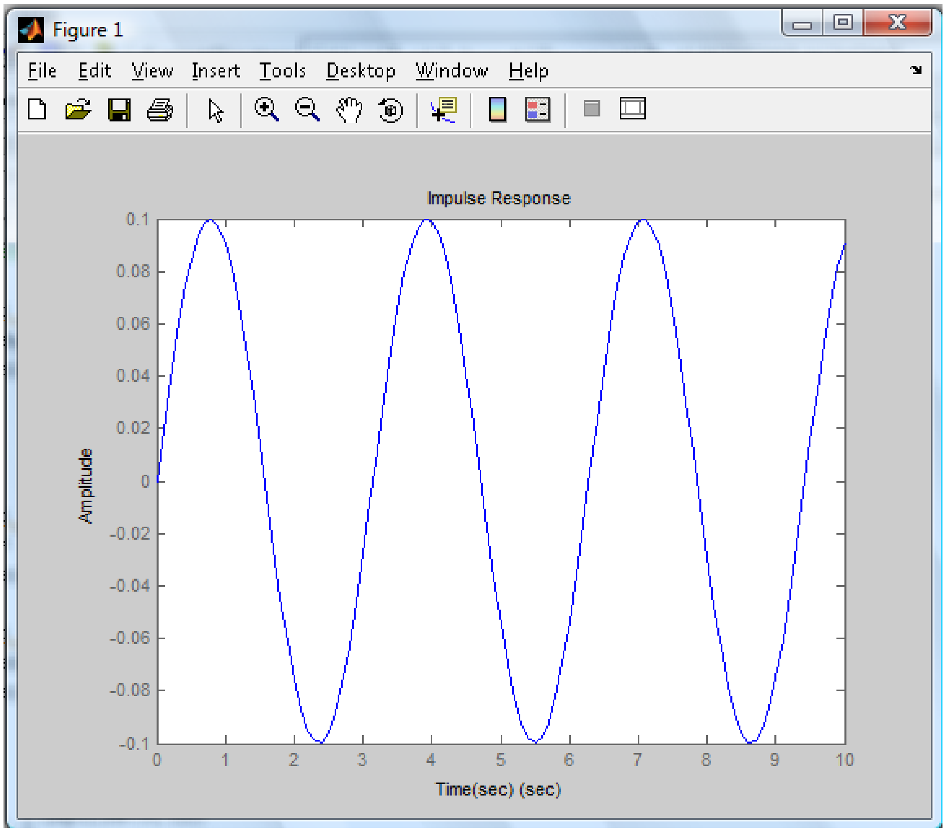
Or:

$$\omega_n = \sqrt{\frac{K}{m + \frac{J}{r^2}}} = r\sqrt{\frac{K}{r^m + J}}$$

d) $G(s) = \dfrac{J}{(ms^2 + K)}$

```
% select values of m, J and K
K=100;
J=5;
m=25;
G=tf([J],[m 0 K])
Pole(G)
impulse(G,10)
xlabel( 'Time(sec)');
ylabel('Amplitude');

Transfer function:
     5
------------
25 s^2 + 100

ans =
        0 + 2.0000i
        0 - 2.0000i
```

Uncontrolled

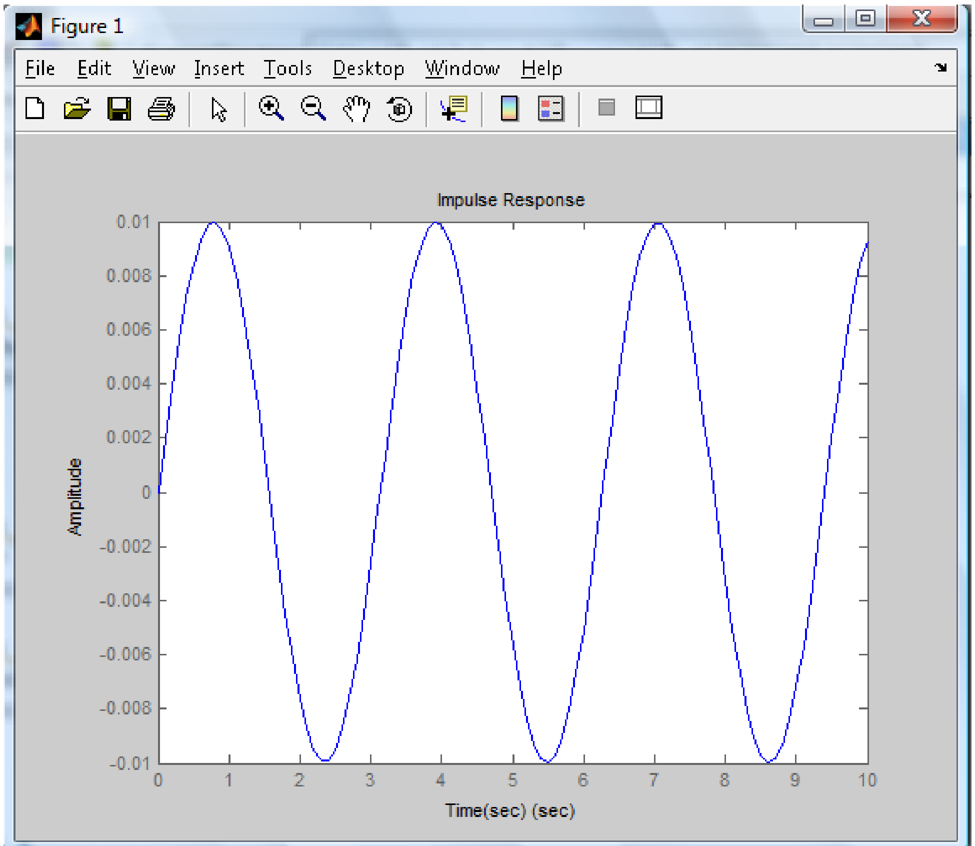


With a proportional controller one can adjust the oscillation amplitude the transfer function is rewritten as:

$$G_{cl}(s) = \frac{JK_p}{(ms^2 + K + JK_p)}$$

```
% select values of m, J and K
Kp=0.1
K=100;
J=5;
m=25;
G=tf([J*Kp],[m 0 (K+J*Kp)])
Pole(G)
impulse(G,10)
xlabel( 'Time(sec)');
ylabel('Amplitude');
```

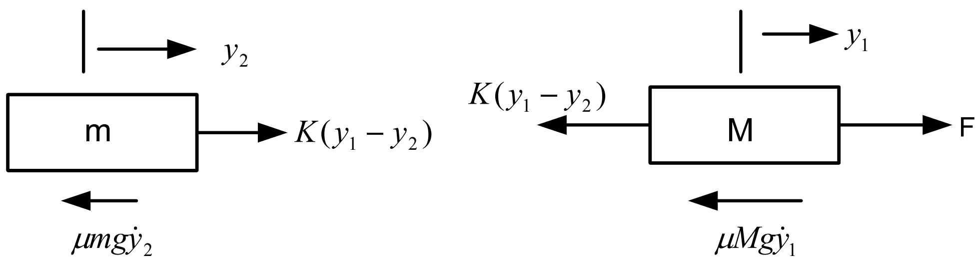
```
Kp =
    0.1000

Transfer function:
     0.5
--------------
25 s^2 + 100.5

ans =
        0 + 2.0050i
        0 - 2.0050i
```



A PD controller must be used to damp the oscillation and reduce overshoot. Use Example 5-11-1 as a guide.

5-69) From Problem 4-6 we have:

a)



b) From Newton's Law:

$$M\ddot{y}_1 = F - K(y_1 - y_2) - \mu Mg\dot{y}_1$$

$$m\ddot{y}_2 = K(y_1 - y_2) - \mu mg\dot{y}_2$$

If y_1 and y_2 are considered as a position and v_1 and v_2 as velocity variables

Then:
$$\begin{cases} \dot{y}_1 = v_1 \\ \dot{y}_2 = v_2 \\ M\dot{v}_1 = F - K(y_1 - y_2) - \mu Mgv_1 \\ m\dot{v}_2 = F - K(y_1 - y_2) - \mu mgv_2 \end{cases}$$

The output equation can be the velocity of the engine, which means $z = v_2$

c)

$$\begin{cases} Ms^2Y_1(s) = F - K\big(Y_1(s) - Y_2(s)\big) - \mu MgsY_1(s) \\ ms^2Y_2(s) = K\big(Y_1(s) - Y_2(s)\big) - \mu mgsY_2(s) \\ Z(s) = V_2(s) = sY(s) \end{cases}$$

Obtaining $\frac{Z(s)}{F(s)}$ requires solving above equation with respect to $Y_2(s)$

From the first equation:

$$(Ms^2 + K + \mu Mgs)Y_1(s) = F + KY_2(s)$$

$$Y_1(s) = \frac{F + KY_2(s)}{Ms^2 + \mu Mgs + K}$$

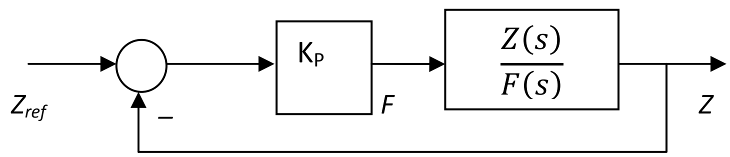
Substituting into the second equation:

$$ms^2Y_2(s) = \frac{KF + K^2Y_2(s)}{Ms^2 + \mu Mgs + K} - KY_2(s) - \mu mgsY_2(s)$$

By solving above equation:

$$\frac{Z(s)}{F(s)} = \frac{sY_2(s)}{F(s)} = \frac{ms^2 + m\,\mu g s + 1}{Mms^3 + (2Mm\,\mu g)s^2 + (Mk + Mm(\mu g)^2 + mK)s + K\mu g(M+m)}$$

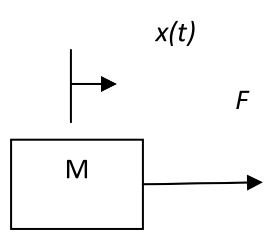
Replace Force F with a proportional controller so that F=K(Z-Z_{ref}):

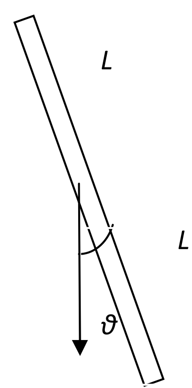


$$\frac{Z(s)}{K(Z_{ref}(s) - Z(s))} = \frac{ms^2 + m\,\mu g s + 1}{Mms^3 + (2Mm\,\mu g)s^2 + (Mk + Mm(\mu g)^2 + mK)s + K\mu g(M+m)}$$

$$\frac{Z(s)}{Z_{ref}(s)} = \frac{K_P(ms^2 + m\,\mu g s + 1)}{Mms^3 + (2Mm\,\mu g)s^2 + (Mk + Mm(\mu g)^2 + mK)s + K\mu g(M+m) + K_P(ms^2 + m\,\mu g s + 1)}$$

5-70) Also see derivations in 4-9.





Here is an alternative representation including friction (damping) μ. In this case the angle *θ* is measured differently.

Let's find the dynamic model of the system:

1) $(M+m)\ddot{x}+\ \mu\dot{x}-ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta=F$
2) $(I+mI^2)\ddot{\theta}+mgl\sin\theta=-ml\ddot{x}\cos\theta$

Let $\theta=\pi+\Phi$. If Φ is small enough then $\cos\Phi\rightarrow 1$ and $\sin\Phi\rightarrow\Phi$, therefore

$$\begin{cases}(M+m)\ddot{x}+\ \mu\dot{x}-ml\ddot{\Phi}=F\\(l+ml^2)\ddot{\Phi}-mgl\Phi=ml\ddot{x}\end{cases}$$

which gives:

$$\frac{\Phi(s)}{F(s)}=\frac{mls^2}{[(M+m)(l+ml^2)-(ml)^2]s^3+\ \mu(l+ml^2)s^2-(M+m)mgls-\mu mgl}$$

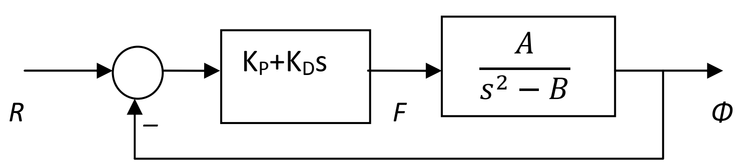
Ignoring friction $\mu=0$.

$$\frac{\Phi(s)}{F(s)}=\frac{ml}{[(M+m)(l+ml^2)-(ml)^2]s^2-(M+m)mgl}=\frac{A}{s^2-B}$$

where

$$A=\frac{ml}{[(M+m)(I+ml^2)-(ml)^2]};B=\frac{(M+m)mgl}{[(M+m)(l+ml^2)-(ml)^2]}$$

Ignoring actuator dynamics (DC motor equations), we can incorporate feedback control using a series PD compensator and unity feedback. Hence,



$$F(s)=K_p\left(R(s)-\Phi\right)-K_Ds\left(R(s)-\Phi\right)$$

The system transfer function is:

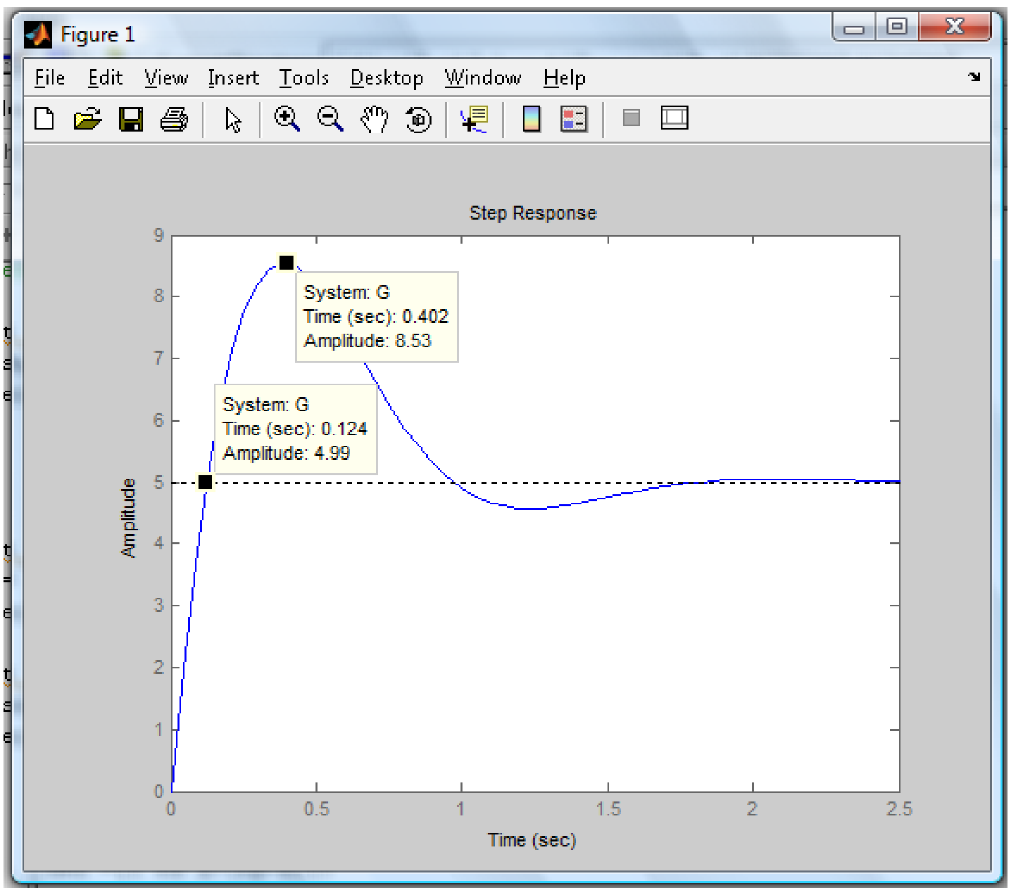
$$\frac{\Phi}{R}=\frac{A\left(K_p+K_Ds\right)}{(s^2+K_Ds+A\left(K_p-B\right)}$$

Control is achieved by ensuring stability (K_p>B)

Use Routh Hurwitz to establish stability first. Use Acsys to do that as demonstrated in this chapter problems. Also Chapter 2 has many examples.

Use MATLAB to simulate response:

```
clear all
Kp=10;
Kd=5;
A=10;
B=8;
num = [A*Kd A*Kp];
den =[1 Kd A*(Kp-B)];
G=tf(num,den)
step(G)

Transfer function:
  50 s + 100
--------------
s^2 + 5 s + 20
```



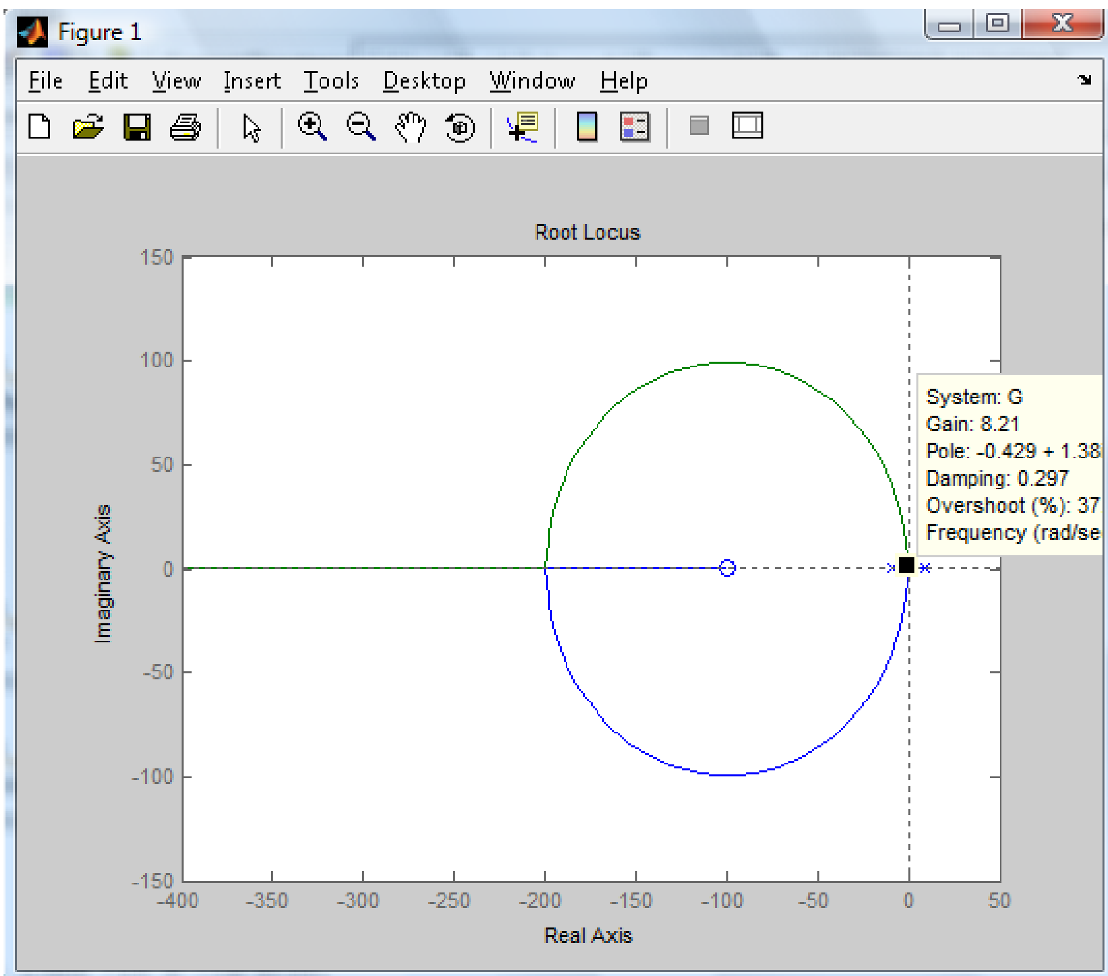
Adjust parameters to achieve desired response. Use THE PROCEDURE in Example 5-11-1.

You may look at the root locus of the forward path transfer function to get a better perspective.

$$\frac{\Phi}{E}=\frac{A\left(K_p+K_D s\right)}{s^2-AB}=\frac{AK_D\left(z+s\right)}{s^2-AB}$$

fix z and vary K_D.

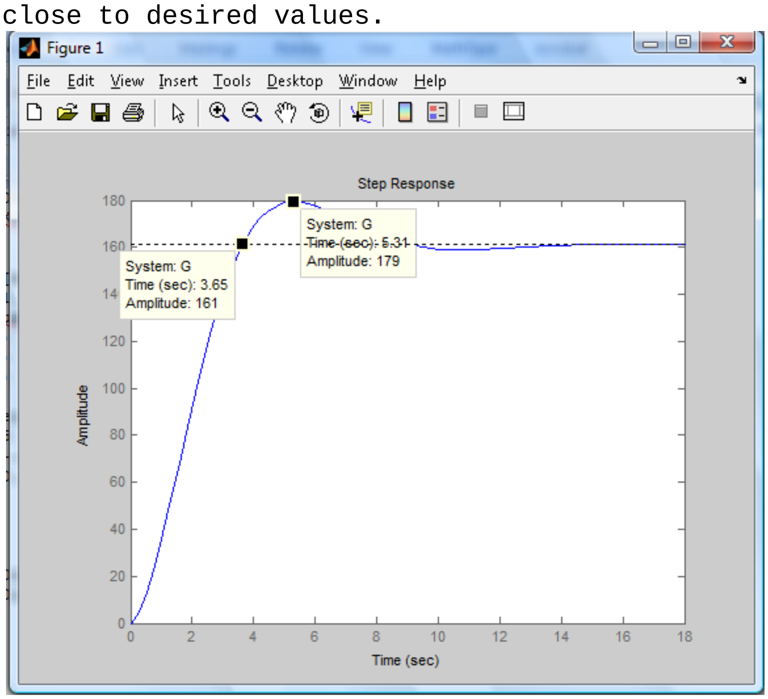
```
clear all
z=100;
Kd=0.01;
A=10;
B=8;
num = [A*Kd A*Kd*z];
den =[1 0 -(A*B)];
G=tf(num,den)
rlocus(G)

Transfer function:
0.1 s + 10
----------
 s^2 - 80
```



For z=10, a large K_D=0.805 results in:

```
clear all
Kd=0.805;
Kp=10*Kd;
A=10;
B=8;
num = [A*Kd A*Kp];
den =[1 Kd A*(Kp-B)];
G=tf(num,den)
pole(G)
zero(G)
step(G)

Transfer function:
   8.05 s + 80.5
-------------------
s^2 + 0.805 s + 0.5


ans =

  -0.4025 + 0.5814i
  -0.4025 - 0.5814i

ans =

   -10
```

Looking at dominant poles we expect to see an oscillatory response with overshoot close to desired values.



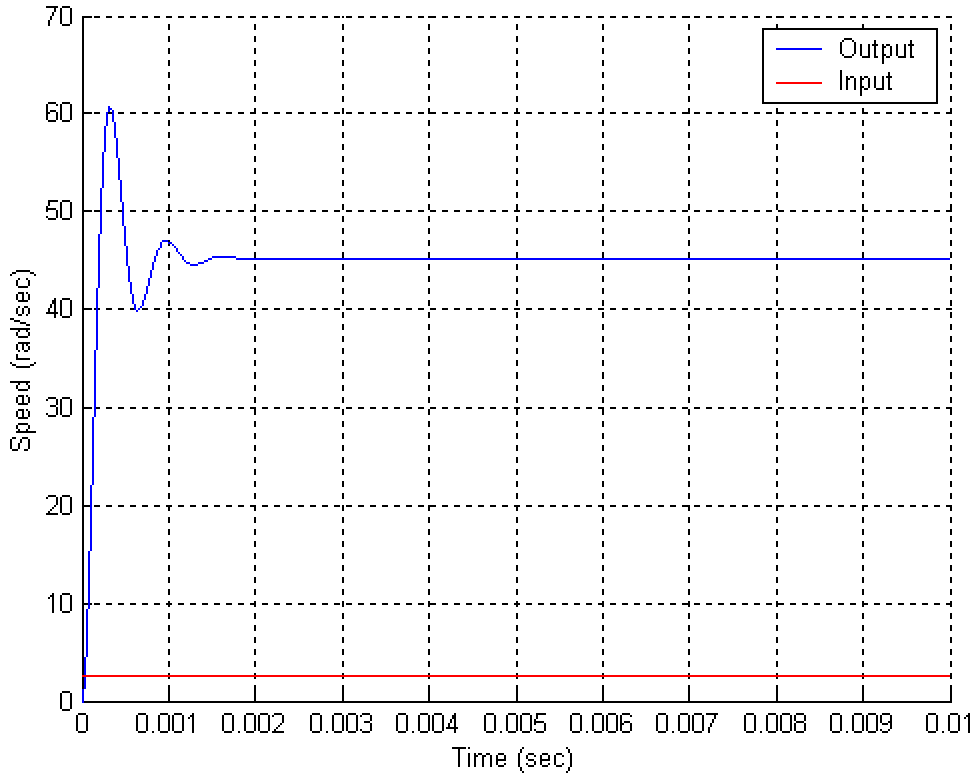
For a better design, and to meet rise time criterion, use Example 5-11-1.

Chapter 6 THE CONTROL LAB

Part 1) Solution to Lab questions within Chapter 6

6-4-1 Open Loop Speed

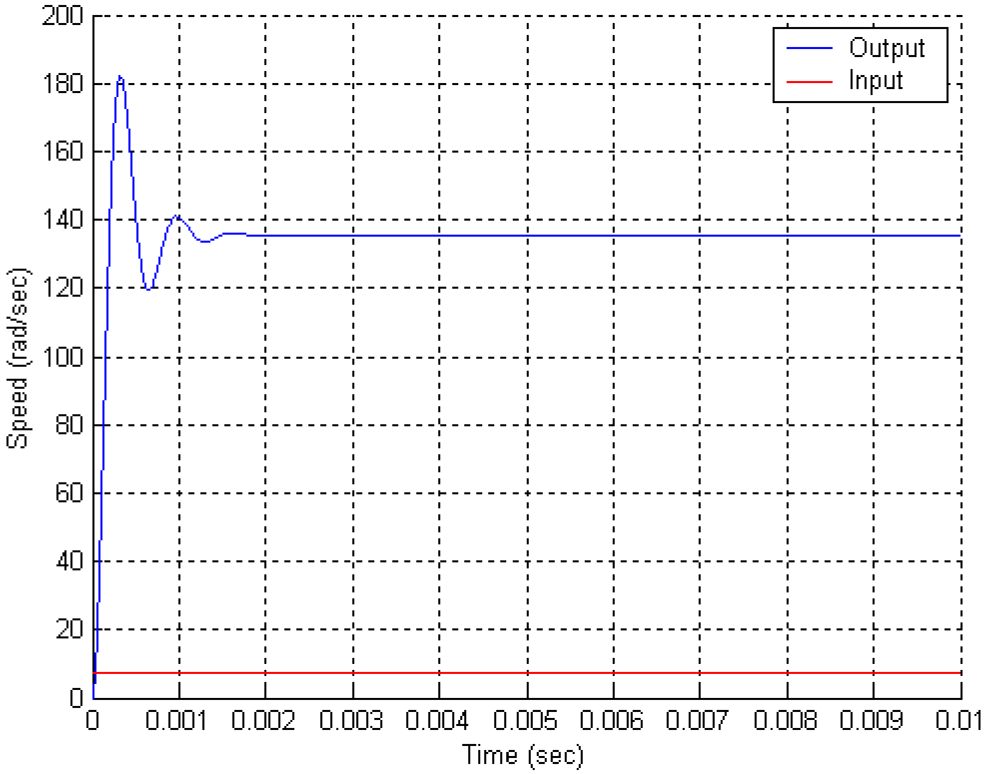
1. Open loop speed response using SIMLab:
 a. +5 V input:



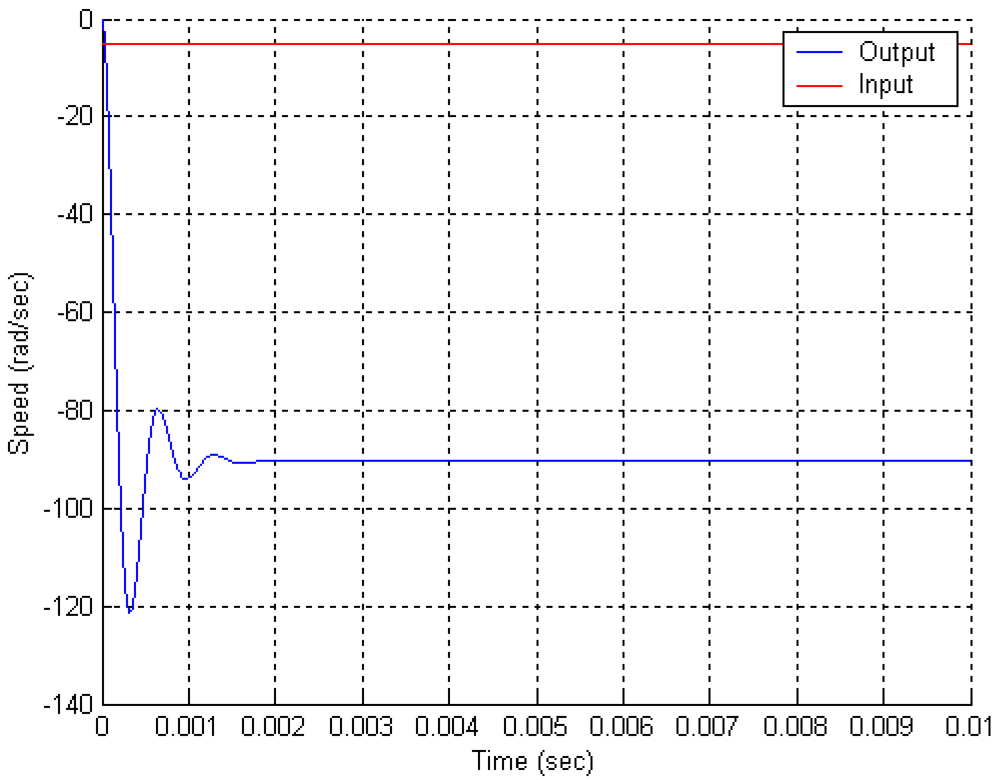
The form of response is like the one that we expected; a second order system response with overshoot and oscillation. Considering an amplifier gain of 2 and $K_b = 0.1$, the desired set point should be set to 2.5 and as seen in the figure, the final value is approximately 50 rad/sec which is armature voltage divided by K_b. To find the above response the systems parameters are extracted from:

$$\tau_m = \frac{R_a J_m}{R_a B + k_b k_m} \quad , \qquad B = \frac{R_a J_m - k_b k_m \tau_m}{R_a \tau_m} = 0.000792 kg \cdot m^2 / \sec$$

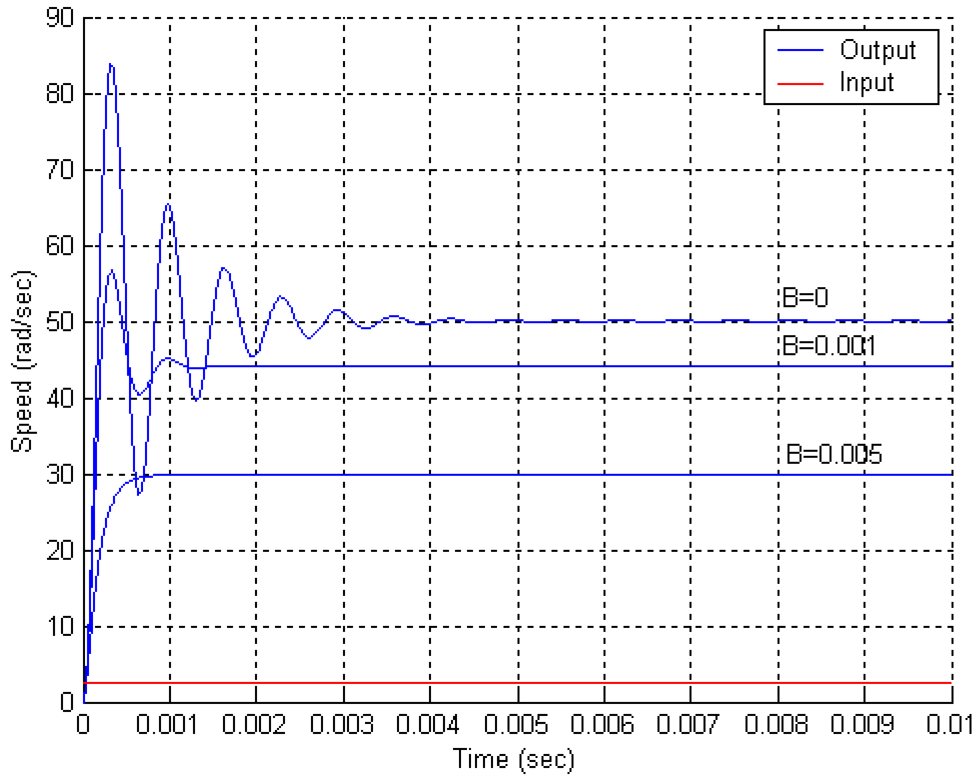
b. +15 V input:



c. –10 V input:

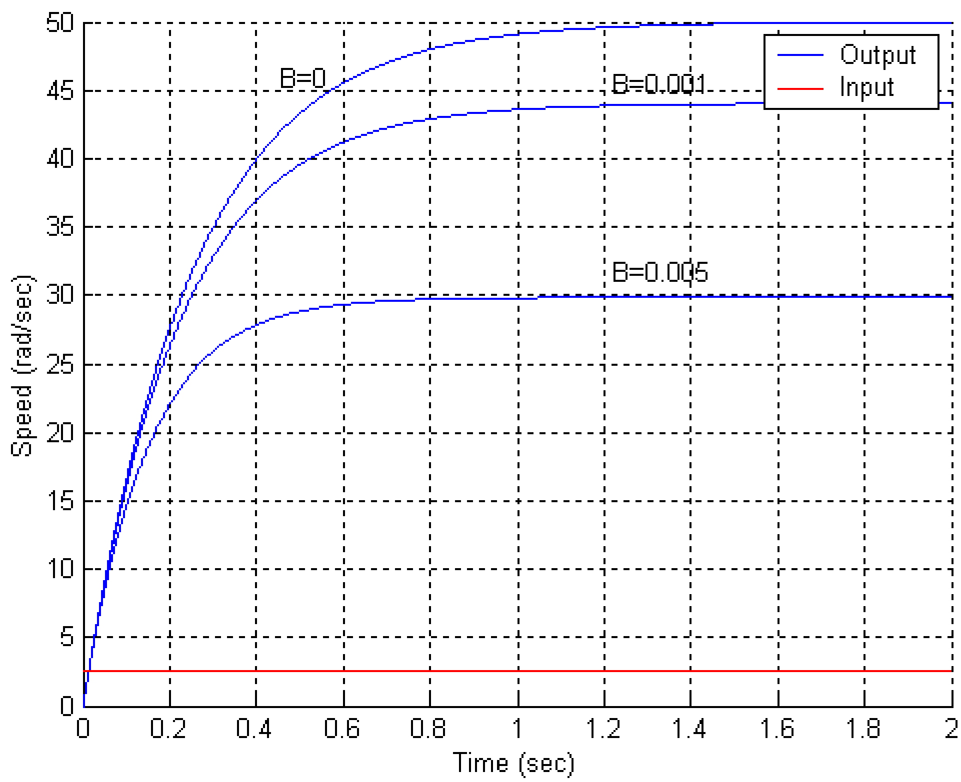


2. Study of the effect of viscous friction:



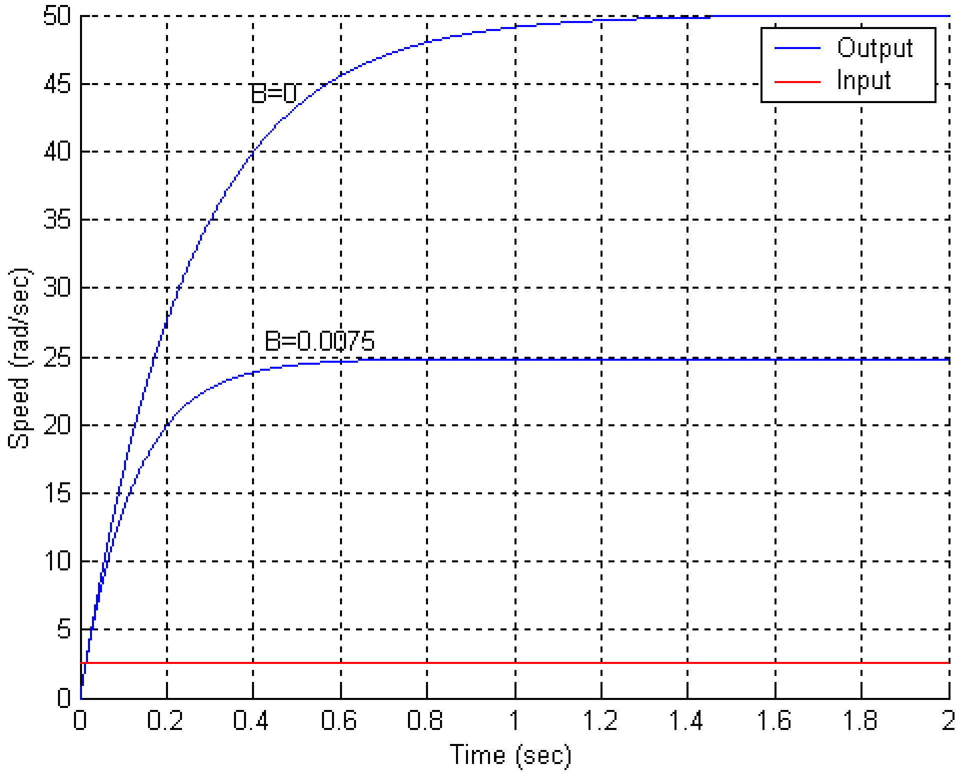
The above figure is plotted for three different friction coefficients (0, 0.001, 0.005) for 5 V armature input. As seen in figure, two important effects are observed as the viscous coefficient is increased. First, the final steady state velocity is decreased and second the response has less oscillation. Both of these effects could be predicted from Eq. (5-114) by increasing damping ratio ζ.

3. Additional load inertia effect:



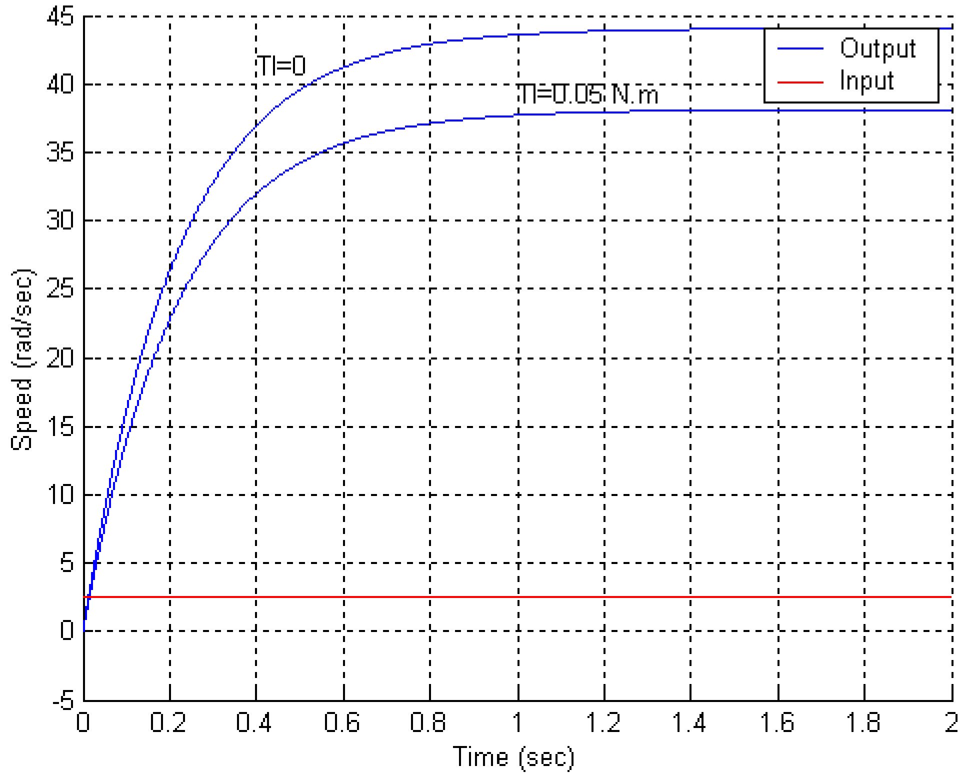
As the overall inertia of the system is increased by $0.005/5.2^2$ and becomes 1.8493×10^{-3} kg.m^2, the mechanical time constant is substantially increased and we can assume the first order model for the motor (ignoring the electrical sub-system) and as a result of this the response is more like an exponential form. The above results are plotted for 5 V armature input.

4. Reduce the speed by 50% by increasing viscous friction:



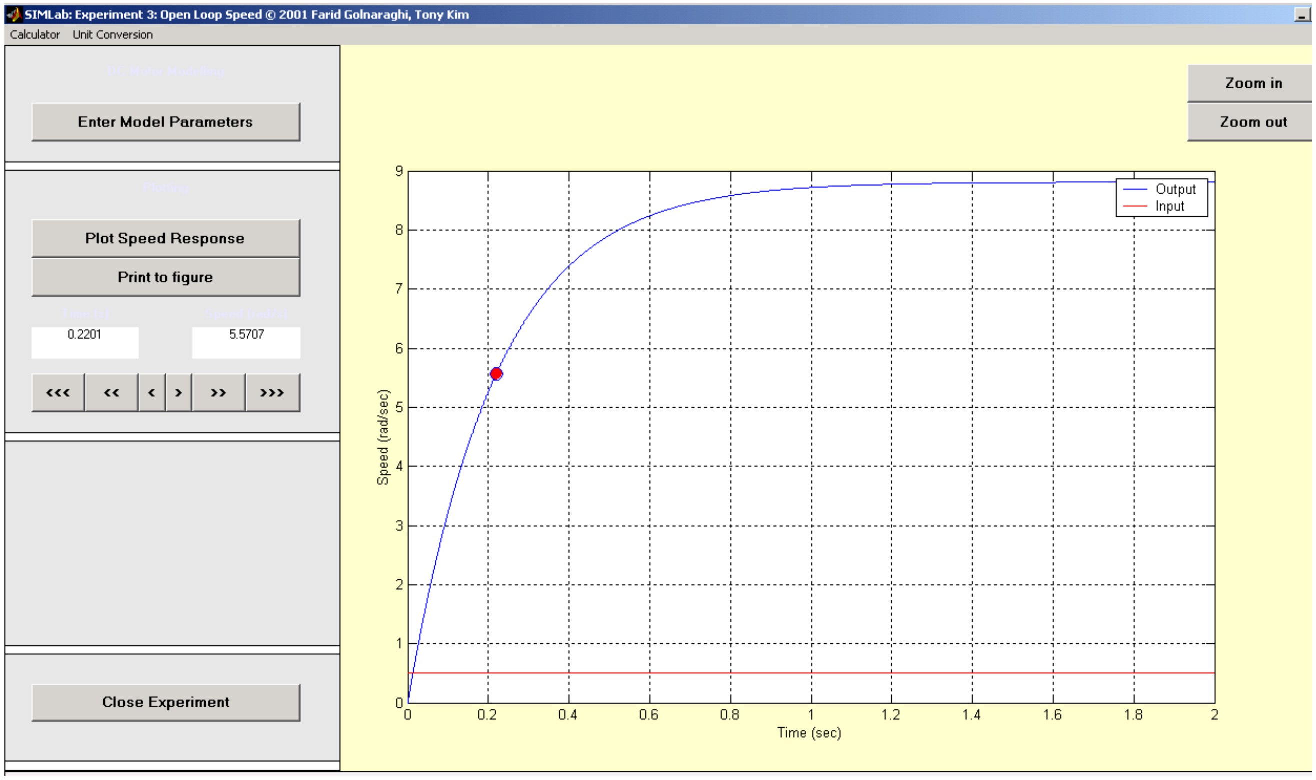
As seen in above figure, if we set *B*=0.0075 *N.s/m* the output speed drop by half comparing with the case that B=0 *N.s/m.* The above results are plotted for 5 V armature input.

5. Study of the effect of disturbance:



Repeating experiment 3 for B=0.001 N.s/m and T_L =0.05 N.m result in above figure. As seen, the effect of disturbance on the speed of open loop system is like the effect of higher viscous friction and caused to decrease the steady state value of speed.

6. Using speed response to estimate motor and load inertia:



Using first order model we are able to identify system parameters based on unit step response of the system. In above plot we repeated the experiments 3 with B=0.001 and set point voltage equal to 1 V. The final value of the speed can be read from the curve and it is 8.8, using the definition of system time constant and the cursor we read 63.2% of speed final value 5.57 occurs at 0.22 sec, which is the system time constant. Considering Eq. (5-116), and using the given value for the rest of parameters, the inertia of the motor and load can be calculated as:

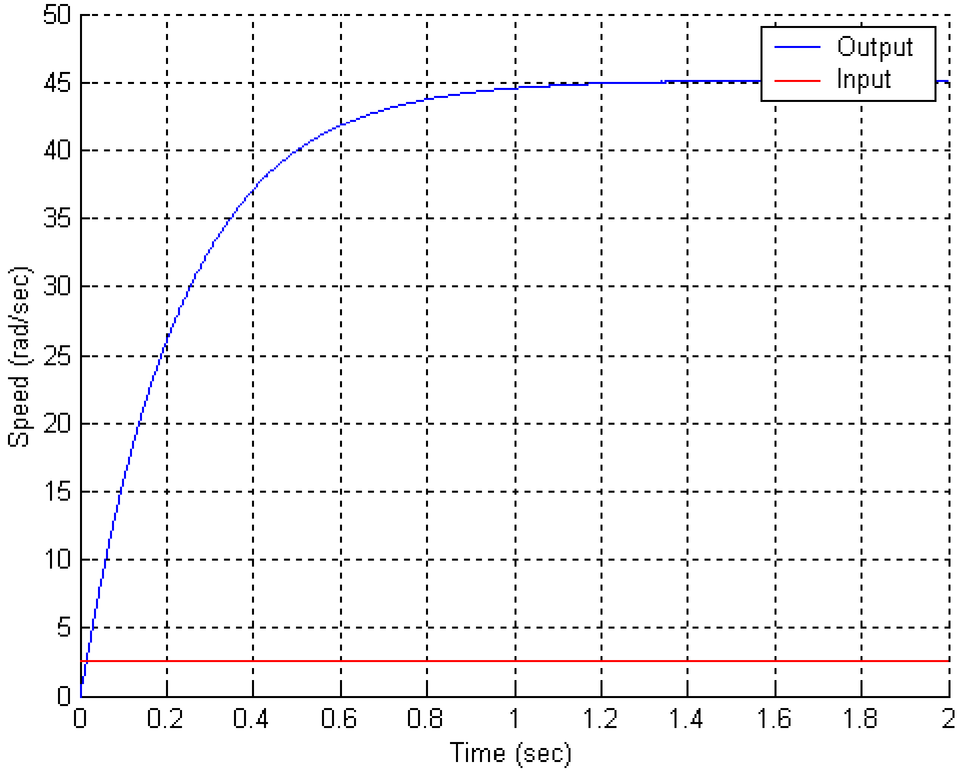
$$J = \frac{\tau_m (R_a B + K_m K_b)}{R_a} = \frac{0.22(1.35 \times 0.001 + 0.01)}{1.35} = 1.8496 \times 10^{-3} \text{ kg.m}^2$$

We also can use the open loop speed response to estimate B by letting the speed to coast down when it gets to the steady state situation and then measuring the required time to get to zero speed. Based on this time and energy conservation principle and knowing the rest of parameters we are able to calculate B. However, this method of identification gives us limited information about the system parameters and we need to measure some parameters directly from motor such as R_a, K_m, K_b and so on.

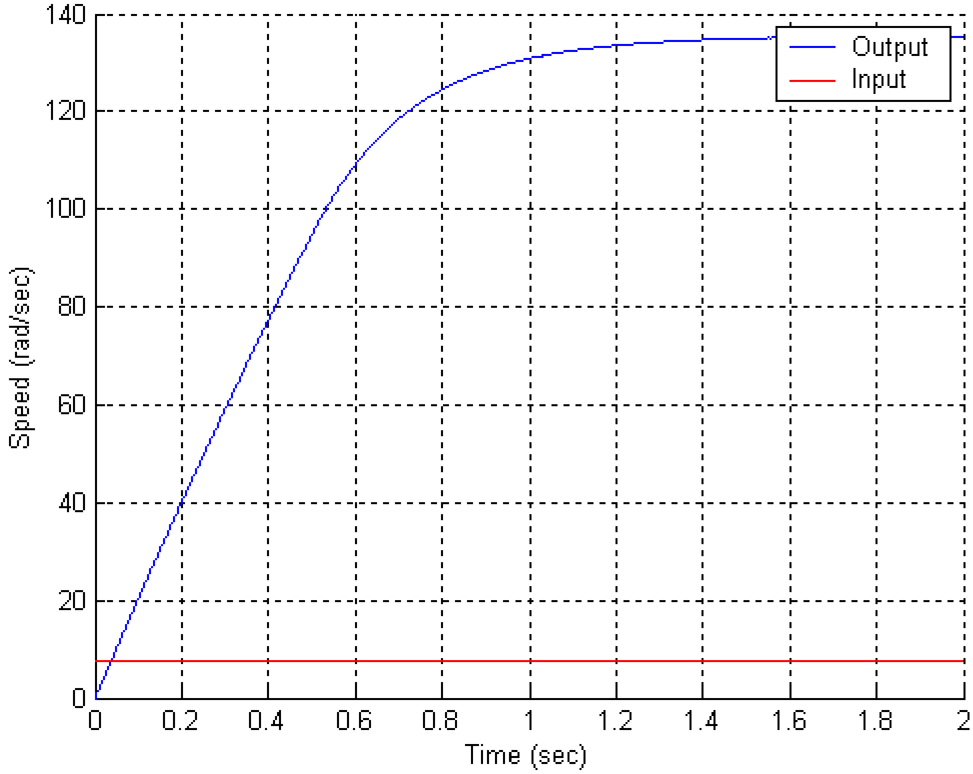
So far, no current or voltage saturation limit is considered for all simulations using SIMLab software.

7. Open loop speed response using Virtual Lab:

a. +5 V:

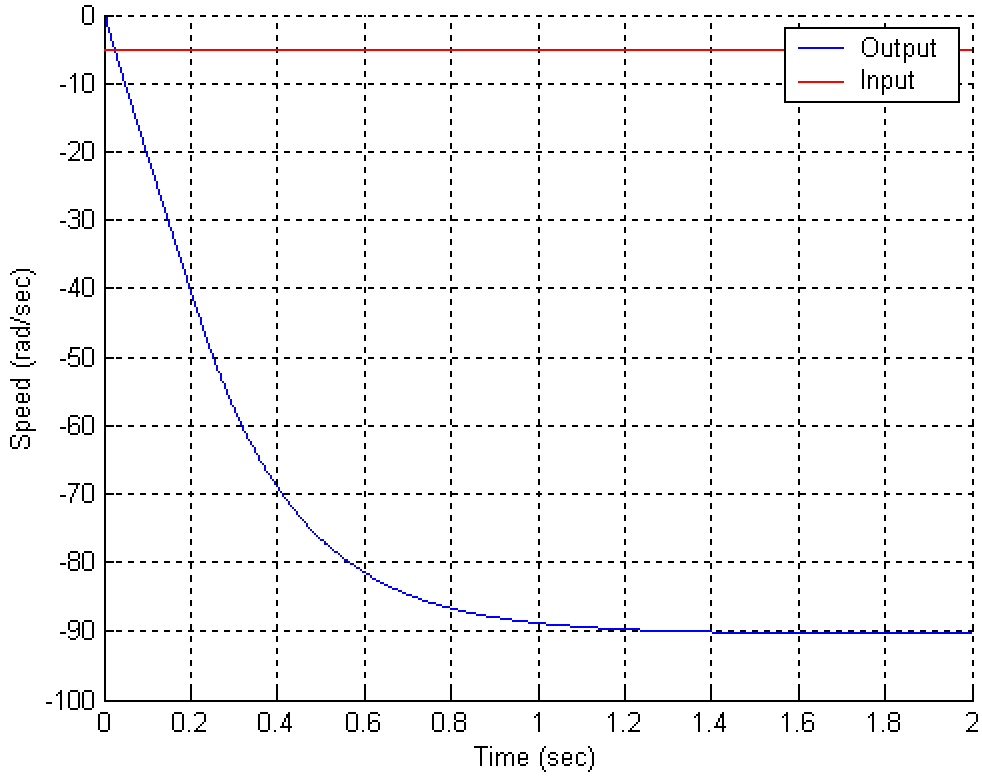


b. +15 V:

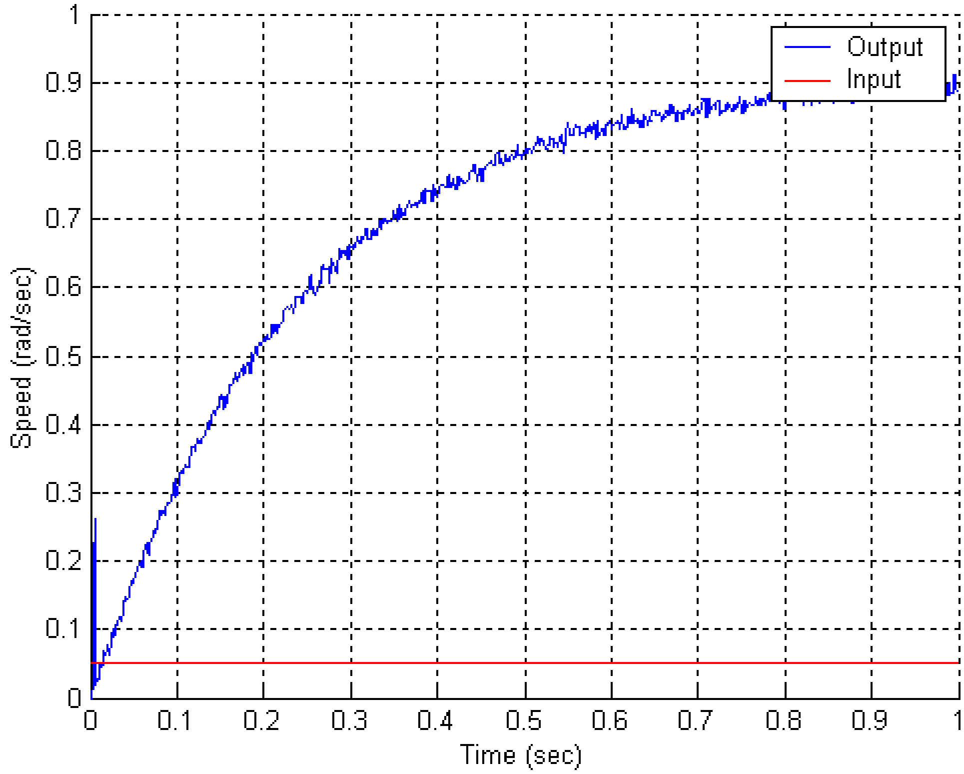


140
120
100
80
60
40
20
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time (sec)
Speed (rad/sec)
Output
Input

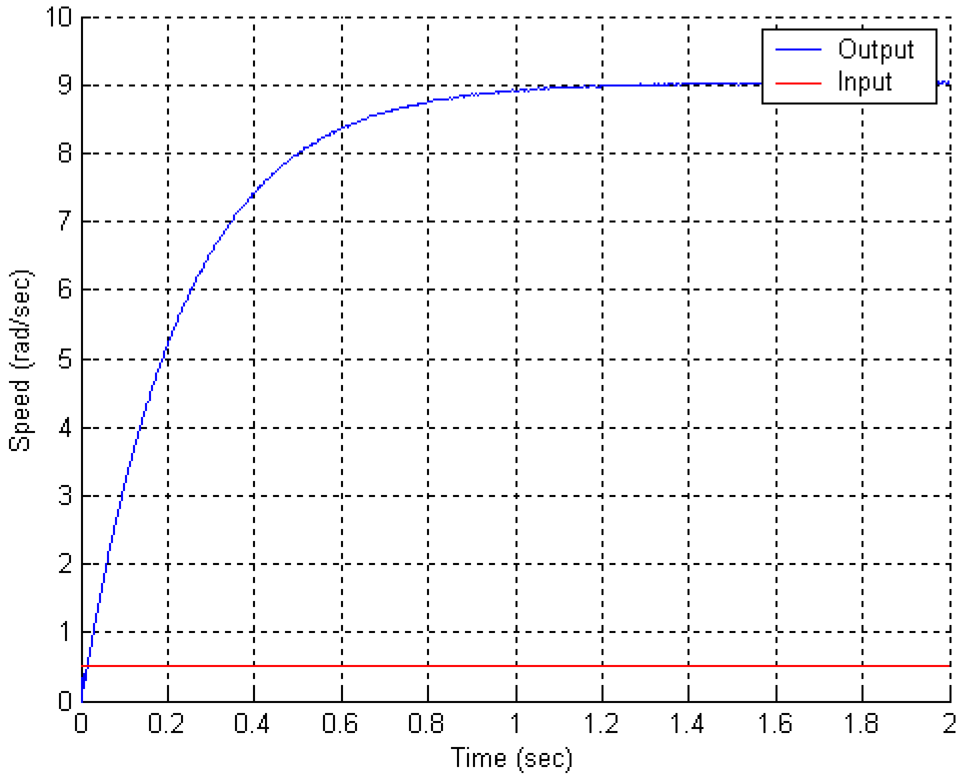
c. –10 V:



Comparing these results with the part 1, the final values are approximately the same but the shape of responses is closed to the first order system behavior. Then the system time constant is obviously different and it can be identified from open loop response. The effect of nonlinearities such as saturation can be seen in +15 V input with appearing a straight line at the beginning of the response and also the effects of noise and friction on the response can be observed in above curves by reducing input voltage for example, the following response is plotted for a 0.1 V step input:


Output
Input
Speed (rad/sec)
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

8. Identifying the system based on open loop response:



Open loop response of the motor to a unit step input voltage is plotted in above figure. Using the definition of time constant and final value of the system, a first order model can be found as:

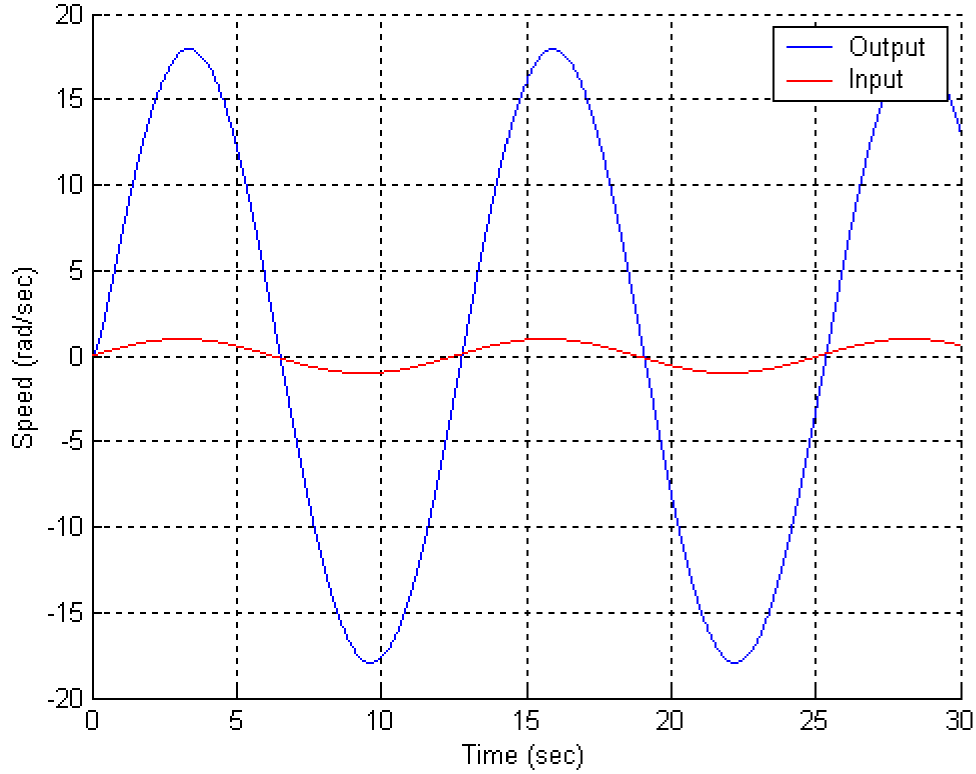
$$G(s) = \frac{9}{0.23s + 1},$$

where the time constant (0.23) is found at 5.68 rad/sec (63.2% of the final value).

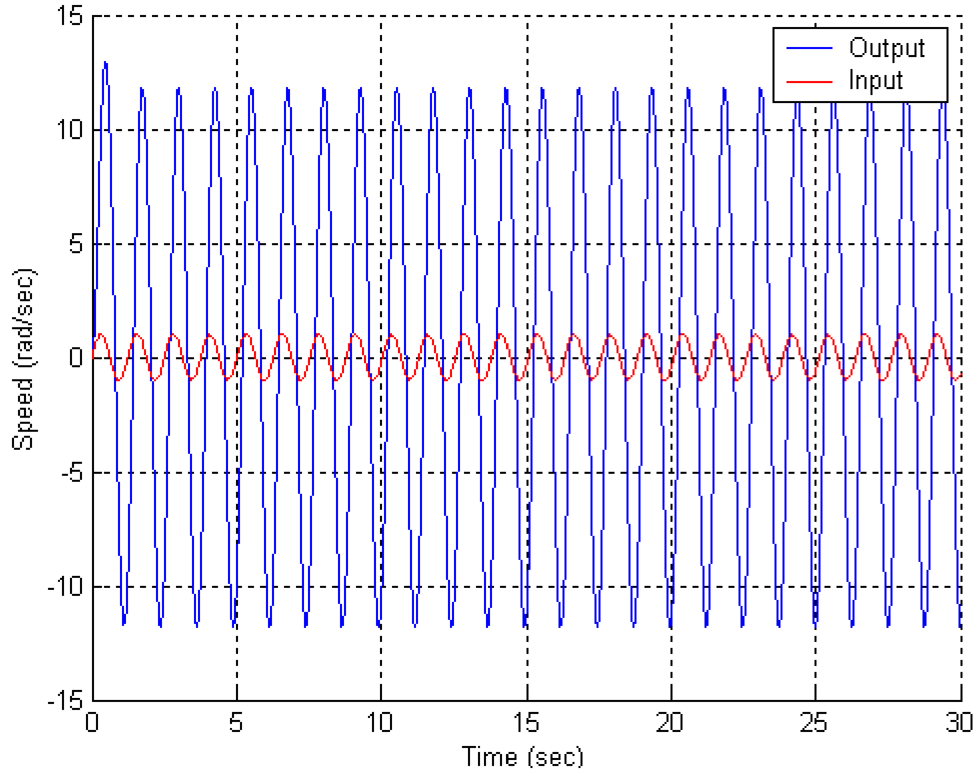
6-4-2 Open Loop Sine Input

9. Sine input to SIMLab and Virtual Lab (1 V. amplitude, and 0.5, 5, and 50 rad/sec frequencies)

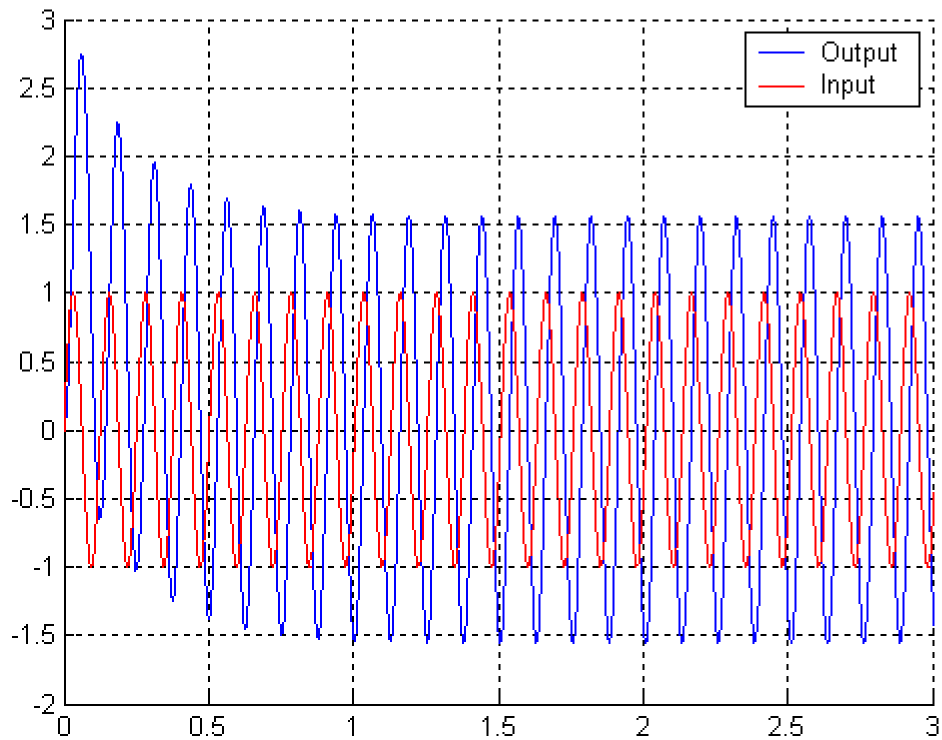
a. 0.5 rad/sec (SIMLab):



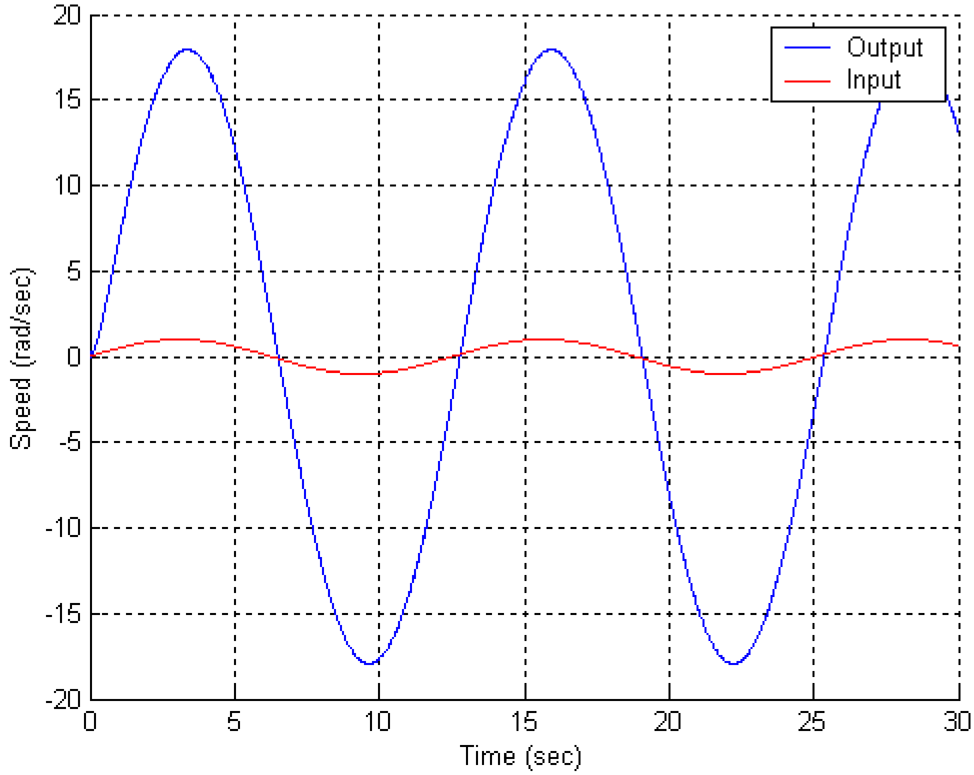
b. 5 rad/sec (SIMLab):



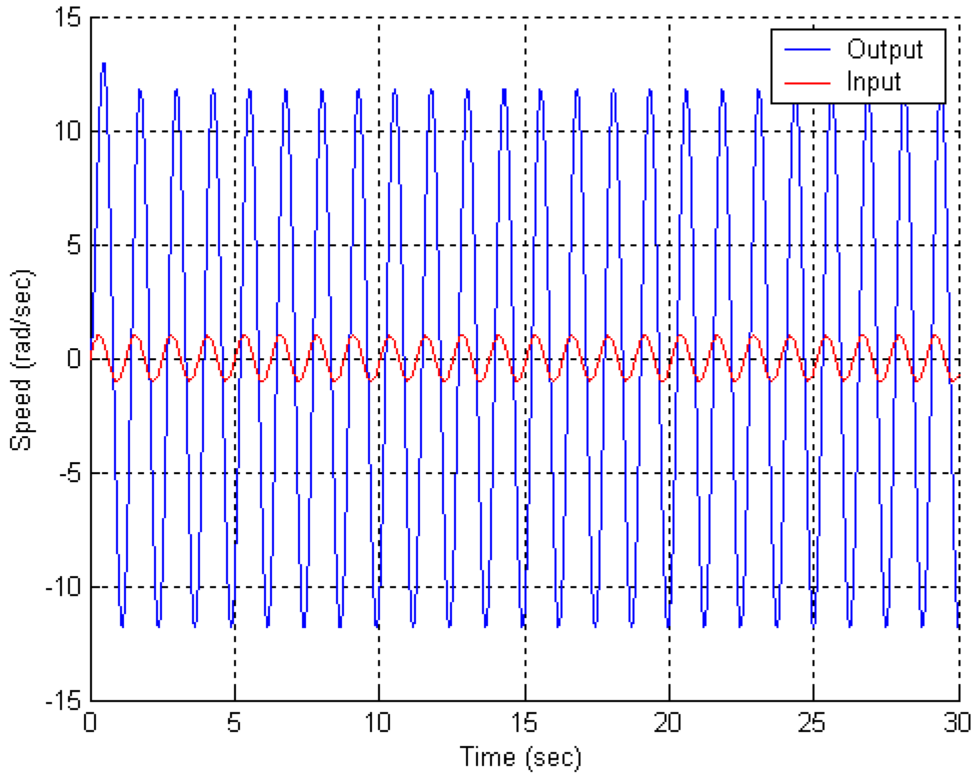
c. 50 rad/sec (SIMLab):



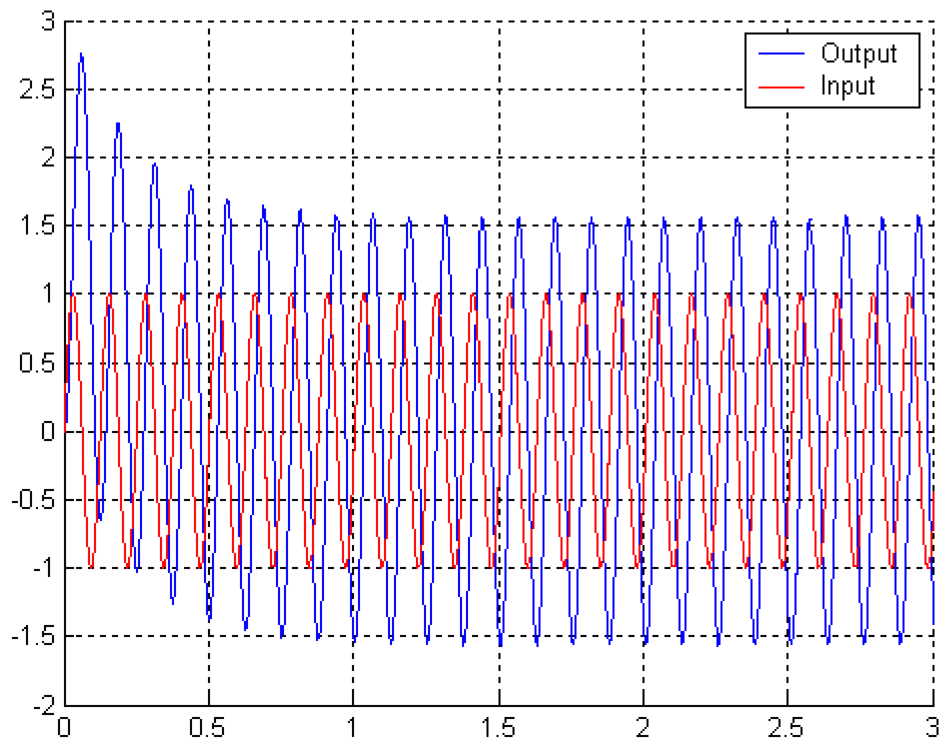
d. 0.5 rad/sec (Virtual Lab):



e. 5 rad/sec (Virtual Lab):

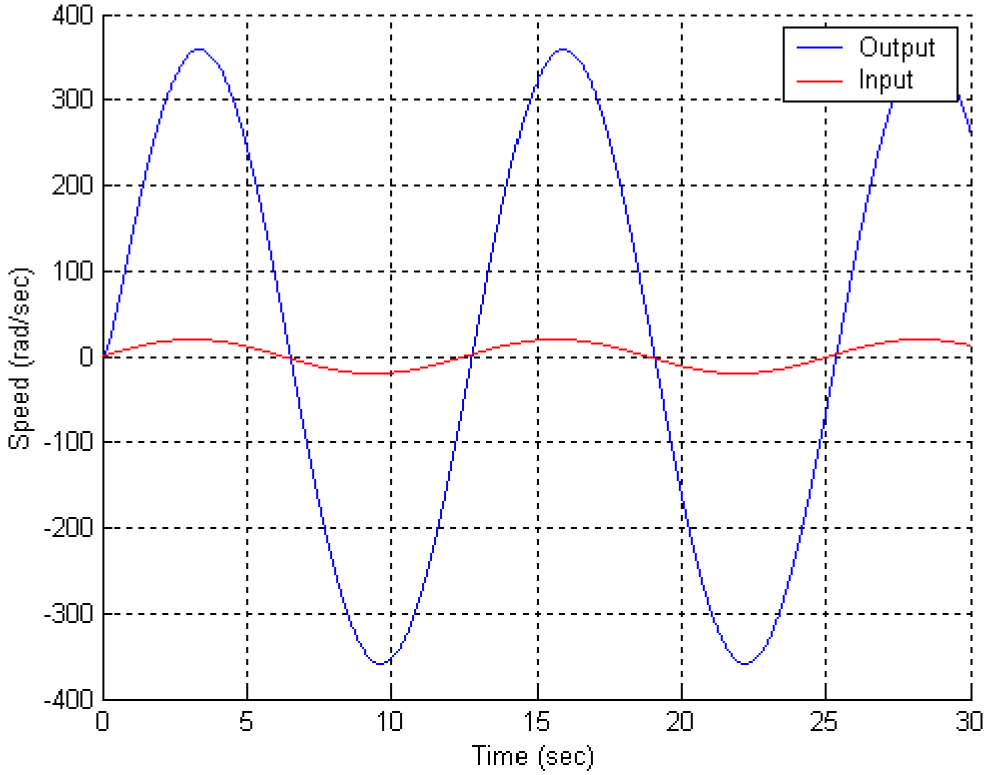


f. 50 rad/sec (Virtual Lab):

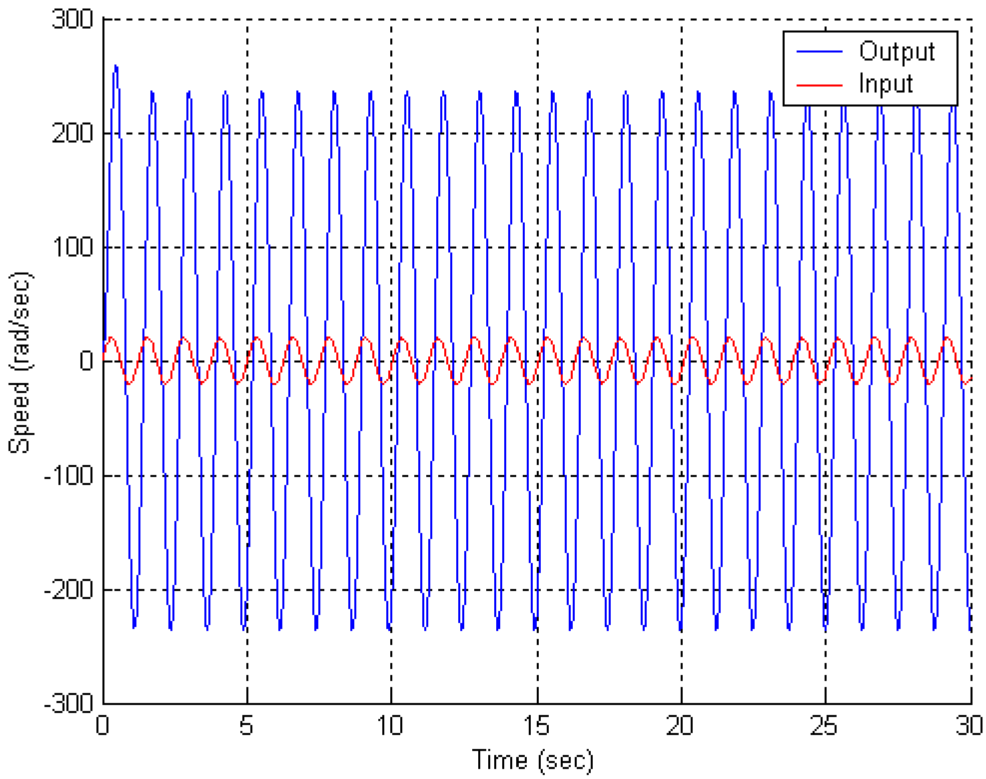


10. Sine input to SIMLab and Virtual Lab (20 V. amplitude)

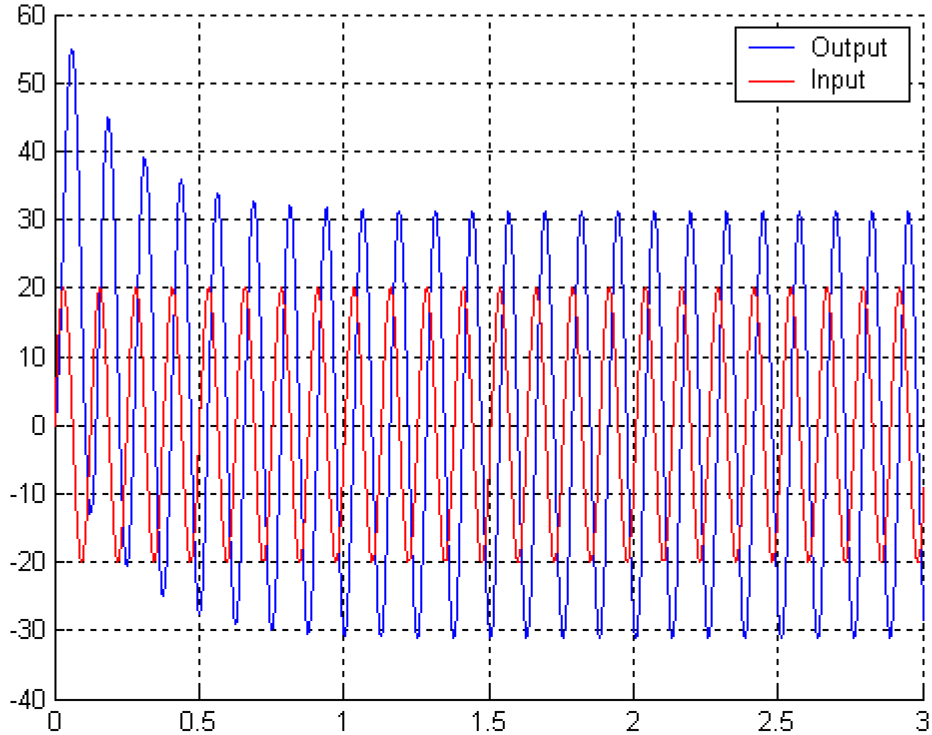
a. 0.5 rad/sec (SIMLab):



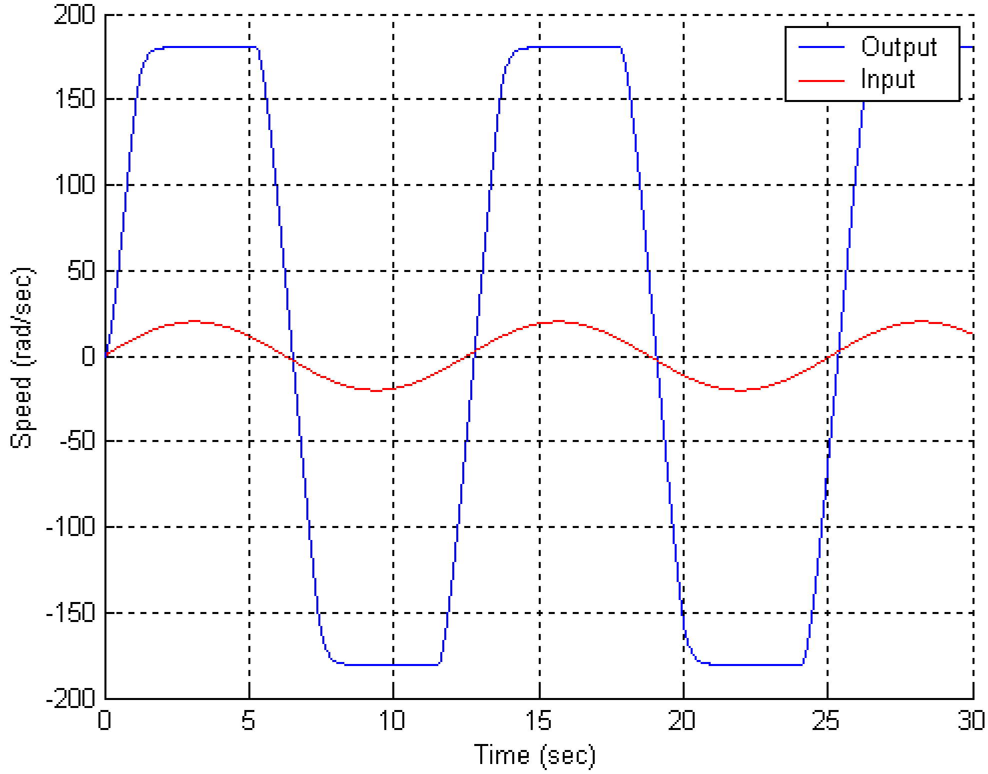
b. 5 rad/sec (SIMLab):



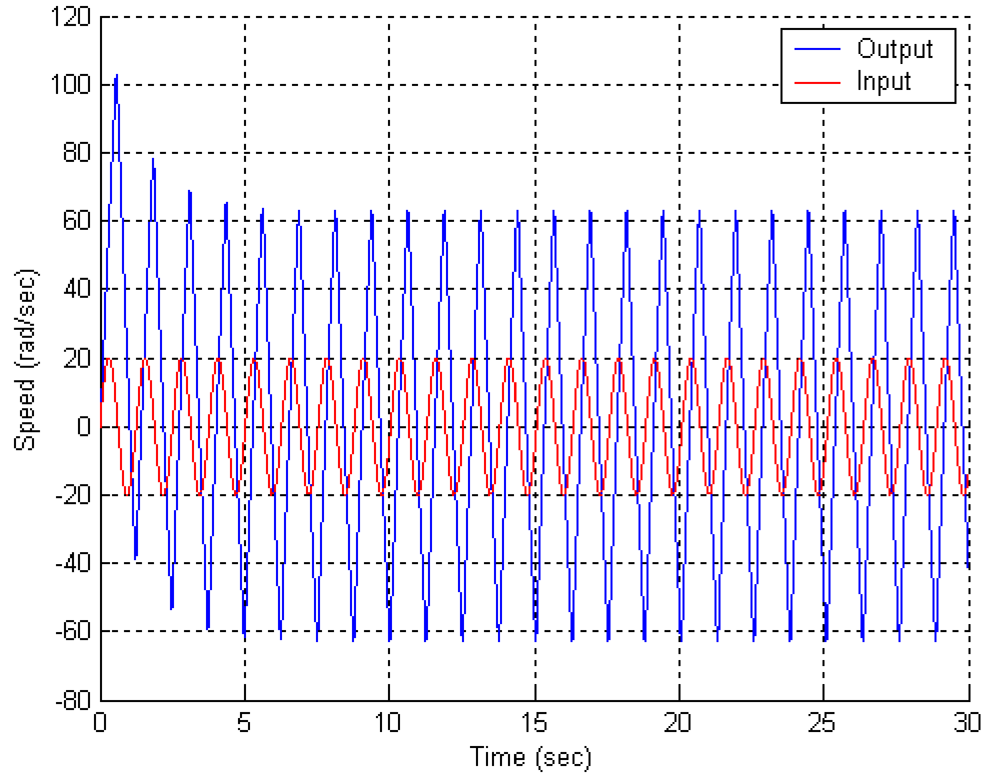
c. 50 rad/sec (SIMLab):



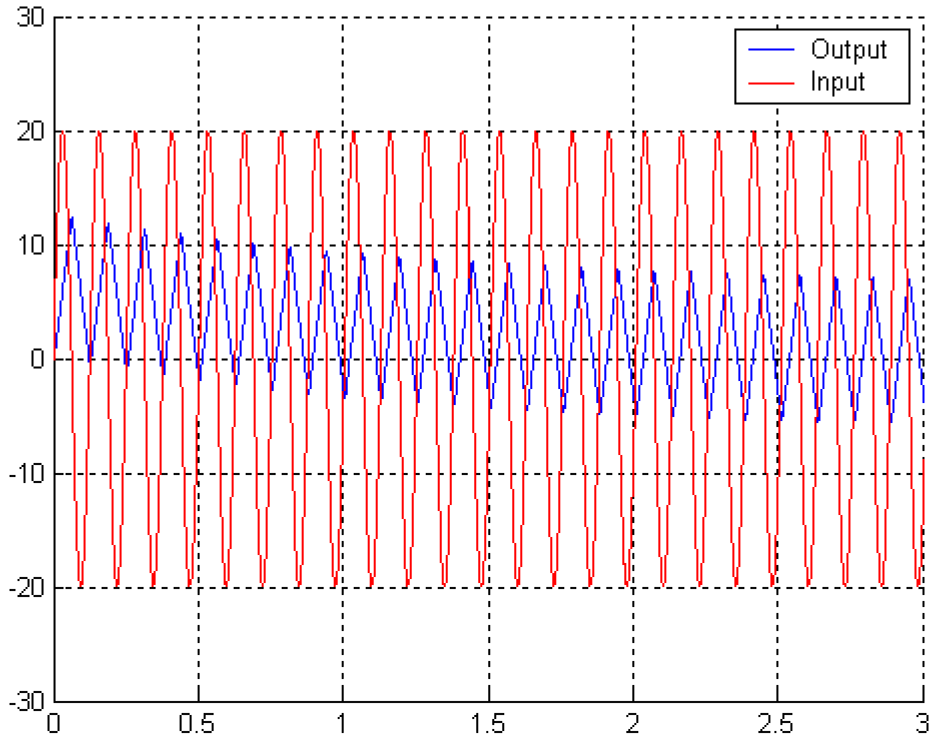
d. 0.5 rad/sec (Virtual Lab):



e. 5 rad/sec (Virtual Lab):



f. 5 rad/sec (Virtual Lab):



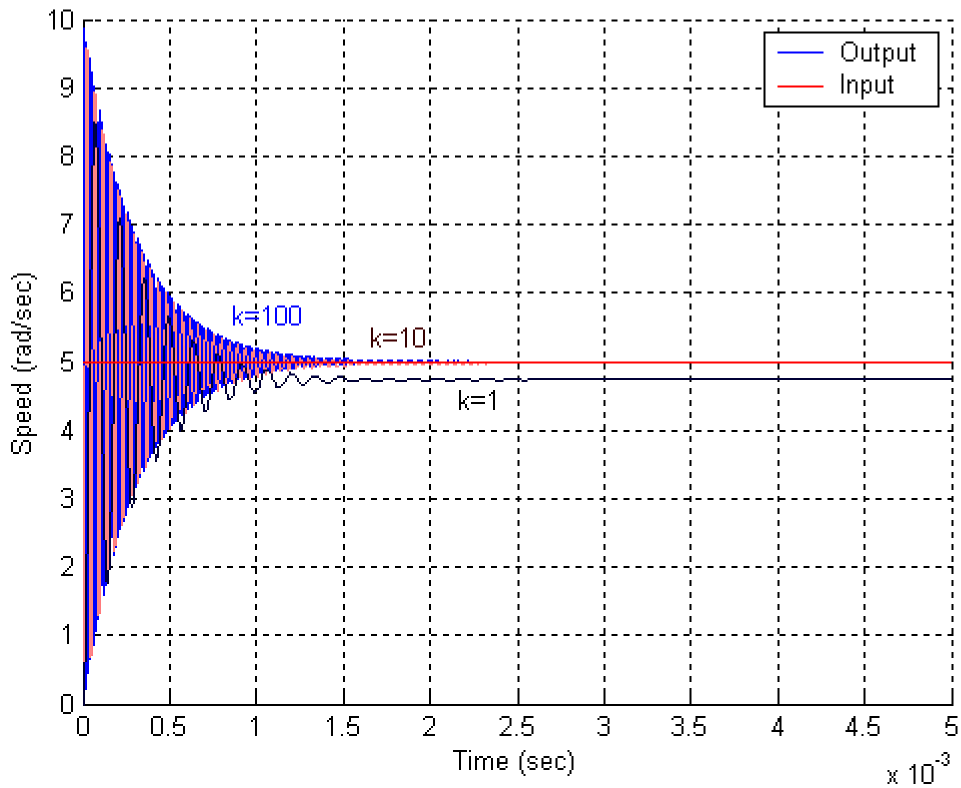
In both experiments 9 and 10, no saturation considered for voltage and current in SIMLab software. If we use the calculation of phase and magnitude in both SIMLab and Virtual Lab we will find that as input frequency increases the magnitude of the output decreases and phase lag increases. Because of existing saturations this phenomenon is more sever in the Virtual Lab experiment (10.f). In this experiments we observe that $M = 0.288$ and $\varphi = -93.82^\circ$ for $\omega = 50$.

6-4-3 Speed Control

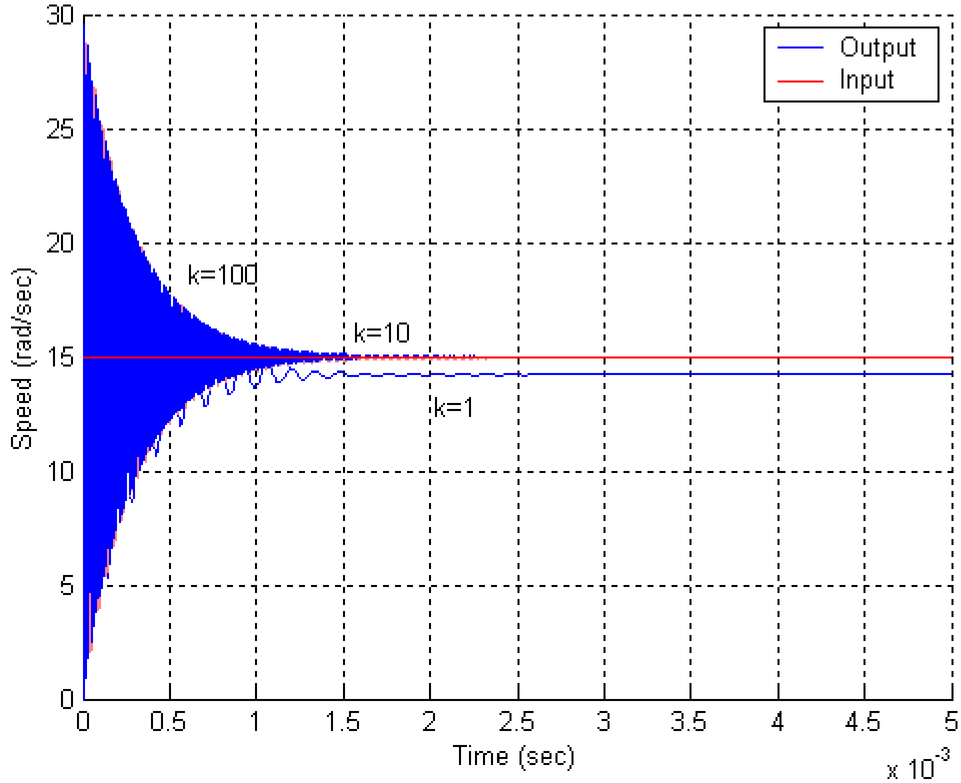
11. Apply step inputs (SIMLab)

In this section no saturation is considered either for current or for voltage.

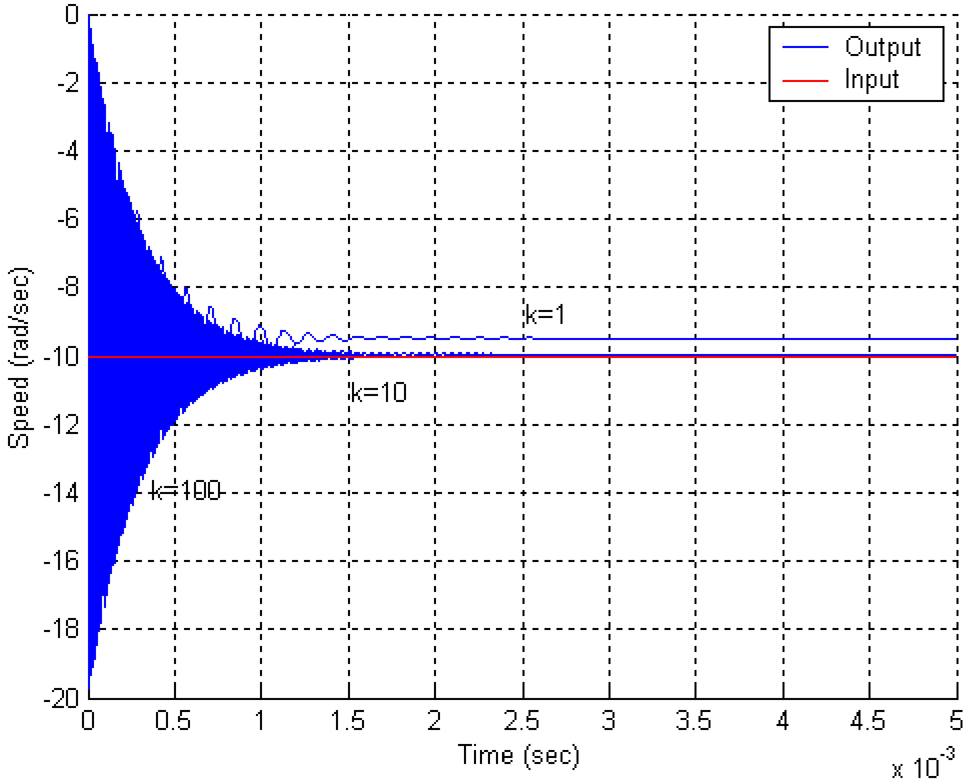
a. +5 V:



b. +15 V:

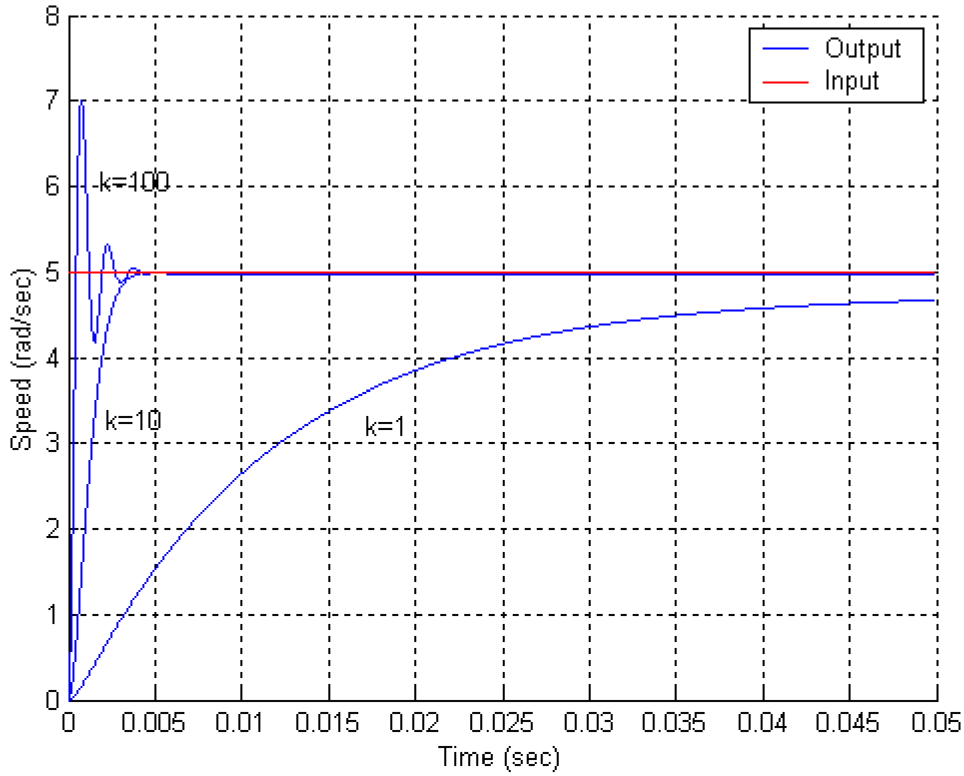


c. -10 V:

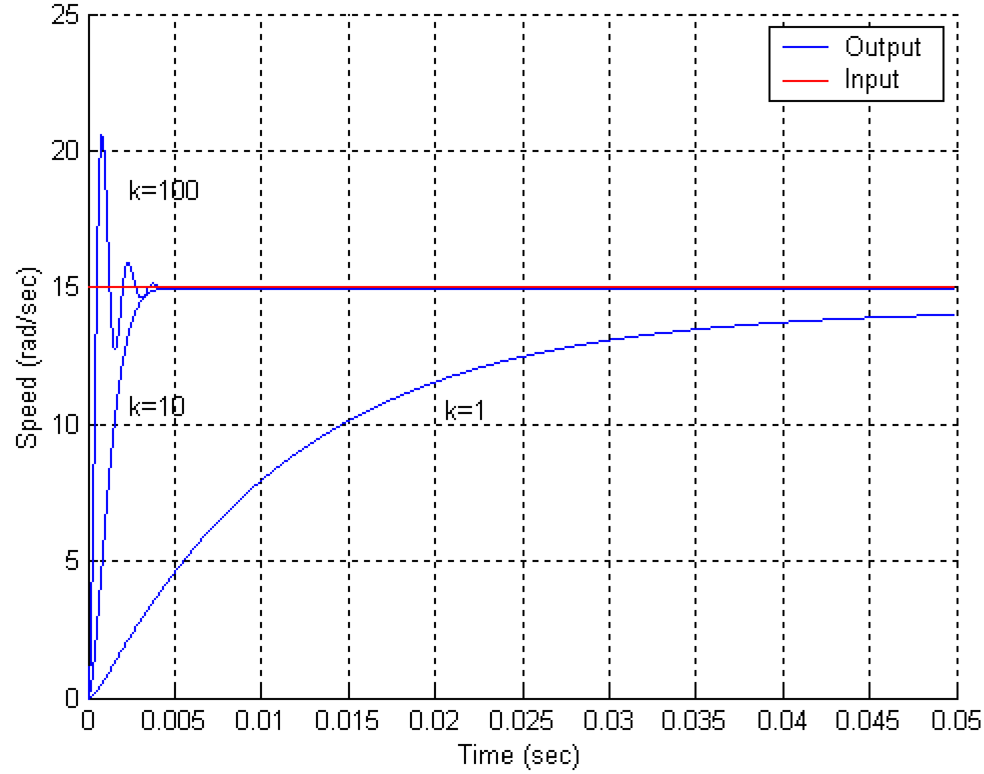

Output
Input
k=1
k=10
k=100
Speed (rad/sec)
Time (sec)
x 10^-3
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

12. Additional load inertia effect:

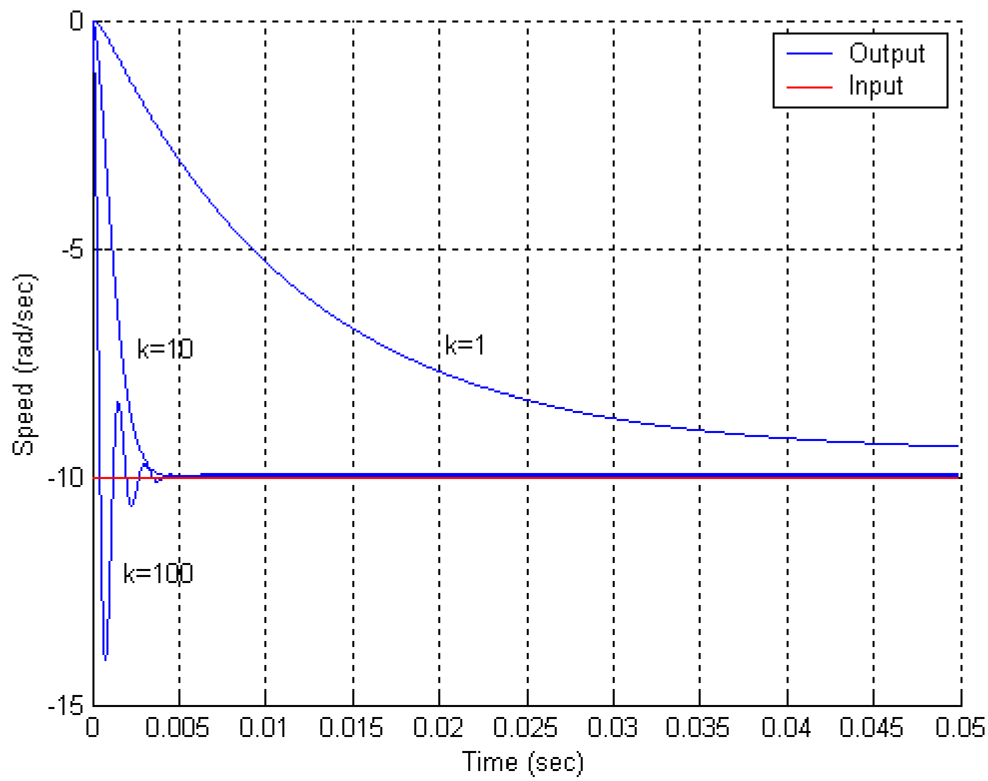
a. +5 V:



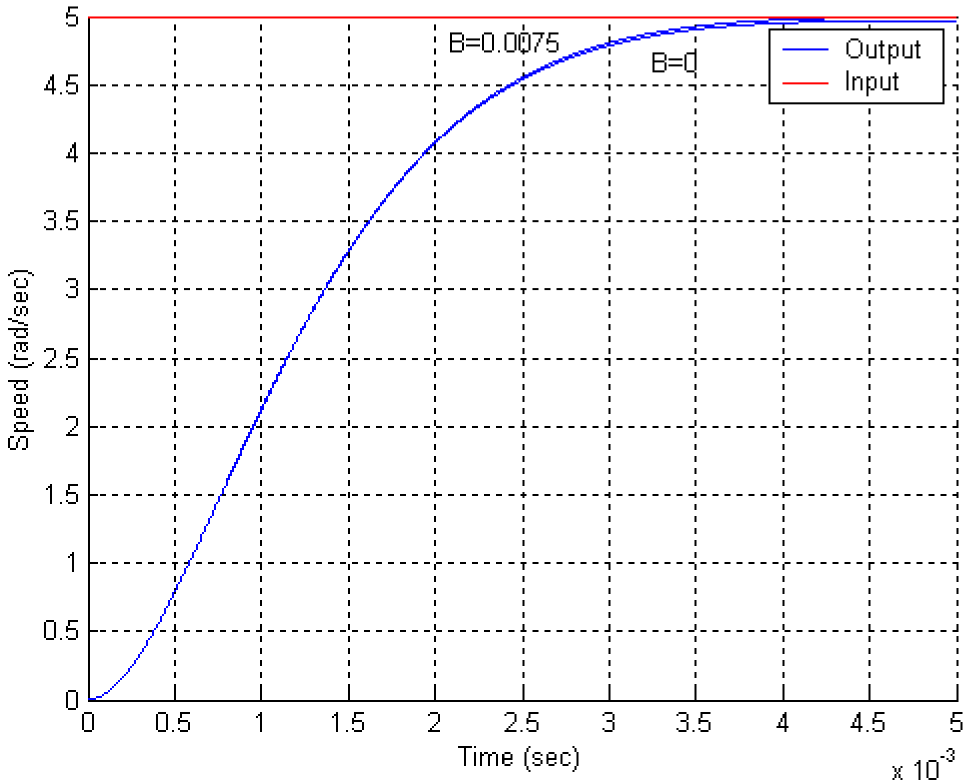
b. +15 V:



c. -10 V:

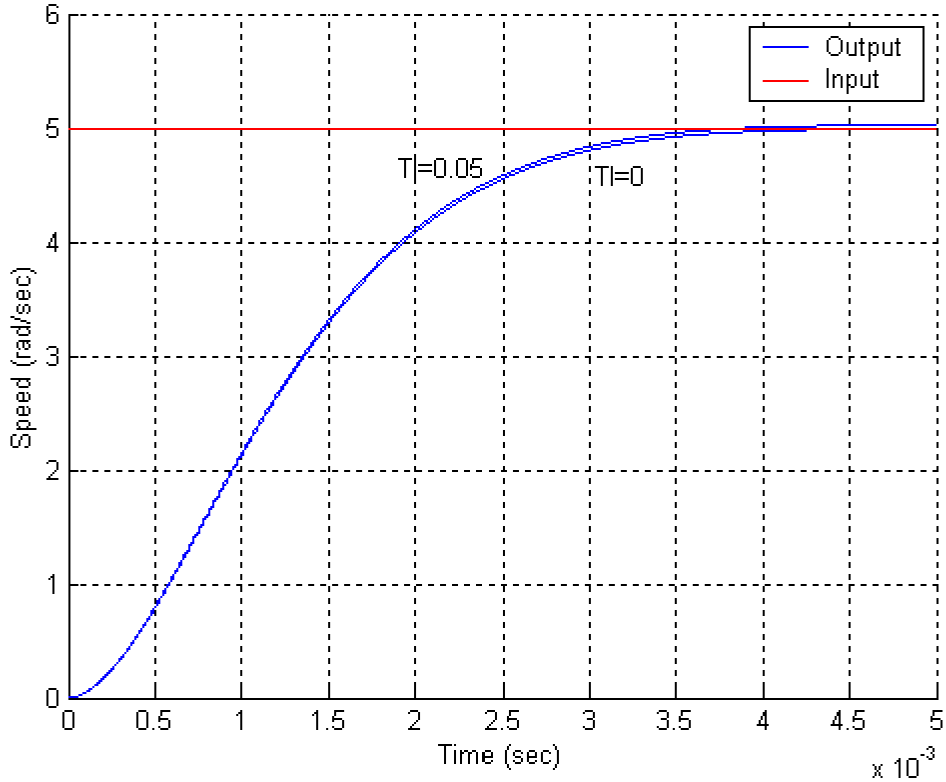


13. Study of the effect of viscous friction:

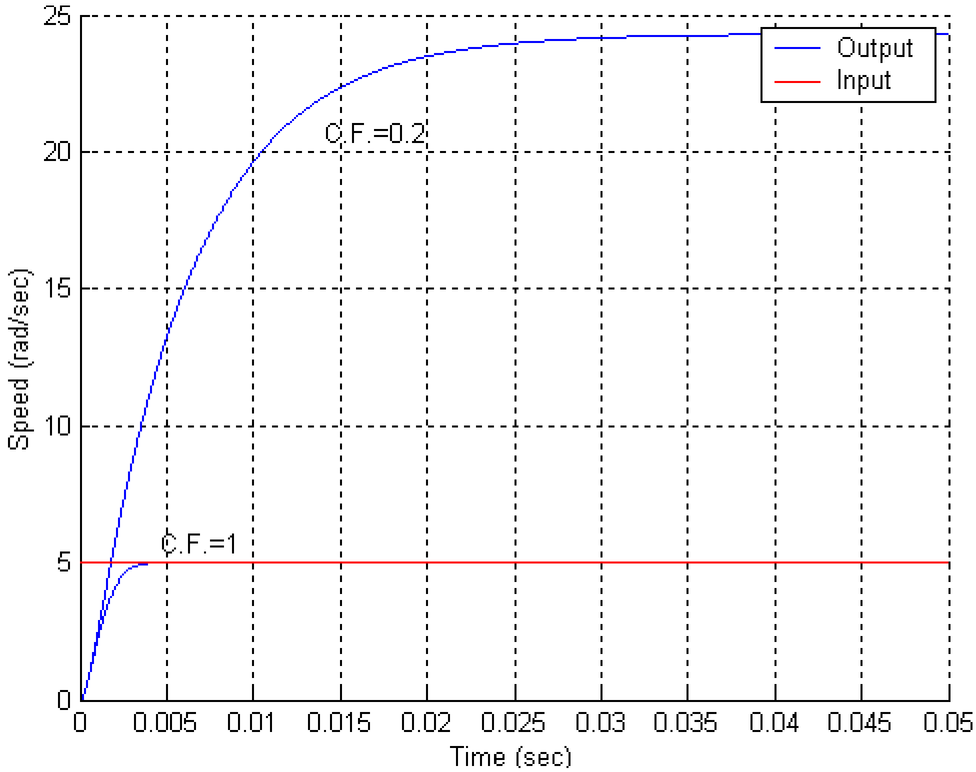


As seen in above figure, two different values for B are selected, zero and 0.0075. We could change the final speed by 50% in open loop system. The same values selected for closed loop speed control but as seen in the figure the final value of speeds stayed the same for both cases. It means that closed loop system is robust against changing in system's parameters. For this case, the gain of proportional controller and speed set point are 10 and 5 rad/sec, respectively.

14. Study of the effect of disturbance:

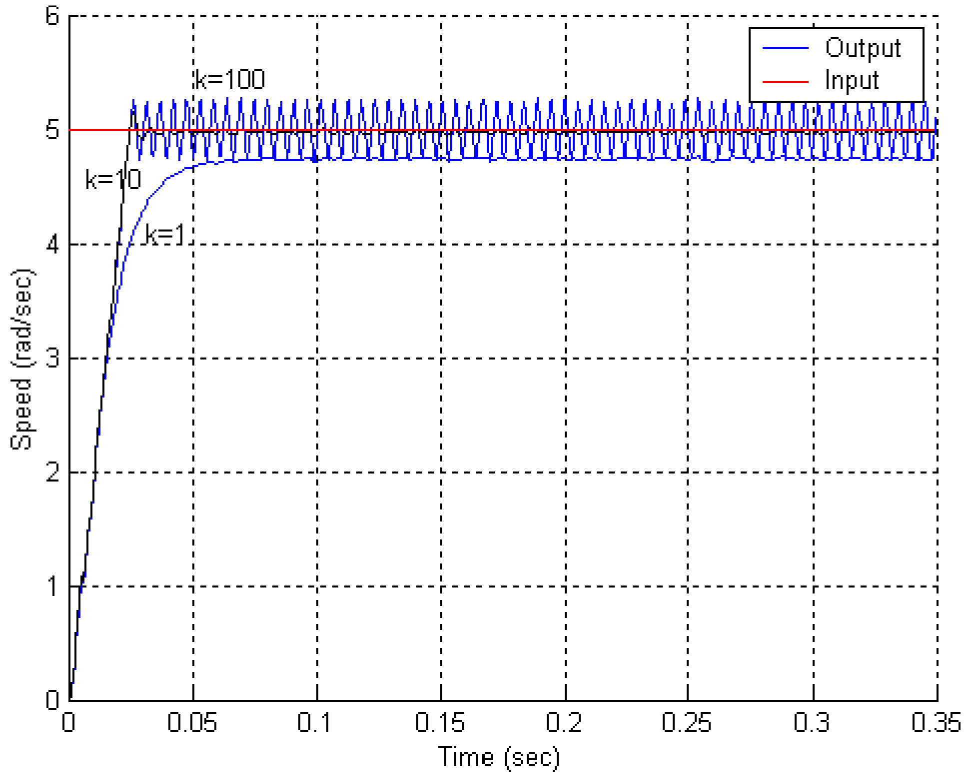


Repeating part 5 in section 6-4-1 for B=0.001 and T_L =0.05 N.m result in above figure. As seen, the effect of disturbance on the speed of closed loop system is not substantial like the one on the open loop system in part 5, and again it is shown the robustness of closed loop system against disturbance. Also, to study the effects of conversion factor see below figure, which is plotted for two different C.F. and the set point is 5 V.

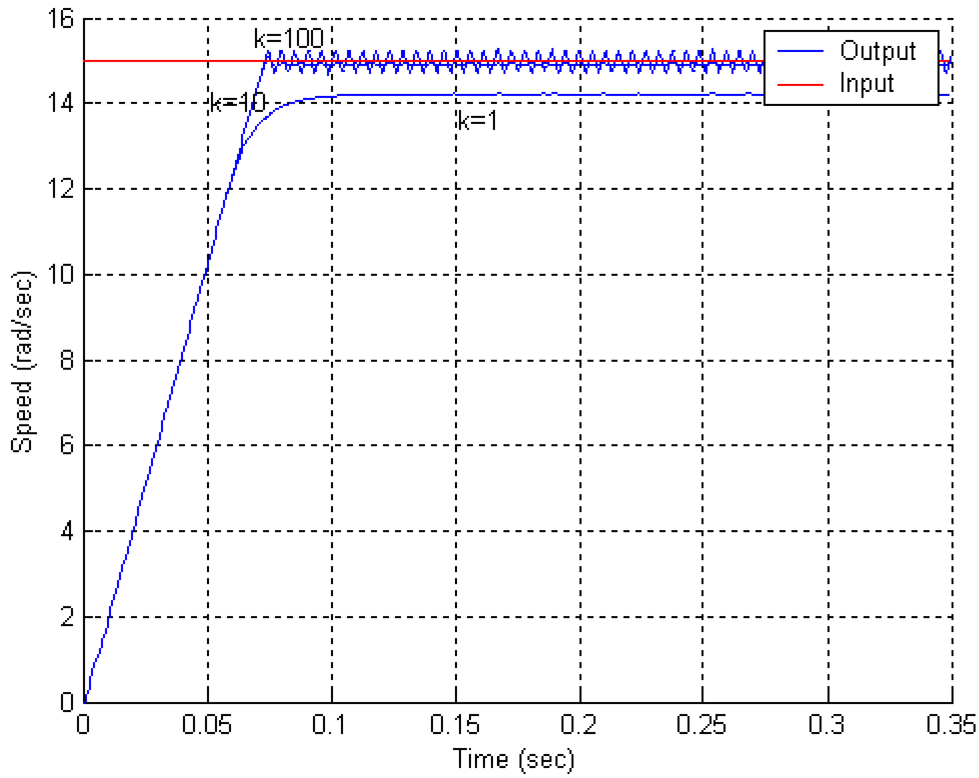


By decreasing the C.F. from 1 to 0.2, the final value of the speed increases by a factor of 5.

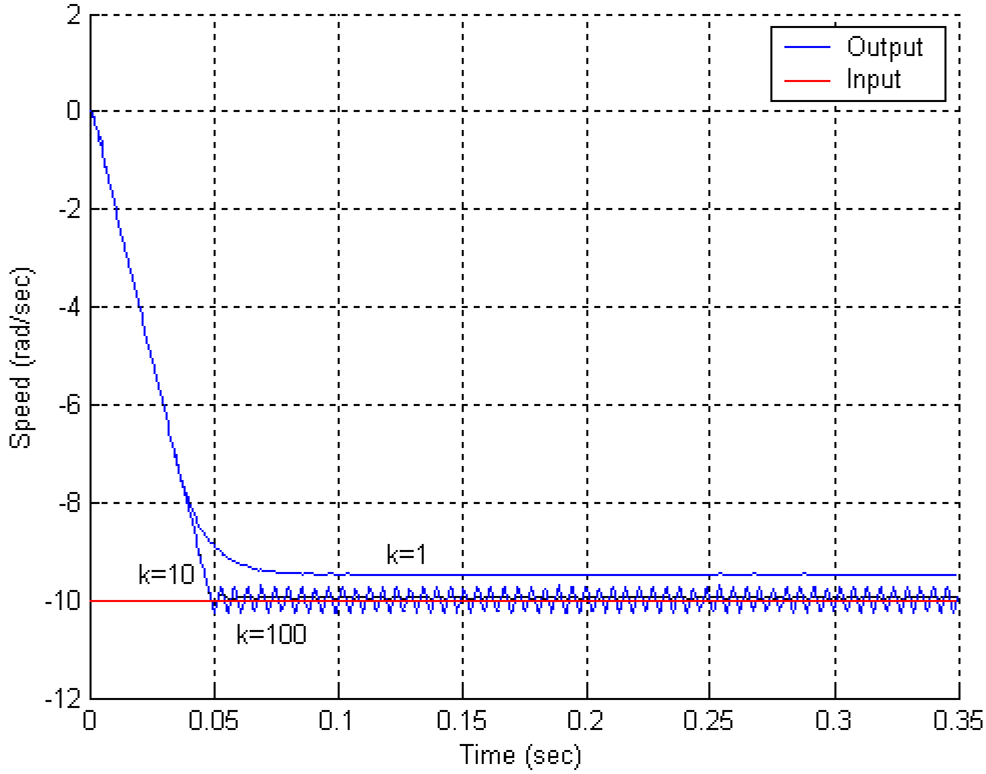
15. Apply step inputs (Virtual Lab)
   a. +5 V:


k=100
k=10
k=1
Output
Input
Speed (rad/sec)
Time (sec)
0
1
2
3
4
5
6
0.05
0.1
0.15
0.2
0.25
0.3
0.35

b. +15 V:


k=100
k=10
k=1
Output
Input
Speed (rad/sec)
Time (sec)
0
2
4
6
8
10
12
14
16
0.05
0.1
0.15
0.2
0.25
0.3
0.35

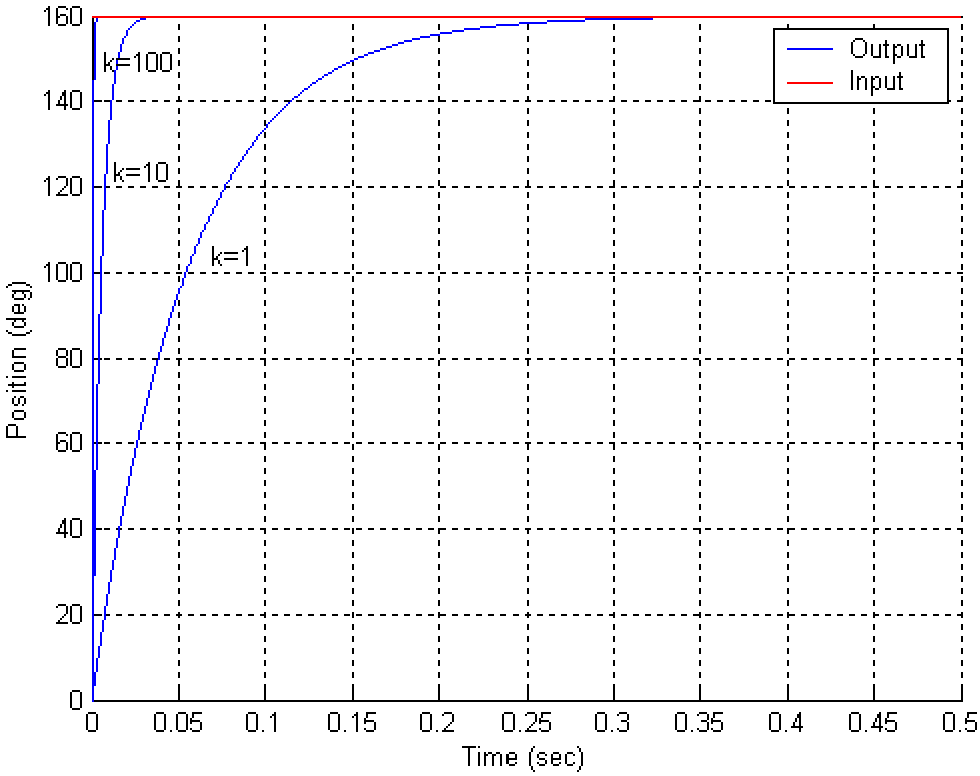
c. –10 V:



As seen the responses of Virtual Lab software, they are clearly different from the same results of SIMLab software. The nonlinearities such as friction and saturation cause these differences. For example, the chattering phenomenon and flatness of the response at the beginning can be considered as some results of nonlinear elements in Virtual Lab software.

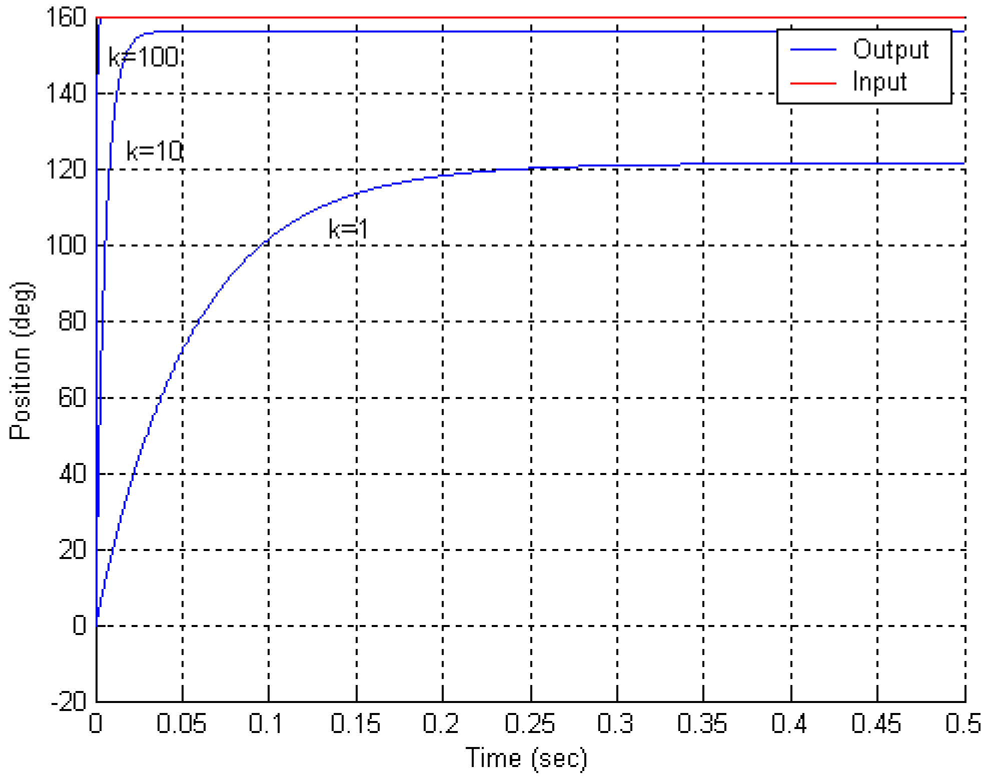
6-4-4 Position Control

16. 160° step input (SIMLab)

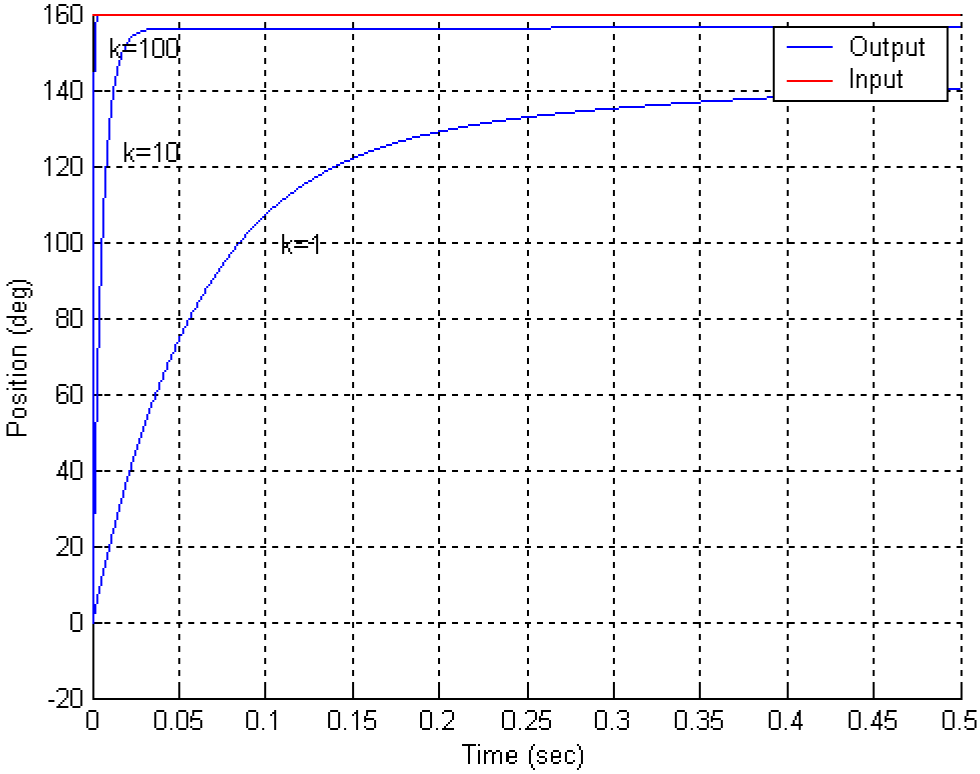


160
140
120
100
80
60
40
20
0
Position (deg)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Time (sec)
Output
Input
k=100
k=10
k=1

17. –0.1 N.m step disturbance

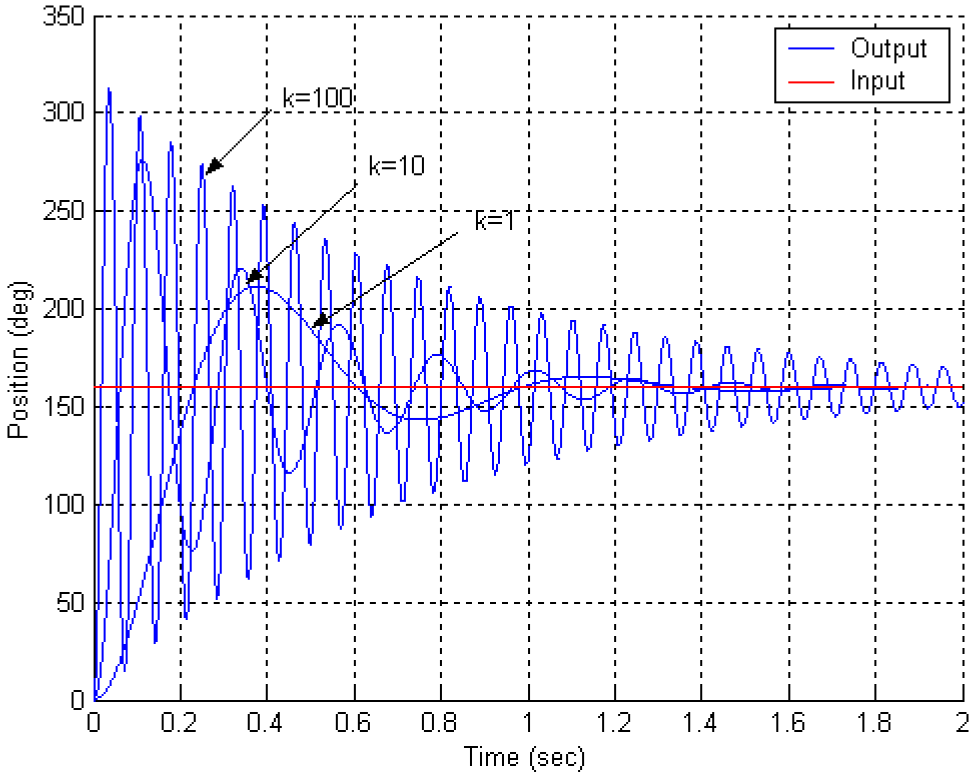


18. Examine the effect of integral control

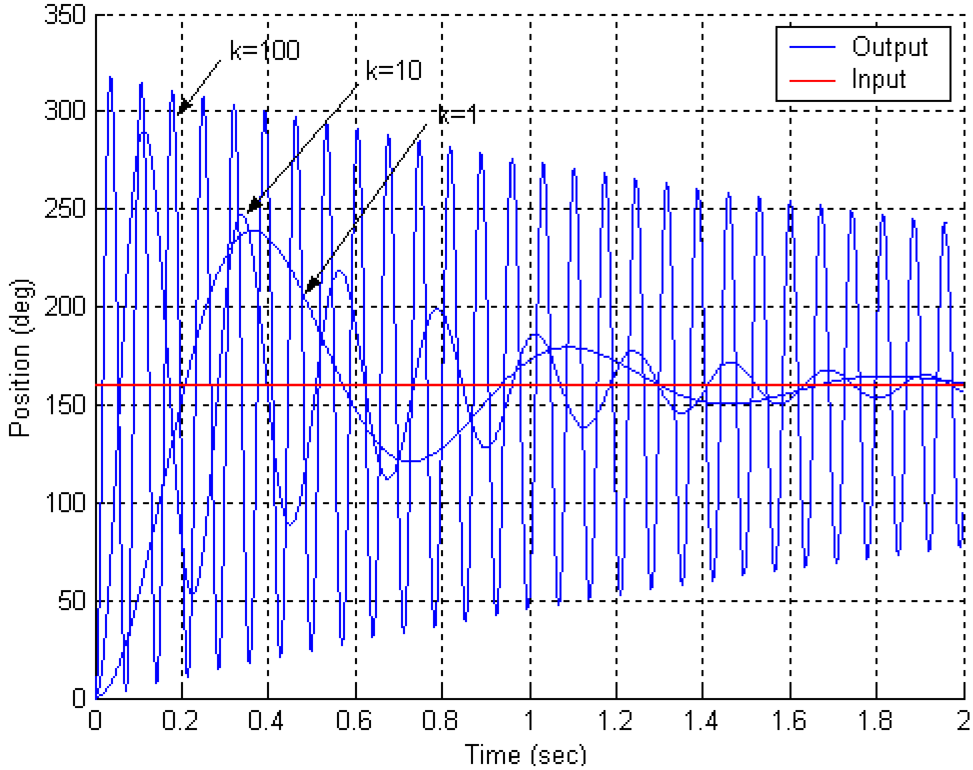


In above figure, an integral gain of 1 is considered for all curves. Comparing this plot with the previous one without integral gain, results in less steady state error for the case of controller with integral part.

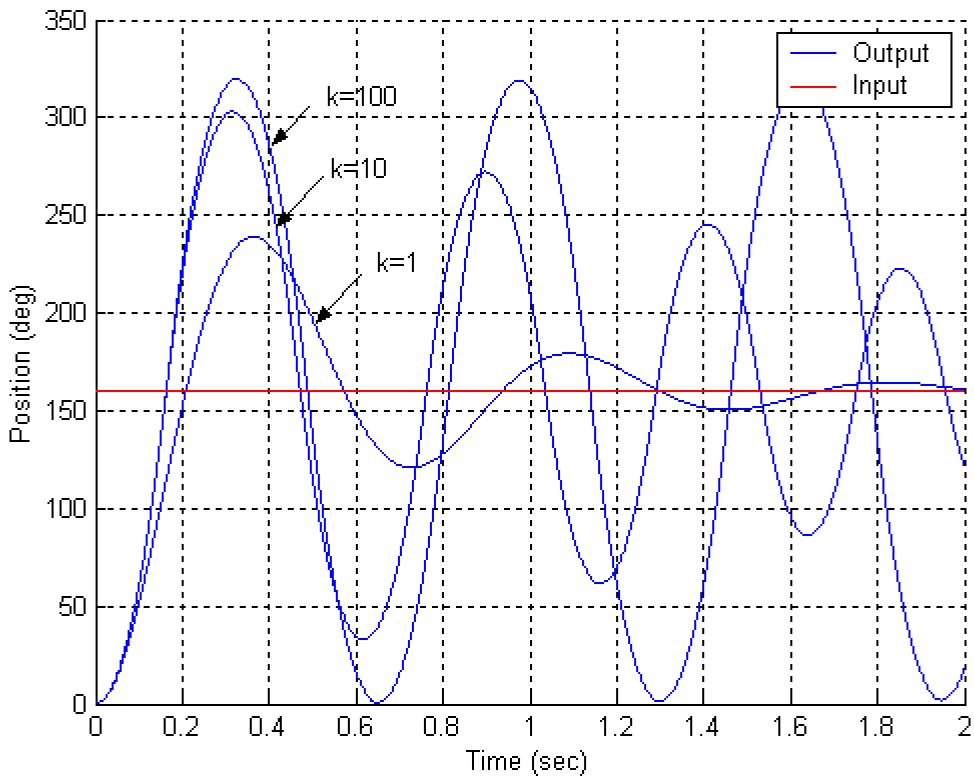
19. Additional load inertia effect (J=0.0019, B=0.004):


Output
Input
k=100
k=10
k=1
Position (deg)
Time (sec)
350
300
250
200
150
100
50
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

20. Set B=0:



21. Study the effect of saturation

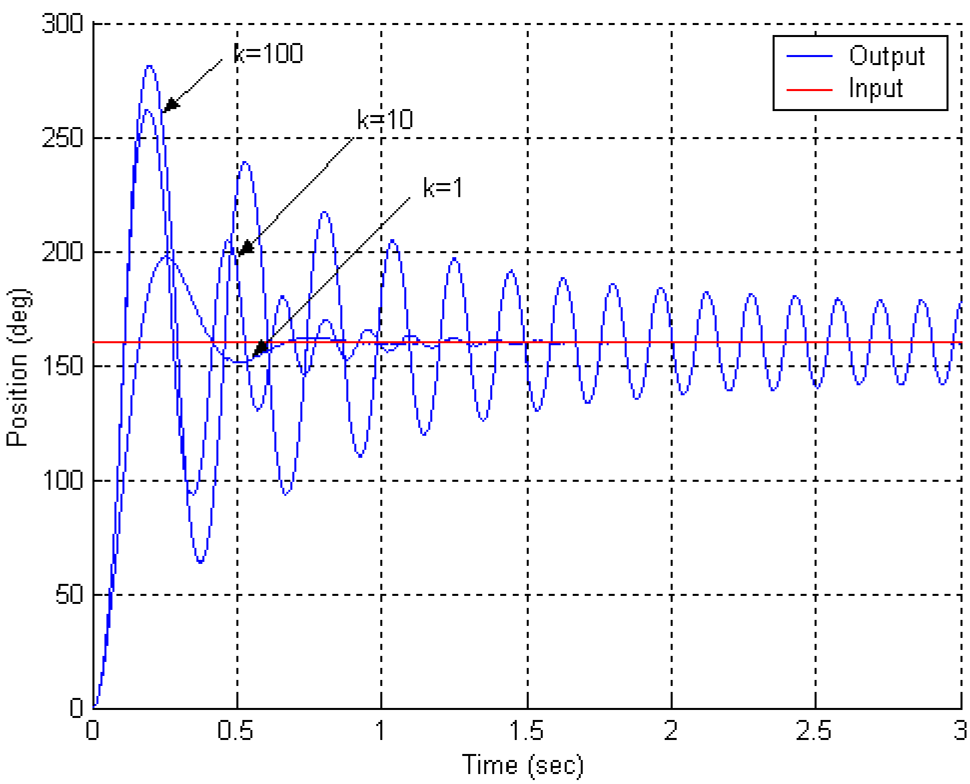


The above figure is obtained in the same conditions of part 20 but in this case we considered ± 10 V. and ± 4 A. as the saturation values for voltage and current, respectively. As seen in the figure, for higher proportional gains the effect of saturations appears by reducing the frequency and damping property of the system.

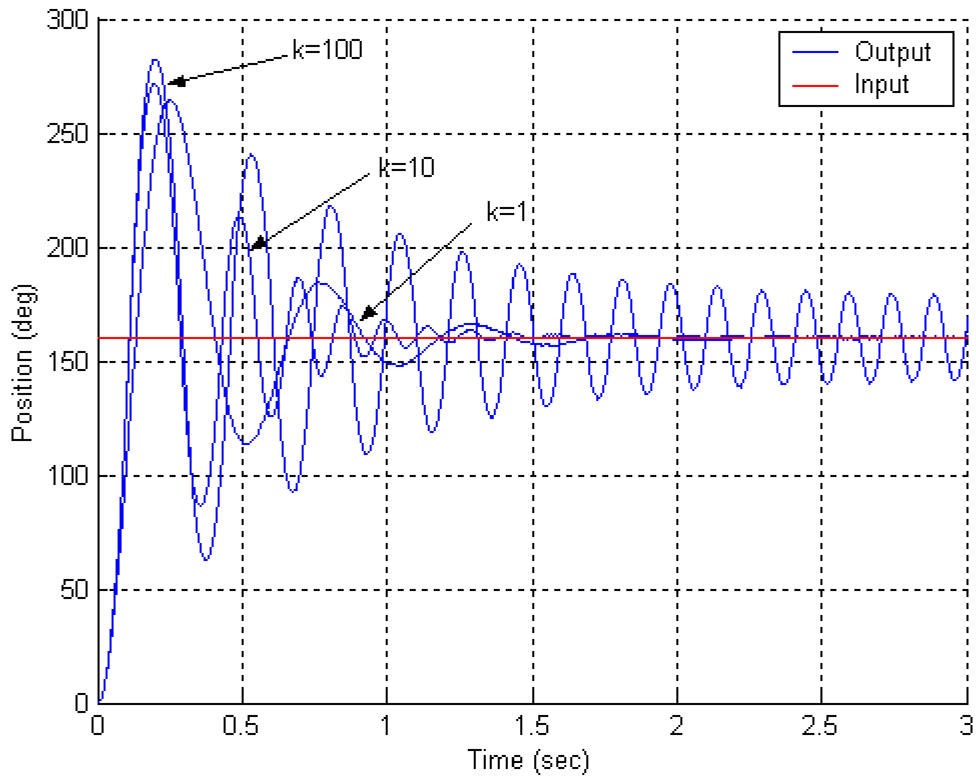
22. Comments on Eq. 5-126

After neglecting of electrical time constant, the second order closed loop transfer function of position control obtained in Eq. 5-126. In experiments 19 through 21 we observe an under damp response of a second order system. According to the equation, as the proportional gain increases, the damped frequency must be increased and this fact is verified in experiments 19 through 21. Experiments16 through 18 exhibits an over damped second order system responses.

23. In following, we repeat parts 16 and 18 using Virtual Lab:



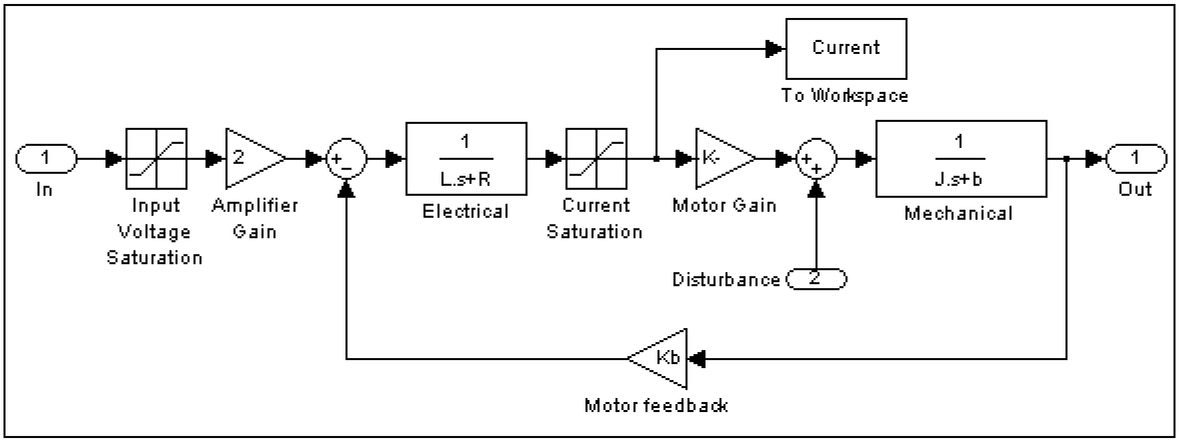
Study the effect of integral gain of 5:


k=100
k=10
k=1
Output
Input
300
250
200
150
100
50
0
0
0.5
1
1.5
2
2.5
3
Position (deg)
Time (sec)

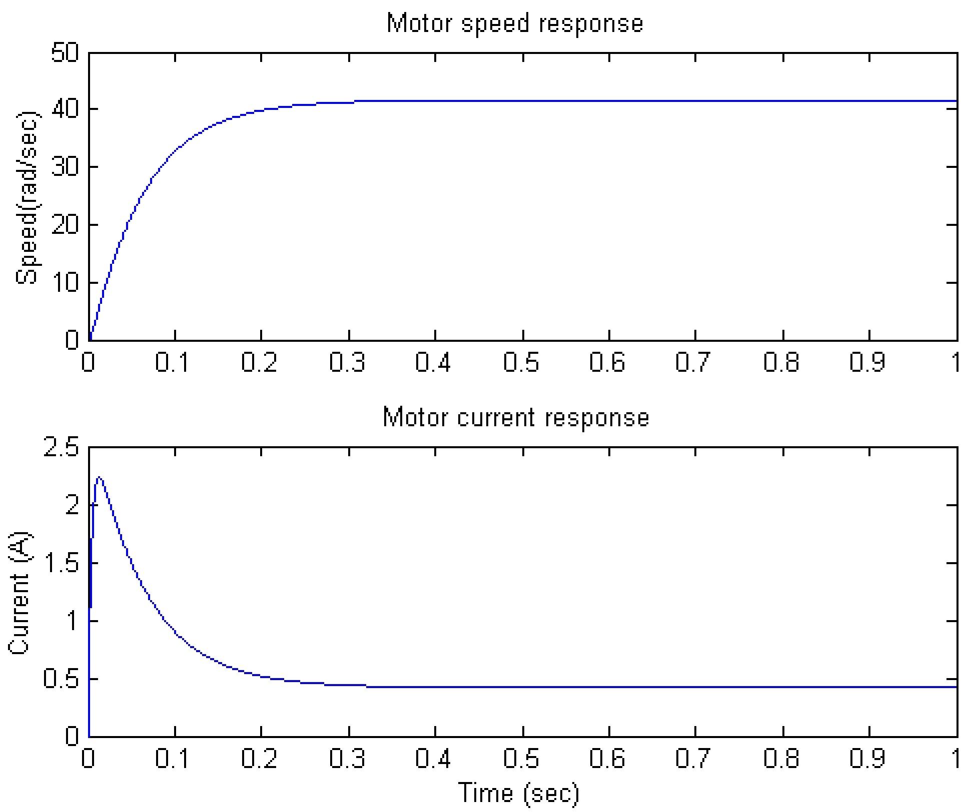
Ch. 6 Problem Solutions

Part 2) Solution to Problems in Chapter 6

6-1. In order to find the current of the motor, the motor constant has to be separated from the electrical component of the motor.

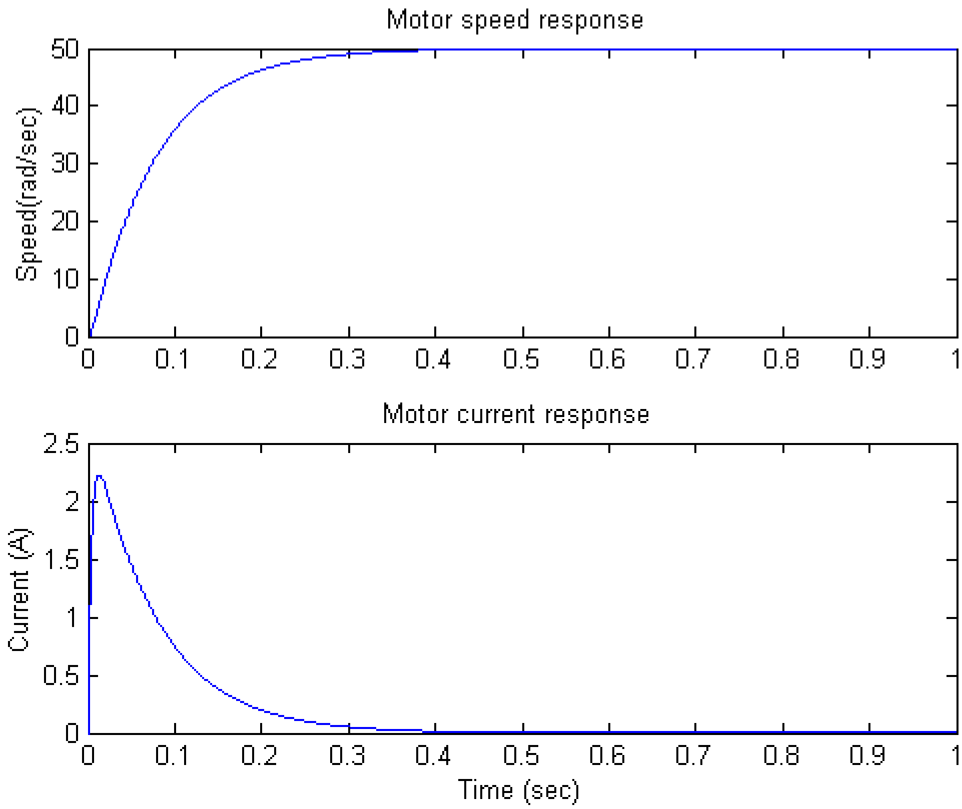


The response of the motor when 5V of step input is applied is:



a) The steady state speed: 41.67rad/sec
b) It takes 0.0678 second to reach 63% of the steady state speed (26.25rad/sec). This is the time constant of the motor.
c) The maximum current: 2.228A

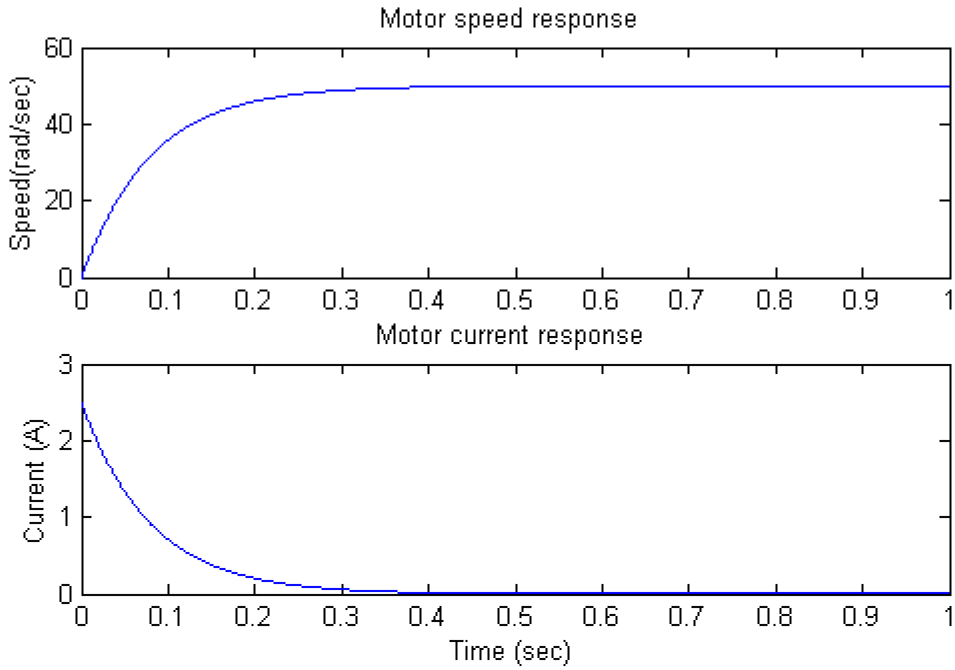
11.2



The steady state speed at 5V step input is 50rad/sec.

a) It takes 0.0797 seconds to reach 63% of the steady state speed (31.5rad/sec).
b) The maximum current: 2.226A
c) 100rad/sec

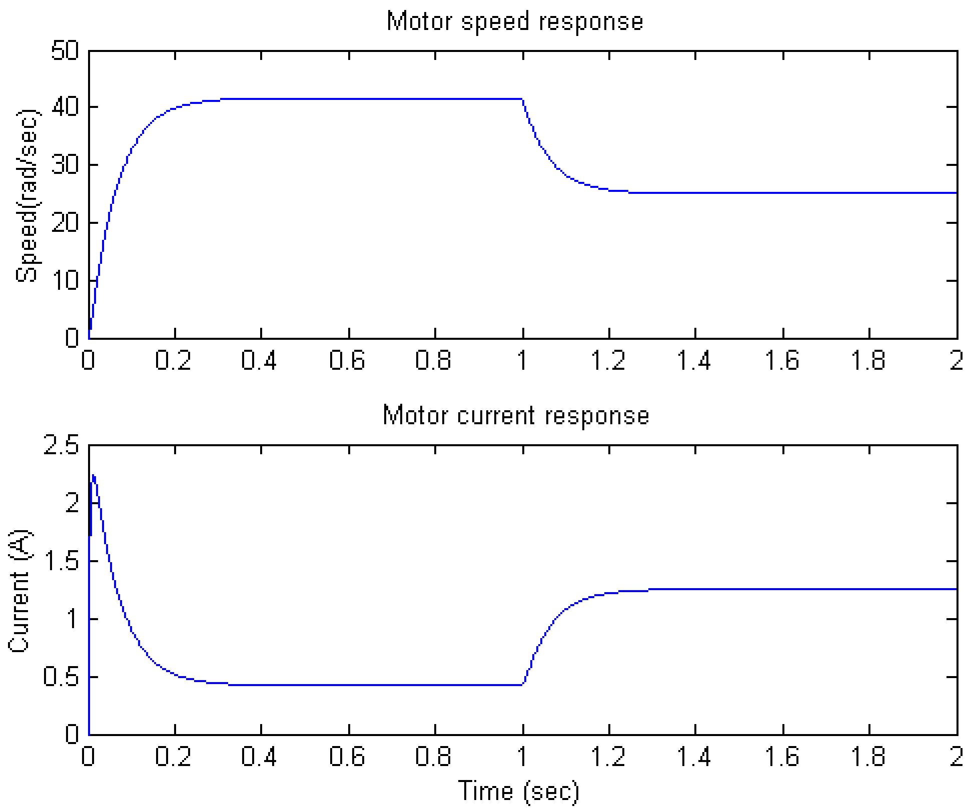
6-3



a) 50rad/sec
b) 0.0795 seconds
c) 2.5A. The current
d) When J_m is increased by a factor of 2, it takes 0.159 seconds to reach 63% of its steady state speed, which is exactly twice the original time period . This means that the time constant has been doubled.

6-4

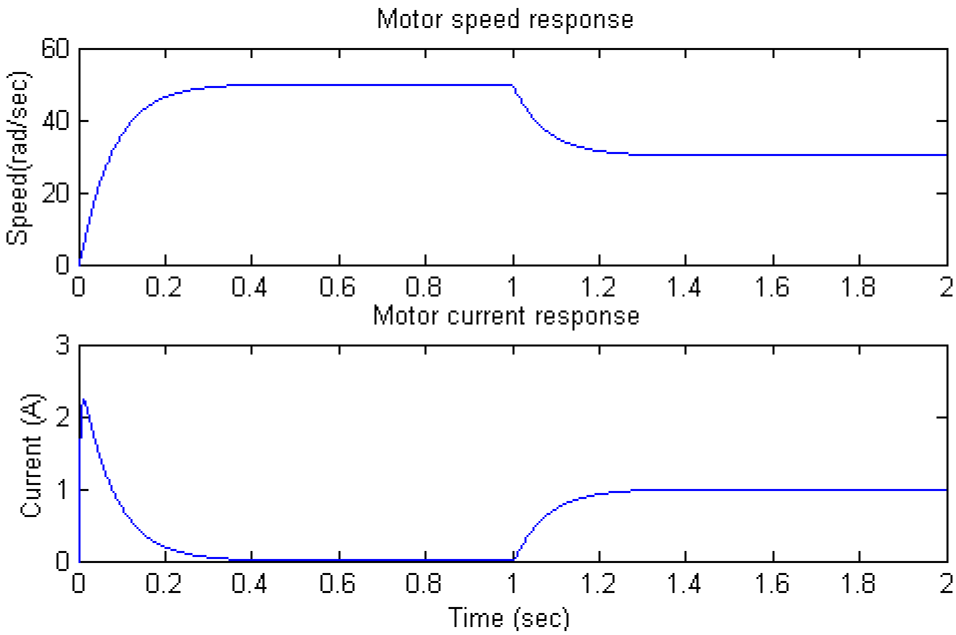
Part 1: Repeat problem 6-1 with T_L = -0.1Nm



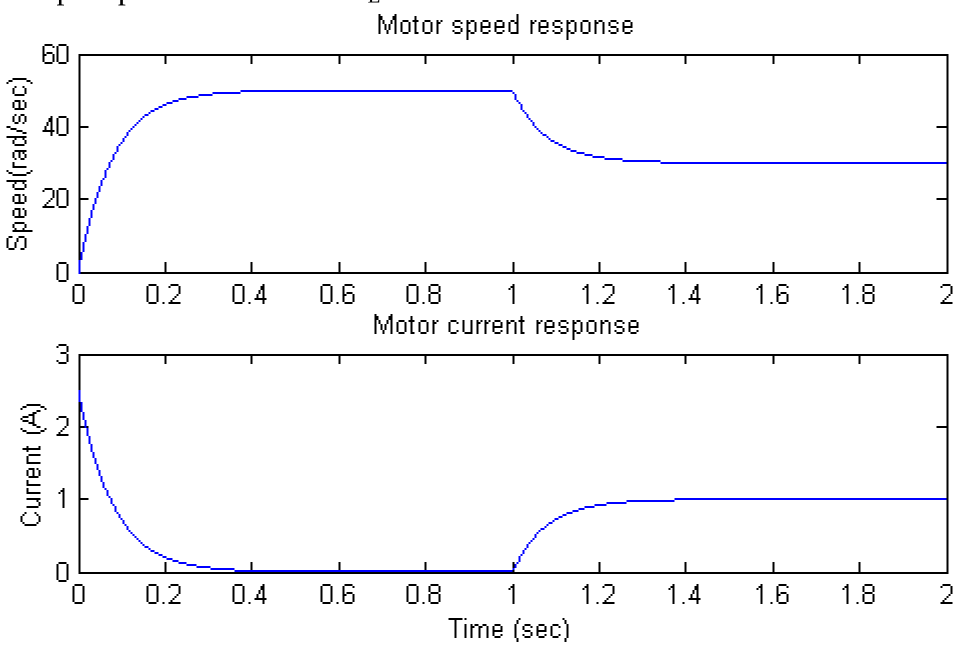
a) It changes from 41.67 rad/sec to 25 rad/sec.
b) First, the speed of 63% of the steady state has to be calculated.
   $41.67 - (41.67 - 25) \times 0.63 = 31.17$ rad/sec.
   The motor achieves this speed 0.0629 seconds after the load torque is applied
c) 2.228A. It does not change

Part 2: Repeat problem 6-2 with T_L = -0.1Nm

a) It changes from 50 rad/sec to 30 rad/sec.
b) The speed of 63% of the steady state becomes
   $50 - (50 - 30) \times 0.63 = 37.4$ rad/sec.
   The motor achieves this speed 0.0756 seconds after the load torque is applied
c) 2.226A. It does not change.



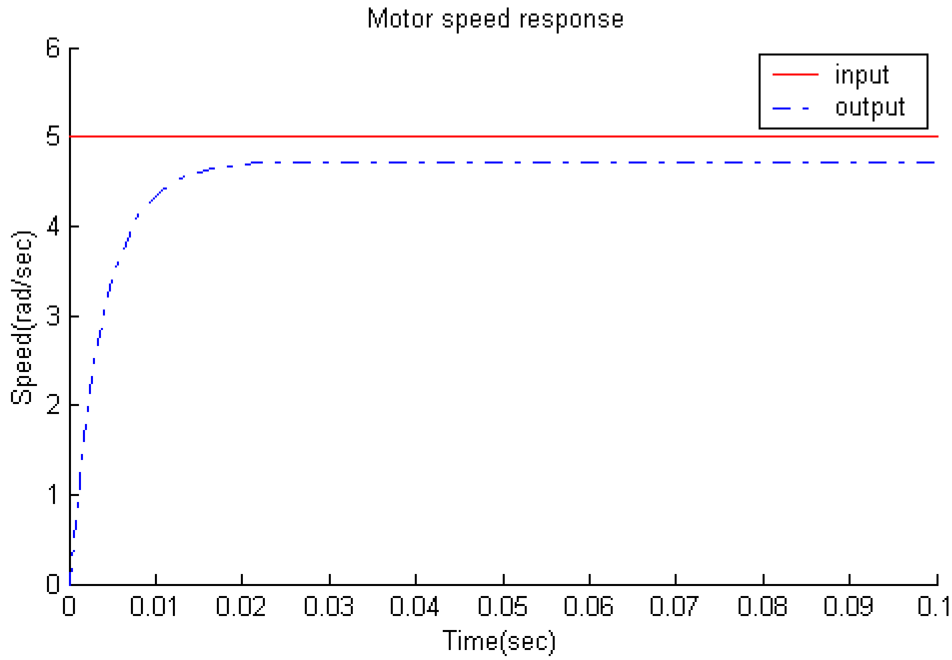
Part 3: Repeat problem 6-3 with T_L = -0.1Nm



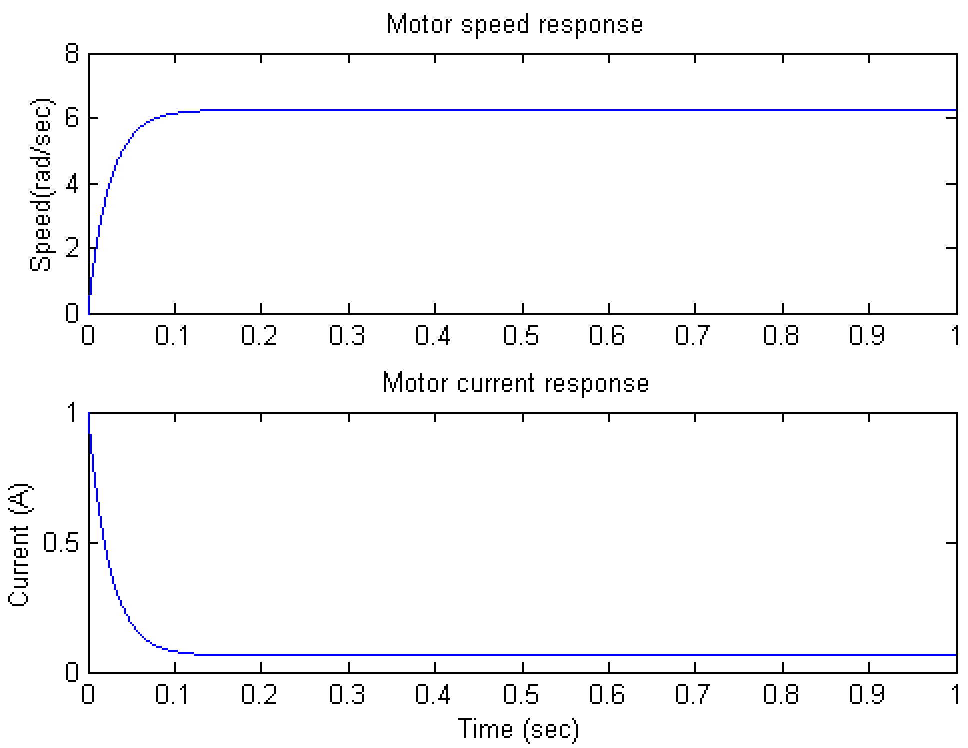
a) It changes from 50 rad/sec to 30 rad/sec.
b) $50 - (50 - 30) \times 0.63 = 37.4$ rad/s

   The motor achieves this speed 0.0795 seconds after the load torque is applied. This is the same as problem 6-3.
c) 2.5A. It does not change
d) As T_L increases in magnitude, the steady state velocity decreases and steady state current increases; however, the time constant does not change in all three cases.

6-4 The steady state speed is 4.716 rad/sec when the amplifier input voltage is 5V:



6-6

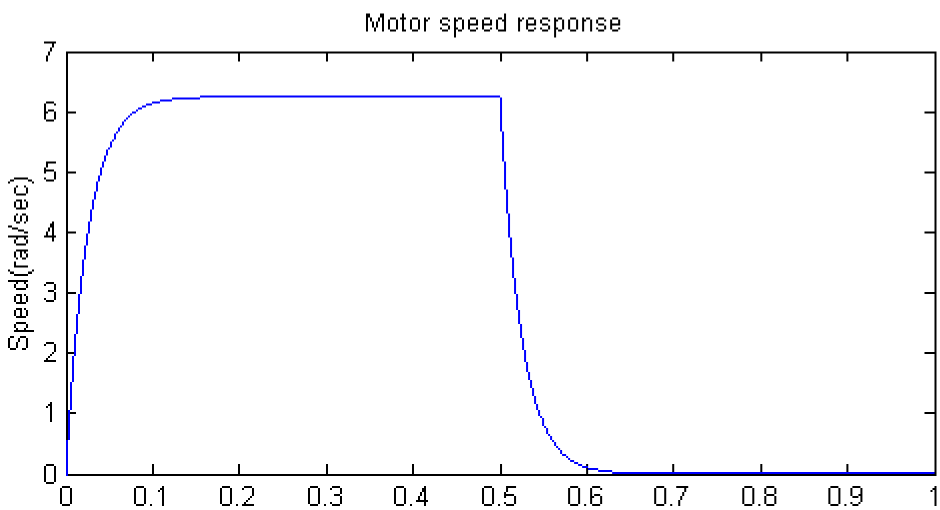


a) 6.25 rad/sec.
b) 63% of the steady state speed: $6.25 \times 0.63 = 3.938$ rad/sec
   It takes 0.0249 seconds to reach 63% of its steady state speed.
c) The maximum current drawn by the motor is 1 Ampere.

6-7

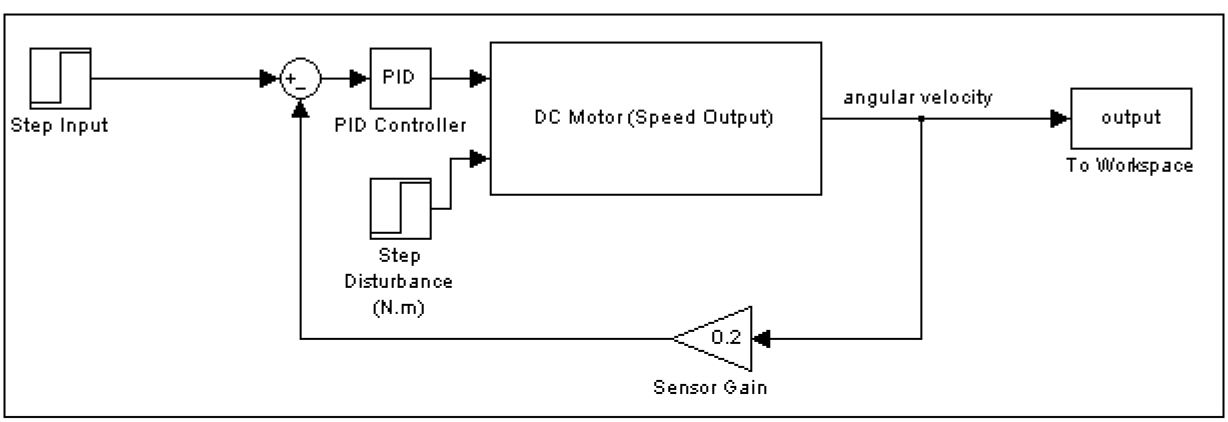
a) 9.434 rad/sec.
b) 63% of the steady state speed: $9.434 \times 0.63 = 5.943$ rad/sec
   It takes 0.00375 seconds to reach 63% of its steady state speed.
c) The maximum current drawn by the motor is 10 Amperes.
d) When there is no saturation, higher K_p value reduces the steady state error and decreases the rise time. If there is saturation, the rise time does not decrease as much as it without saturation. Also, if there is saturation and K_p value is too high, chattering phenomenon may appear.

6-8

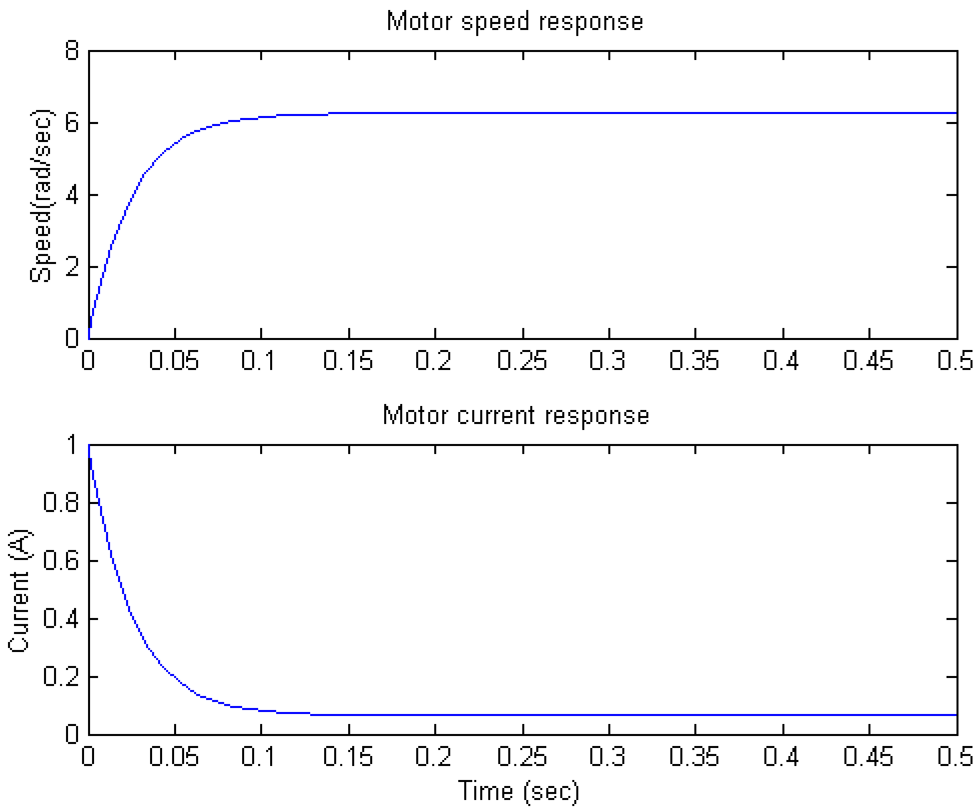


a) The steady state becomes zero. The torque generated by the motor is 0.1 Nm.
b) $6.25 - (6.25 - 0) \times 0.63 = 2.31$ rad/sec. It takes 0.0249 seconds to reach 63% of its new steady state speed. It is the same time period to reach 63% of its steady state speed without the load torque (compare with the answer for the Problem 6-6 b).

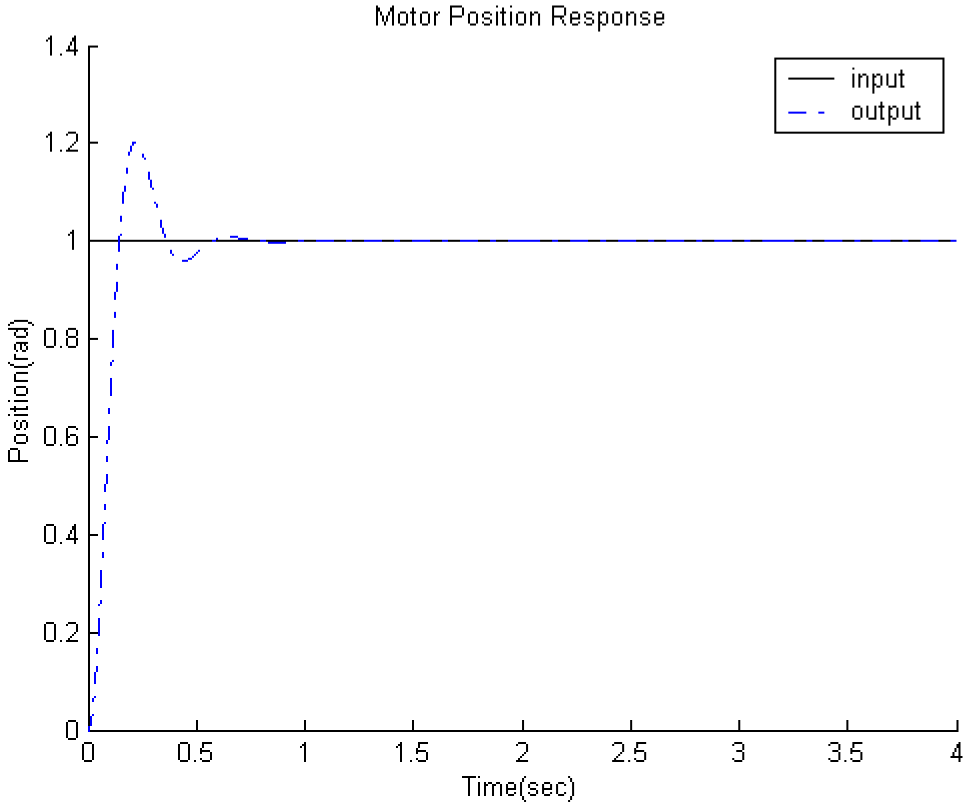
11-9 The SIMLab model becomes



The sensor gain and the speed input are reduced by a factor of 5. In order to get the same result as Problem 6-6, the K_p value has to increase by a factor of 5. Therefore, $K_p = 0.5$.
The following graphs illustrate the speed and current when the input is 2 rad/sec and $K_p = 0.5$.

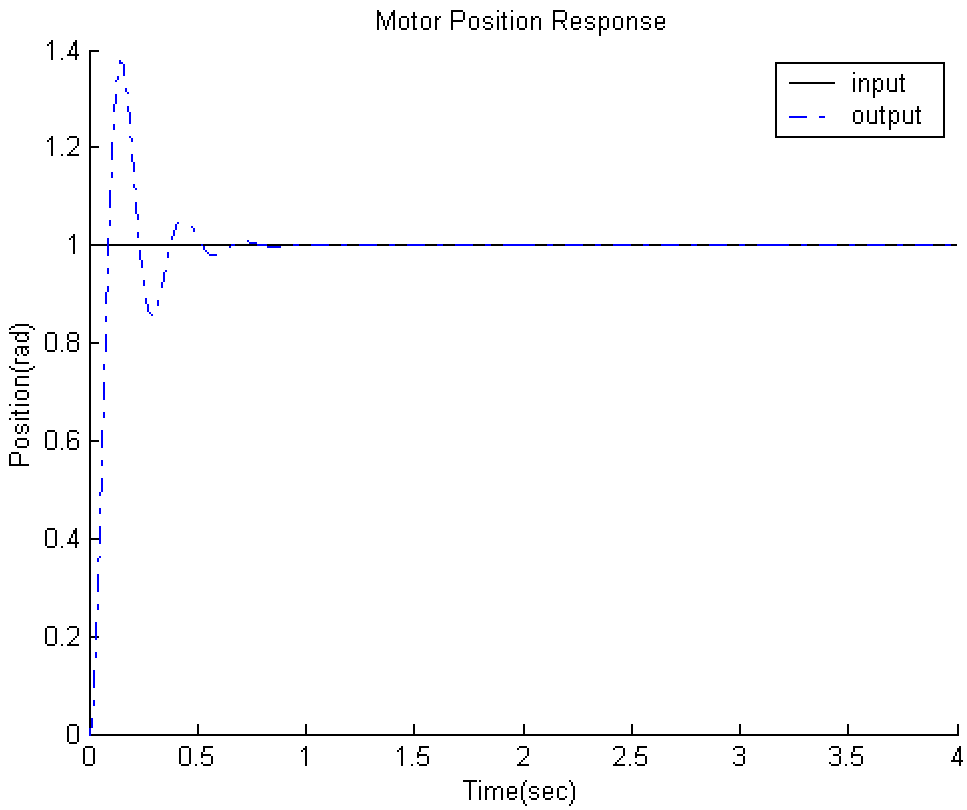

Motor speed response
Speed(rad/sec)
8
6
4
2
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Motor current response
Current (A)
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Time (sec)

6-10



a) 1 radian.
b) 1.203 radians.
c) 0.2215 seconds.

6-11

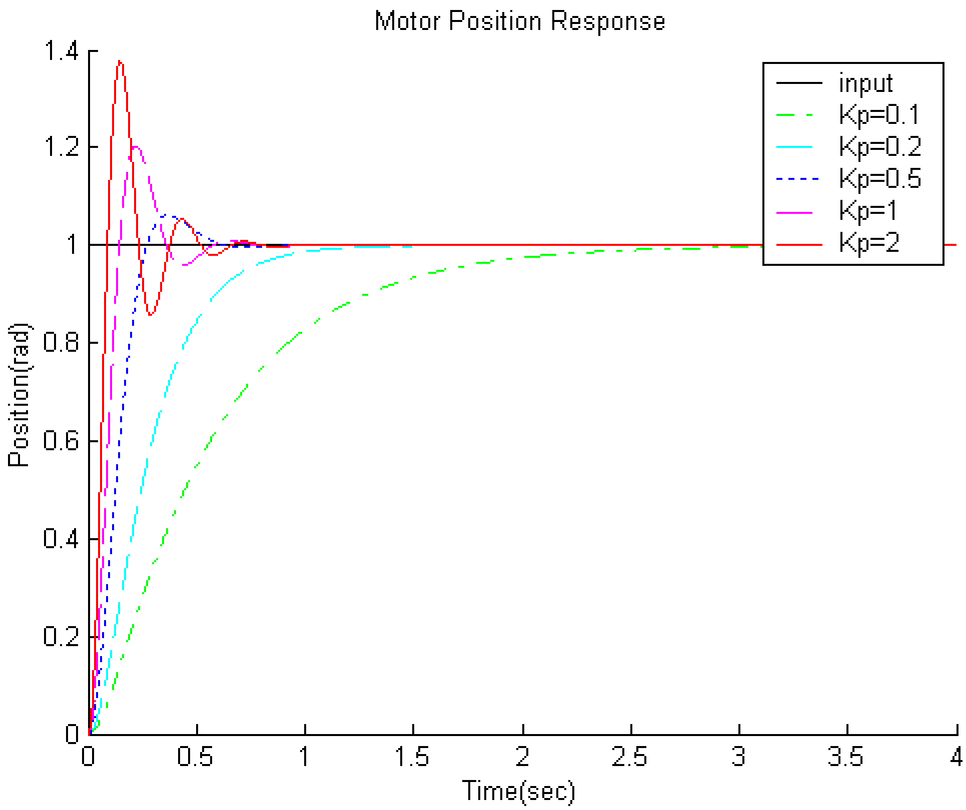


a) The steady state position is very close to 1 radian.
b) 1.377 radians.
c) 0.148 seconds.

It has less steady state error and a faster rise time than Problem 6-10, but has larger overshoot.

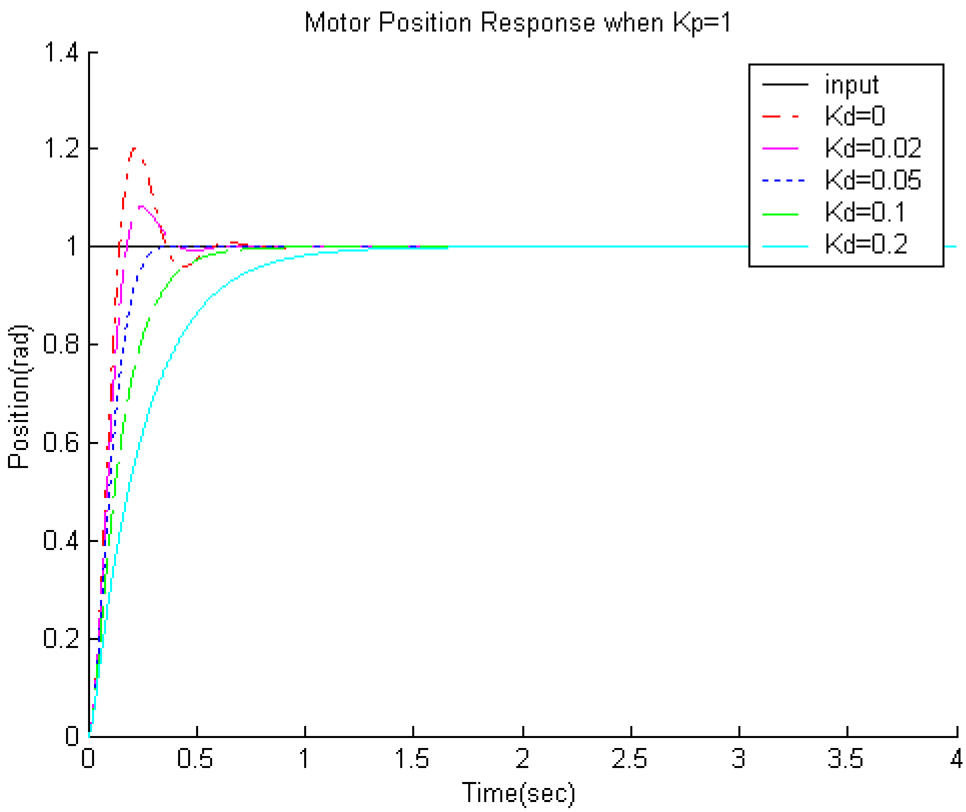
6-12

Different proportional gains and their corresponding responses are shown on the following graph.



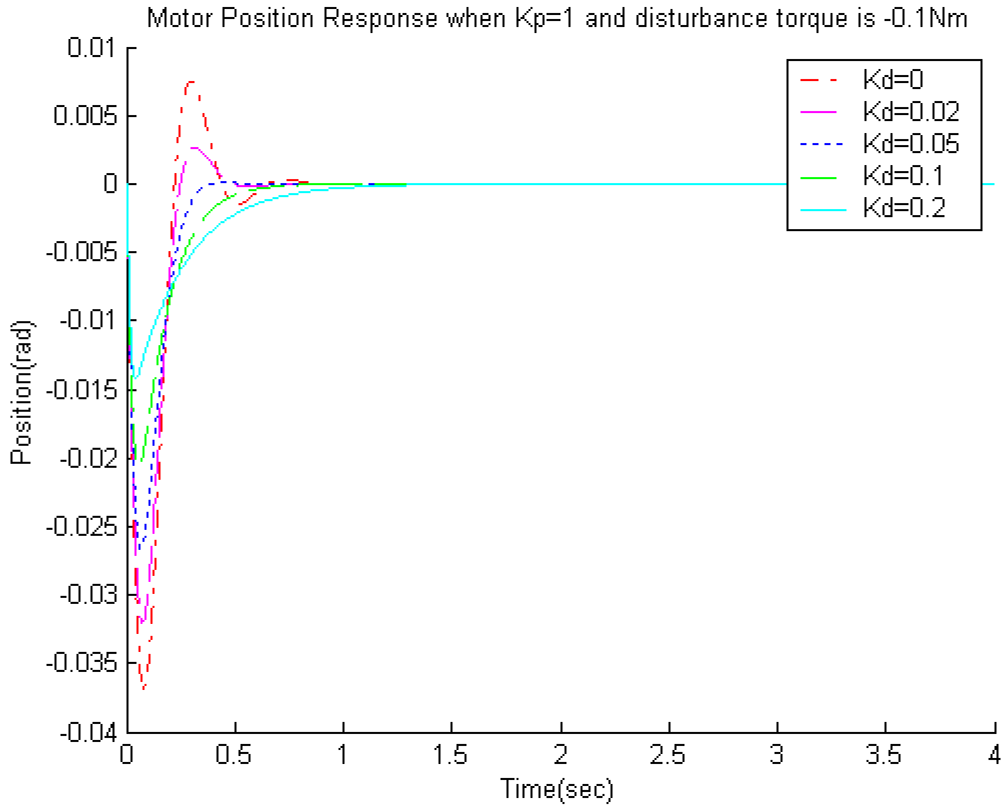
As the proportional gain gets higher, the motor has a faster response time and lower steady state error, but if it the gain is too high, the motor overshoot increases. If the system requires that there be no overshoot, $K_p = 0.2$ is the best value. If the system allows for overshoot, the best proportional gain is dependant on how much overshoot the system can have. For instance, if the system allows for a 30% overshoot, $K_p = 1$ is the best value.

6-13 Let $K_p = 1$ is the best value.



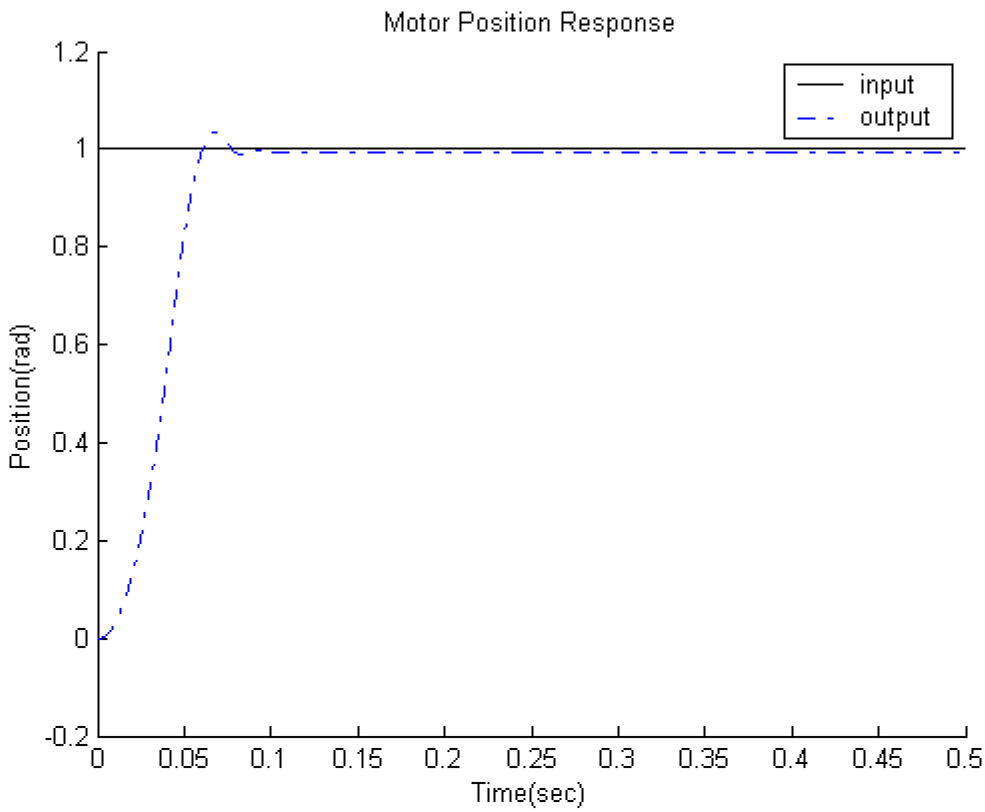
As the derivative gain increases, overshoot decreases, but rise time increases.

6-14


Motor Position Response when Kp=1 and disturbance torque is -0.1Nm
Kd=0
Kd=0.02
Kd=0.05
Kd=0.1
Kd=0.2
Position(rad)
Time(sec)
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
-0.035
-0.04
0
0.5
1
1.5
2
2.5
3
3.5
4

6-15

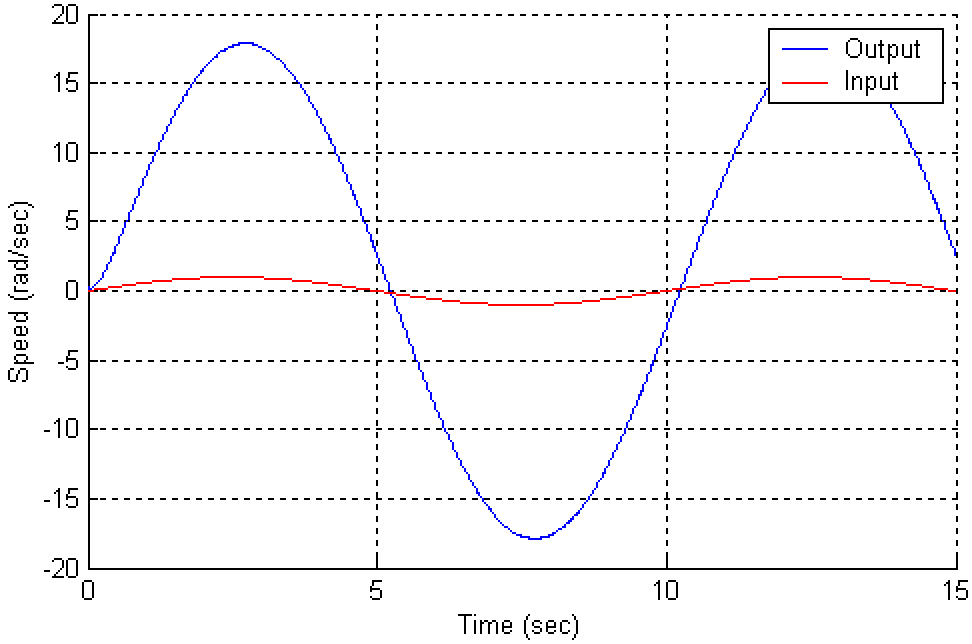
There could be many possible answers for this problem. One possible answer would be
K_p= 100
K_i= 10
K_d= 1.4



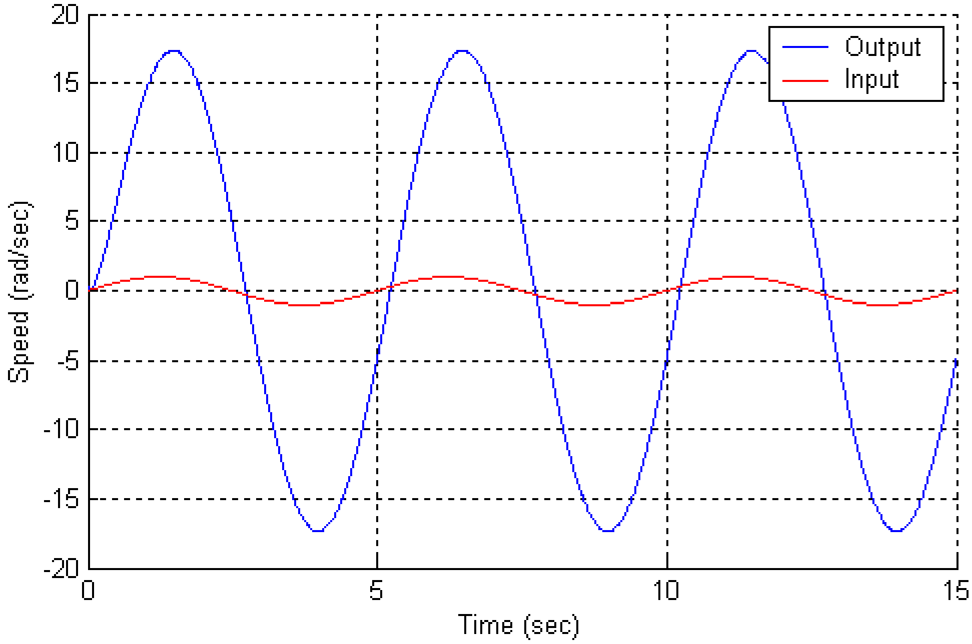
The Percent Overshoot in this case is 3.8%.

6-16

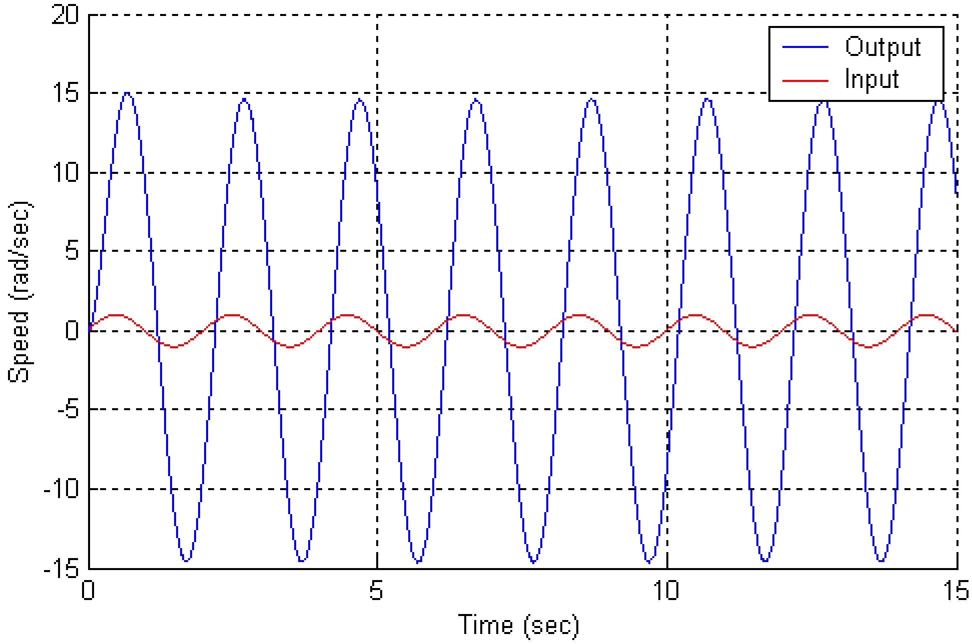
0.1 Hz



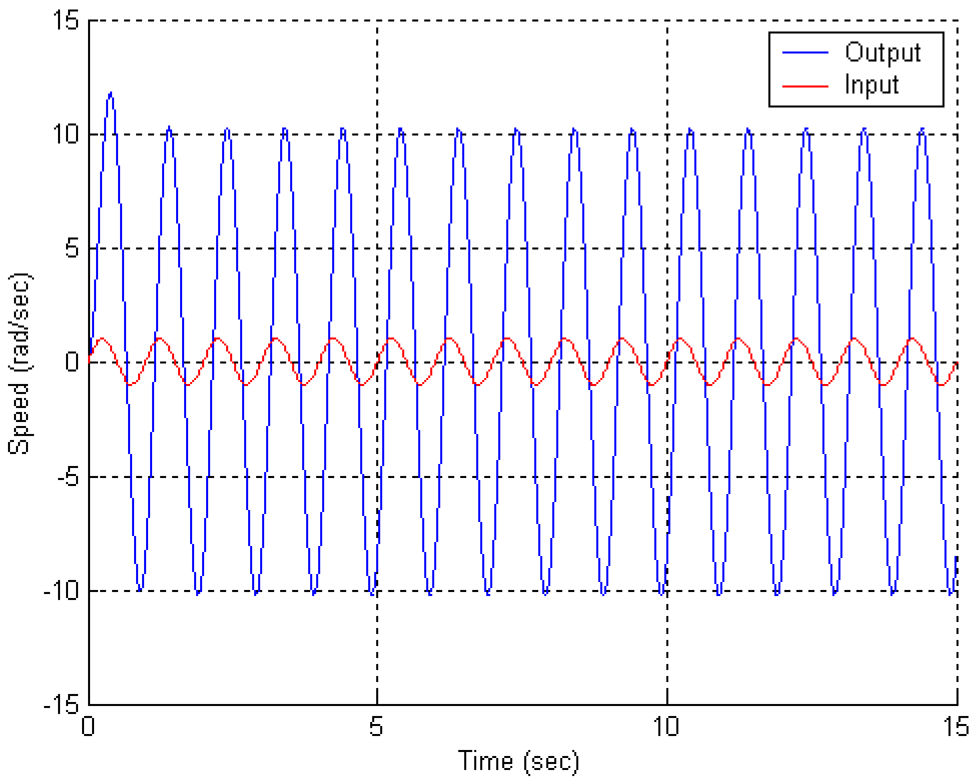
0.2 Hz



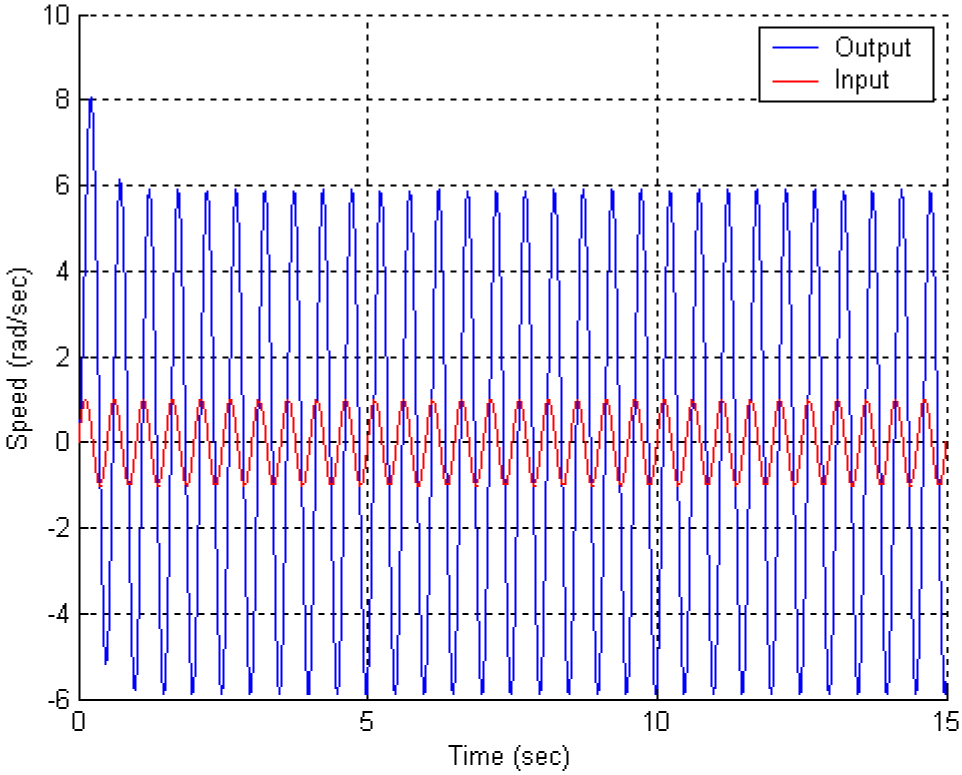
0.5 Hz


Output
Input
20
15
10
5
0
-5
-10
-15
Speed (rad/sec)
0
5
10
15
Time (sec)

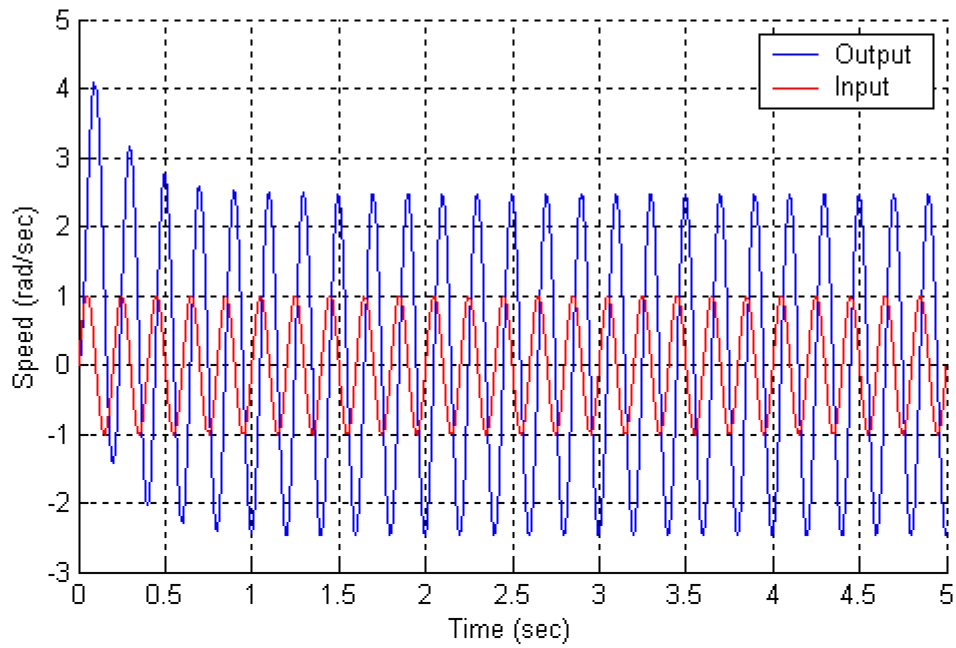
1 Hz


Output
Input
15
10
5
0
-5
-10
-15
Speed (rad/sec)
0
5
10
15
Time (sec)

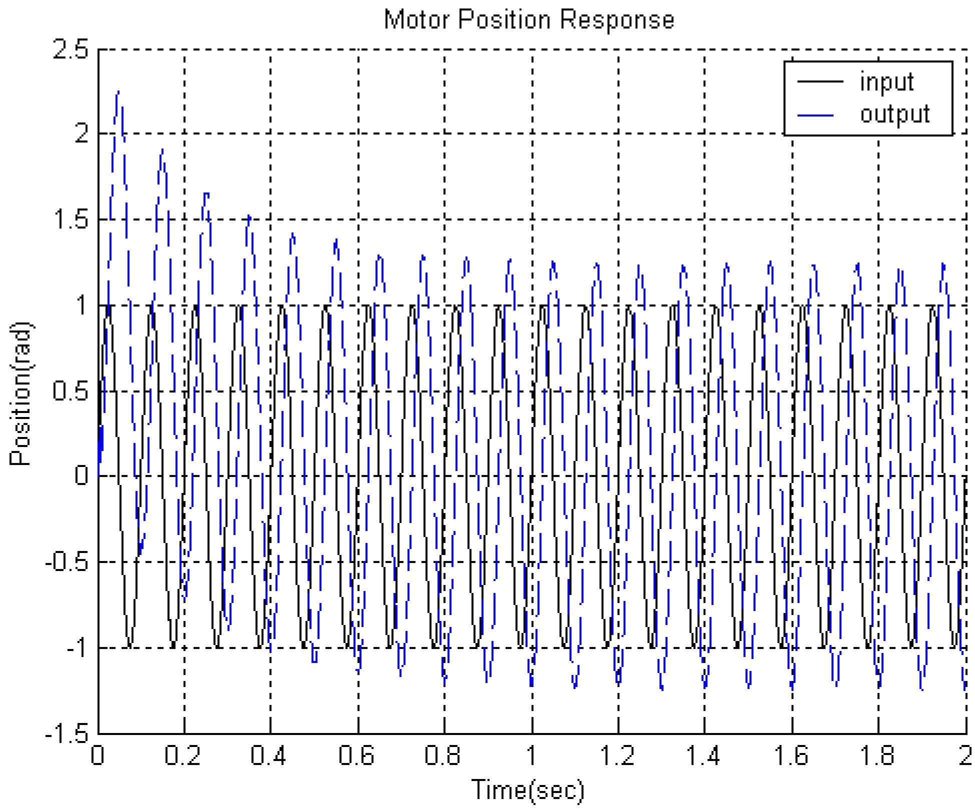
2Hz


Output
Input
Speed (rad/sec)
Time (sec)
10
8
6
4
2
0
-2
-4
-6
0
5
10
15

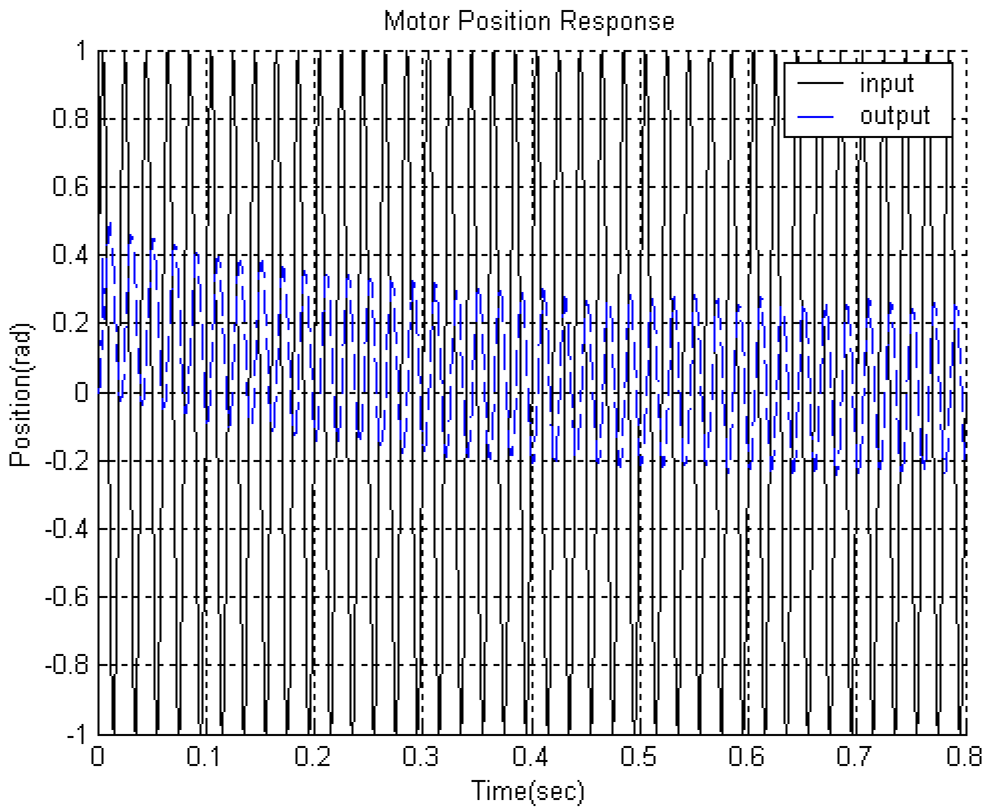
5Hz


Output
Input
Speed (rad/sec)
Time (sec)
5
4
3
2
1
0
-1
-2
-3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

10Hz

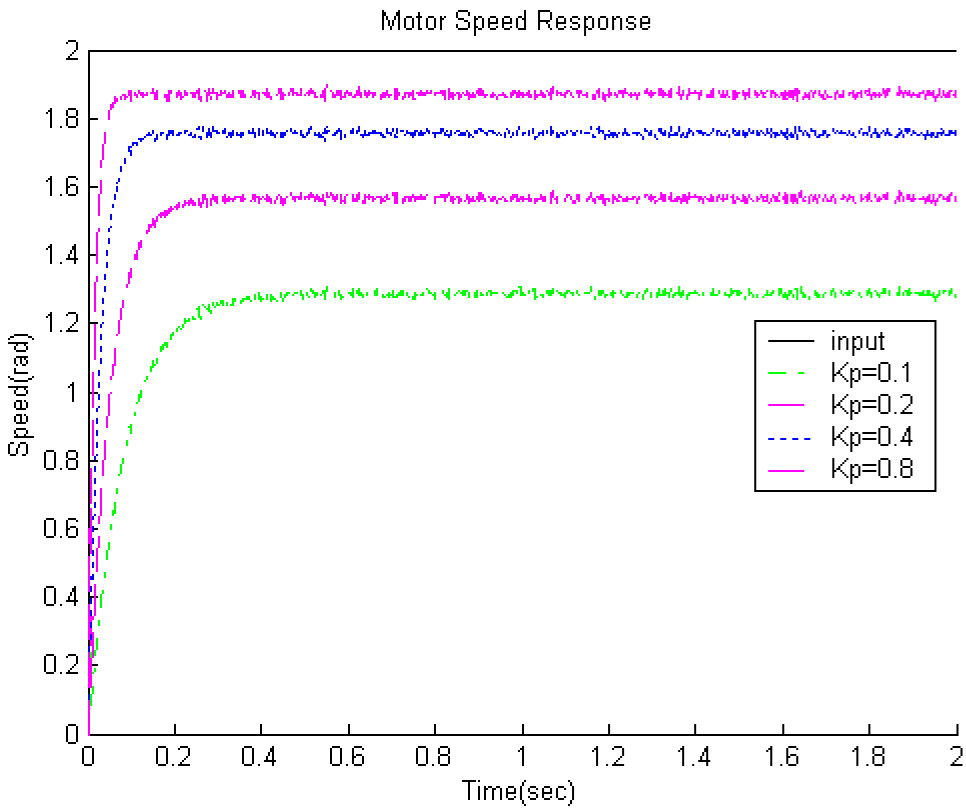

Motor Position Response
input
output
Position(rad)
Time(sec)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

50Hz



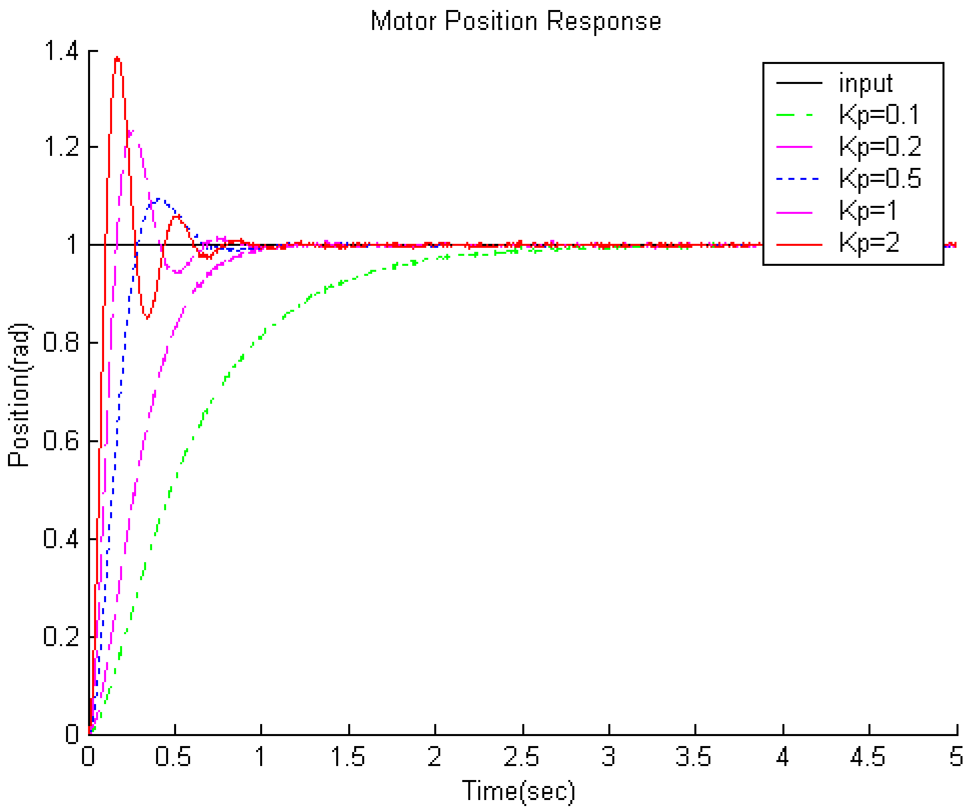
As frequency increases, the phase shift of the input and output also increase. Also, the amplitude of the output starts to decrease when the frequency increases above 0.5Hz.

6-17



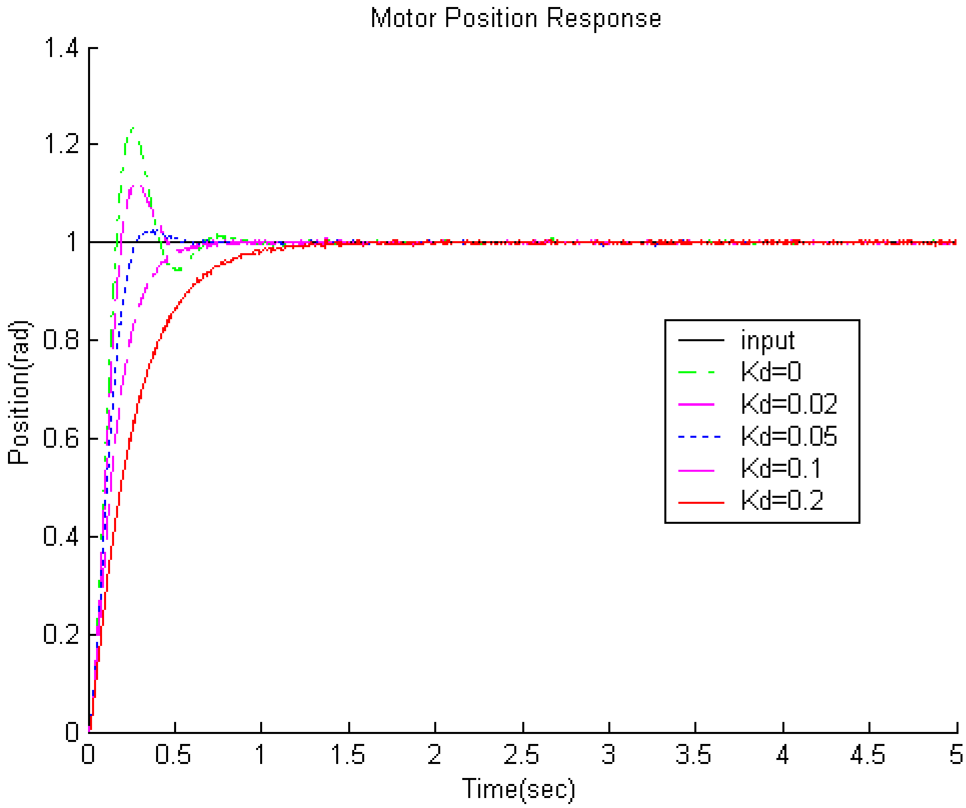
As proportional gain increases, the steady state error decreases.

6-18



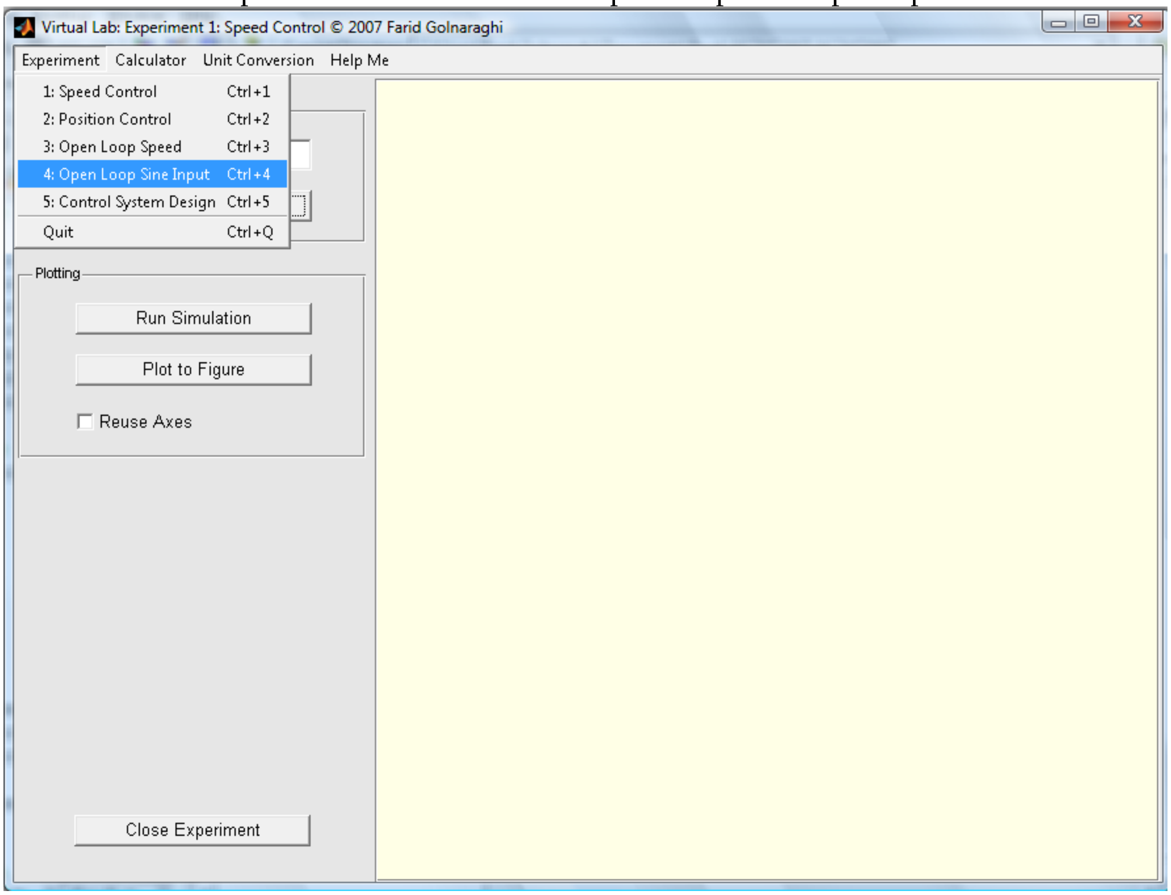
Considering fast response time and low overshoot, $K_p=1$ is considered to be the best value.

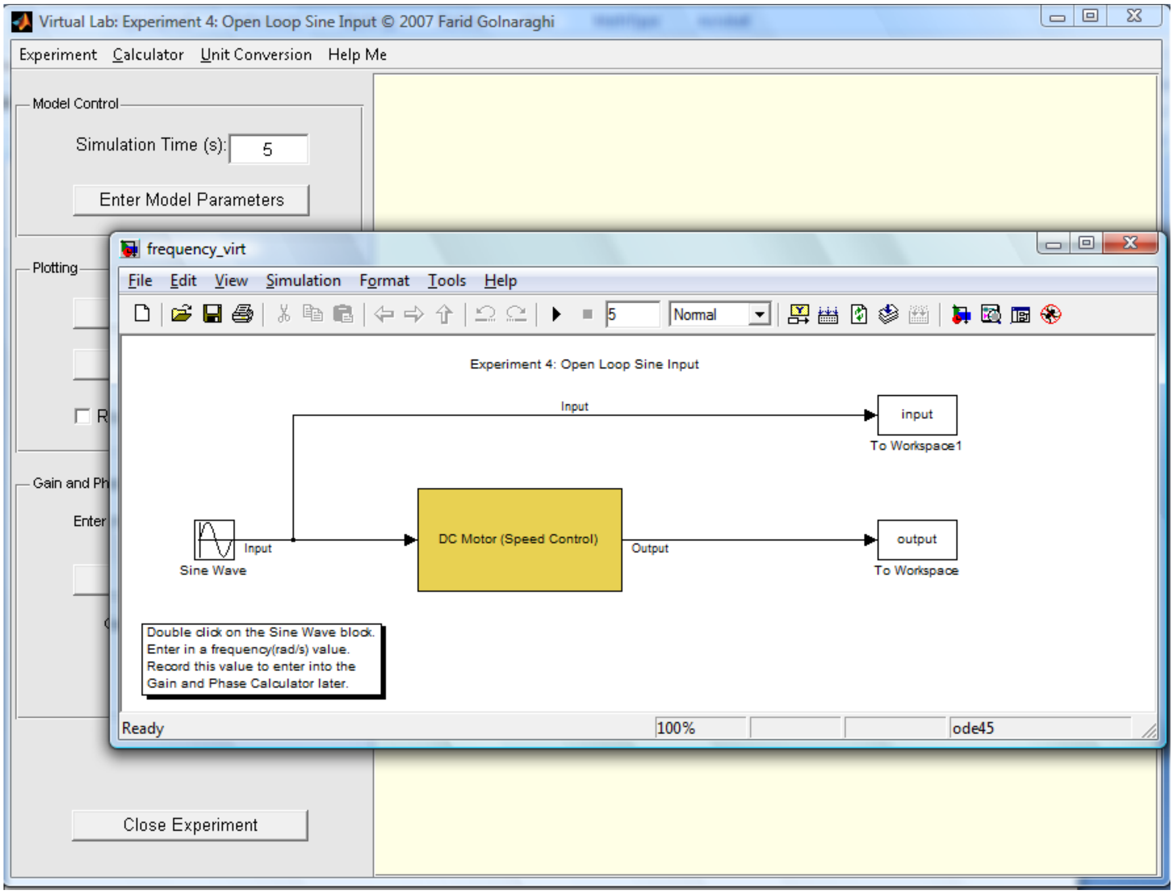
6-19 It was found that the best $K_p = 1$



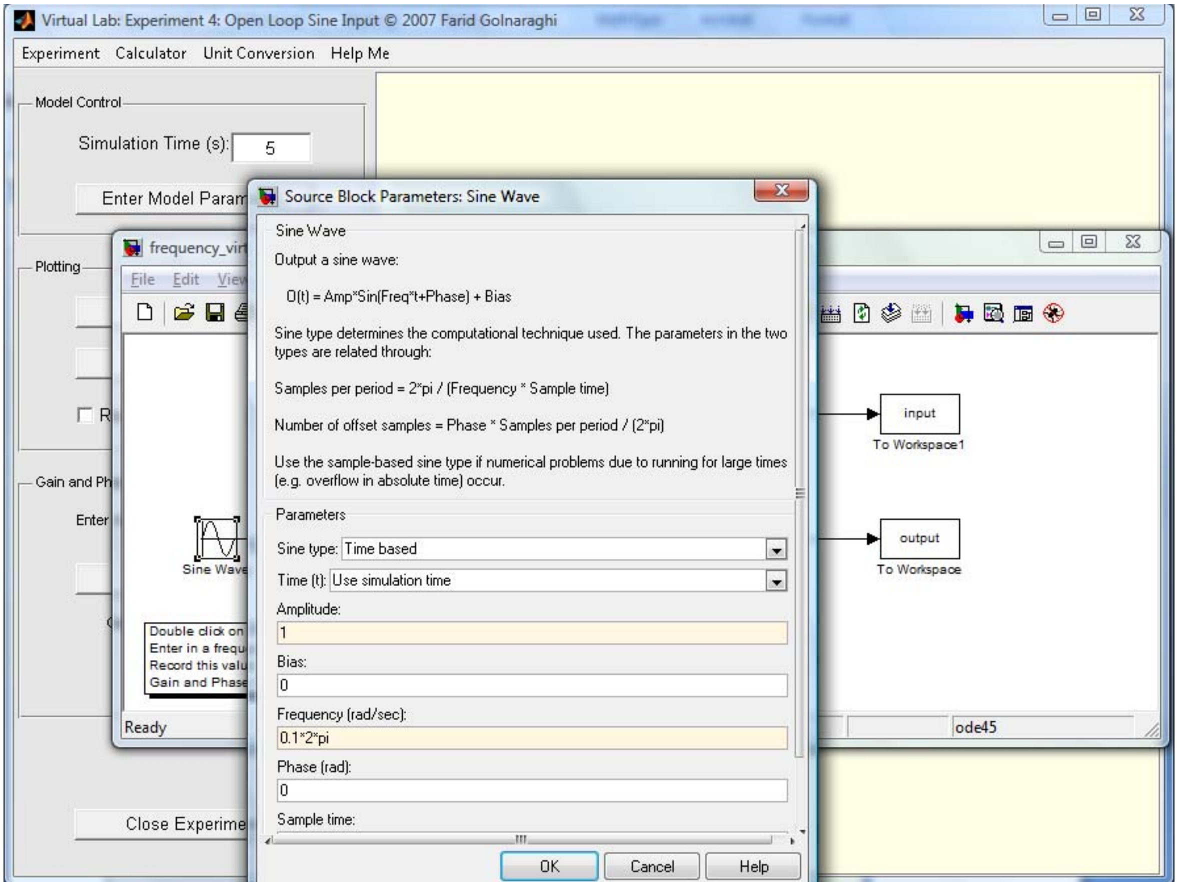
As K_d value increases, the overshoot decreases and the rise time increases.

6-20 From the "Experiment" Menu select the Open Loop Sine Input" option.

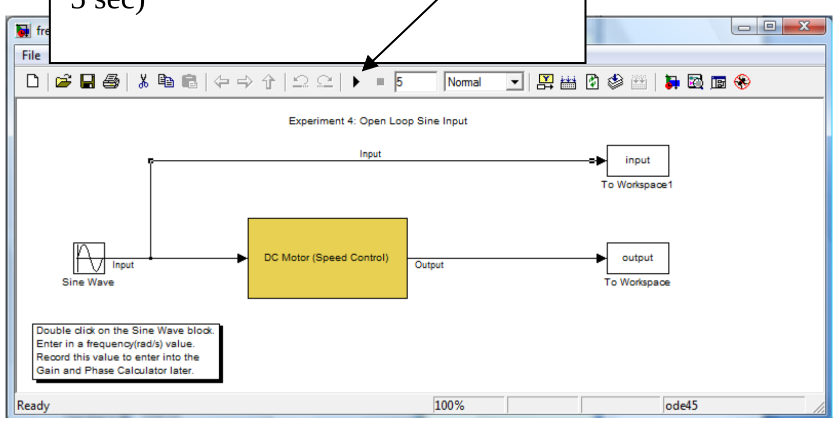


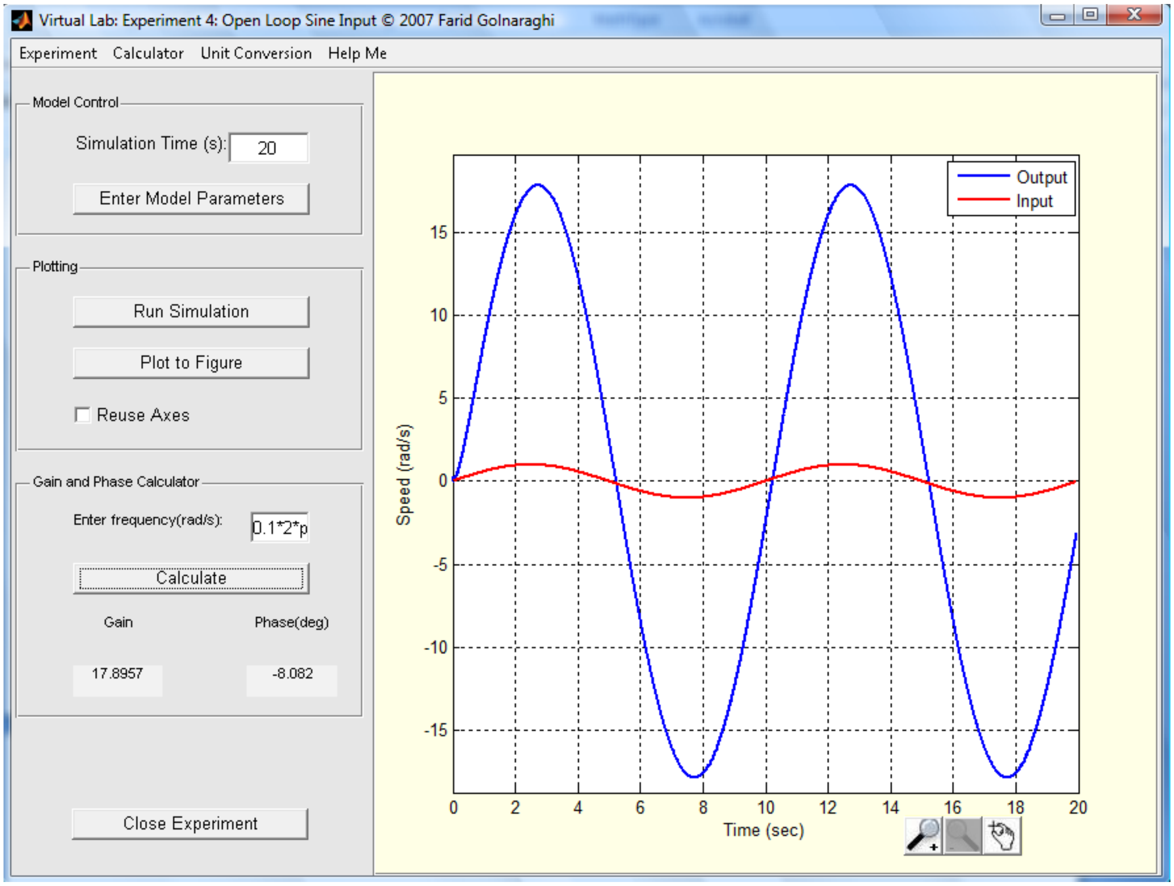


Double click on the "Sine wave" block and choose the input values.



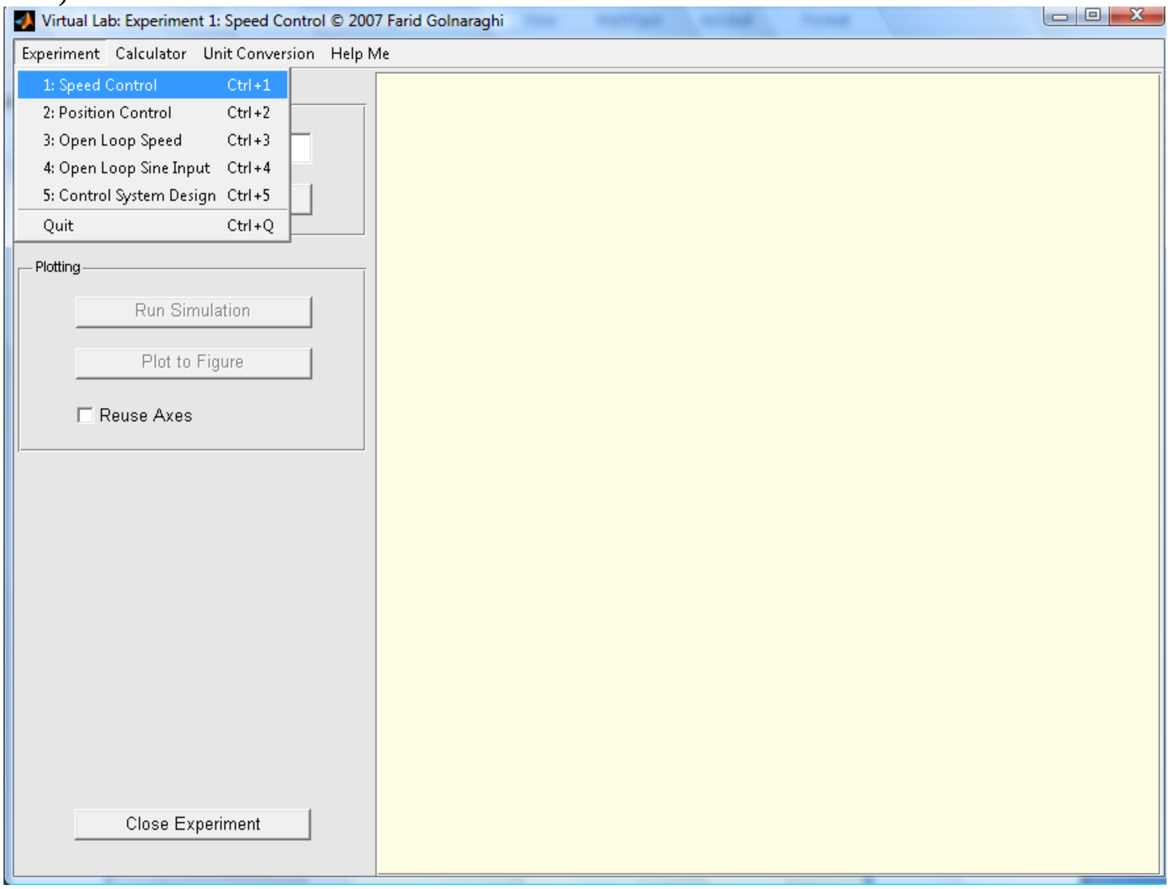
Run simulation in SIMULINK. Change run time to 20 sec (default is 5 sec)



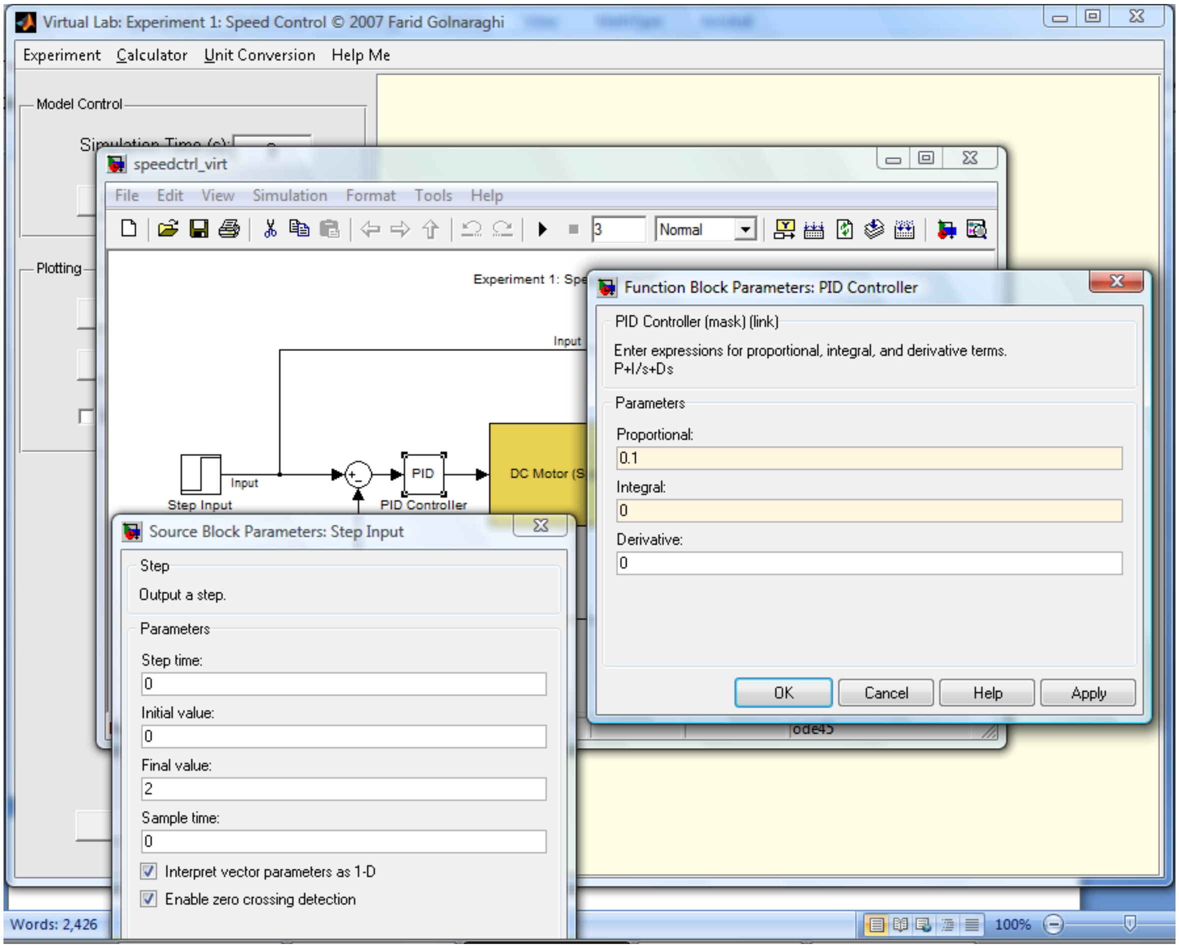


Calculate the Gain and Phase values by entering the input frequency. Repeat the process for other frequency values and use the calculated gain and phase values to plot the frequency response of the system.

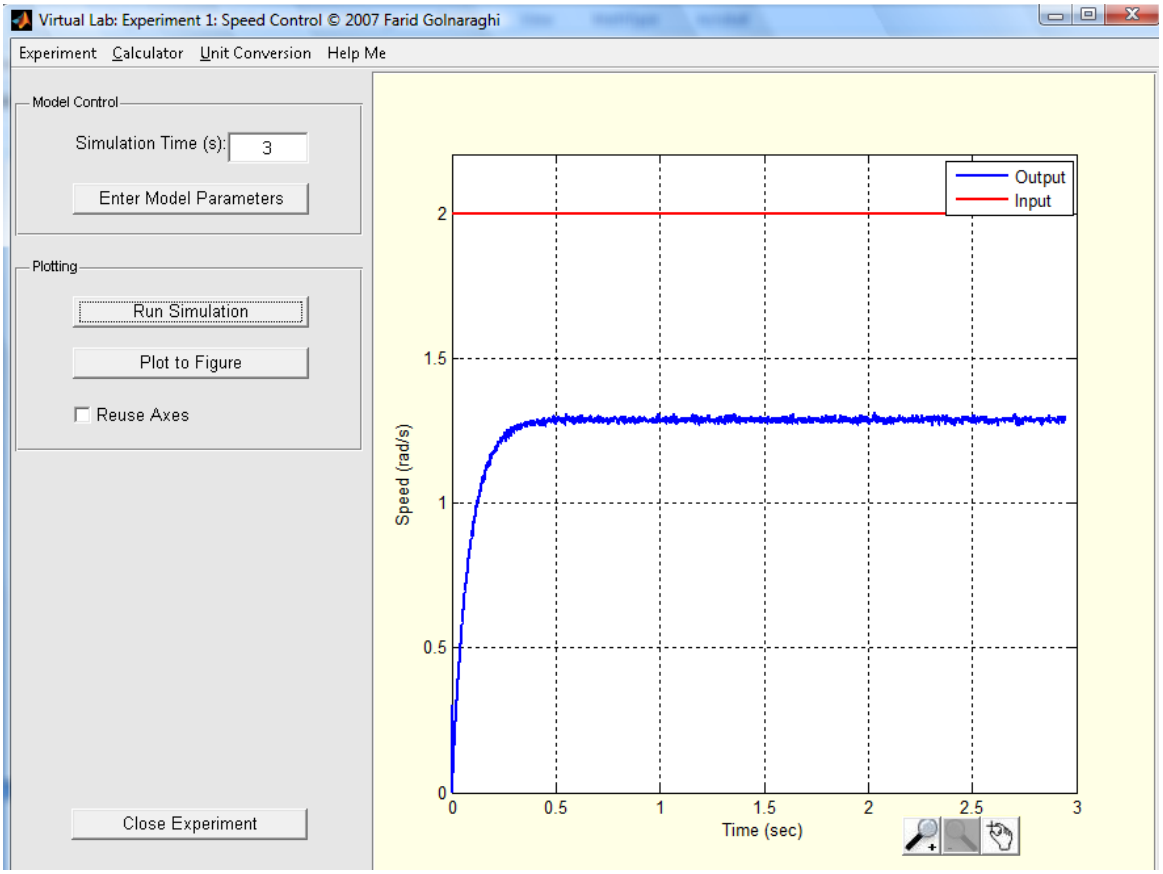
6-21)



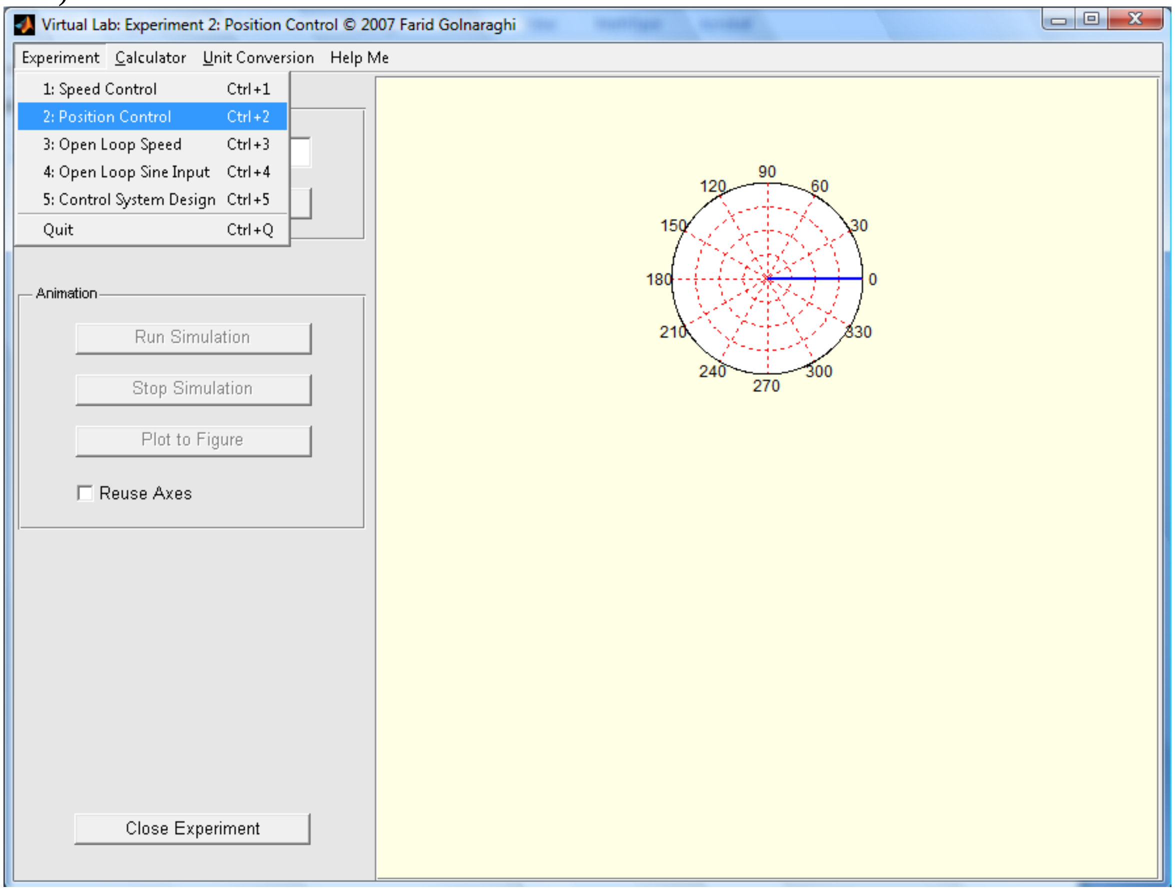
Select the Speed Control option in Virtual Lab Experiment Window.
Enter the Step Input and Controller Gain values by double clicking on their respective blocks.



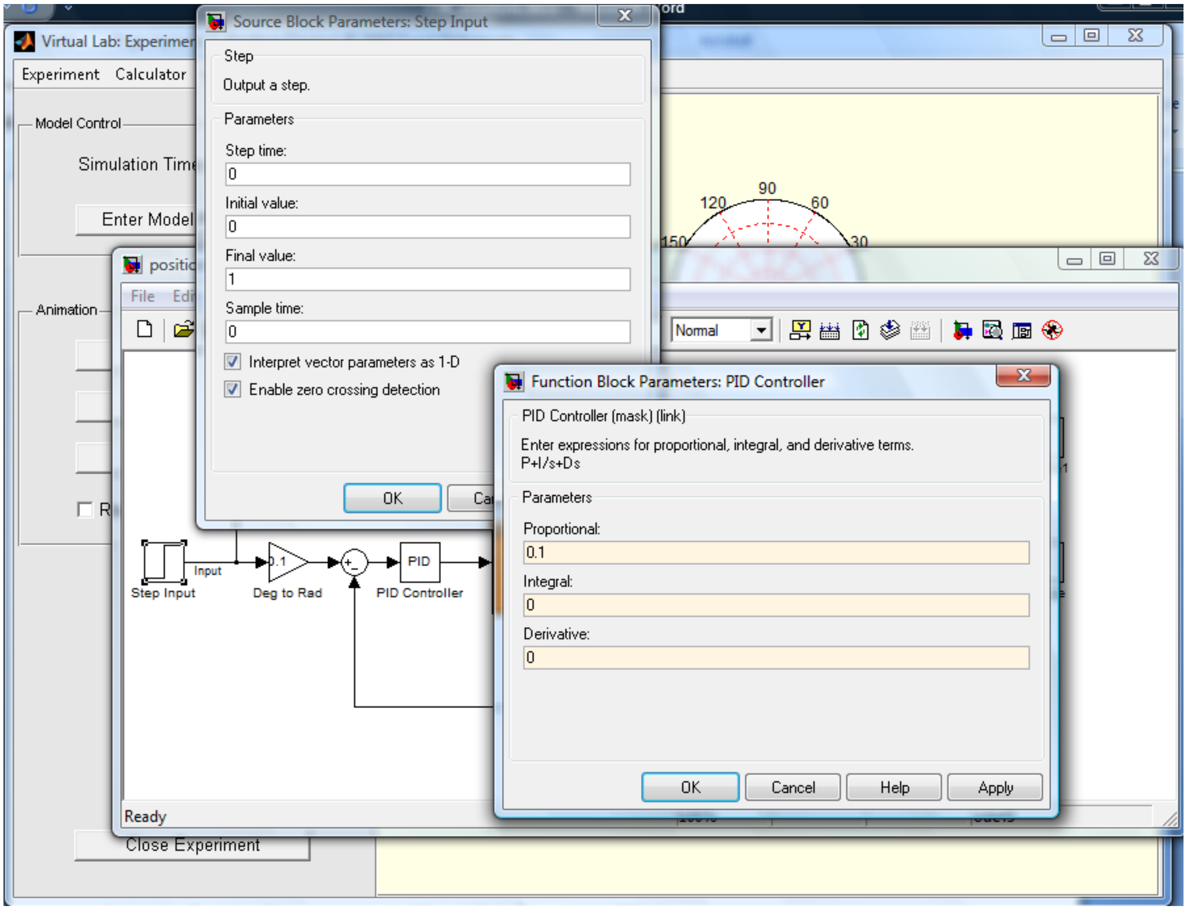
Run simulation, and repeat the process for different gain values. Observe the steady state value change with K.


Virtual Lab: Experiment 1: Speed Control © 2007 Farid Golnaraghi
Experiment Calculator Unit Conversion Help Me
Model Control
Simulation Time (s): 3
Enter Model Parameters
Plotting
Run Simulation
Plot to Figure
Reuse Axes
Close Experiment
Output
Input
Speed (rad/s)
Time (sec)
0
0.5
1
1.5
2
2.5
3

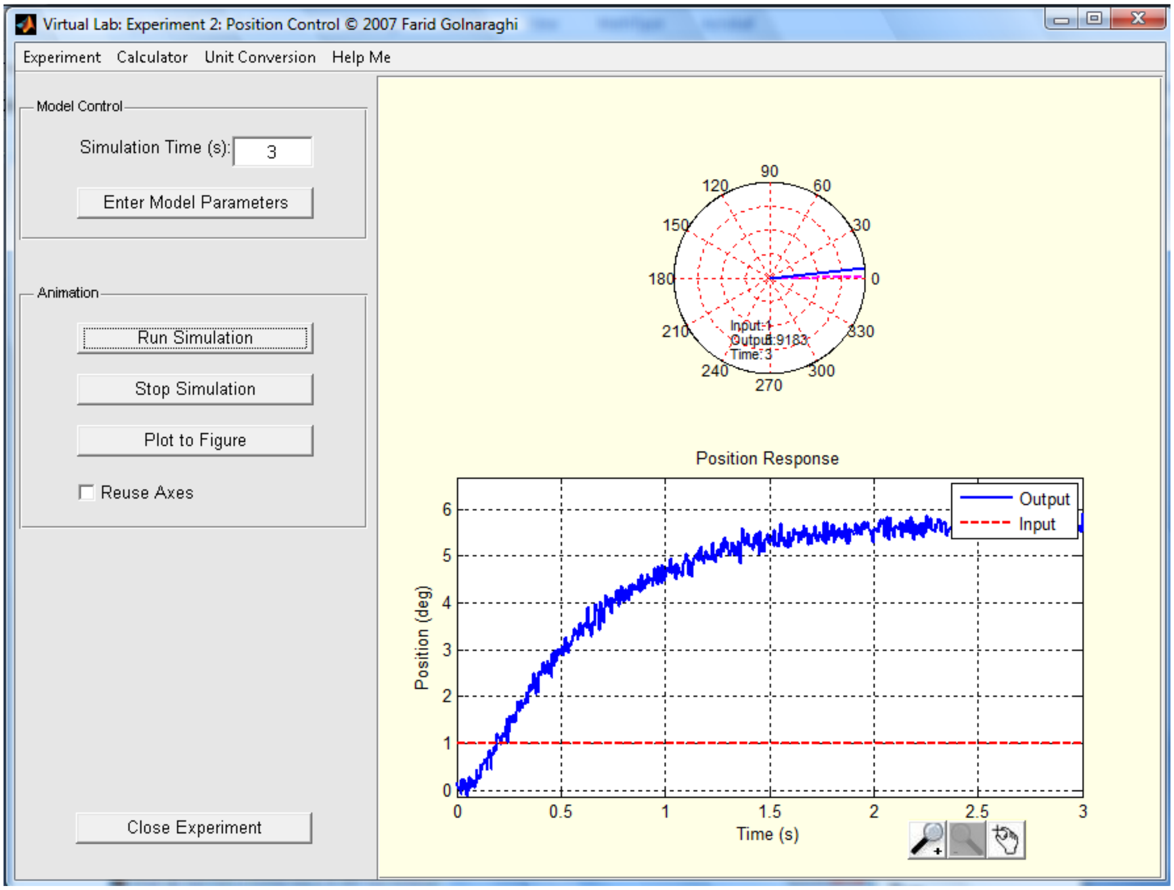
6-22)



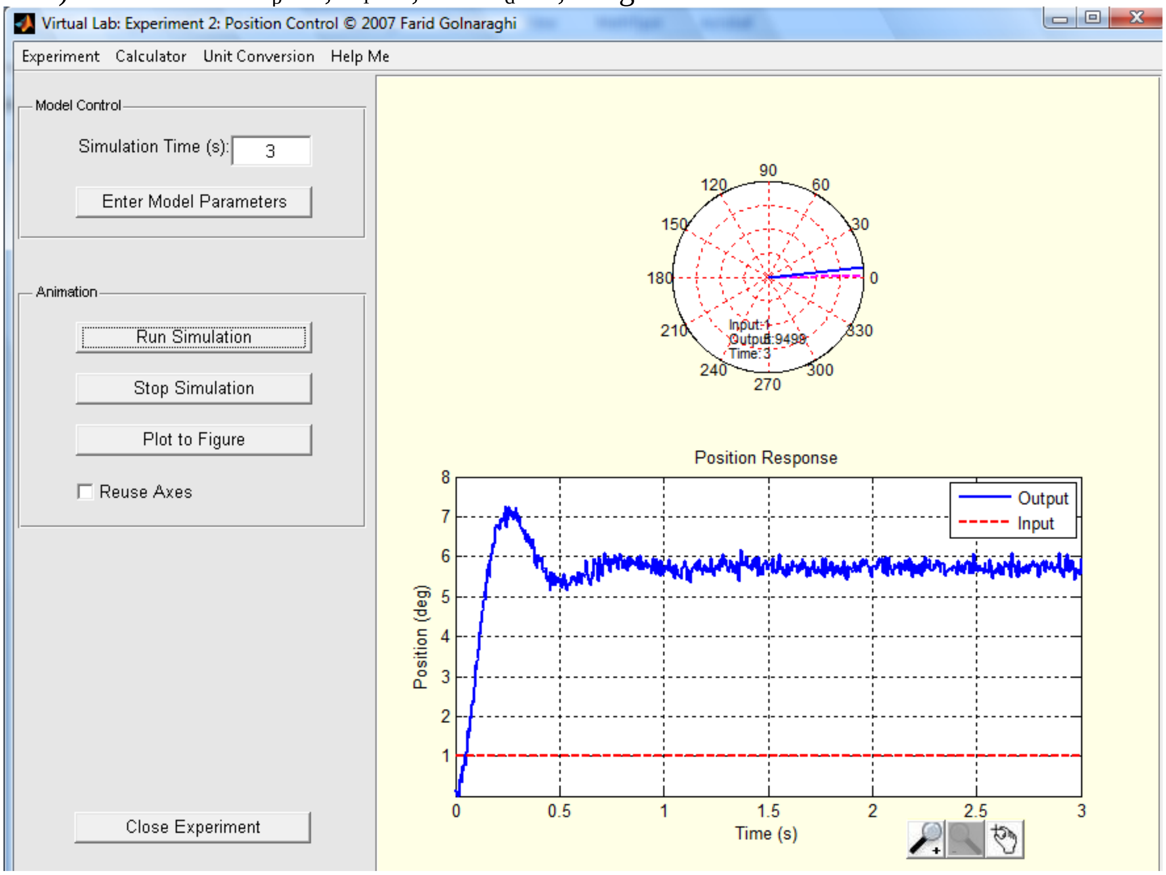
Select the Position Control option in Virtual Lab Experiment Window.
Enter the Step Input and Controller Gain values by double clicking on their respective blocks.



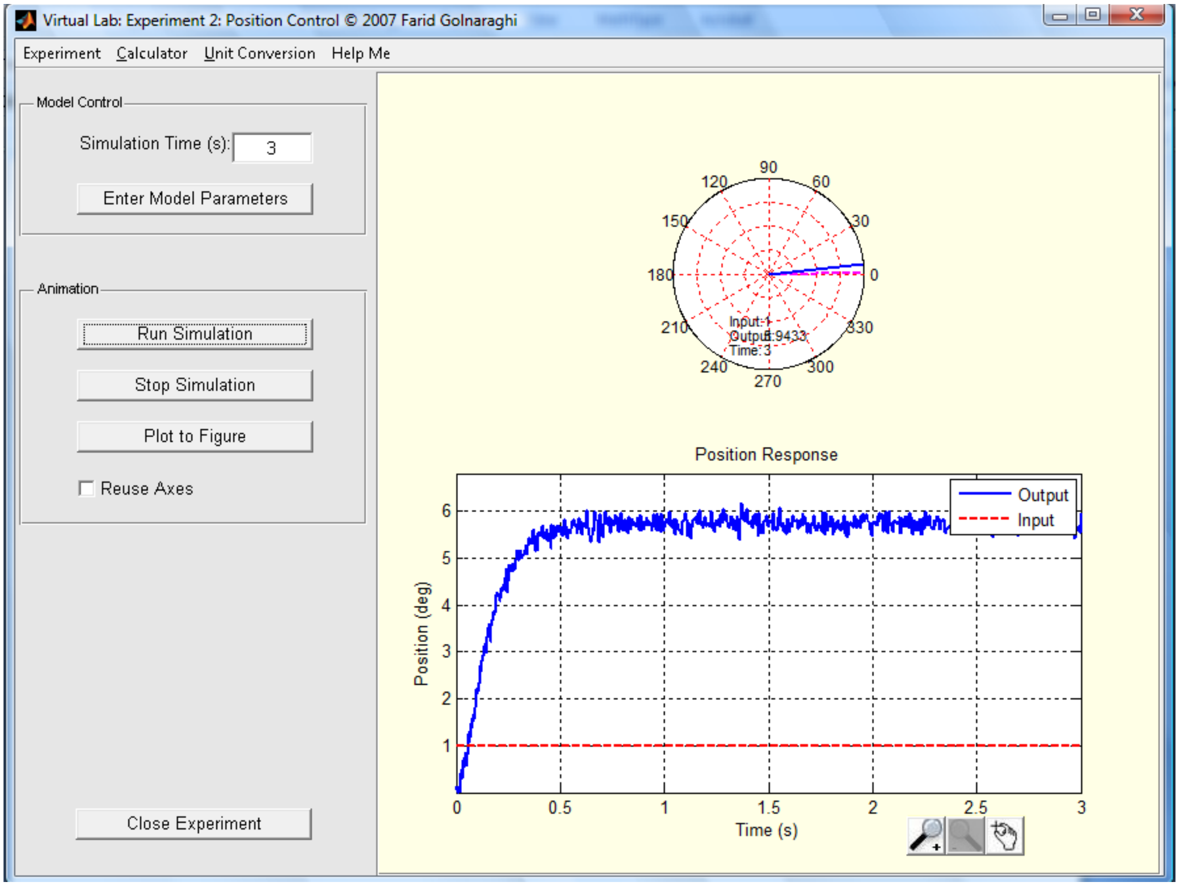
Run simulation, and repeat the process for different gain values.


Virtual Lab: Experiment 2: Position Control © 2007 Farid Golnaraghi
Experiment Calculator Unit Conversion Help Me
Model Control
Simulation Time (s): 3
Enter Model Parameters
Animation
Run Simulation
Stop Simulation
Plot to Figure
Reuse Axes
Close Experiment
Input: 1
Output: 5.9183
Time: 3
Position Response
Output
Input
Position (deg)
Time (s)

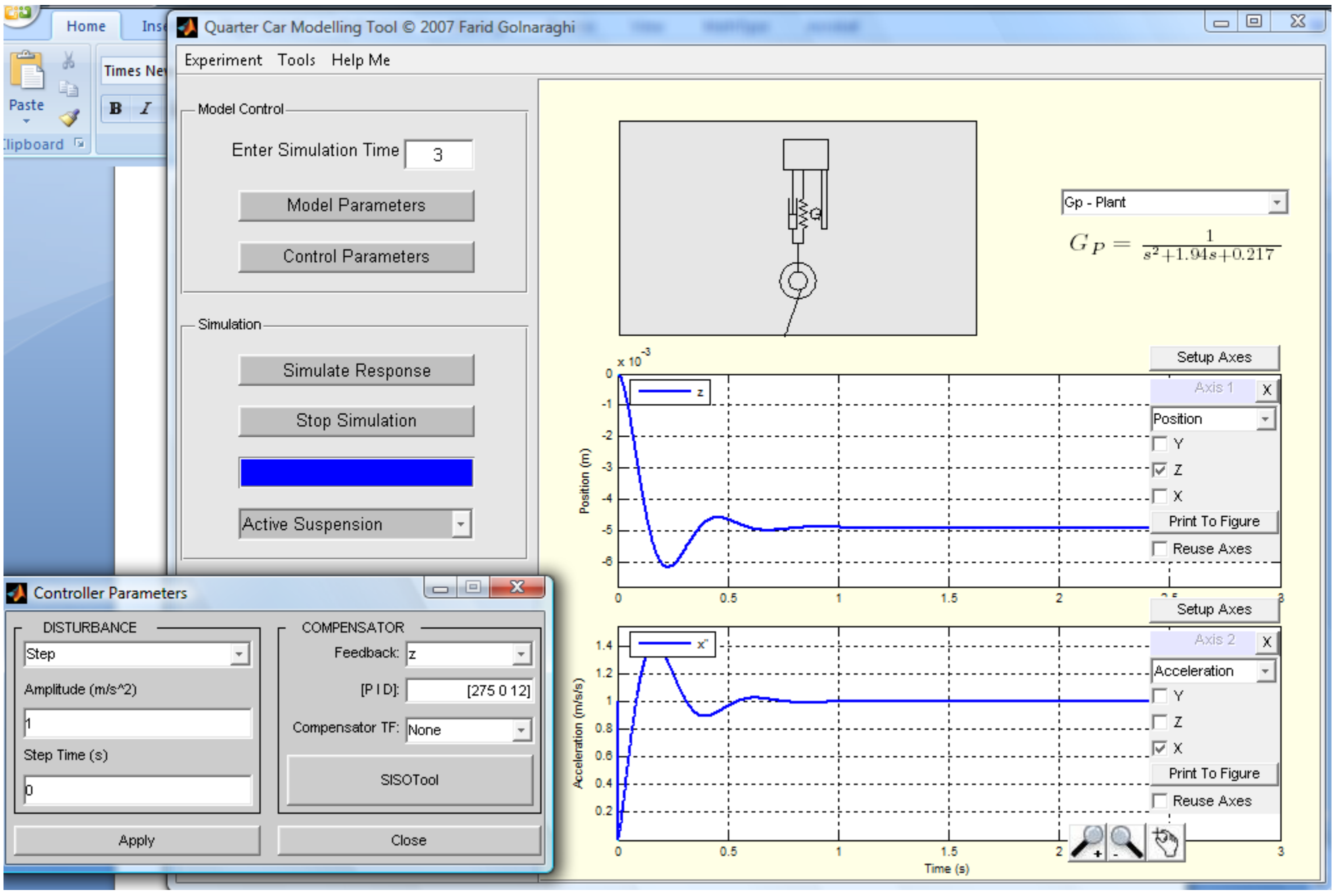
6-23) If we choose K_p= 1, K_i= 0, and K_d= 0, we get



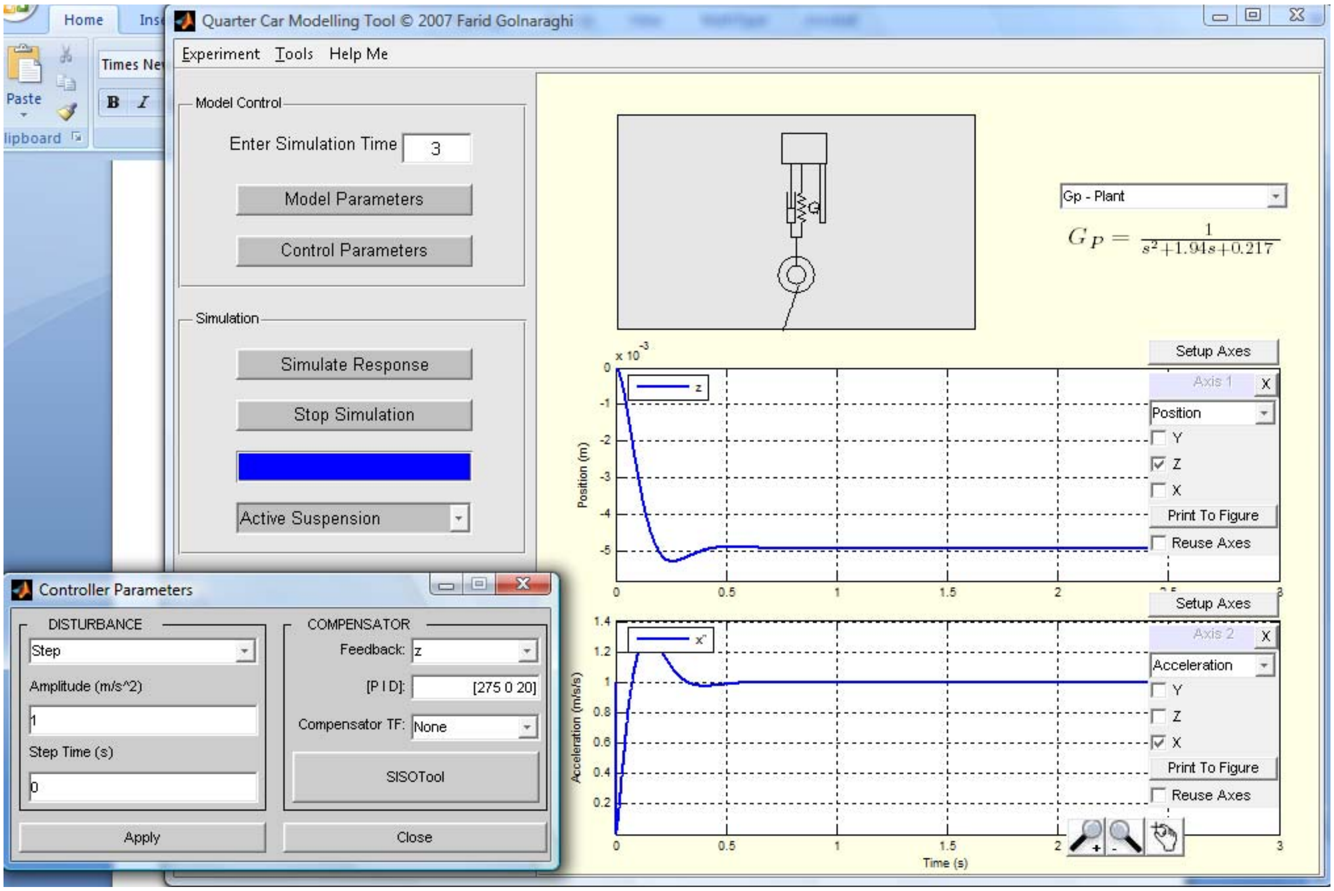
Increase K_d=0.1 we get:


Virtual Lab: Experiment 2: Position Control © 2007 Farid Golnaraghi
Experiment
Calculator
Unit Conversion
Help Me
Model Control
Simulation Time (s):
3
Enter Model Parameters
Animation
Run Simulation
Stop Simulation
Plot to Figure
Reuse Axes
Close Experiment
90
120
60
150
30
180
0
210
330
240
300
270
Position Response
Output
Input
Position (deg)
Time (s)
0
0.5
1
1.5
2
2.5
3
1
2
3
4
5
6

6-24) Here Kp=275 and Kd=12.

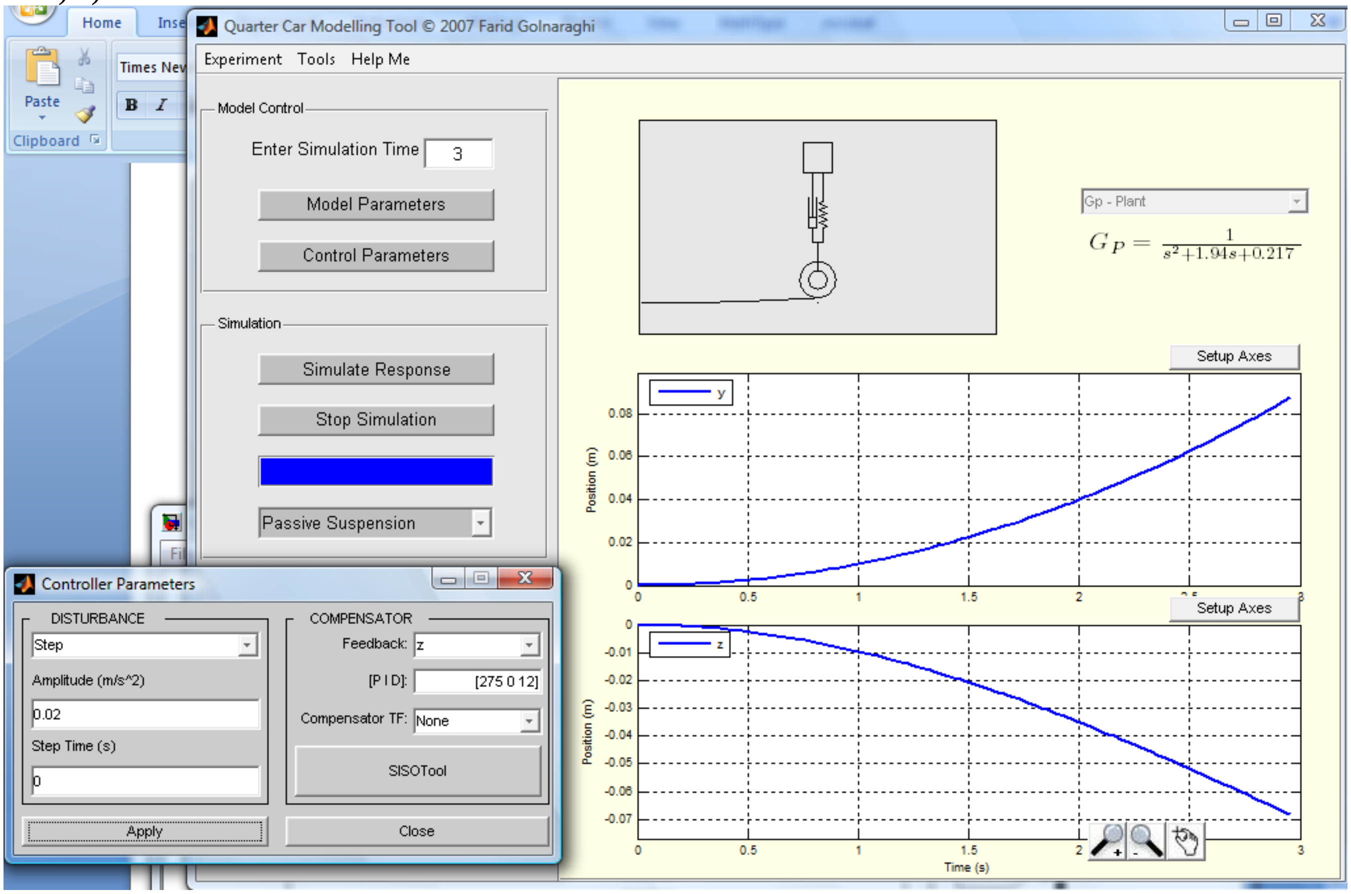


Next Kp=275 and Kd=20.



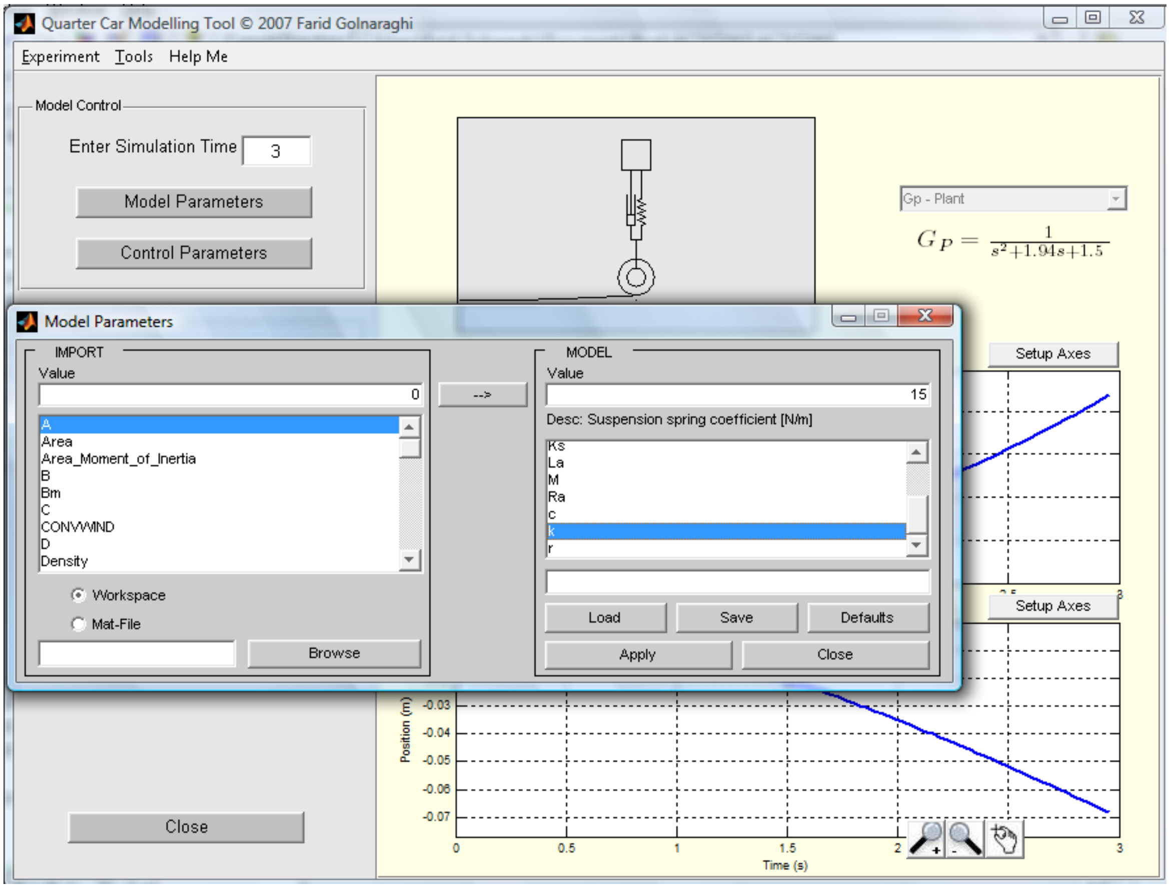
You may try other parts and make observations.

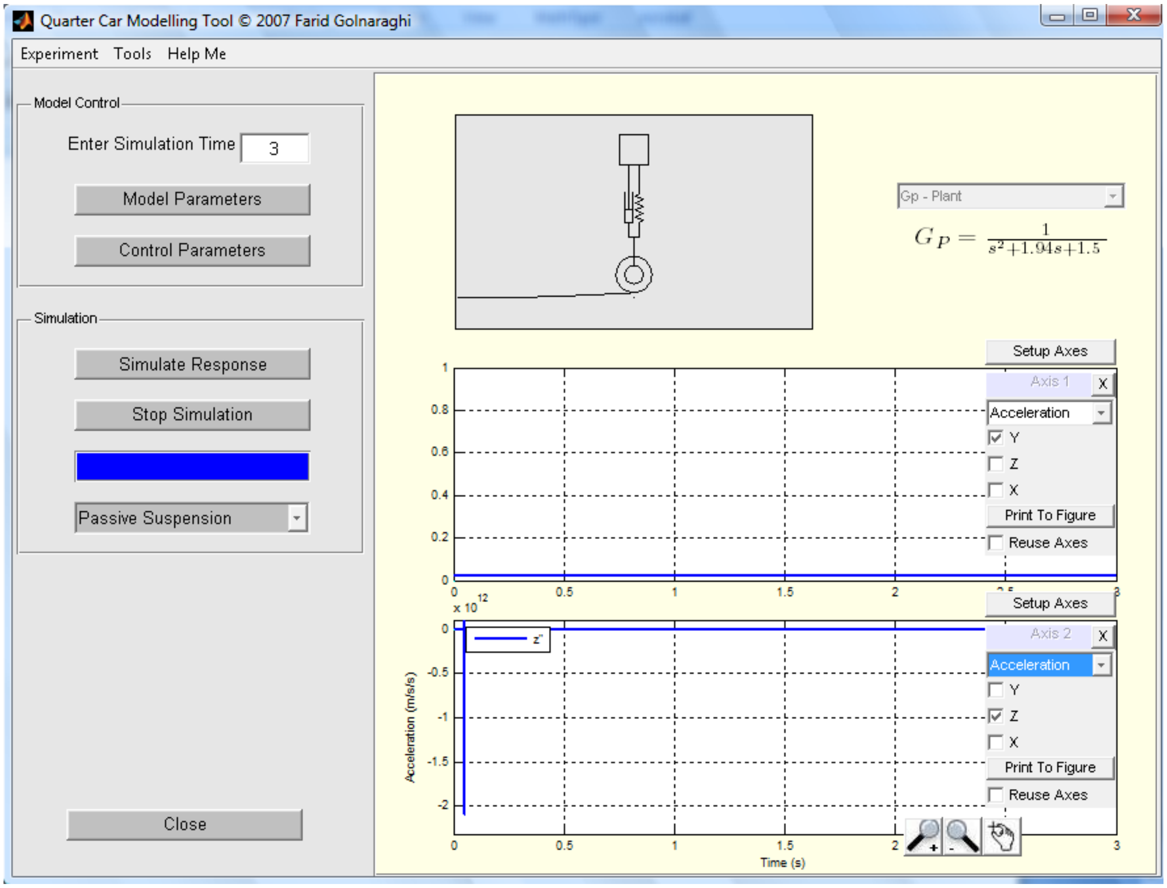
6-25) a)



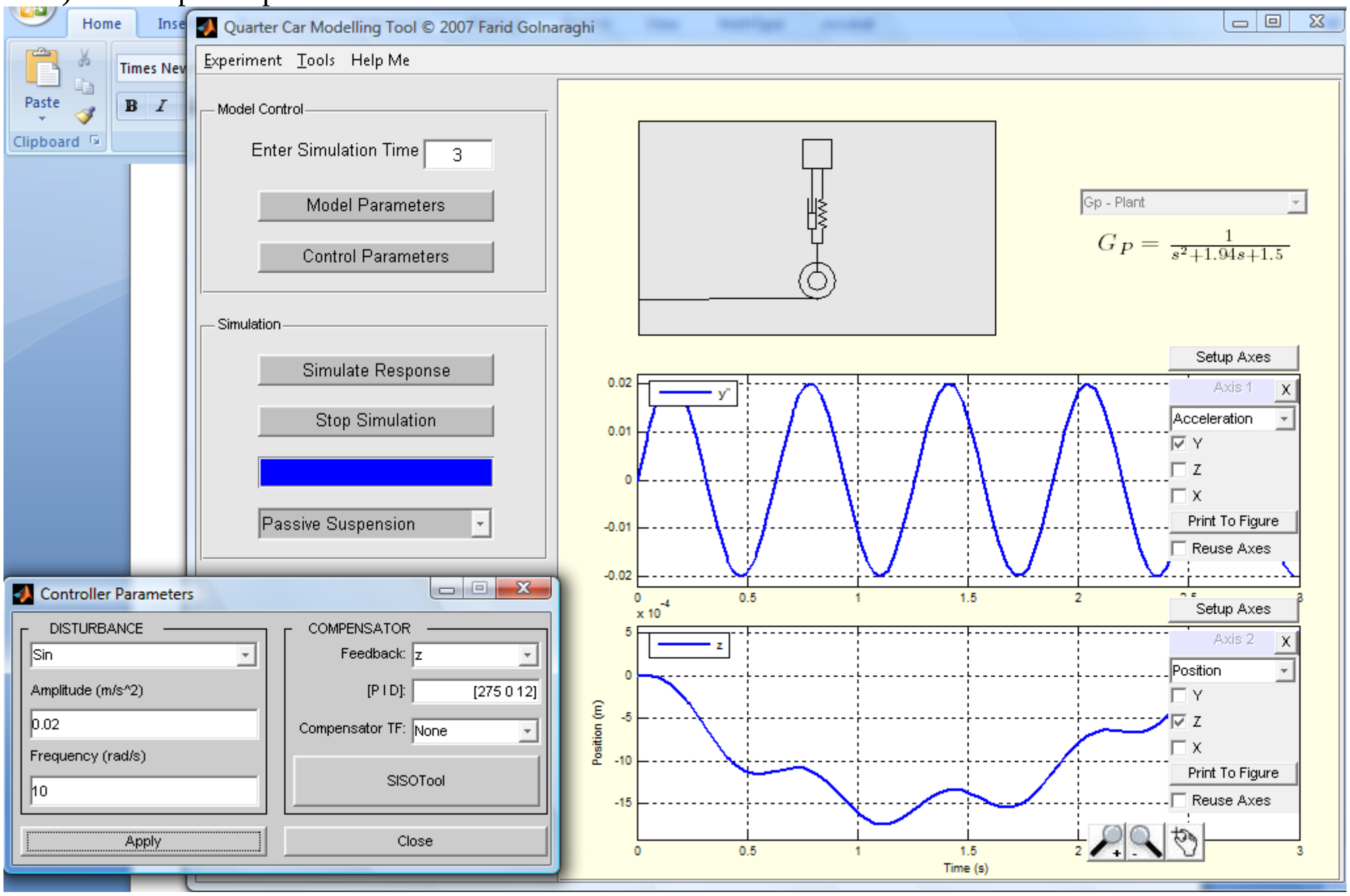
Change variables displayed using “Setup Axes” if desired.

b) Next use the Model Parameters button and change k to 15 as shown. Simulate the response and show the desire variables.

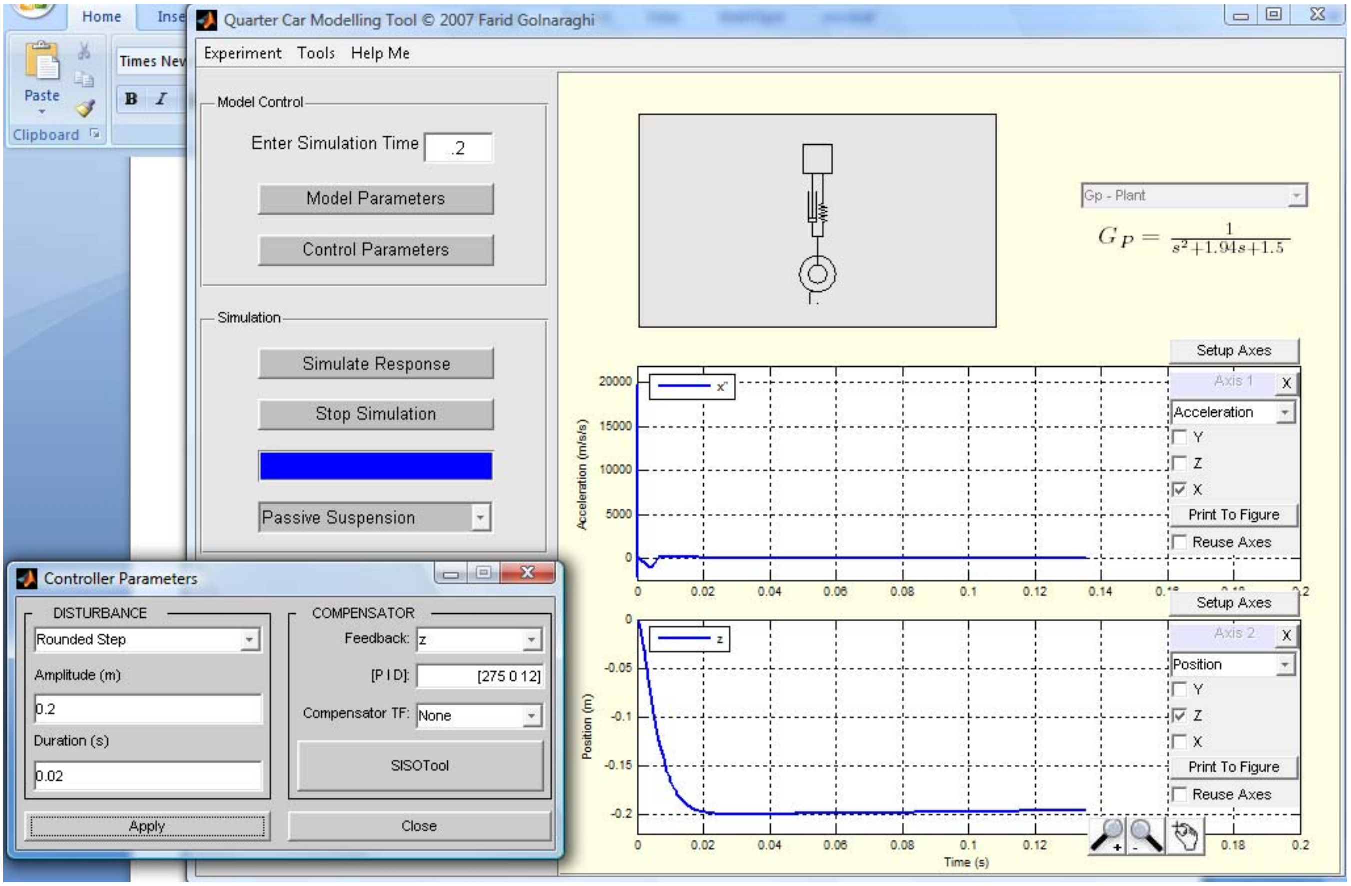

Quarter Car Modelling Tool © 2007 Farid Golnaraghi
Experiment Tools Help Me
Model Control
Enter Simulation Time 3
Model Parameters
Control Parameters
Close
Gp - Plant
$G_P = \frac{1}{s^2+1.94s+1.5}$
Setup Axes
Setup Axes
Model Parameters
IMPORT
Value
0
A
Area
Area_Moment_of_Inertia
B
Bm
C
CONVWIND
D
Density
Workspace
Mat-File
Browse
-->
MODEL
Value
15
Desc: Suspension spring coefficient [N/m]
Ks
La
M
Ra
c
k
r
Load
Save
Defaults
Apply
Close
Position (m)
Time (s)


Quarter Car Modelling Tool © 2007 Farid Golnaraghi
Experiment
Tools
Help Me
Model Control
Enter Simulation Time
3
Model Parameters
Control Parameters
Simulation
Simulate Response
Stop Simulation
Passive Suspension
Close
Gp - Plant
$G_P = \frac{1}{s^2+1.94s+1.5}$
Setup Axes
Axis 1
Acceleration
Y
Z
X
Print To Figure
Reuse Axes
Axis 2
x 10^12
Acceleration (m/s/s)
Time (s)

c) A sample response



d)



Other parts are trivial and follow section 6-6.

Chapter 7

7-1 (a) $P(s) = s^4 + 4s^3 + 4s^2 + 8s \qquad Q(s) = s + 1$

Finite zeros of *P*(*s*): $0, \ -3.5098, \ -0.24512 \pm j1.4897$

Finite zeros of *Q*(*s*): -1

Asymptotes: ***K* > 0:** $60^o, \ 180^o, \ 300^o$ ***K* < 0:** $0^o, \ 120^o, \ 240^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-3.5 - 0.24512 - 0.24512 - (-1)}{4 - 1} = -1$$

(b) $P(s) = s^3 + 5s^2 + s \qquad Q(s) = s + 1$

Finite zeros of *P*(*s*) : $0, \ -4.7912, \ -0.20871$

Finite zeros of *Q*(*s*): -1

Asymptotes: ***K* > 0:** $90^o, \ 270^o$ ***K* < 0:** $0^o, \ 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-4.7913 - 0.2087 - (-1)}{3 - 1} = -2$$

(c) $P(s) = s^2 \qquad Q(s) = s^3 + 3s^2 + 2s + 8$

Finite zeros of *P*(*s*): $0, \ 0$

Finite zeros of *Q*(*s*): $-3.156, \quad 0.083156 \pm j1.5874$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

(d) $P(s) = s^3 + 2s^2 + 3s \qquad Q(s) = \left(s^2 - 1\right)\left(s + 3\right)$

Finite zeros of *P*(*s*): $0, \quad -1 \pm j1.414$

Finite zeros of *Q*(*s*): $1, -1, -3$

Asymptotes: **There are no asymptotes, since the number of zeros of *P*(*s*) and *Q*(*s*) are equal.**

(e) $P(s) = s^5 + 2s^4 + 3s^3 \qquad Q(s) = s^2 + 3s + 5$

Finite zeros of *P*(*s*): $0, \quad 0, \quad 0, \quad -1 \pm j1.414$

Finite zeros of *Q*(*s*): $-1.5 \pm j1.6583$

Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-1-(-1.5)-(-1.5)}{5-2} = \frac{1}{3}$$

(f) $P(s) = s^4 + 2s^2 + 10 \qquad Q(s) = s + 5$

Finite zeros of *P*(*s*): $-1.0398 \pm j1.4426, \quad 1.0398 \pm j1.4426$

Finite zeros of *Q*(*s*): -5

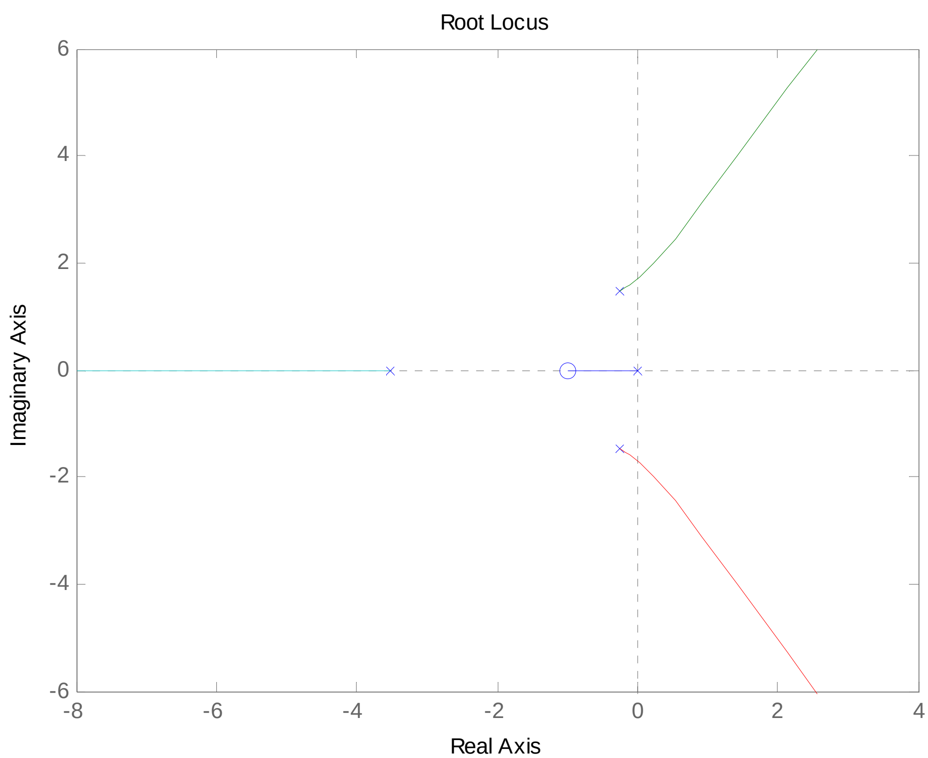
Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1.0398-1.0398+1.0398+1.0398-(-5)}{4-1} = \frac{-5}{3}$$

7-2(a) MATLAB code:

```
s = tf('s')
num_GH=(s+1);
den_GH=(s^4+4*s^3+4*s^2+8*s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



Assymp1_angle = 60

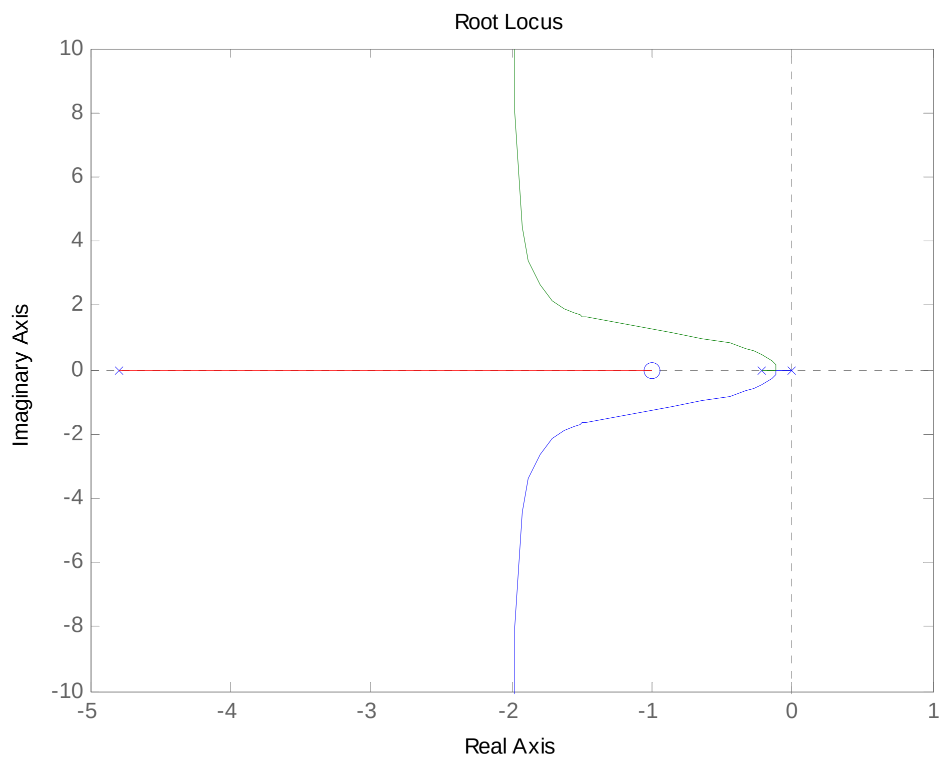
Assymp2_angle = -60

Assymp3_angle = 180

sigma = -1.0000 (intersect of asymptotes)

7-2(b) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s+1);
den_GH=(s^3+5*s^2+s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



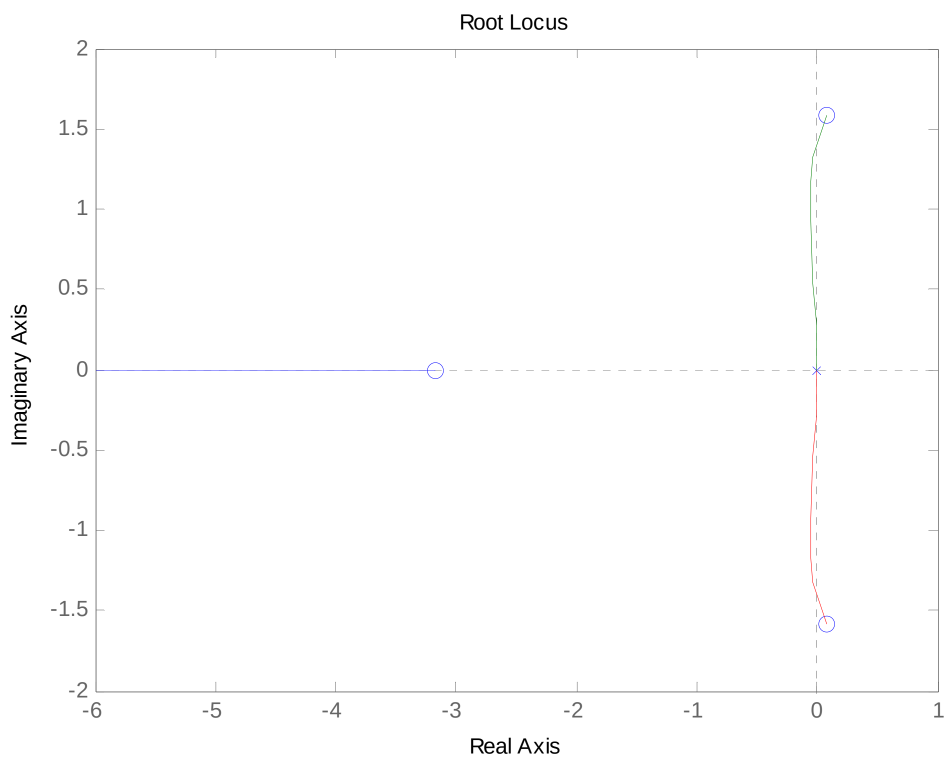
Assymp1_angle = 90

Assymp2_angle = -90

sigma = -2 (intersect of asymptotes)

7-2(c) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s^3+3*s^2+2*s+8);
den_GH=(s^2);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```

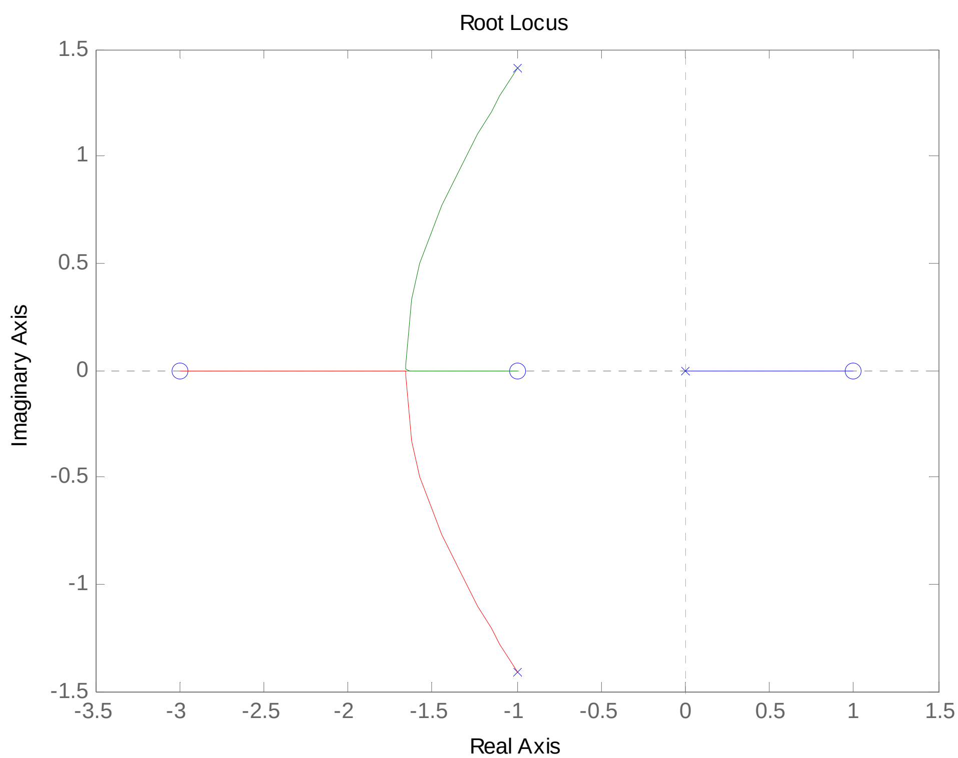


Assymp1_angle = 180

sigma = -3.0000 (intersect of asymptotes)

7-2(d) MATLAB code:

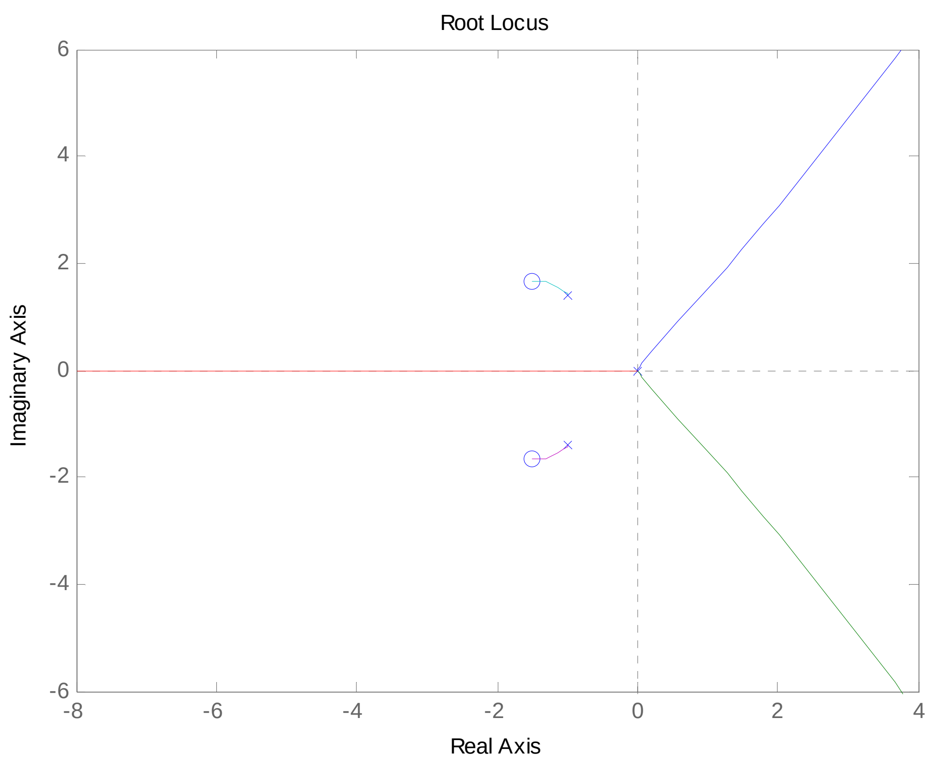
```
s = tf('s')
'Generating the transfer function:'
num_GH=((s^2-1)*(s+3));
den_GH=(s^3+2*s^2+3*s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
```



No asymptotes

7-2(e) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s^2+3*s+5);
den_GH=(s^5+2*s^4+3*s^3);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



Assymp1_angle = 60

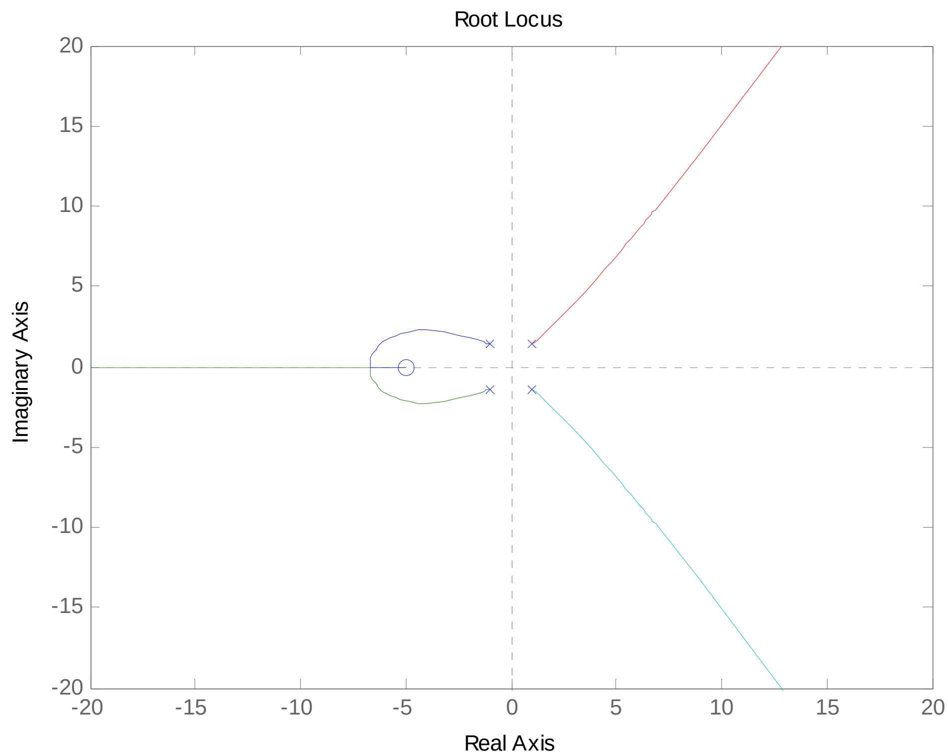
Assymp2_angle = -60

Assymp3_angle = 180

sigma = 0.3333 (intersect of asymptotes)

7-2(f) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s+5);
den_GH=(s^4+2*s^2+10);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
xlim([-20 20])
ylim([-20 20])
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



Assymp1_angle = 60

Assymp2_angle = -60

Assymp3_angle = 180

sigma = 1.6667 (intersect of asymptotes)

7-3) Consider

$$G(s)H(s) = K\frac{Q(s)}{P(s)} = K\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

As the asymptotes are the behavior of G(s)H(s) when |s|→∞ , then

$|s| > |z_i|$ for i = 1,2,…,m and $|s| > |p_i|$ for i = 1,2,…,n

therefore ∠ G(s)H(s) $= m\arg(s) - n\arg(s) = -(n-m)\arg(s)$

According to the condition on angles:

$$\angle G(s)H(s) = \begin{cases} 2(i+1)\pi & K \geq 0 \\ 2i\,\pi & K \leq 0 \end{cases}$$

If we consider $\arg(s) = \theta_i$, then:

$$\angle G(s)H(s) = \begin{cases} -(n-m)\theta_i = (2i+1)\pi & K \geq 0 \\ -(n-m)\theta_i = 2i\,\pi & K \leq 0 \end{cases}$$

or

$$\begin{cases} \theta_i = \dfrac{2i+1}{|n-m|}\pi & K \geq 0 \\ \theta_i = \dfrac{2i}{|n-m|}\pi & K \leq 0 \end{cases}$$

7-4) If $G(s)H(s) = K\dfrac{Q(s)}{P(s)}$, then each point on root locus must satisfy the characteristic equation of $P(s) + KQ(s) = 0$

If $P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ and $Q(s) = s^m + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$, then

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0 + K(s^m - b_{m-1}s^{m-1} + \cdots + b_0) = 0$$

or

$$s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} + \cdots + K = 0$$

If the roots of above expression is considered as s_i for i = 1,2,…,(n-m), then

$$a_{n-1} - b_{m-1} = -\sum_{i=1}^{n-m} s_i = -\sum_{i=1}^{n} s_i - \sum_{i=1}^{m} s_i = -\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i$$

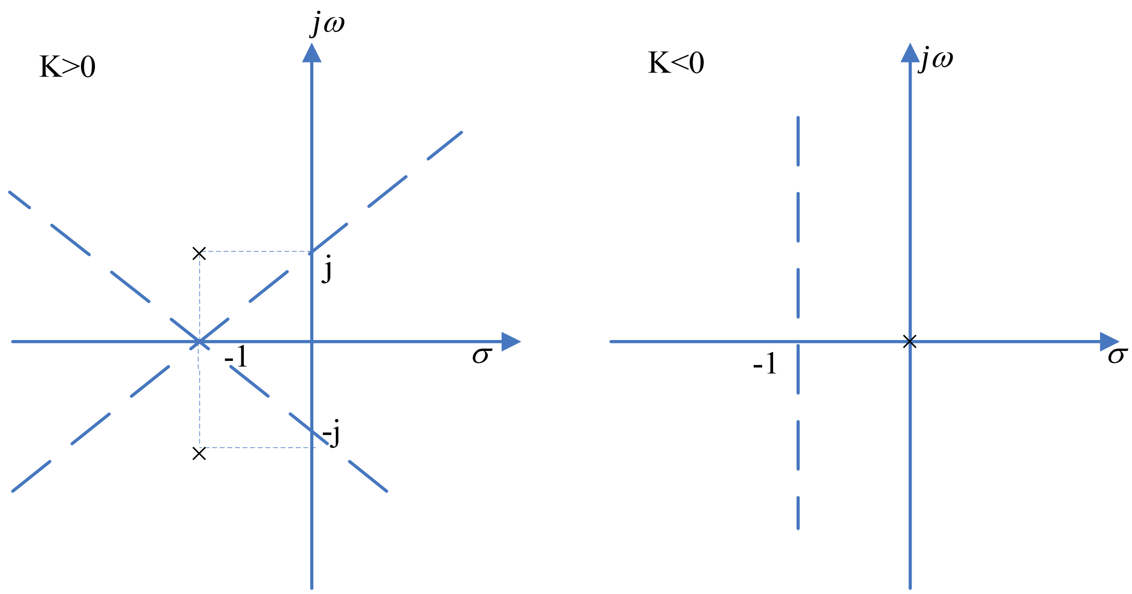
since the intersect of (n-m) asymptotes lies on the real axis of the s-plane and $-(a_{n-1} - b_{m-1})$ is real, therefore

$$\sigma_1 = -\frac{a_{n-1} \quad b_{m-1}}{n-m} = -\frac{\sum_{i=1}^{n} p_i \quad \sum_{i=1}^{m} z_i}{n-m}$$

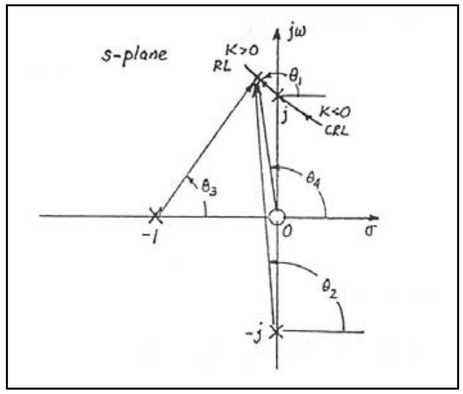
7-5) Poles of GH is s = 0, -2, -1 + j, -1 – j, therefore the center of asymptotes:

$$\sigma_1 = \frac{\sum p_i - \sum z_i}{n \quad m} = -1$$

The angles of asymptotes: $\begin{cases} \theta_i = 45^o, 135^o, 225^o, 315^o & K > 0 \\ \theta_i = 0^o, 90^o, 180^o, 270^o & K < 0 \end{cases}$


K>0
jω
j
-1
σ
-j
K<0
jω
-1
σ

7-6 (a) Angles of departure and arrival.



K > 0: $-\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^o$

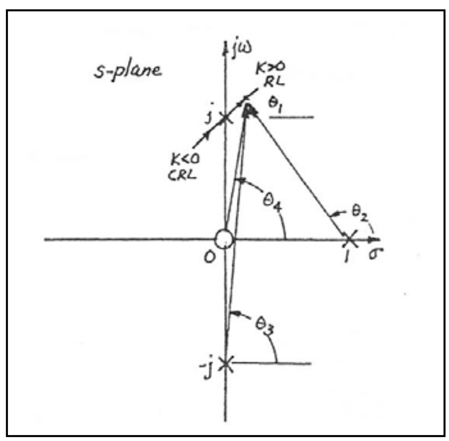
$-\theta_1 - 90^o - 45^o + 90^o = -180^o$

$\theta_1 = 135^o$

K < 0: $-\theta_1 - 90^o - 45^o + 90^o = 0^o$

$\theta_1 = -45^o$

(b) Angles of departure and arrival.

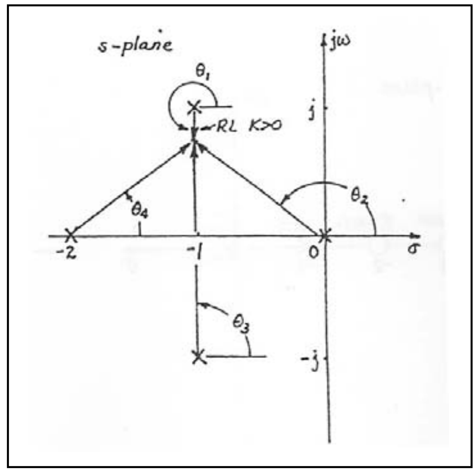


K > 0: $-\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^o$

K < 0: $-\theta_1 - 135^o - 90^o + 90^o = 0^o$

$\theta_1 = -135^o$

(c) Angle of departure:

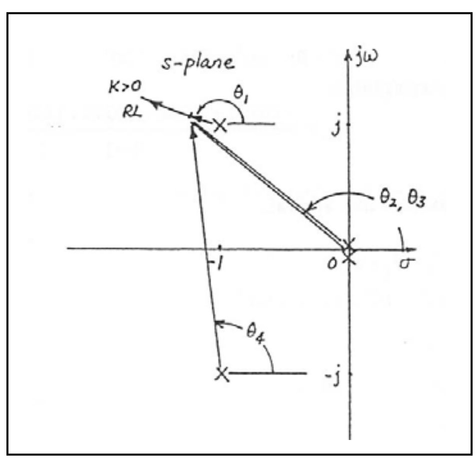


***K* > 0:** $-\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^o$

$-\theta_1 - 135^o - 90^o - 45^o = -180^o$

$\theta_1 = -90^o$

(d) Angle of departure

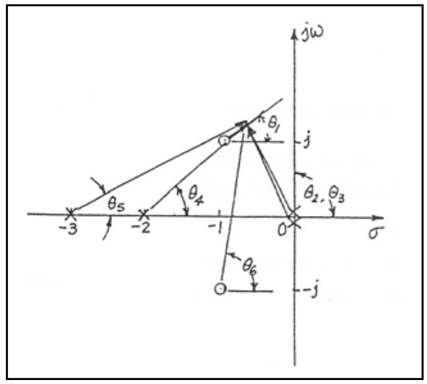


***K* > 0:** $-\theta_1 - \theta_2 - \theta_3 - \theta_4 = -180^o$

$-\theta_1 - 135^o - 135^o - 90^o = -180^o$

$\theta_1 = -180^o$

(e) Angle of arrival



***K* < 0:** $\theta_1 + \theta_6 - \theta_2 - \theta_3 - \theta_4 - \theta_5 = -360^o$

$$\theta_1 + 90^o - 135^o - 135^o - 45^o - 26.565^o = -360^o$$

$$\theta_1 = -108.435^o$$

7-7) a) $\angle G(s)H(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$

$= \sum_{i=1}^{m} \angle(s+z_i) + \sum_{i=1, i\neq j}^{n} \angle(s+p_i) - \angle(s+p_j)$

$= \angle G(s)H'(s) - \angle(s+p_j)$

$= \angle G(s)H'(s) - \theta_D$

we know that $\angle G(s)H(s) = \begin{cases} (2l+1)\times 180 & K \geq 0 \quad l = 0, \pm 1, \ldots \\ (2l)\times 180 & K \leq 0 \quad l = 0, \pm 1, \ldots \end{cases}$

therefore

$\begin{cases} \angle G(s)H'(s) - \theta_D = 180 & K \geq 0 \\ \angle G(s)H'(s) - \theta_D = 0 & K \leq 0 \end{cases}$

As a result, $\theta_D = \angle G(s)H'(s) - 180^o = 180 + \angle G(s)H'(s)$, when $-180^o = 180^o$

b) Similarly:

$\angle G(s)H(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$

$= \sum_{i=1, i\neq j}^{m} \angle(s+z_i) + \sum_{i=1}^{n} \angle(s-p_i) + \angle(s+z_j)$

$= \angle G(s)H''(s) + \angle(s+z_j)$

$= \angle G(s)H''(s) + \theta$

Therefore:

$\begin{cases} \angle G(s)H''(s) + \theta = 180 & K \geq 0 \\ \angle G(s)H''(s) + \theta = 0 & K \leq 0 \end{cases}$

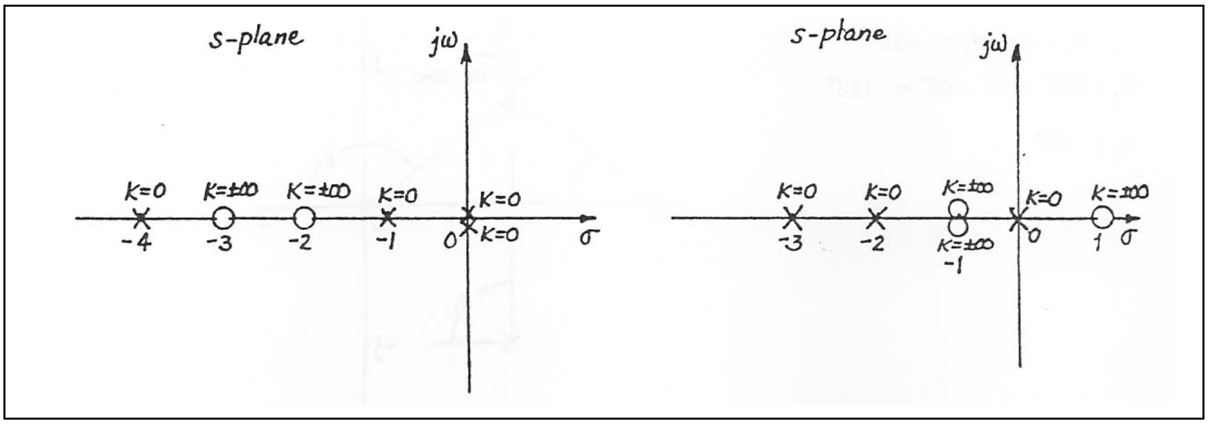
As a result, $\theta = 180 - \angle G(s)H''(s)$

7-8) zeros: $s = -1-j, -1+j$ and poles: $s = 0, -2j, +2j$

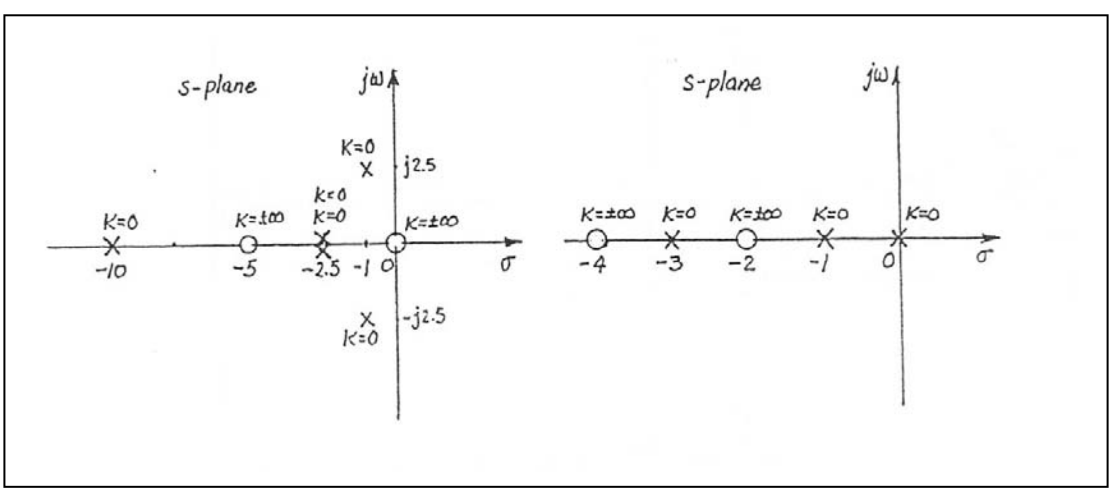
Departure angles from: $\begin{cases} s = 2j & : \quad \theta = 180 - 63.4 = 116.6 \\ s = -2j & : \quad \theta = -198.4 \end{cases}$

Arrival angles at $\begin{cases} s = -1 + j & : \quad \theta = 180 - (-18.4) = 198.4 \\ s = -1 - j & : \quad \theta = -198.4 \end{cases}$

7-9) (a) **(b)**



(c) **(d)**



7-10) The breaking points are on the real axis of $1 + KG(s)H(s) = 0$ and must satisfy

$$\frac{dG(s)H(s)}{ds} = 0$$

If $G(s)H(s) = \frac{Q(s)}{P(s)}$ and α is a breakaway point, then

$$1 + K\frac{Q(\alpha)}{P(\alpha)} = 0 \rightarrow K = -\frac{P(\alpha)}{Q(\alpha)}$$

Finding α where K is maximum or minimum $\frac{dK}{d\alpha} = 0$, therefore

$$\frac{d}{d\alpha}\left[\frac{P(\alpha)}{Q(\alpha)}\right] = 0$$

or

$$\frac{d}{d\alpha}\left[\frac{(\alpha + p_1)(\alpha + p_2)\ldots(\alpha + p_n)}{(\alpha + z_1)(\alpha + z_2)\ldots(\alpha + z_m)}\right] = 0$$

$$\sum_{i=1}^{n}\frac{1}{\alpha + p_i}\left[\frac{P(\alpha)}{Q(\alpha)}\right] - \sum_{i=1}^{m}\frac{1}{\alpha + z_i}\left[\frac{P(\alpha)}{Q(\alpha)}\right] = 0$$

$$\sum_{i=1}^{n}\frac{1}{\alpha + p_i} = \sum_{i=1}^{m}\frac{1}{\alpha + z_i}$$

7-11) (a) Breakaway-point Equation: $2s^5 + 20s^4 + 74s^3 + 110s^2 + 48s = 0$

Breakaway Points: $-0.7275, \quad -2.3887$

(b) Breakaway-point Equation: $3s^6 + 22s^5 + 65s^4 + 100s^3 + 86s^2 + 44s + 12 = 0$

Breakaway Points: $-1, \quad -2.5$

(c) Breakaway-point Equation: $3s^6 + 54s^5 + 347.5s^4 + 925s^3 + 867.2s^2 - 781.25s - 1953 = 0$

Breakaway Points: $-2.5, \quad 1.09$

(d) Breakaway-point Equation: $-s^6 - 8s^5 - 19s^4 + 8s^3 + 94s^2 + 120s + 48 = 0$

Breakaway Points: $-0.6428, \quad 2.1208$

7-12) (a)

$$G(s)H(s) = \frac{K(s+8)}{s(s+5)(s+6)}$$

Asymptotes: *K* > 0: 90^o and 270^o ***K* < 0:** 0^o and 180^o

Intersect of Asymptotes:

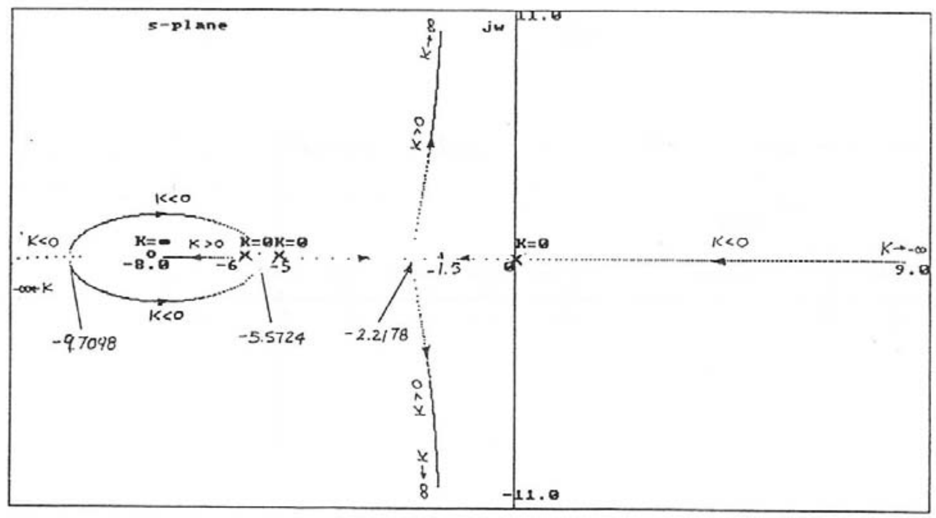
$$\sigma_1 = \frac{0-5-6-(-8)}{3-1} = -1.5$$

Breakaway-point Equation:

$$2s^3 + 35s^2 + 176s + 240 = 0$$

Breakaway Points: $-2.2178, \quad -5.5724, \quad -9.7098$

Root Locus Diagram:



7-12 (b)

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)(s+4)}$$

Asymptotes: *K* > 0: $45^o,\ 135^o,\ 225^o,\ 315^o$ ***K* < 0:** $0^o,\ 90^o,\ 180^o,\ 270^o$

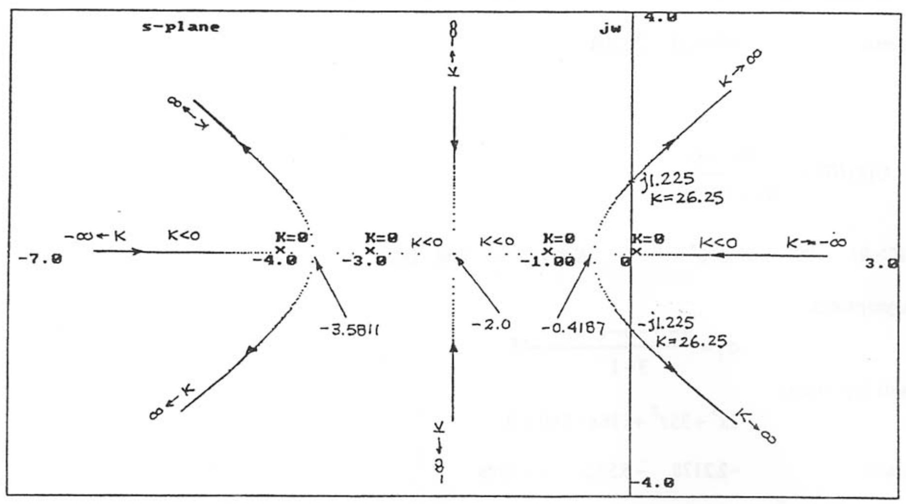
Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-3-4}{4} = -2$$

Breakaway-point Equation: $4s^3 + 24s^2 + 38s + 12 = 0$

Breakaway Points: $-0.4189,\ -2,\ -3.5811$

Root Locus Diagram:



7-12 (c)

$$G(s)H(s) = \frac{K(s+4)}{s^2(s+2)^2}$$

Asymptotes: ***K* > 0:** $60^o,\ 180^o,\ 300^o$ ***K* < 0:** $0^o,\ 120^o,\ 240^o$

Intersect of Asymptotes:

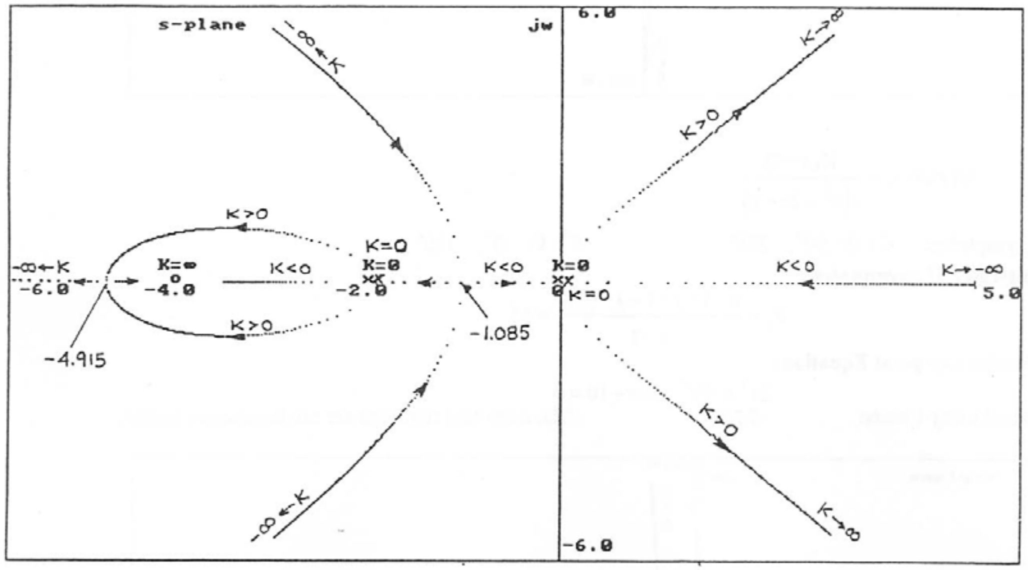
$$\sigma_1 = \frac{0+0-2-2-(-4)}{4-1} = 0$$

Breakaway-point Equation:

$$3s^4 + 24s^3 + 52s^2 + 32s = 0$$

Breakaway Points: $0,\ -1.085,\ -2,\ -4.915$

Root Locus Diagram:



7-12 (d)

$$G(s)H(s)=\frac{K(s+2)}{s(s^2+2s+2)}$$

Asymptotes: ***K* > 0:** $90^o,\quad 270^o$ ***K* < 0:** $0^o,\quad 180^o$

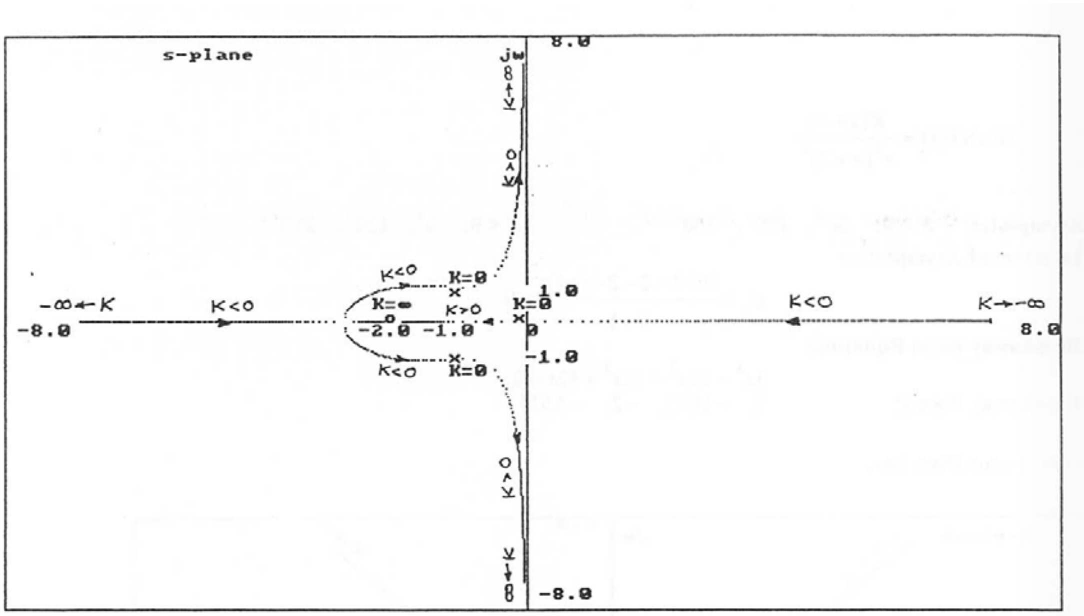
Intersect of Asymptotes:

$$\sigma_1=\frac{0-1-j-1-j-(-2)}{3-1}=0$$

Breakaway-point Equation: $2s^3+8s^2+8s+4=0$

Breakaway Points: -2.8393 The other two solutions are not breakaway points.

Root Locus Diagram



7-12 (e)

$$G(s)H(s) = \frac{K(s+5)}{s\left(s^2+2s+2\right)}$$

Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

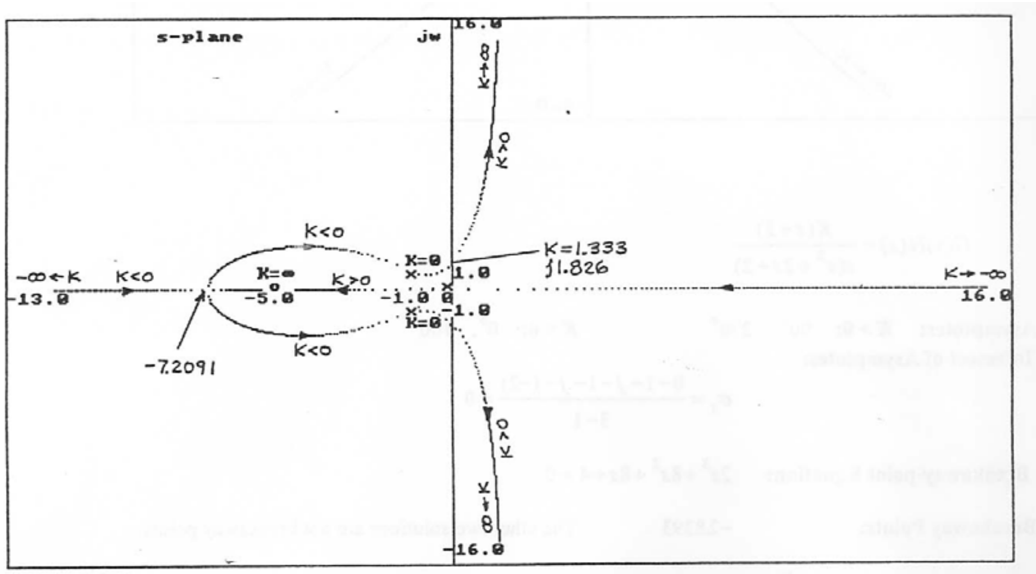
Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-j-1-j-(-5)}{3-1} = 1.5$$

Breakaway-point Equation:

$$2s^3 + 17s^2 + 20s + 10 = 0$$

Breakaway Points: -7.2091 The other two solutions are not breakaway points.



7-12 (f)

$$G(s)H(s)=\frac{K}{s(s+4)\left(s^2+2s+2\right)}$$

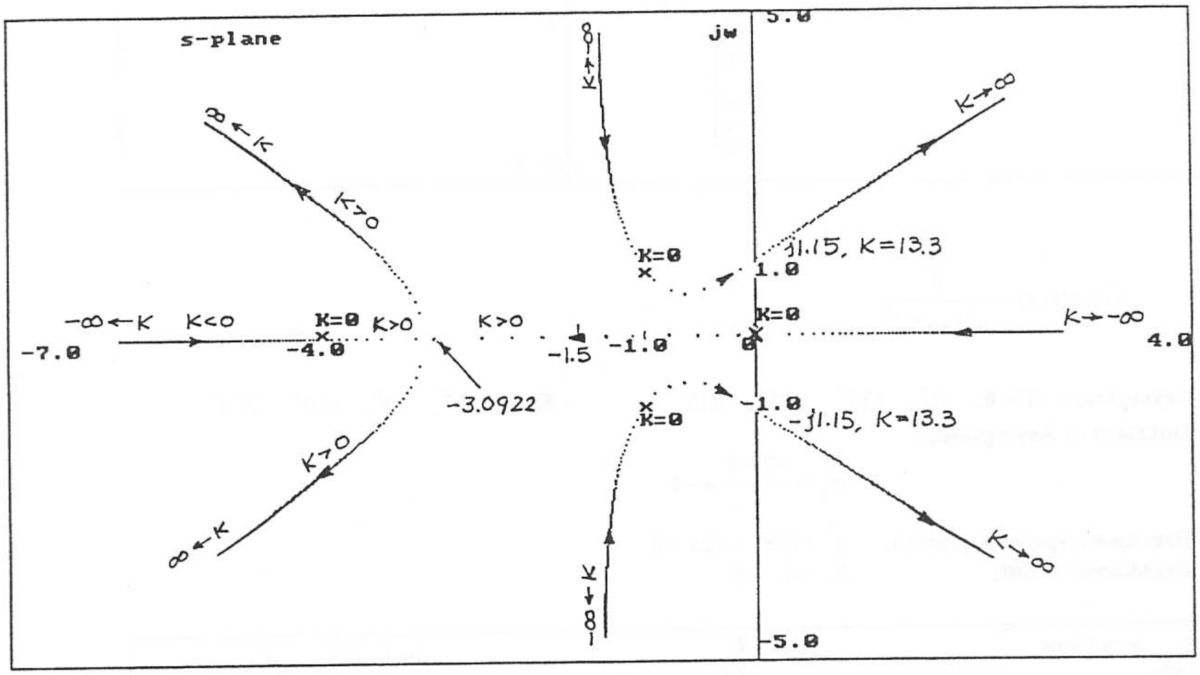
Asymptotes: ***K* > 0:** $45^o,\ 135^o,\ 225^o,\ 315^o$ ***K* < 0:** $0^o,\ 90^o,\ 180^o,\ 270^o$

Intersect of Asymptotes:

$$\sigma_1=\frac{0-1-j-1+j-4}{4}=-1.5$$

Breakaway-point Equation: $4s^3+18s^2+20s+8=0$

Breakaway Point: -3.0922 The other solutons are not breakaway points.



7-12 (g)

$$G(s)H(s) = \frac{K(s+4)^2}{s^2(s+8)^2}$$

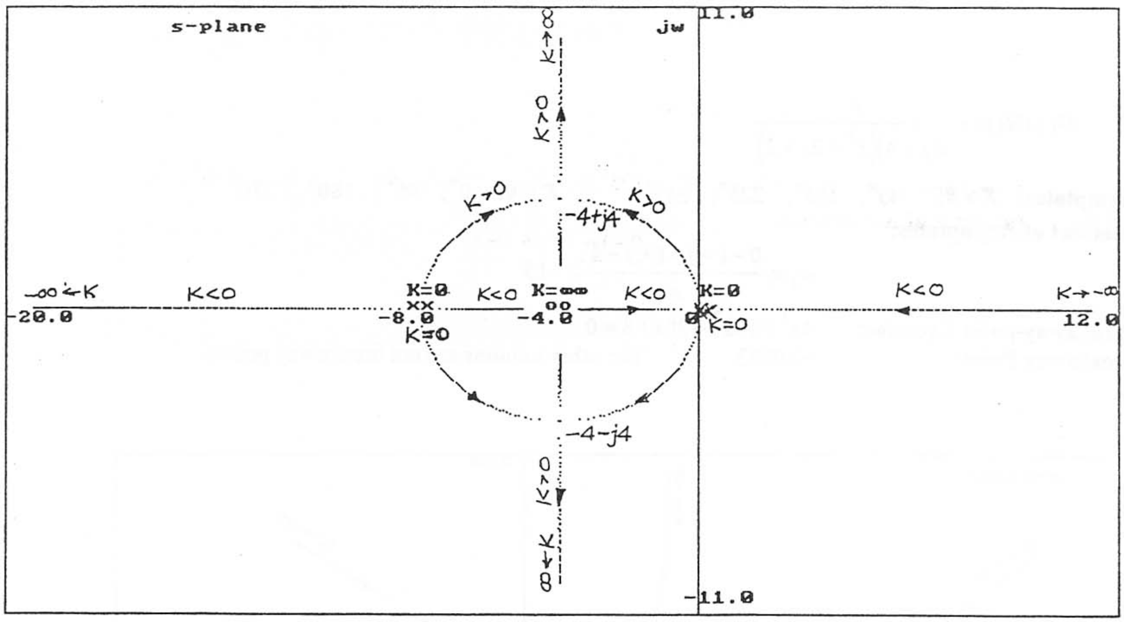
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intesect of Asymptotes:

$$\sigma_1 = \frac{0+0-8-8-(-4)-(-4)}{4-2}$$

Breakaway-point Equation: $s^5 + 20s^4 + 160s^3 + 640s^2 + 1040s = 0$

Breakaway Points: 0, −4, −8, $-4 - j4$, $-4 + j4$



7-12 (h)

$$G(s)H(s) = \frac{K}{s^2(s+8)^2}$$

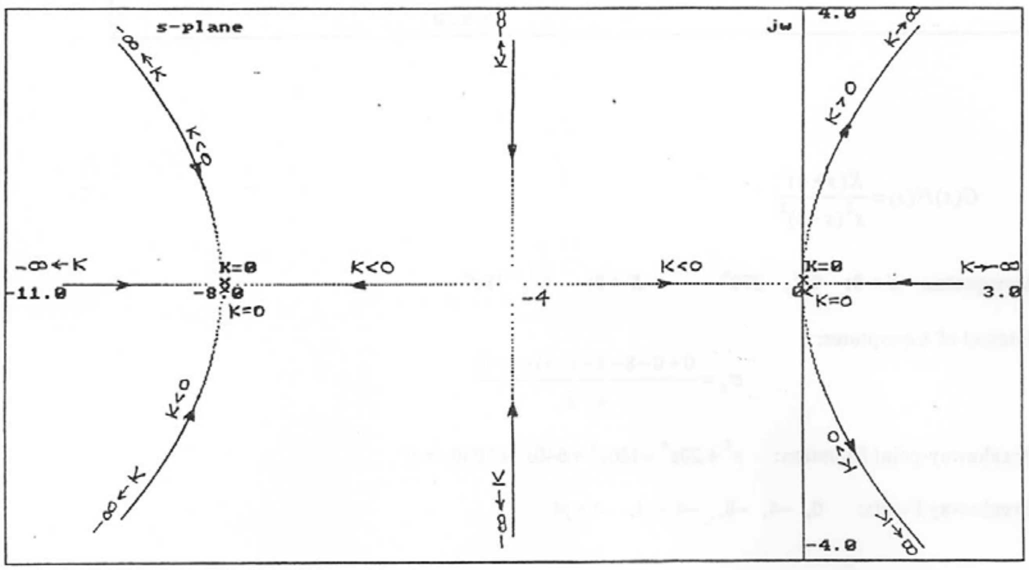
Asymptotes: ***K*** **> 0:** $45^o, \quad 135^o, \quad 225^o, \quad 315^o$ ***K*** **< 0:** $0^o, \quad 90^o, \quad 180^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8-8}{4} = -4$$

Breakaway-point Equation: $s^3 + 12s^2 + 32s = 0$

Breakaway Point: 0, −4, −8



7-12 (i)

$$G(s)H(s) = \frac{K\left(s^2+8s+20\right)}{s^2(s+8)^2}$$

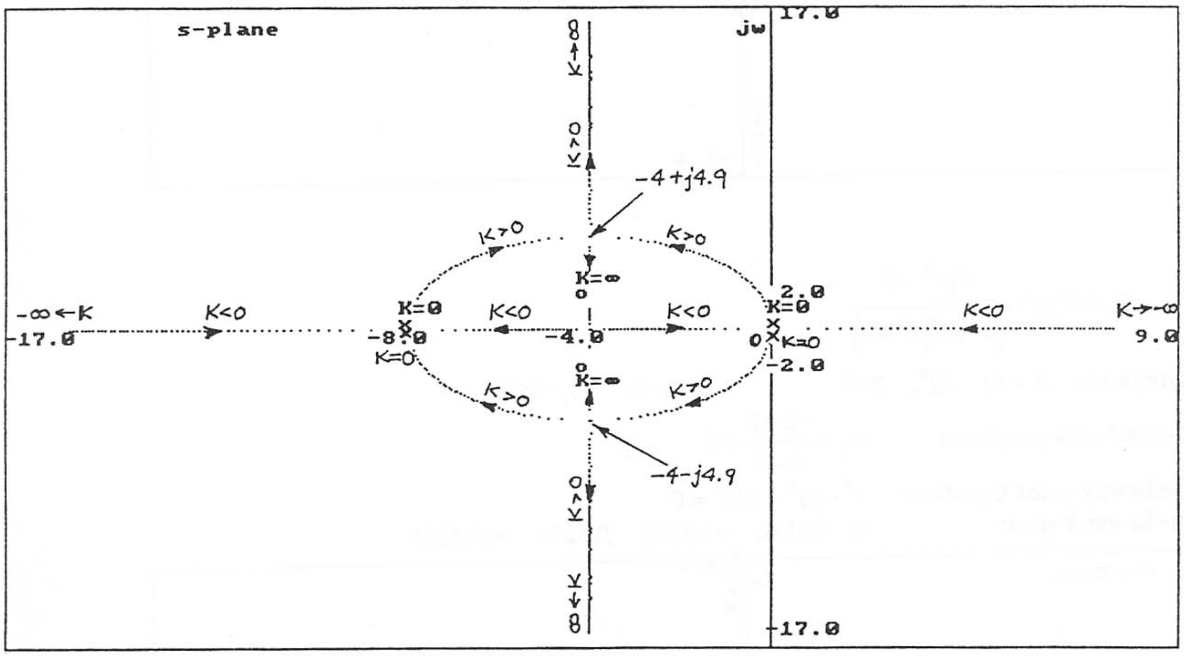
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8-8-(-4)-(-4)}{4-2} = -4$$

Breakaway-point Equation: $s^5 + 20s^4 + 128s^3 + 736s^2 + 1280s = 0$

Breakaway Points: $-4, \quad -8, \quad -4 + j4.9, \quad -4 - j4.9$



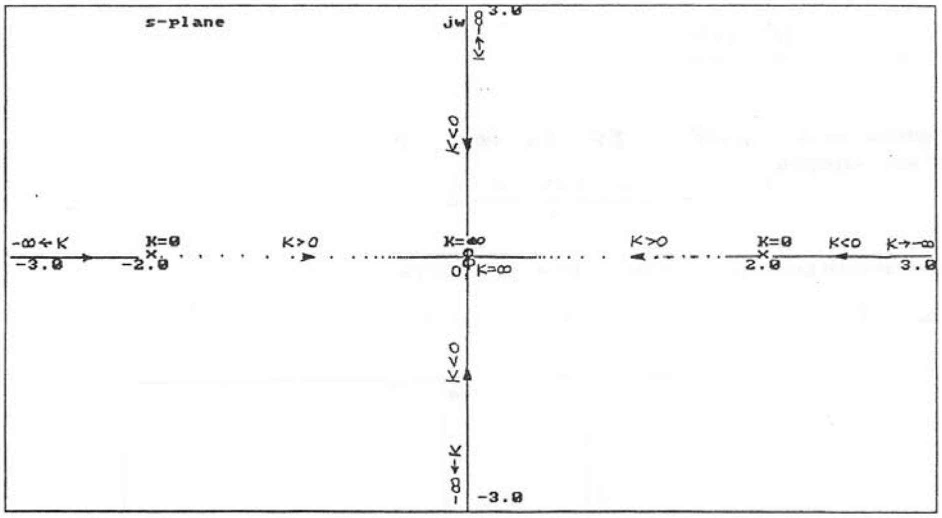
(j)

$$G(s)H(s) = \frac{Ks^2}{\left(s^2 - 4\right)}$$

Since the number of finite poles and zeros of $G(s)H(s)$ are the same, there are no asymptotes.

Breakaway-point Equation: $8s = 0$

Breakaway Points: $s = 0$



7-12 (k)

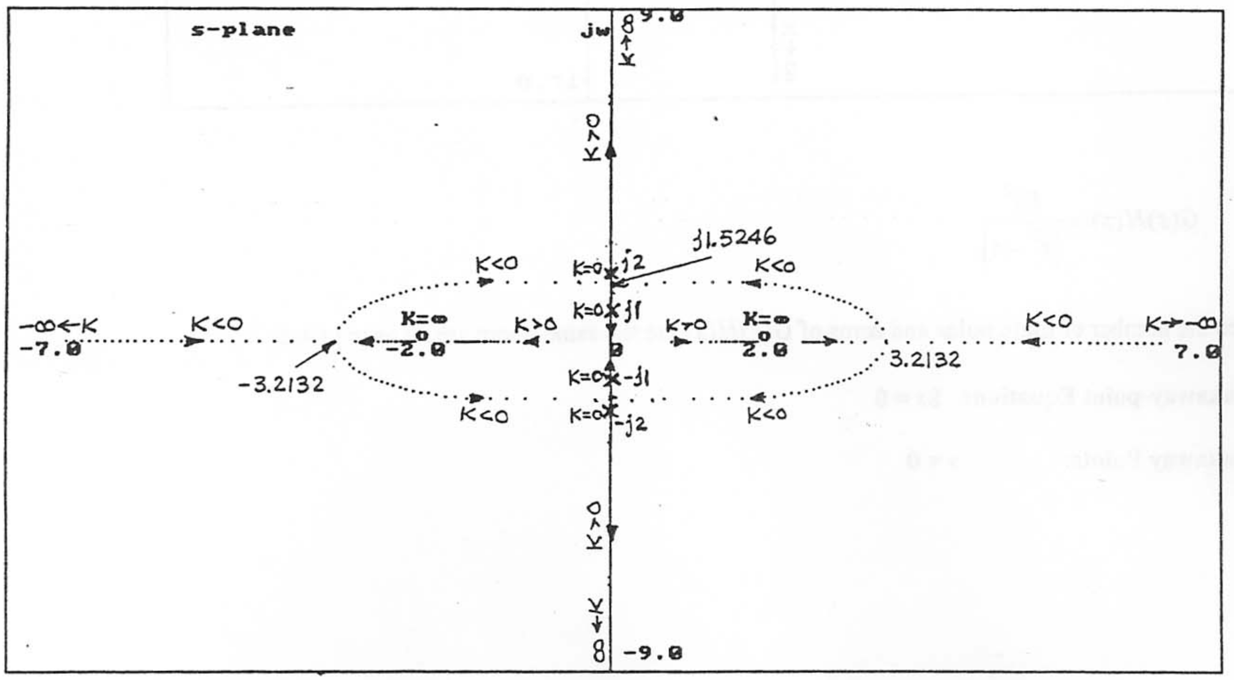
$$G(s)H(s) = \frac{K\left(s^2 - 4\right)}{\left(s^2 + 1\right)\left(s^2 + 4\right)}$$

Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes: $\sigma_1 = \dfrac{-2+2}{4-2} = 0$

Breakaway-point Equation: $s^6 - 8s^4 - 24s^2 = 0$

Breakaway Points: $0, \quad 3.2132, \quad -3.2132, \quad j1.5246, \quad -j1.5246$



7-12 (l)

$$G(s)H(s) = \frac{K(s^2 - 1)}{\left(s^2 + 1\right)\left(s^2 + 4\right)}$$

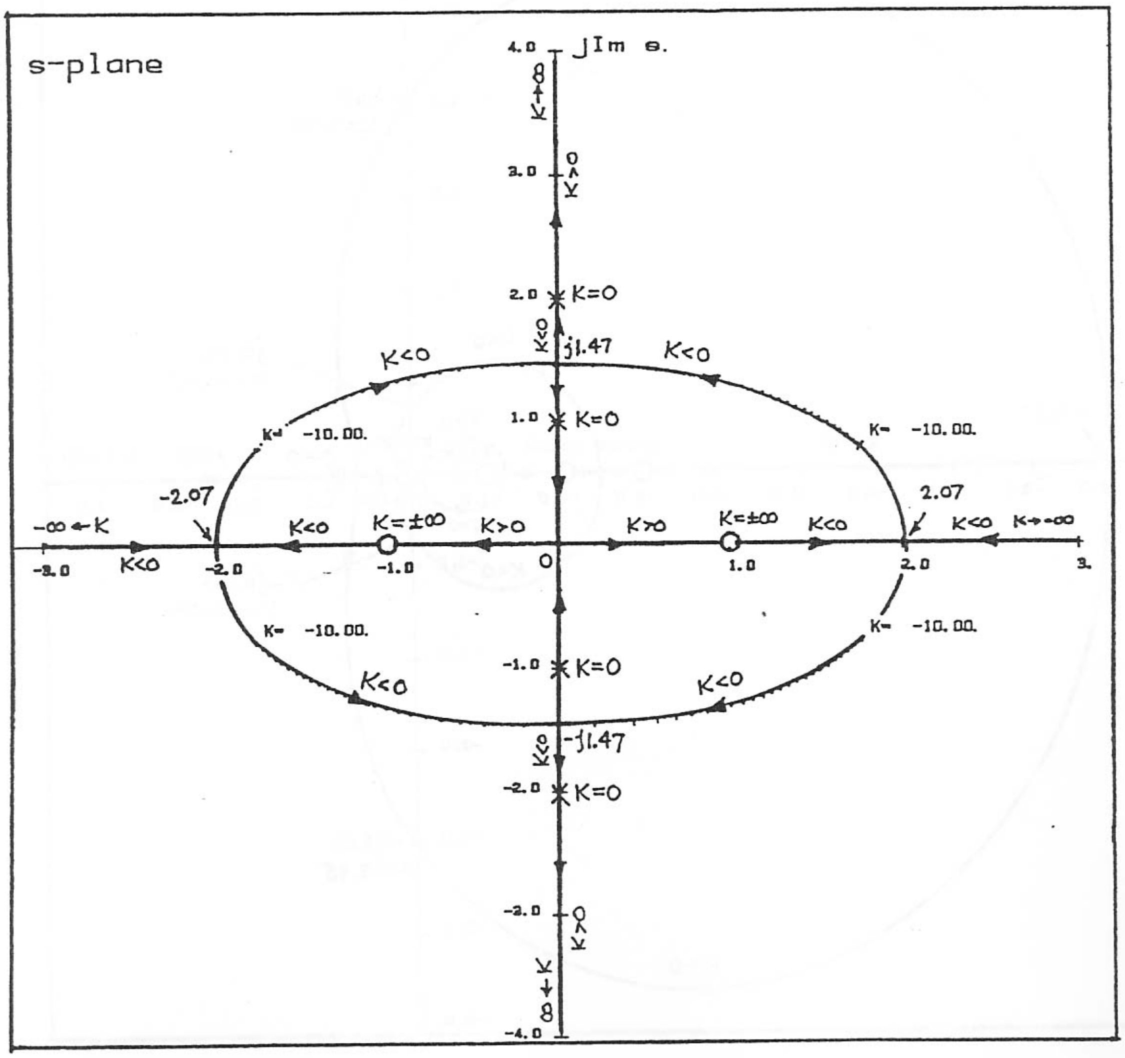
Asymptotes: *K* > 0: $90^o, \quad 270^o$ **K < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1 + 1}{4 - 2} = 0$$

Breakaway-point Equation: $s^5 - 2s^3 - 9s = 0$

Breakaway Points: $-2.07, \quad 2.07, \quad -j1.47, \quad j1.47$



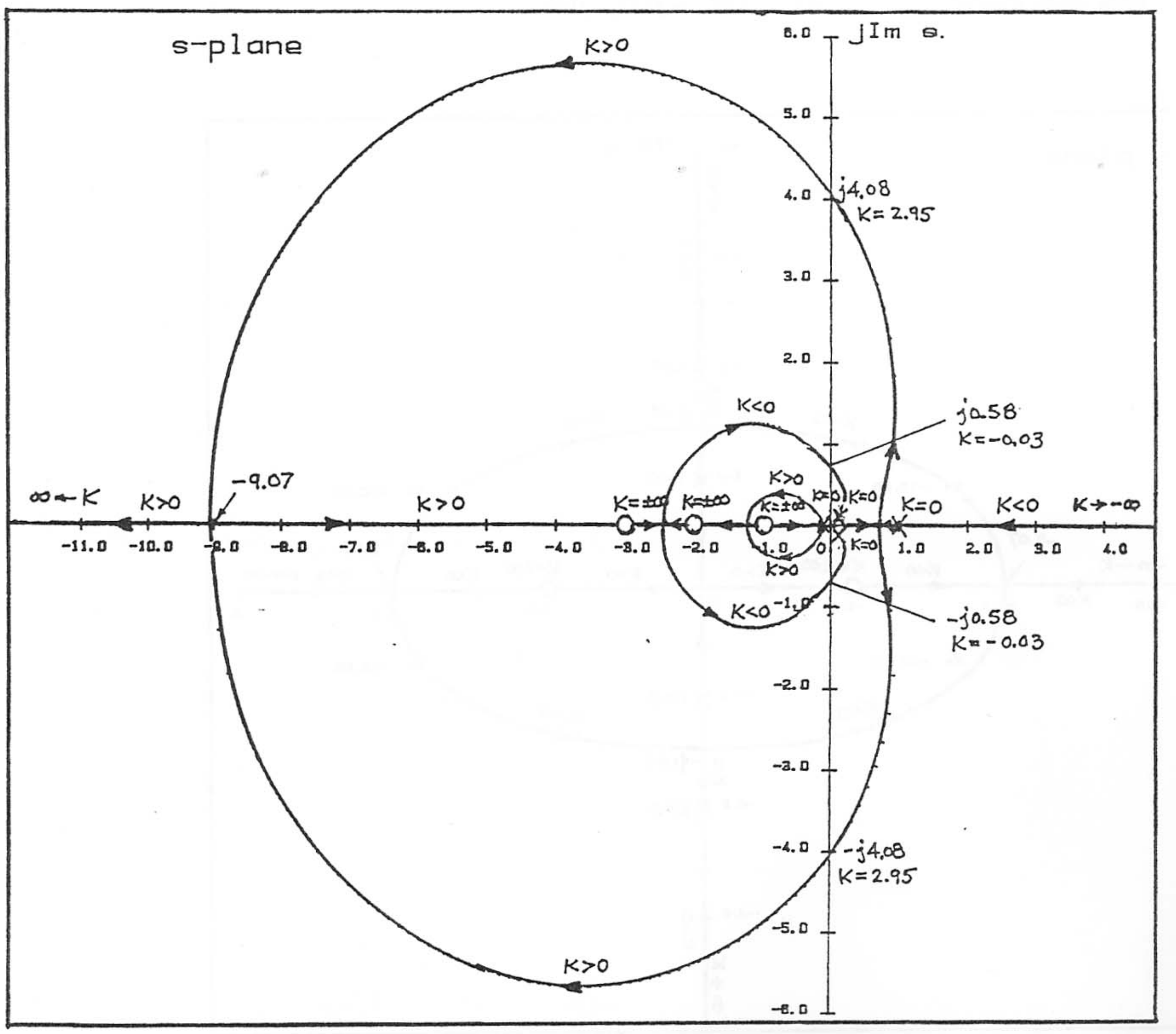
(m)

$$G(s)H(s) = \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)}$$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $s^6 + 12s^5 + 27s^4 + 2s^3 - 18s^2 = 0$

Breakaway Points: –1.21, –2.4, –9.07, 0.683, 0, 0



(n)

$$G(s)H(s) = \frac{K(s+5)(s+40)}{s^3(s+250)(s+1000)}$$

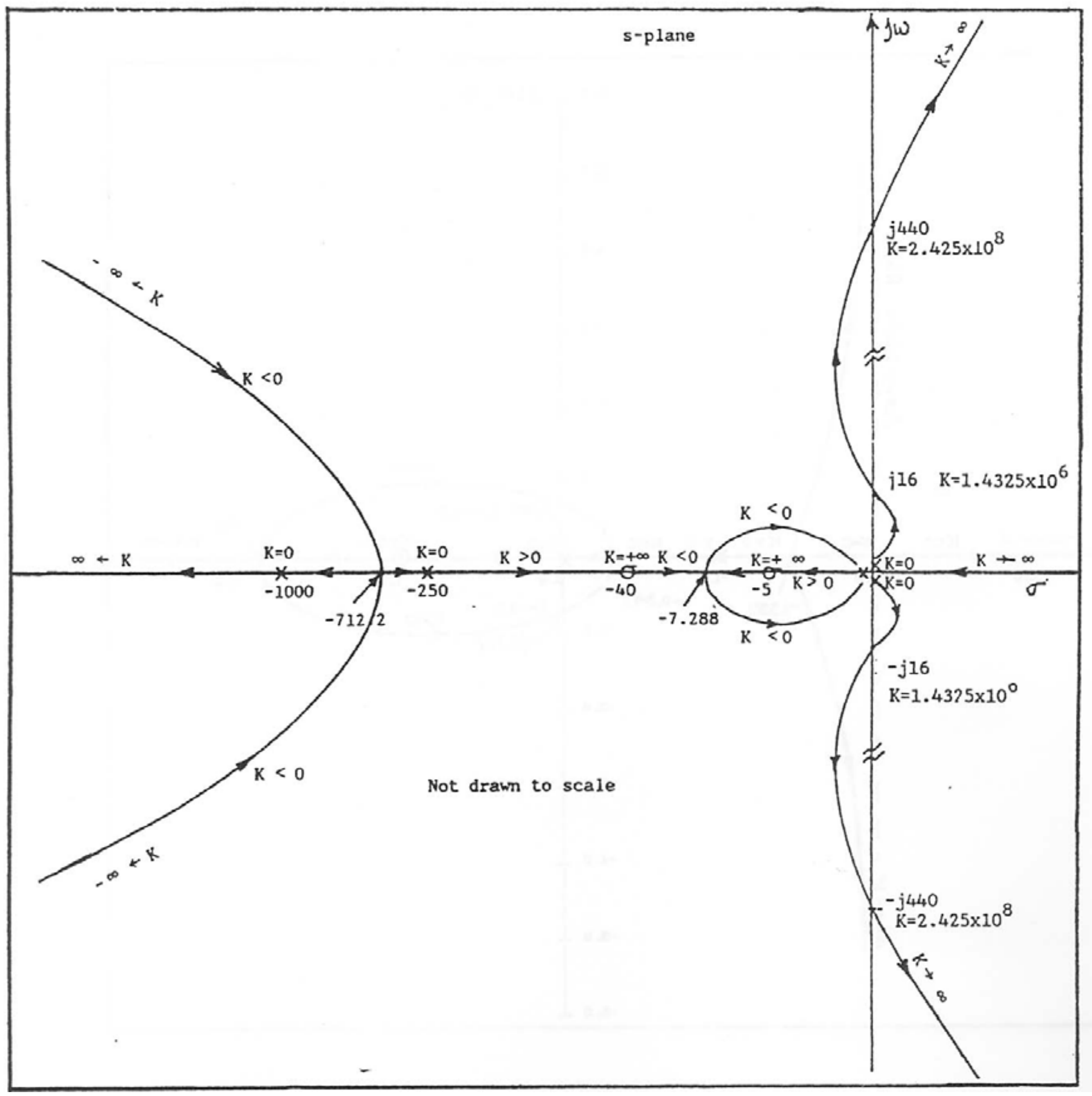
Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of asymptotes:

$$\sigma_1 = \frac{0+0+0-250-1000-(-5)-(-40)}{5-2} = -401.67$$

Breakaway-point Equation: $3750s^6 + 335000s^5 + 5.247\times10^8 s^4 + 2.9375\times10^{10} s^3 + 1.875\times10^{11} s^2 = 0$

Breakaway Points: –7.288, –712.2, 0, 0



7-12 (o)

$$G(s)H(s) = \frac{K(s-1)}{s(s+1)(s+2)}$$

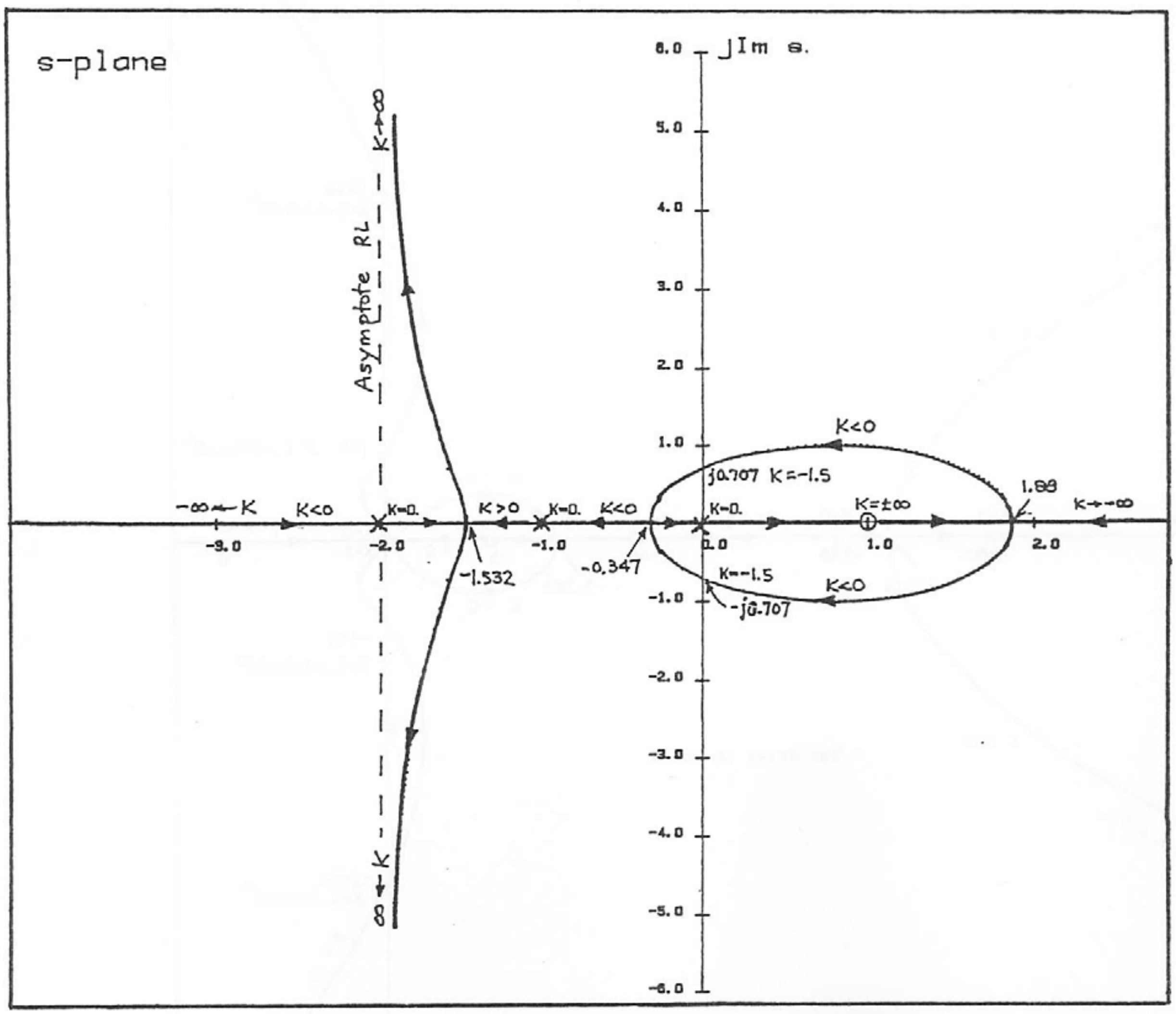
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-2-1}{3-1} = -2$$

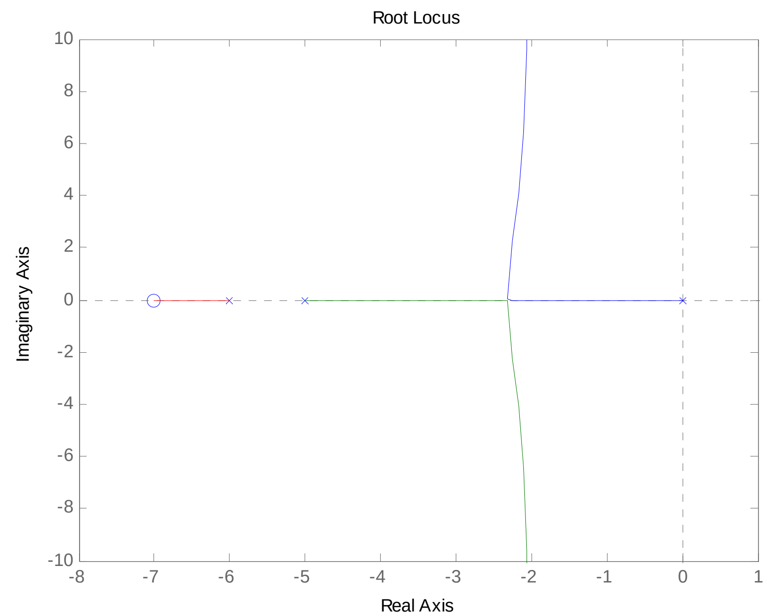
Breakaway-point Equation: $s^3 - 3s - 1 = 0$

Breakaway Points; –0.3473, –1.532, 1.879



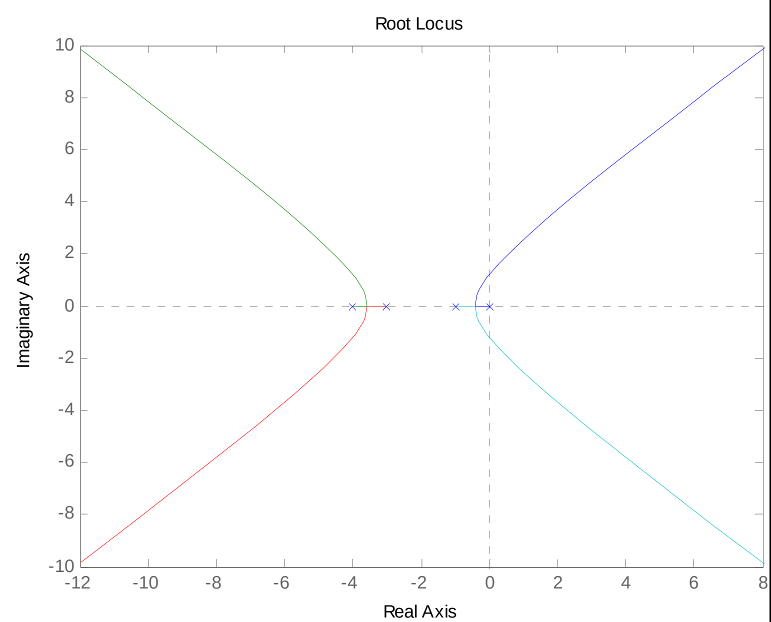
7-13(a) MATLAB code:

```
num=[1 7];
den=conv([1 0],[1 5]);
den=conv(den,[1 6]);
mysys=tf(num,den)
rlocus(mysys);
```



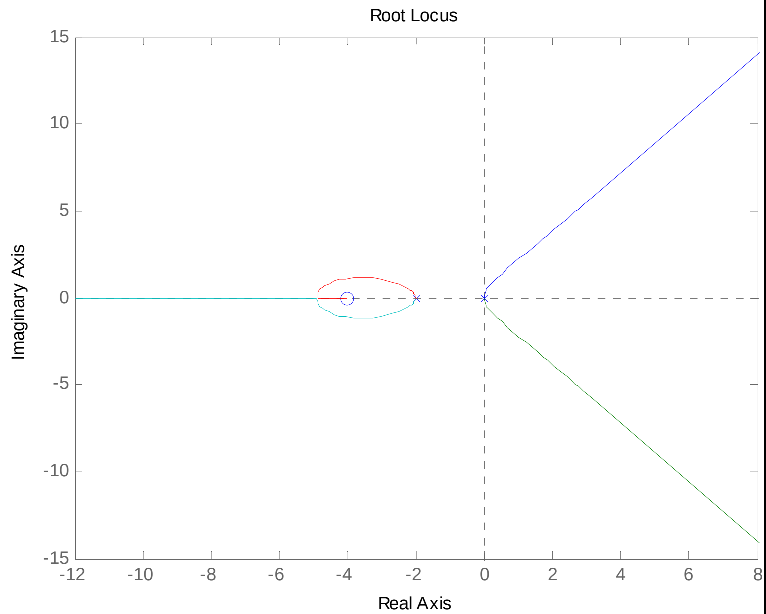
7-13(b) MATLAB code:

```
num=[0 1];
den=conv([1 0],[1 1]);
den=conv(den,[1 3]);
den=conv(den,[1 4]);
mysys=tf(num,den)
rlocus(mysys);
```



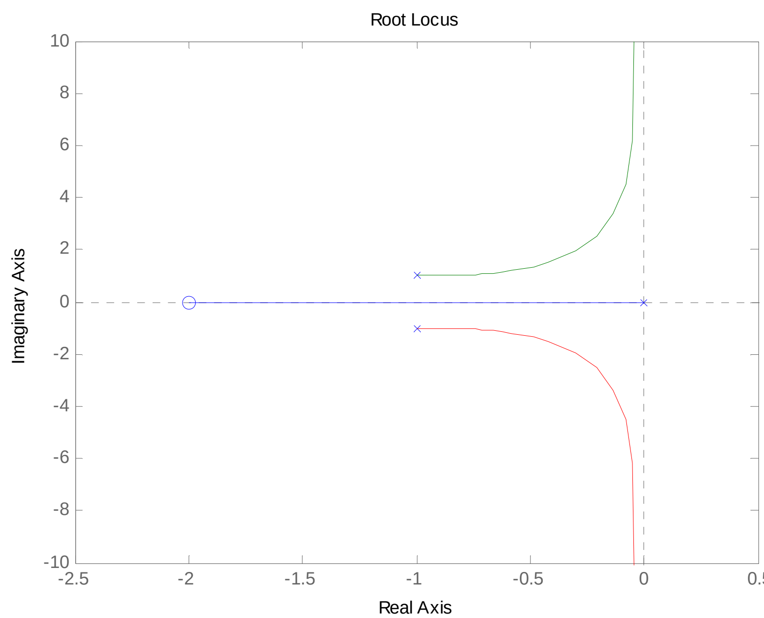
7-13(c) MATLAB code:

```
num=[1 4];
den=conv([1 0],[1 0]);
den=conv(den,[1 2]);
den=conv(den,[1 2]);
mysys=tf(num,den)
rlocus(mysys);
```



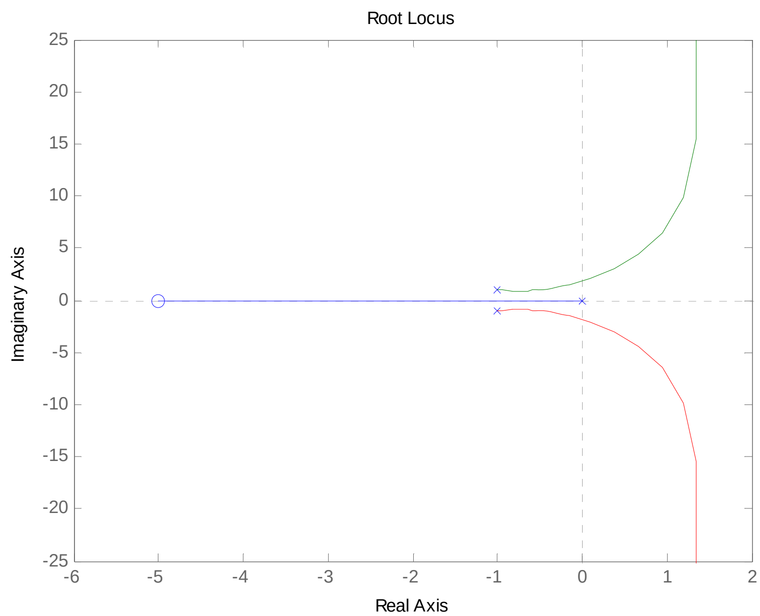
7-13(d) MATLAB code:

```
num=[1 2];
den=conv([1 0],[1
(1+j)]);
den=conv(den,[1 (1-
j)]);
mysys=tf(num,den)
rlocus(mysys);
```



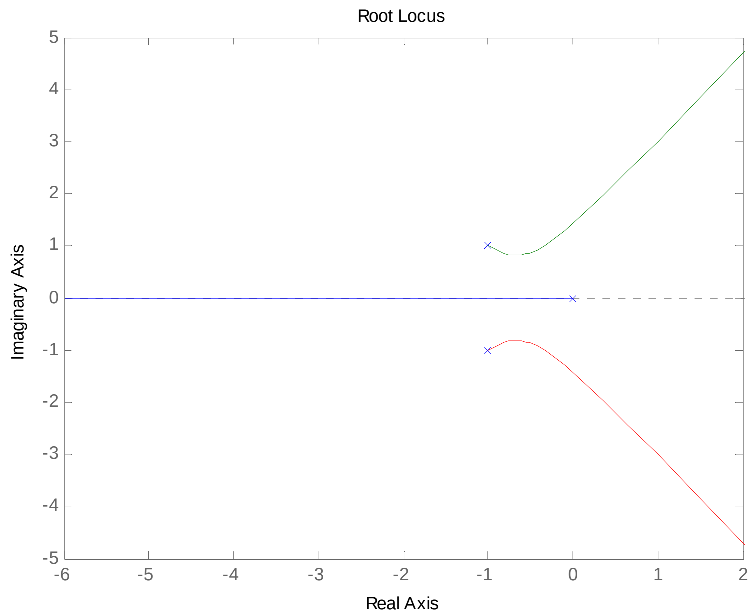
7-13(e) MATLAB code:

```
num=[1 5];
den=conv([1 0],[1
(1+j)]);
den=conv(den,[1 (1-
j)]);
mysys=tf(num,den)
rlocus(mysys);
```



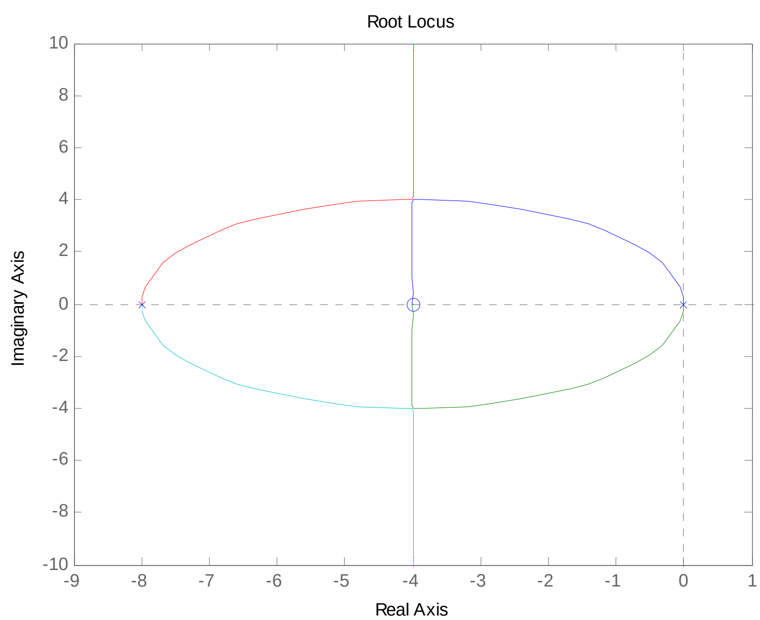
7-13(f) MATLAB code:

```
num=conv([1 4],[1 4]);
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```



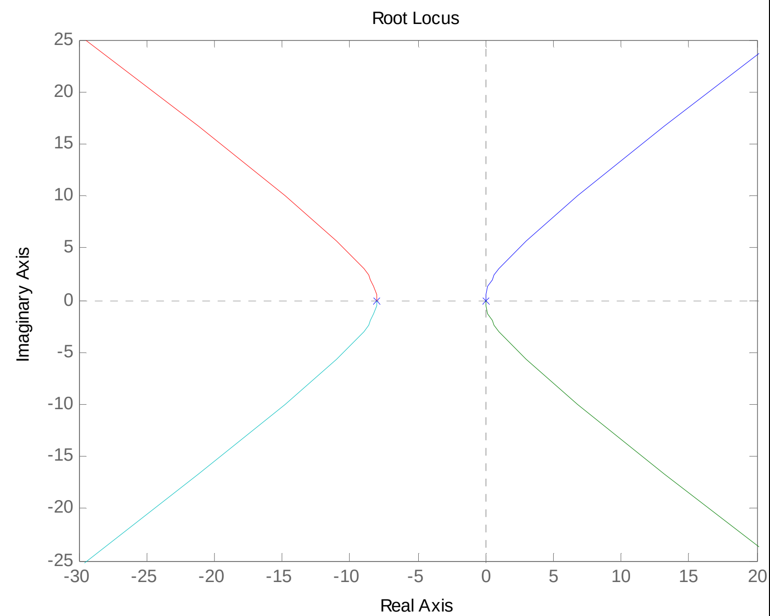
7-13(g) MATLAB code:

```
num=conv([1 4],[1 4]);
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```



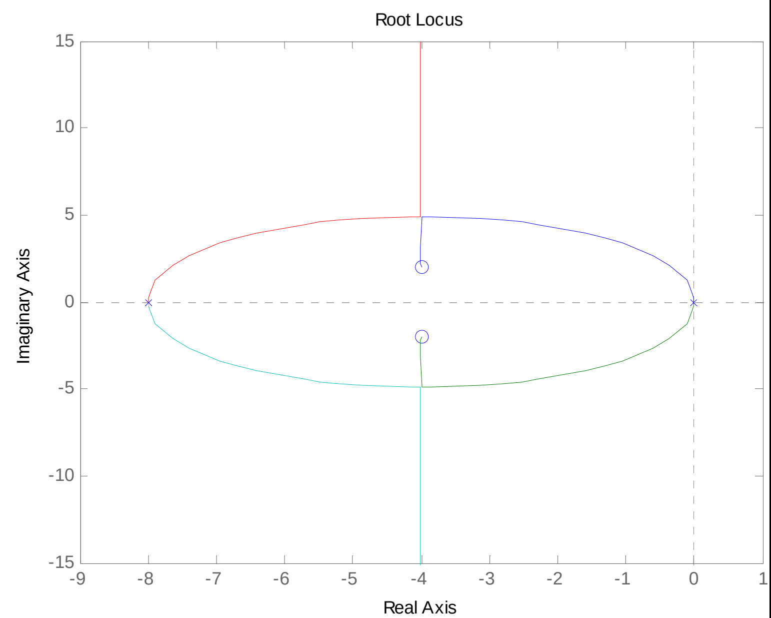
7-13(h) MATLAB code:

```
num=[0 1];
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```



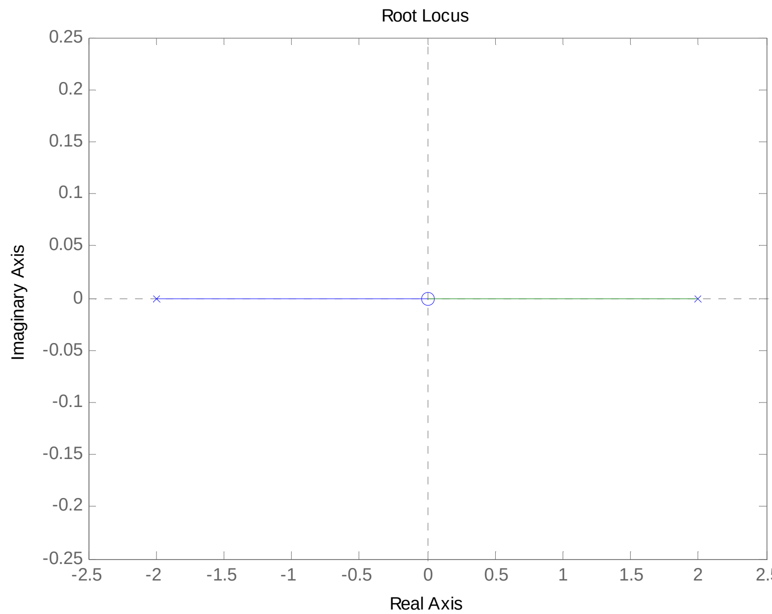
7-13(i) MATLAB code:

```
num=conv([1 4-2j],[1
4+2j])
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```



7-13(j) MATLAB code:

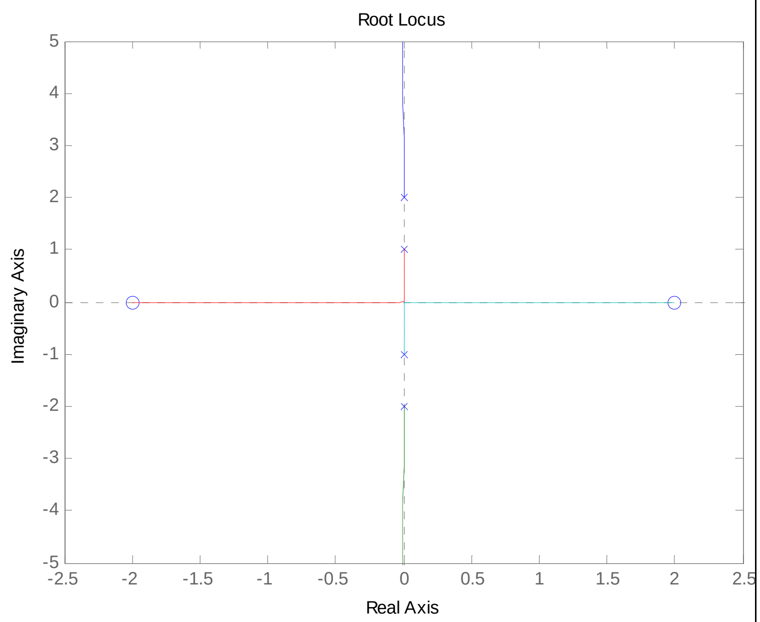
```
num=conv([1 0],[1 0]);
den=conv([1 2],[1 -2]);
mysys=tf(num,den)
rlocus(mysys);
```



7-13(k) MATLAB code:

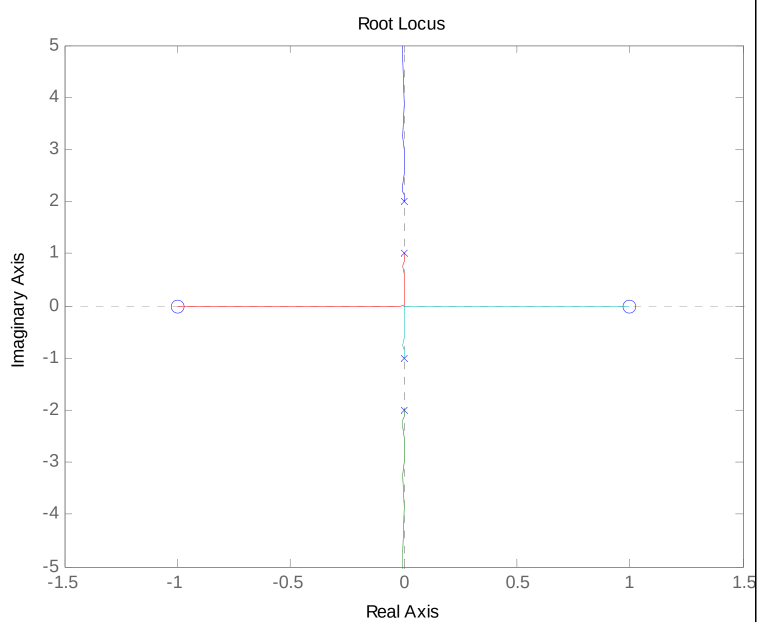
```
num=conv([1 2],[1 -2]);
den=conv([1 -j],[1 j]);
den=conv(den,[1 -2j]);
den=conv(den,[1 2j]);
mysys=tf(num,den)
```

```
rlocus(mysys);
```



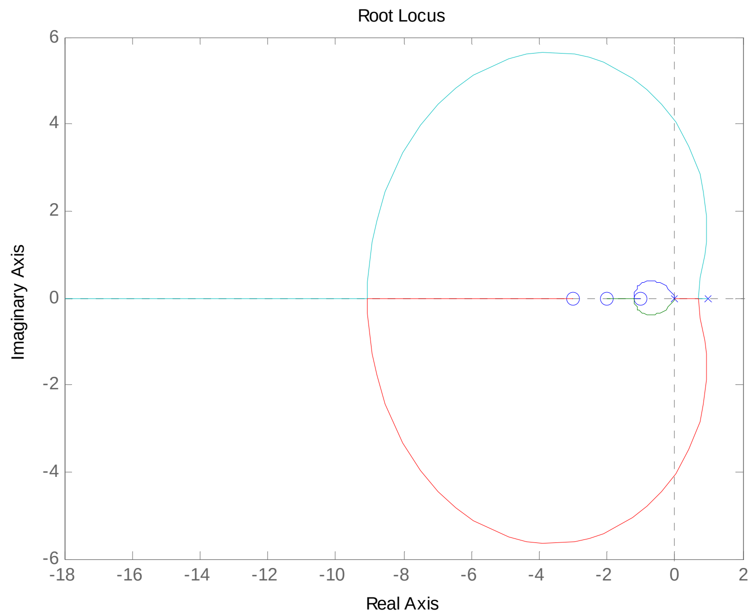
7-13(l) MATLAB code:

```
num=conv([1 1],[1 -1]);
den=conv([1 -j],[1 j]);
den=conv(den,[1 -2j]);
den=conv(den,[1 2j]);
mysys=tf(num,den)
rlocus(mysys);
```



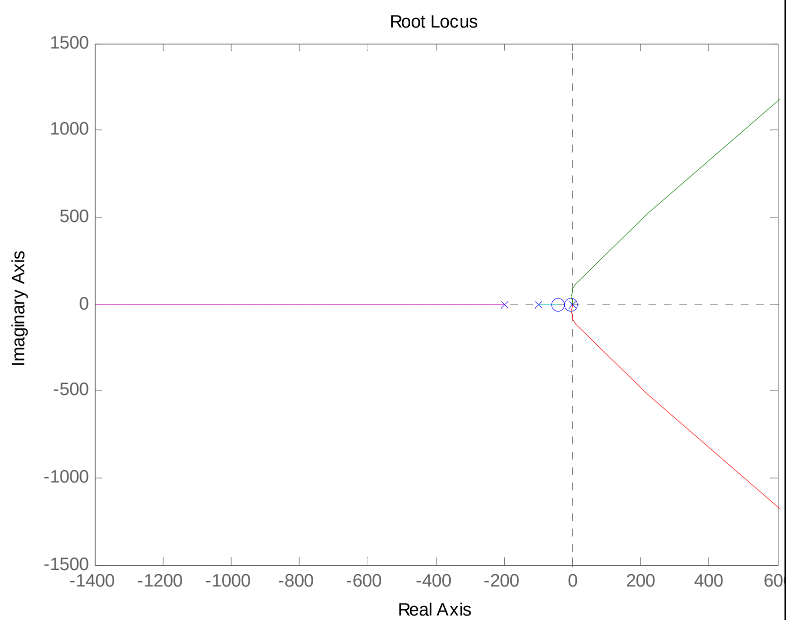
7-13(m) MATLAB code:

```
num=conv([1 1],[1 2]);
num=conv(num,[1 3]);
den=conv([1 0],[1 0]);
den=conv(den,[1 0]);
den=conv(den,[1 -1]);
mysys=tf(num,den)
rlocus(mysys);
```



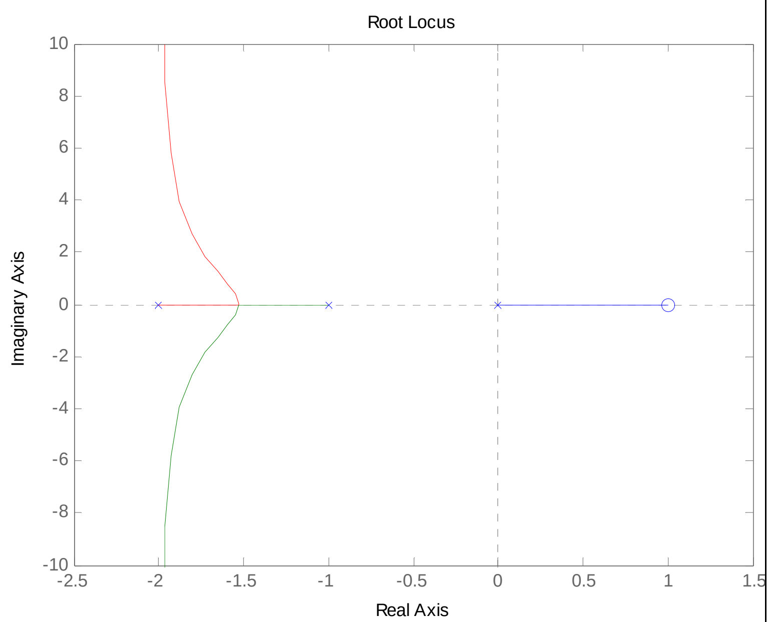
7-13(n) MATLAB code:

```
num=conv([1 5],[1 40]);
den=conv([1 0],[1 0]);
den=conv(den,[1 0]);
den=conv(den,[1 100]);
den=conv(den,[1 200]);
mysys=tf(num,den)
rlocus(mysys);
```



7-13(o) MATLAB code:

```
num=conv([1 5],[1 40]);
den=conv([1 0],[1 0]);
den=conv(den,[1 0]);
den=conv(den,[1 100]);
den=conv(den,[1 200]);
mysys=tf(num,den)
rlocus(mysys);
```



7-14) (a) $Q(s) = s+5 \qquad P(s) = s\left(s^2+3s+2\right) = s(s+1)(s+2)$

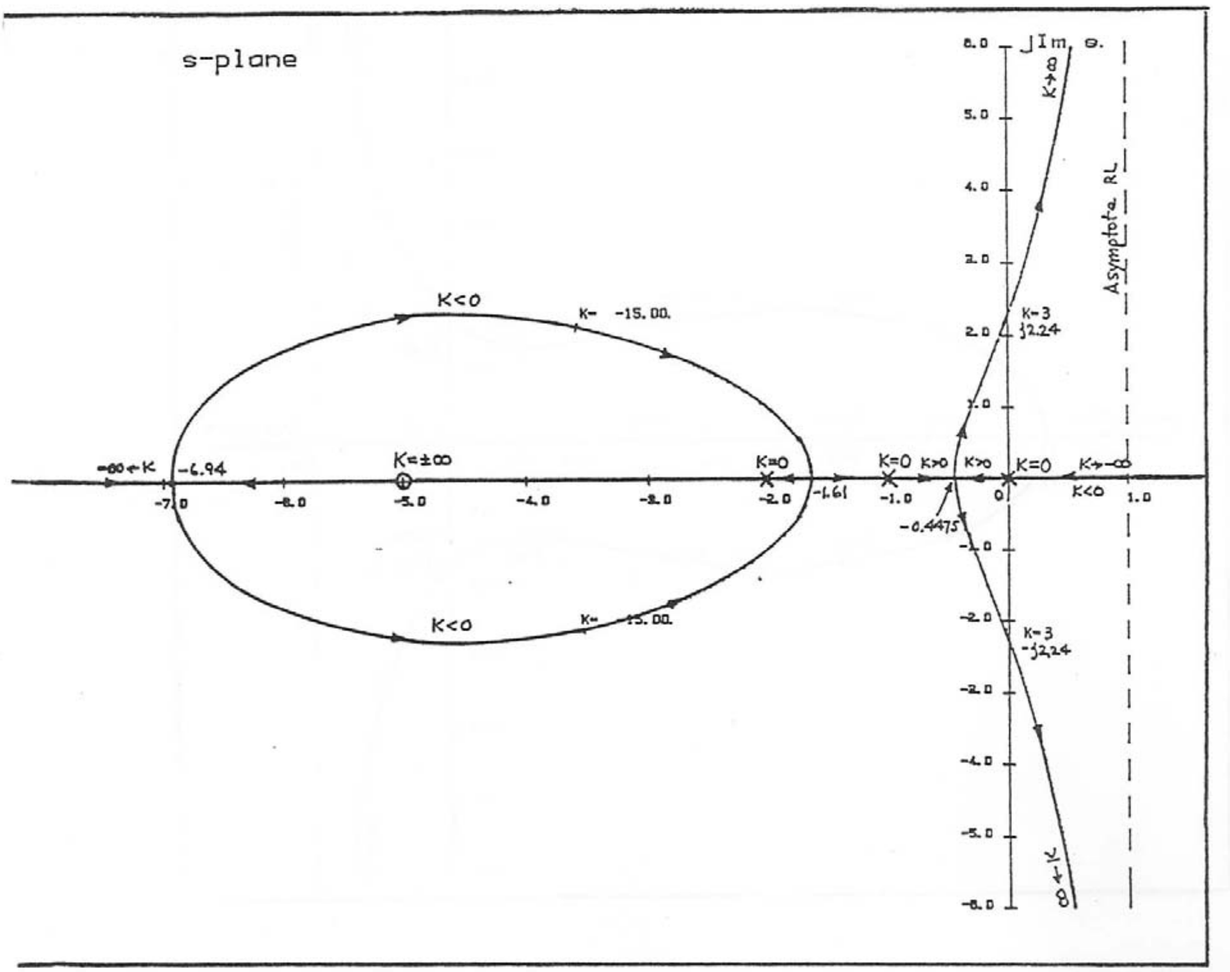
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-2-(-5)}{3-1} = 1$$

Breakaway-point Equation: $s^3 + 9s^2 + 15s + 5 = 0$

Breakaway Points: $-0.4475, \quad -1.609, \quad -6.9434$



7-14 (b) $Q(s) = s + 3 \qquad P(s) = s(s^2 + s + 2)$

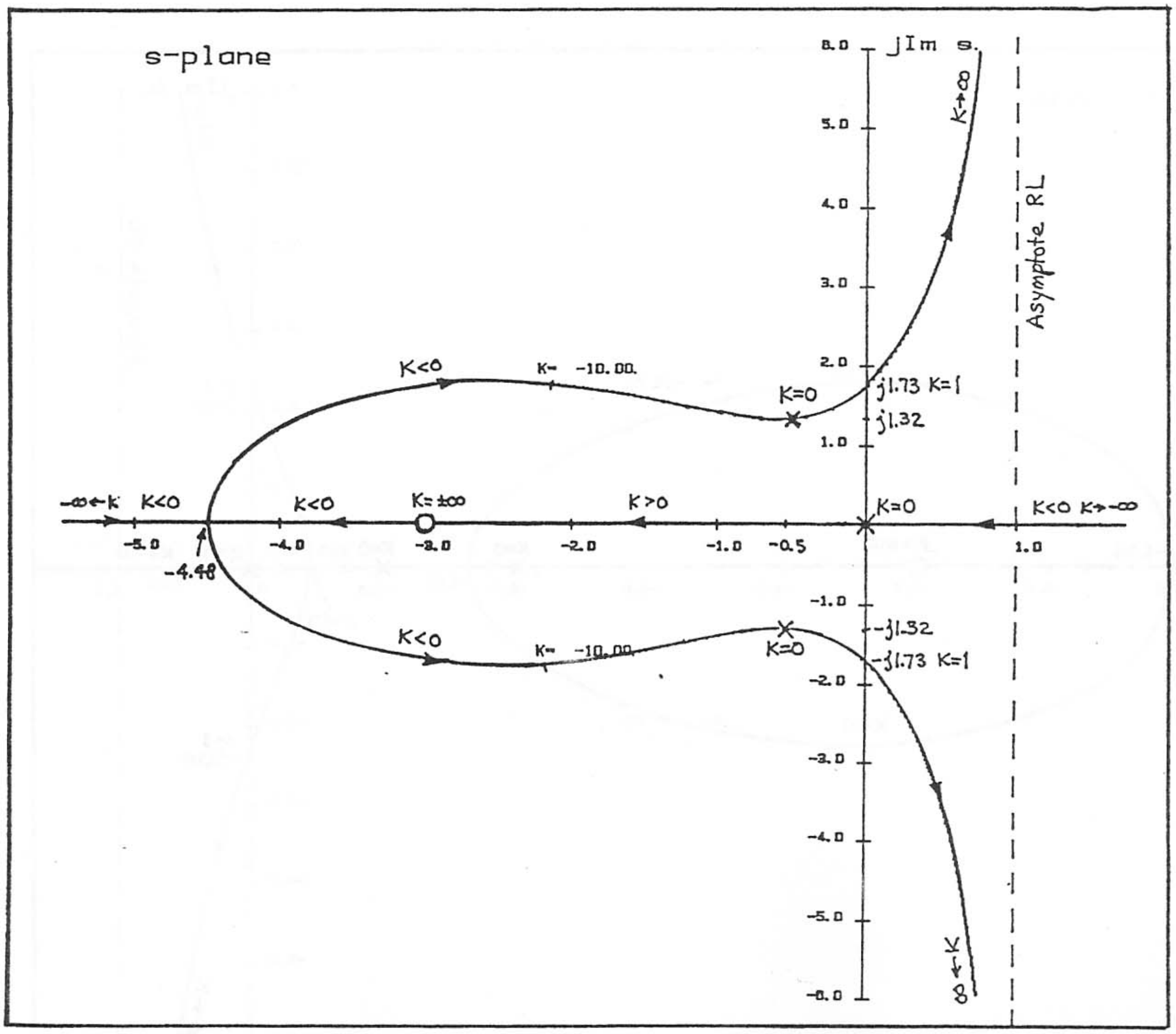
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-(-3)}{3-1} = 1$$

Breakaway-point Equation: $s^3 + 5s^2 + 3s + 3 = 0$

Breakaway Points: –4.4798 The other solutions are not breakaway points.

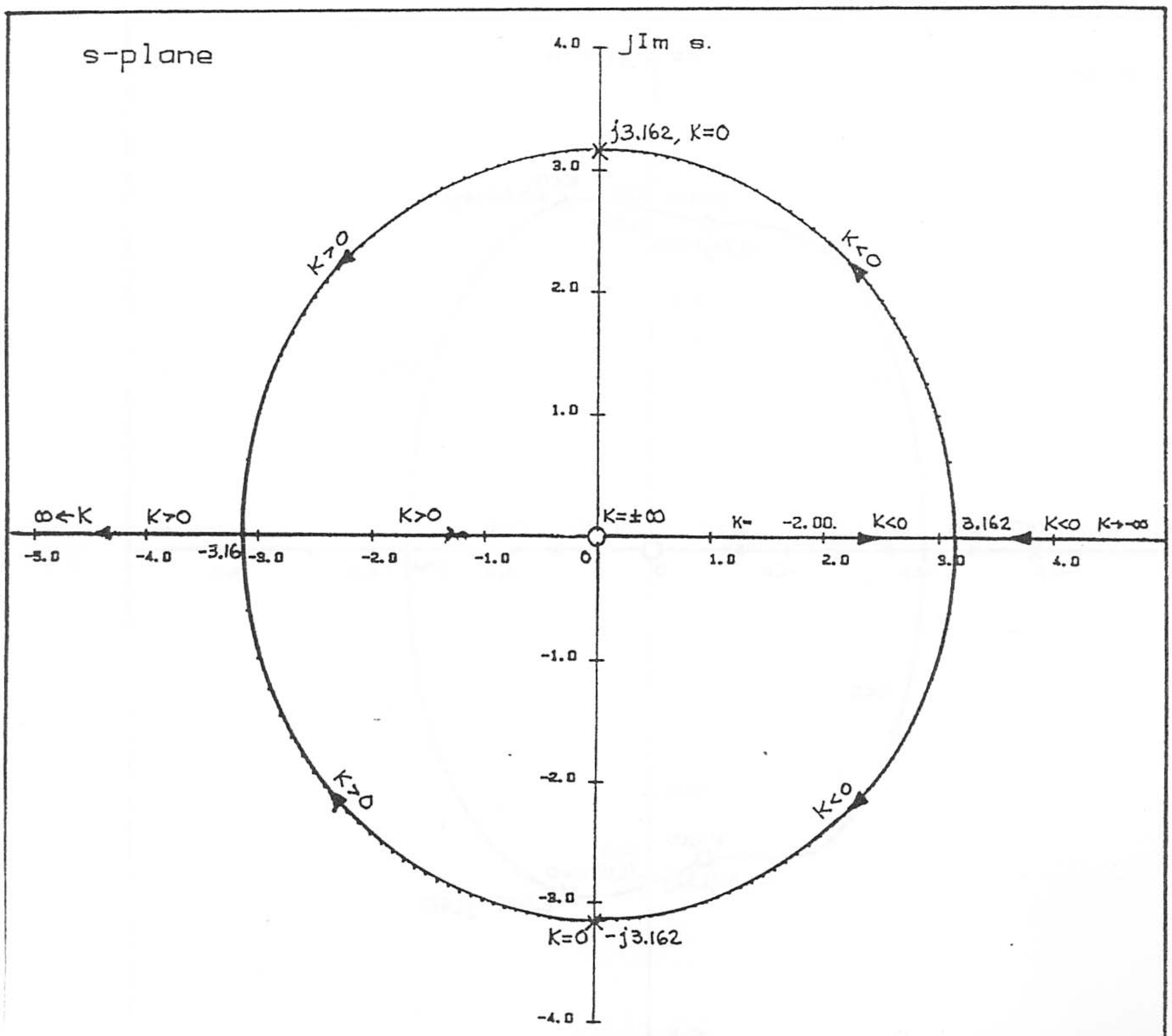


7-14 (c) $Q(s) = 5s \qquad P(s) = s^2 + 10$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $5s^2 - 50 = 0$

Breakaway Points: –3.162, 3.162

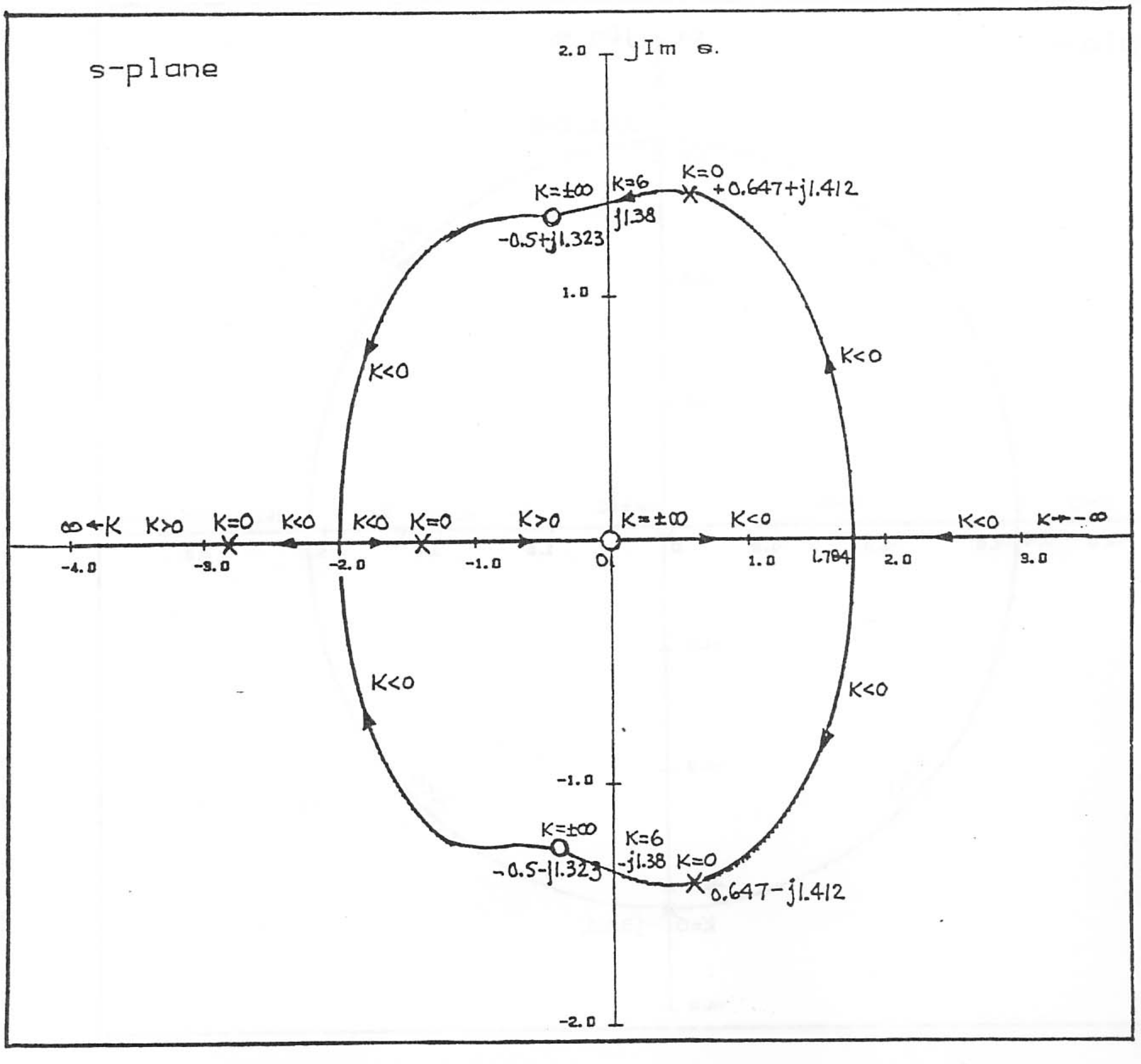


7-14 (d) $Q(s) = s\left(s^2 + s + 2\right)$ $P(s) = s^4 + 3s^3 + s^2 + 5s + 10$

Asymptotes: *K* > 0: 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $s^6 + 2s^5 + 8s^4 + 2s^3 - 33s^2 - 20s - 20 = 0$

Breakaway Points: –2, 1.784. The other solutions are not breakaway points.

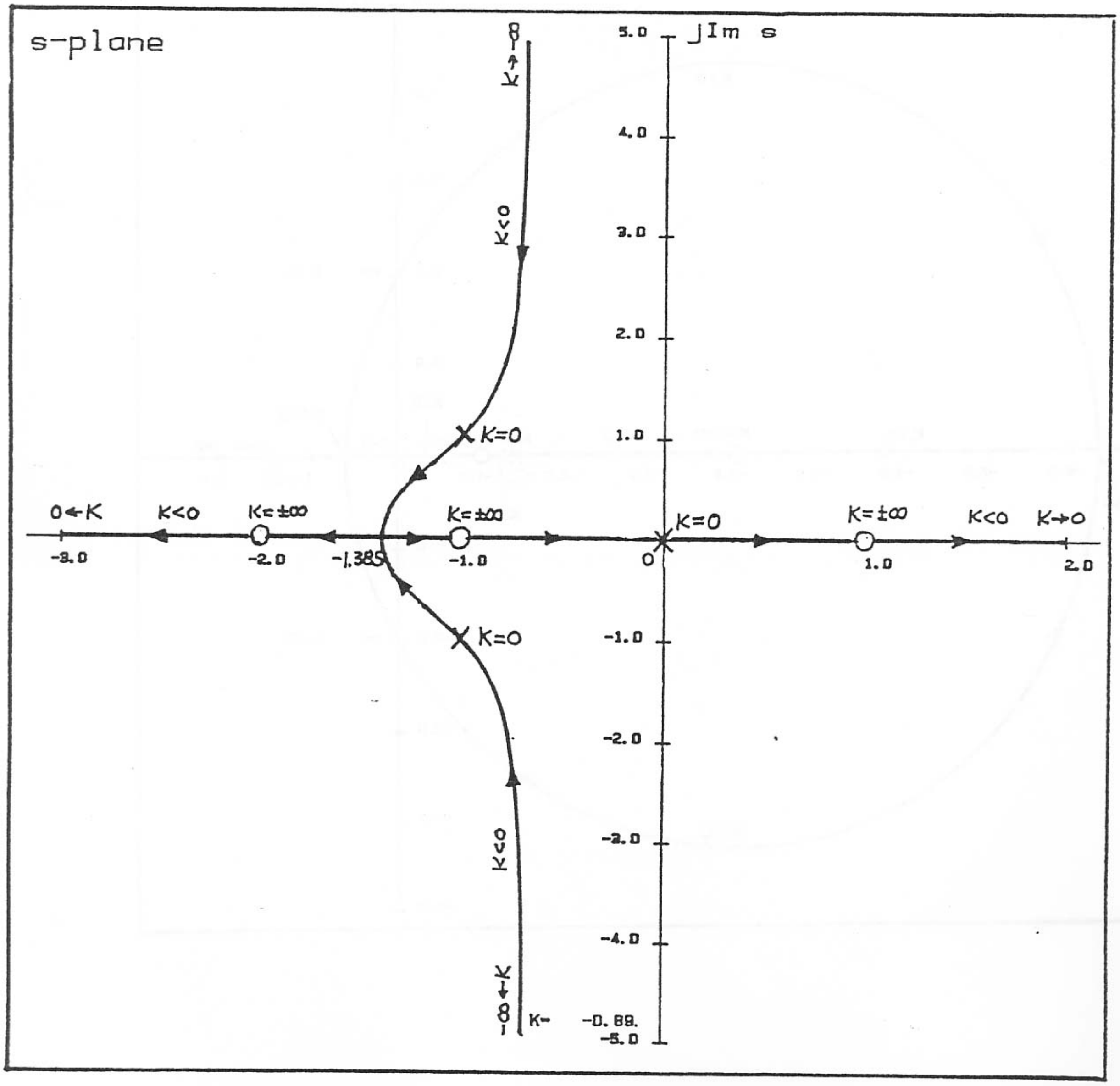


7-14 (e) $Q(s) = \left(s^2 - 1\right)\left(s + 2\right) \qquad P(s) = s\left(s^2 + 2s + 2\right)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway-point Equation: $6s^3 + 12s^2 + 8s + 4 = 0$

Breakaway Points: –1.3848

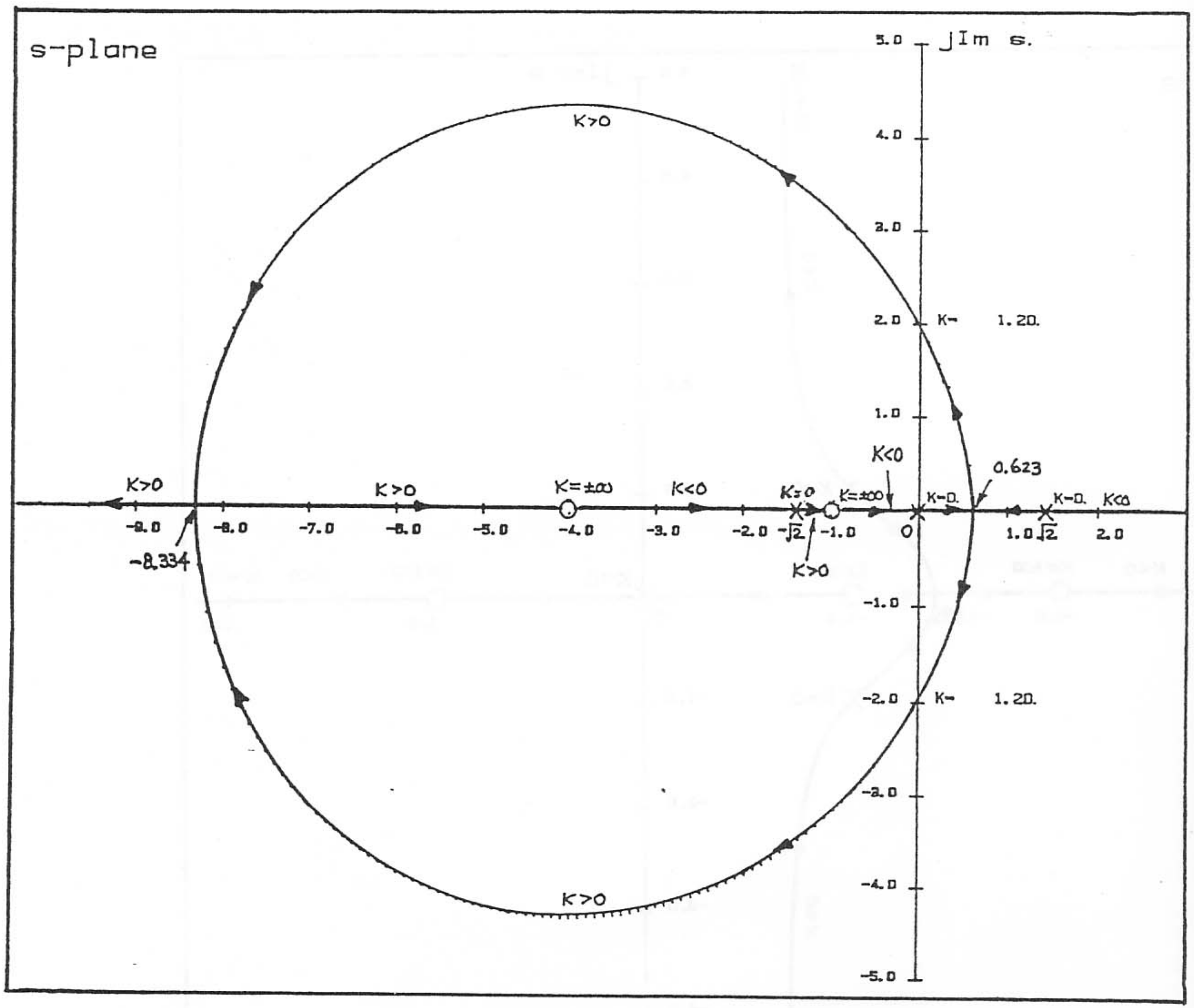


7-14 (f) $Q(s) = (s+1)(s+4) \qquad P(s) = s\left(s^2 - 2\right)$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point equations: $s^4 + 10s^3 + 14s^2 - 8 = 0$

Breakaway Points: –8.334, 0.623



7-14 (g) $Q(s) = s^2 + 4s + 5 \qquad P(s) = s^2\left(s^2 + 8s + 16\right)$

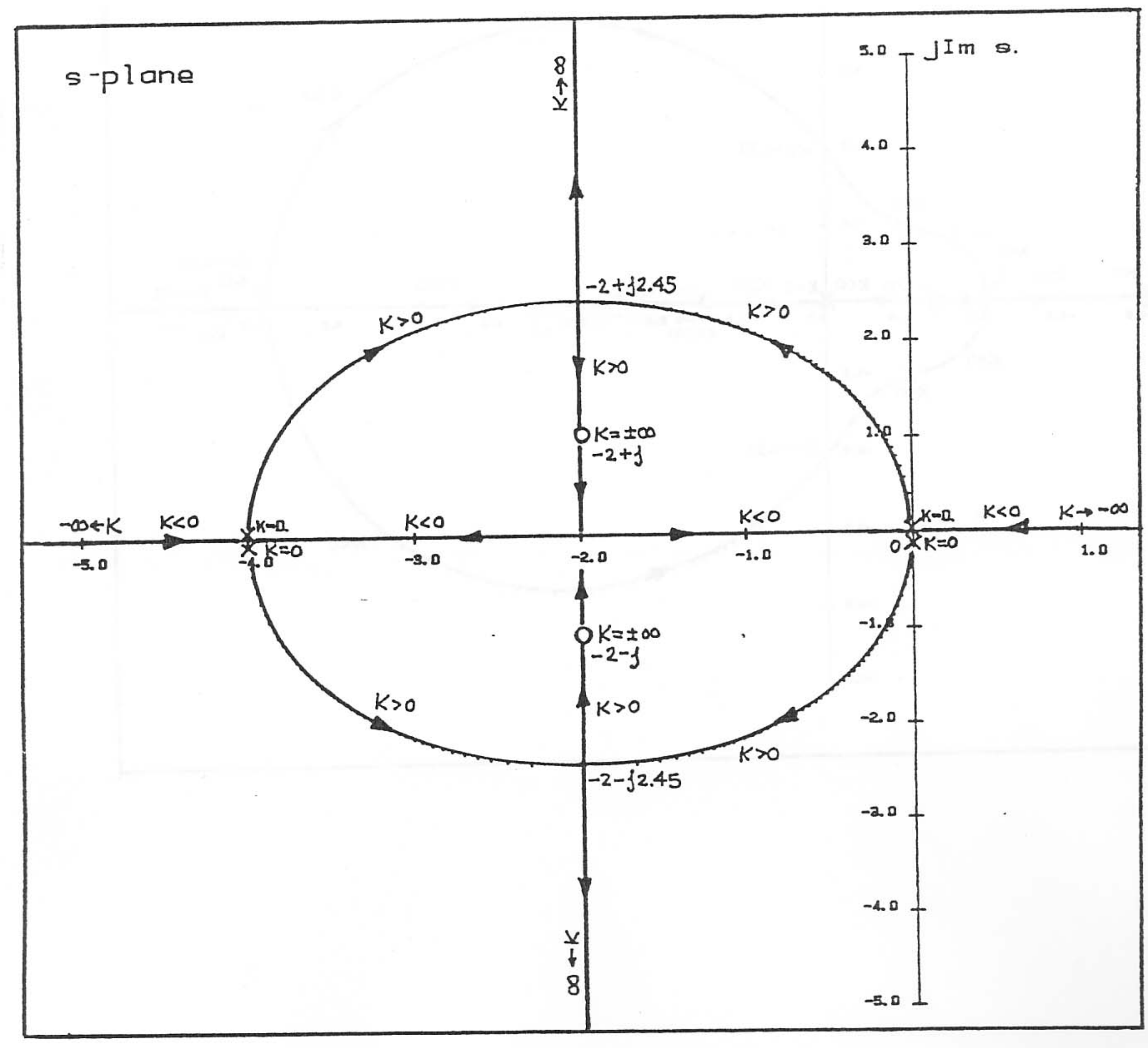
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8-(-4)}{4-2} = -2$$

Breakaway-point Equation: $s^5 + 10s^4 + 42s^3 + 92s^2 + 80s = 0$

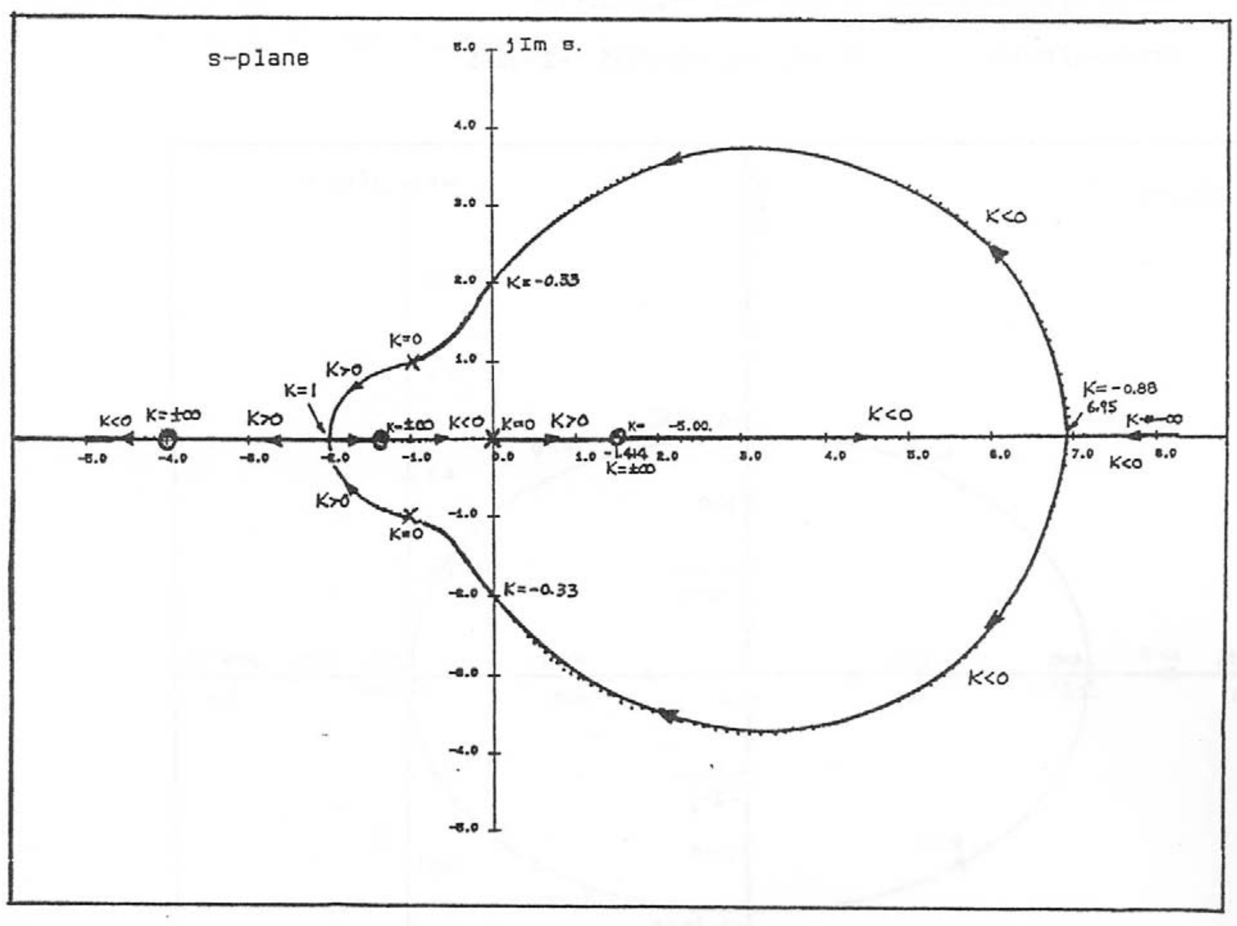
Breakaway Points: $0, \quad -2, \quad -4, \quad -2 + j2.45, \quad -2 - j2.45$



7-14 (h) $Q(s) = \left(s^2 - 2\right)\left(s + 4\right) \qquad P(s) = s\left(s^2 + 2s + 2\right)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway Points: –2, 6.95

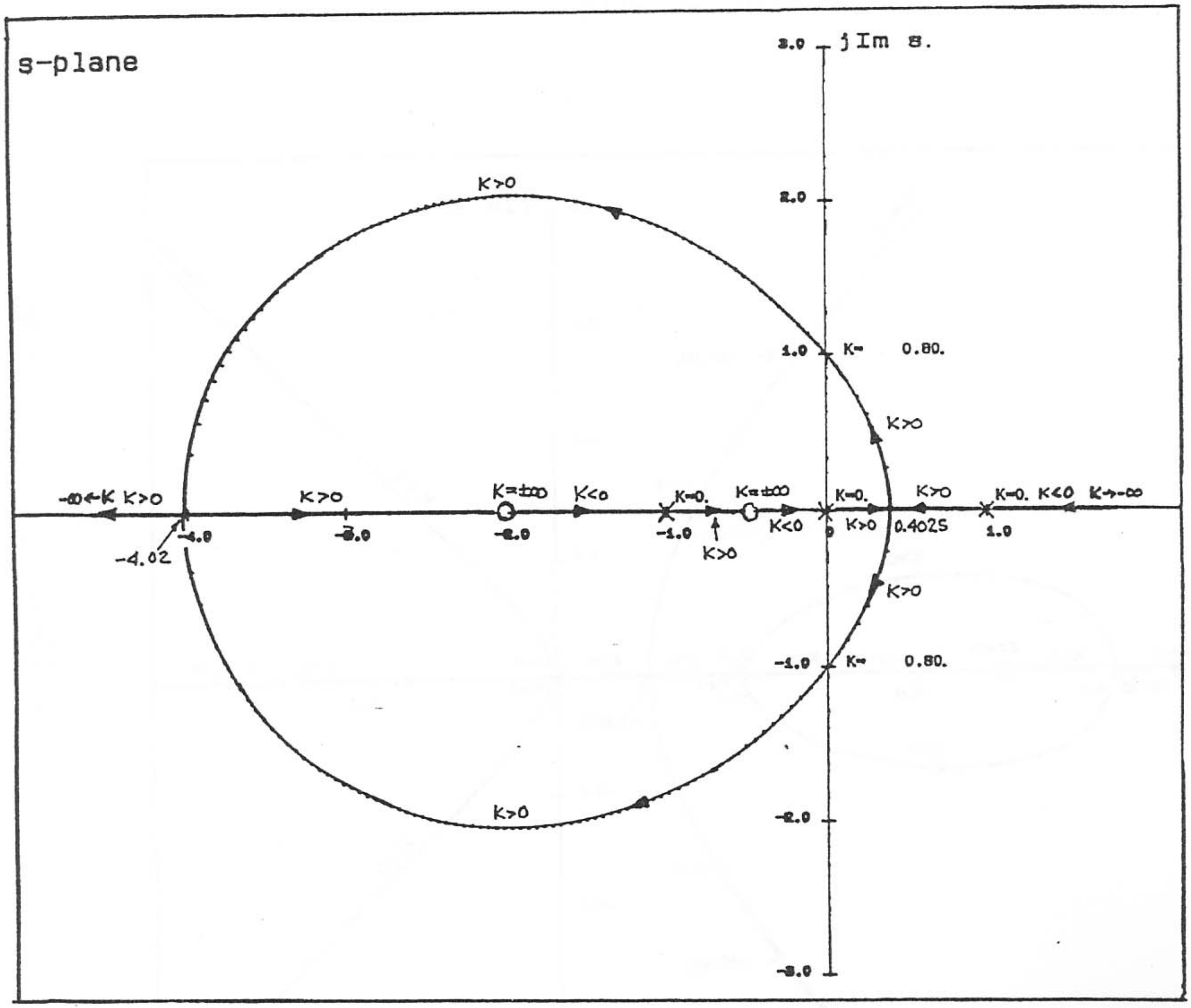


7-14 (i) $Q(s) = (s+2)(s+0.5) \qquad P(s) = s(s^2 - 1)$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $s^4 + 5s^3 + 4s^2 - 1 = 0$

Breakaway Points: –4.0205, 0.40245 The other solutions are not breakaway points.


s-plane
jIm s.
3.0
2.0
1.0
K= 0.80.
-1.0
K= 0.80.
-2.0
-3.0
K>0
K>0
K>0
K>0
-∞←K K>0
K>0
K=±∞
K<0
K=0.
K=±∞
K=0.
K>0
K=0. K<0 K→-∞
-4.02
-4.0
-3.0
-2.0
-1.0
K>0
K<0
0
K>0
0.4025
1.0

7-14 (j) $Q(s) = 2s + 5 \qquad P(s) = s^2\left(s^2 + 2s + 1\right) = s^2\left(s + 1\right)^2$

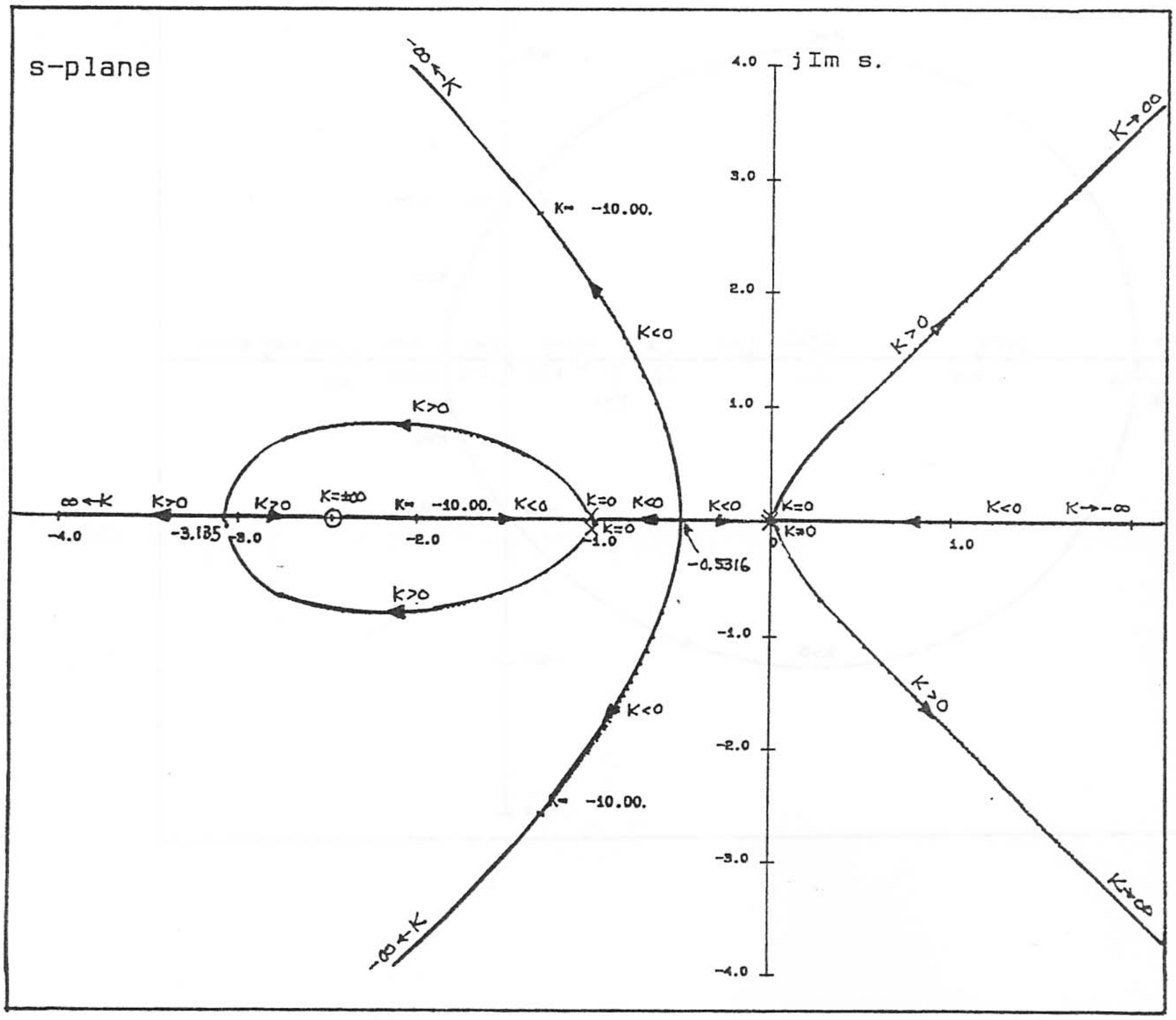
Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of Asymptotes;

$$\sigma_1 = \frac{0 + 0 - 1 - 1 - (-2.5)}{4 - 1} = \frac{0.5}{3} = 0.167$$

Breakaway-point Equation: $6s^4 + 28s^3 + 32s^2 + 10s = 0$

Breakaway Points: 0, –0.5316, –1, –3.135



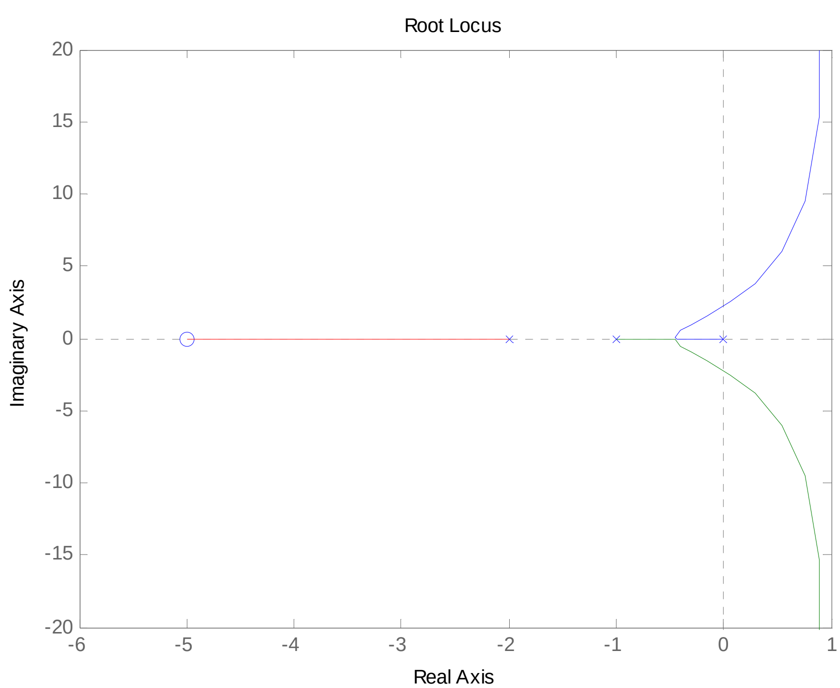
7-15) MATLAB code:

```
clear all;
close all;
s = tf('s')

%a)
num_GH_a=(s+5);
den_GH_a=(s^3+3*s^2+2*s);
GH_a=num_GH_a/den_GH_a;
figure(1);
rlocus(GH_a)
```

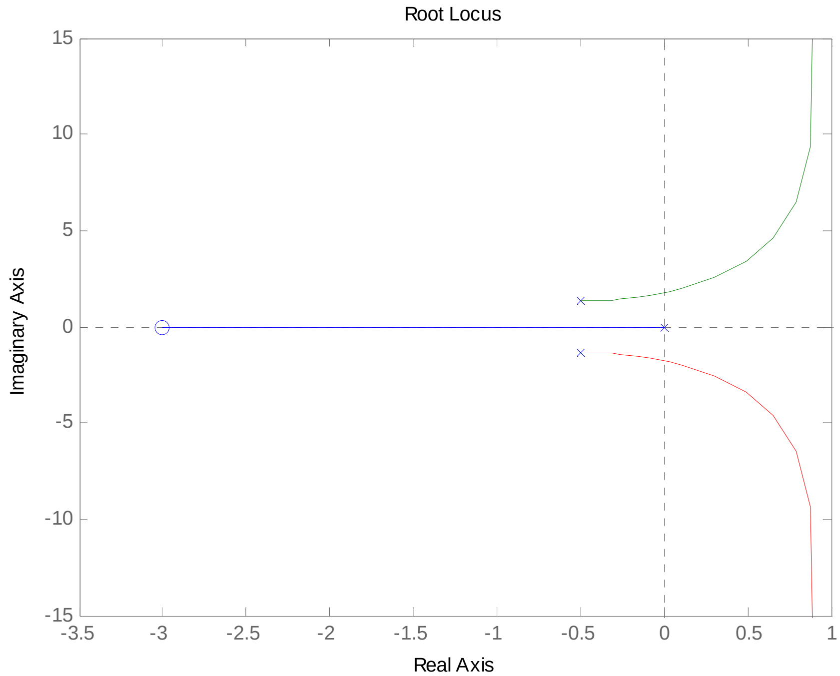
```
%b)
num_GH_b=(s+3);
den_GH_b=(s^3+s^2+2*s);
GH_b=num_GH_b/den_GH_b;
figure(2);
rlocus(GH_b)

%c)
num_GH_c= 5*s^2;
den_GH_c=(s^3+10);
GH_c=num_GH_c/den_GH_c;
figure(3);
rlocus(GH_c)

%d)
num_GH_d=(s^3+s^2+2);
den_GH_d=(s^4+3*s^3+s^2+15);
GH_d=num_GH_d/den_GH_d;
figure(4);
rlocus(GH_d)

%e)
num_GH_e=(s^2-1)*(s+2);
den_GH_e=(s^3+2*s^2+2*s);
GH_e=num_GH_e/den_GH_e;
figure(5);
rlocus(GH_e)

%f)
num_GH_f=(s+4)*(s+1);
den_GH_f=(s^3-2*s);
GH_f=num_GH_f/den_GH_f;
figure(6);
rlocus(GH_f)

%g)
num_GH_g=(s^2+4*s+5);
den_GH_g=(s^4+6*s^3+9*s^2);
GH_g=num_GH_g/den_GH_g;
figure(7);
rlocus(GH_g)


%h)
num_GH_h=(s^2-2)*(s+4);
den_GH_h=(s^3+2*s^2+2*s);
GH_h=num_GH_h/den_GH_h;
```

```
figure(8);
rlocus(GH_h)

%i)
num_GH_i=(s+2)*(s+0.5);
den_GH_i=(s^3-s);
GH_i=num_GH_i/den_GH_i;
figure(9);
rlocus(GH_i)

%j)
num_GH_j=(2*s+5);
den_GH_j=(s^4+2*s^3+2*s^2);
GH_j=num_GH_j/den_GH_j;
figure(10);
rlocus(GH_j)

%k)
num_GH_k=1;
den_GH_k=(s^5+2*s^4+3*s^3+2*s^2+s);
GH_k=num_GH_k/den_GH_k;
figure(11);
rlocus(GH_k)
```

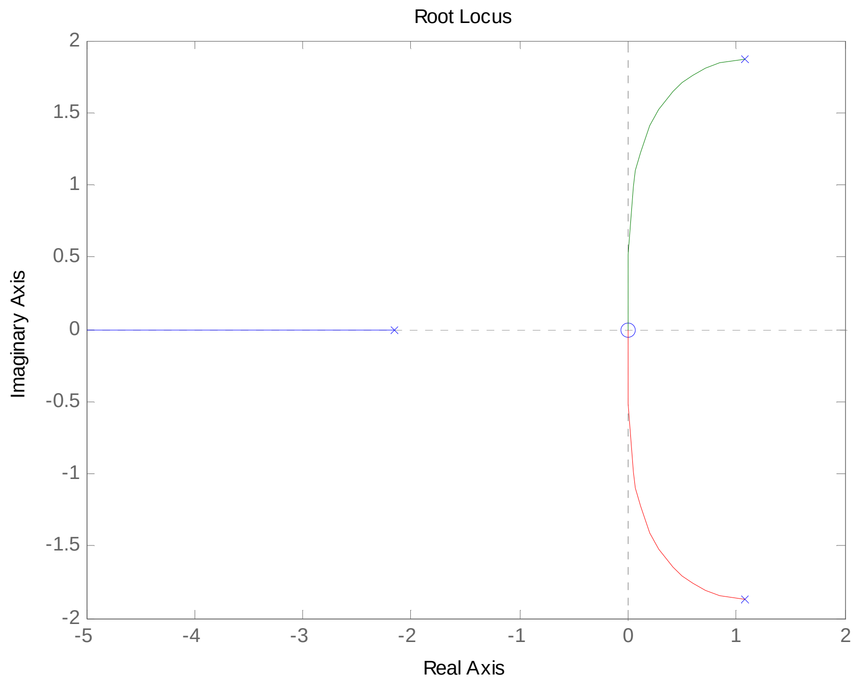
Root Locus diagram – 7-15(a):



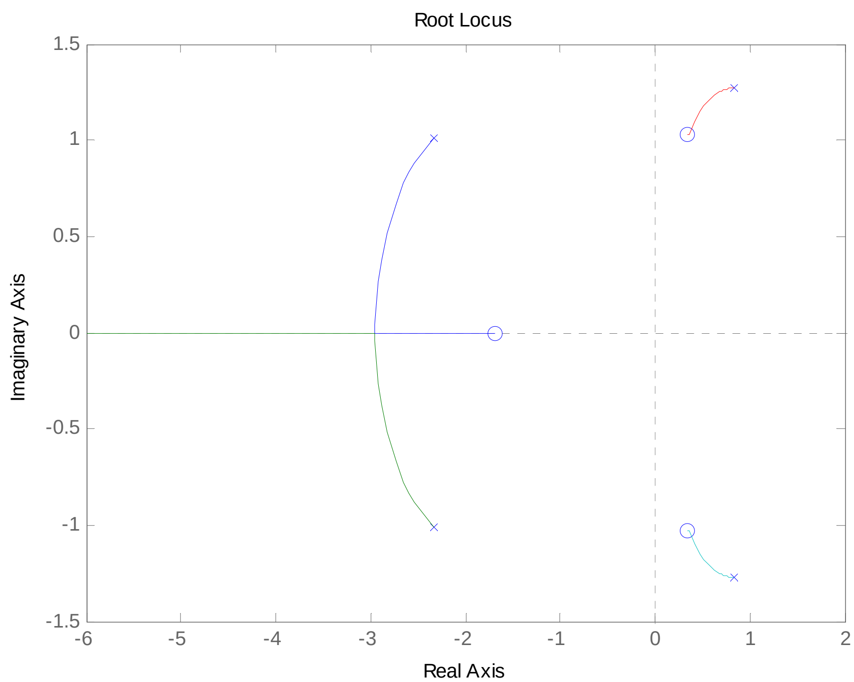
Root Locus diagram – 7-15(b):



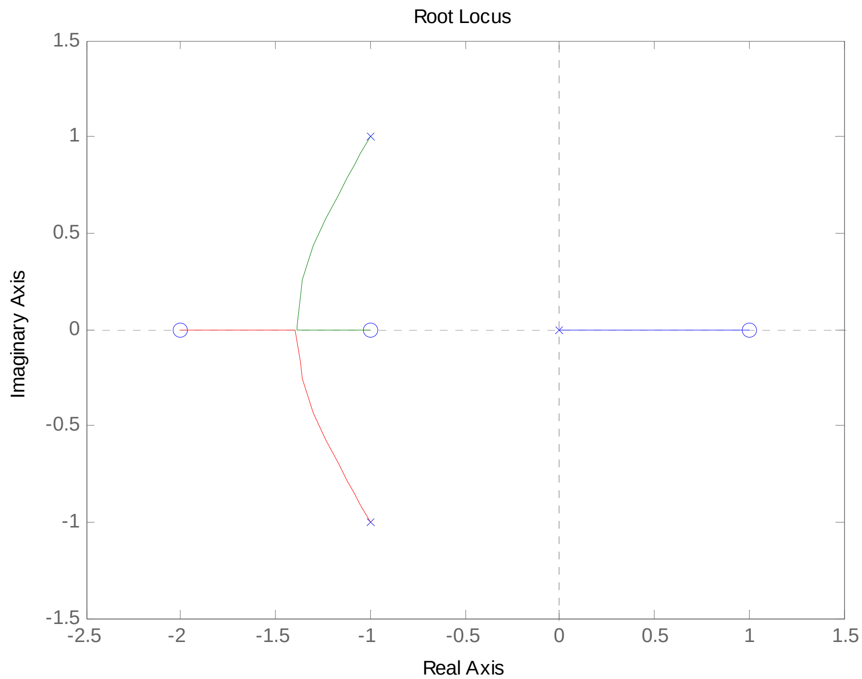
Root Locus diagram – 7-15(c):



Root Locus diagram – 7-15(d):



Root Locus diagram – 7-15(e):



Root Locus diagram – 7-15(f):

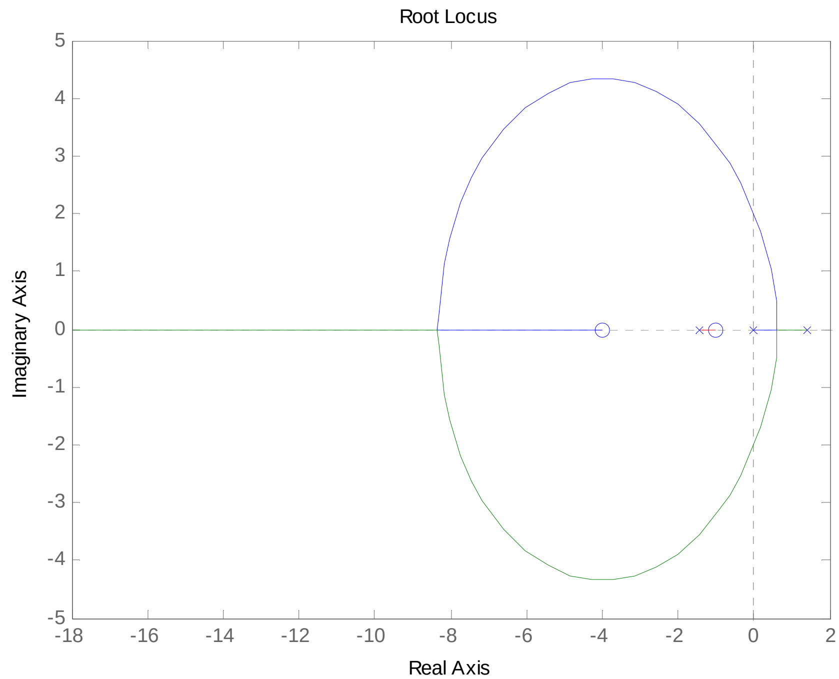
poles: $s = -1, -2 + j, -2 - j$

$$\sigma_1 = \frac{-1 - 2 - j - 2 + j}{3} = -1.67$$

Asymptotes angle: $\theta_i - \frac{2i+1}{|n-m|} \times 100 - \frac{2i+1}{3} \times 100$

Therefore, $\theta_i - 60, 180, 300$

Departure angle from: $\begin{cases} s = -2 - j & : \quad \theta = 45 \\ s = -2 + j & : \quad \theta = -45 \end{cases}$



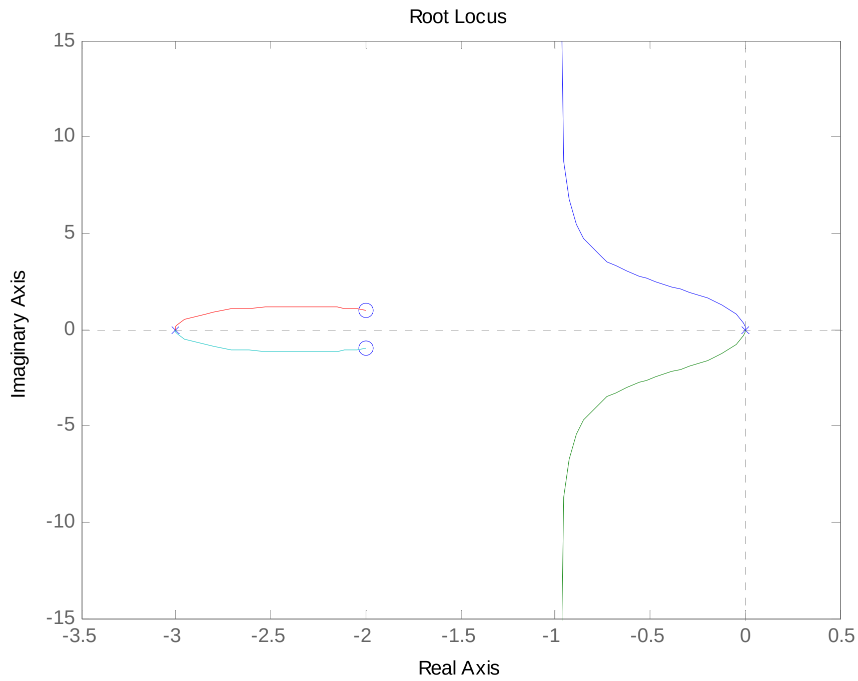
Root Locus diagram – 7-15(g):

Poles: $s = -1, -5 - j, 3 + j$ and zeroes: $s = -2$

$$\sigma_1 = \frac{-1 - 3 - j - 3 + j + 2}{2} = -2.5$$

Asymptotes angles: $\begin{cases} \theta_i = \dfrac{2i+1}{n-m}180 = \dfrac{2i-1}{3-1}180 \\ \theta_i = 90, 270 \end{cases}$

Departure angles from: $\begin{cases} s = -3 - j & : \quad \theta = -72^o \\ s = -2 + j & : \quad \theta = 72^o \end{cases}$



Root Locus diagram – 7-15(h):

Poles: $s = 0, -1$ and zeros: $s = -2, -3$

The break away points:

$$-\frac{d}{ds}\left[\frac{P(s)}{Q(s)}\right] = 0$$

which means:

$$-\frac{d}{ds}\left[\frac{s(s+1)}{(s+2)(s+3)}\right] = 0$$

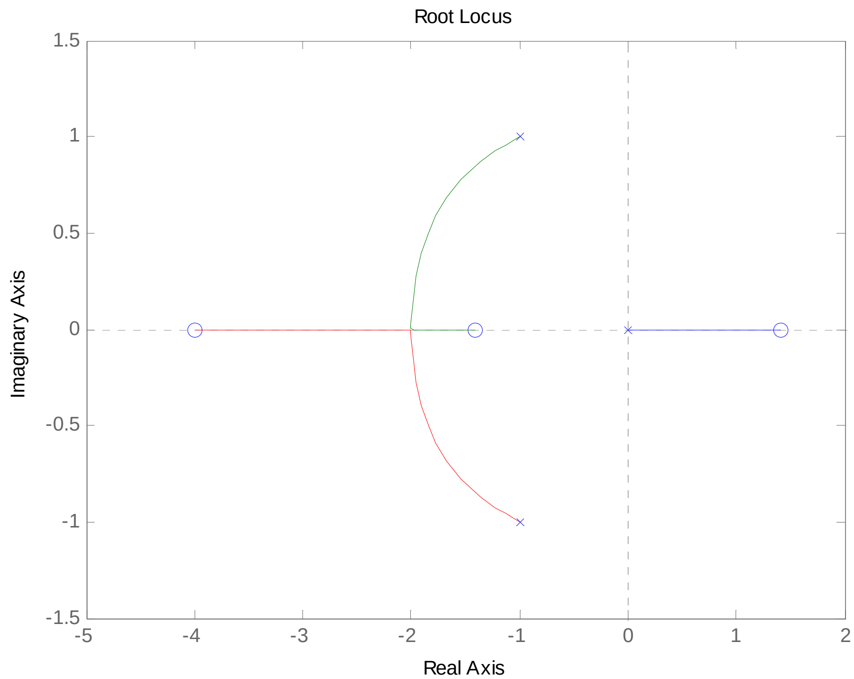
or

$$\frac{1}{s+1} + \frac{1}{s} = \frac{1}{s+2} + \frac{1}{s+3}$$

$$(2s+1)(s^2+5s+6) - (2s+5)(s^2+s) = 0$$

$$4s^2 + 12s + 6 = 0$$

$$\begin{cases} s = -0.634 \\ s = -2.366 \end{cases}$$



Root Locus diagram – 7-15(i):

Poles: $s = 0, -2 - j, -2 + j$

breaking points: $-\frac{d}{ds}(s^3 + 4s^2 + 5s) = 0$

which means :

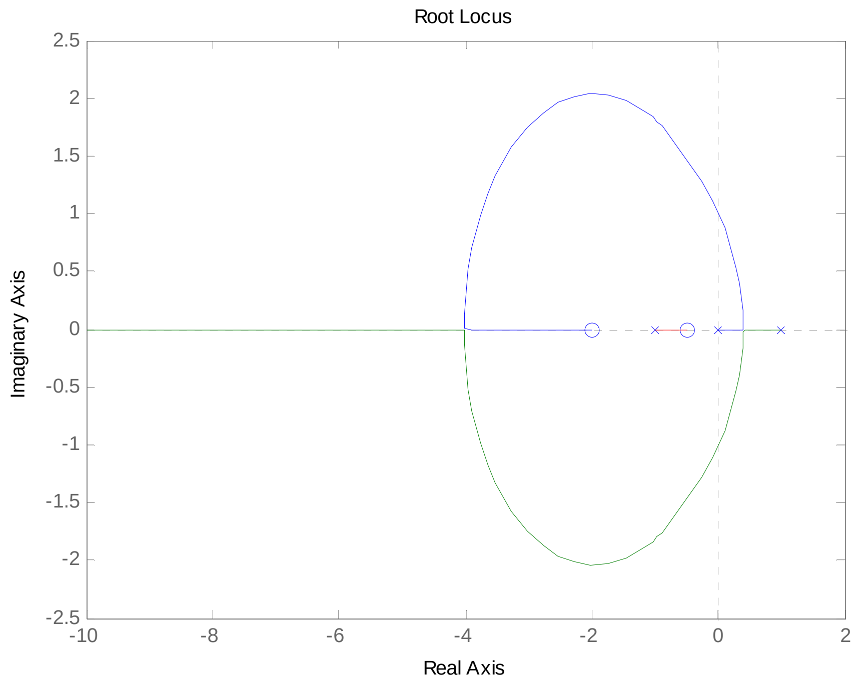
$$\begin{cases} s = -1 \\ s = -1.67 \end{cases}$$

Departure angles from: $\begin{cases} s = -2j & : \quad \theta = -63.43 \\ s = -2 + j & : \quad \theta = 63.43 \end{cases}$

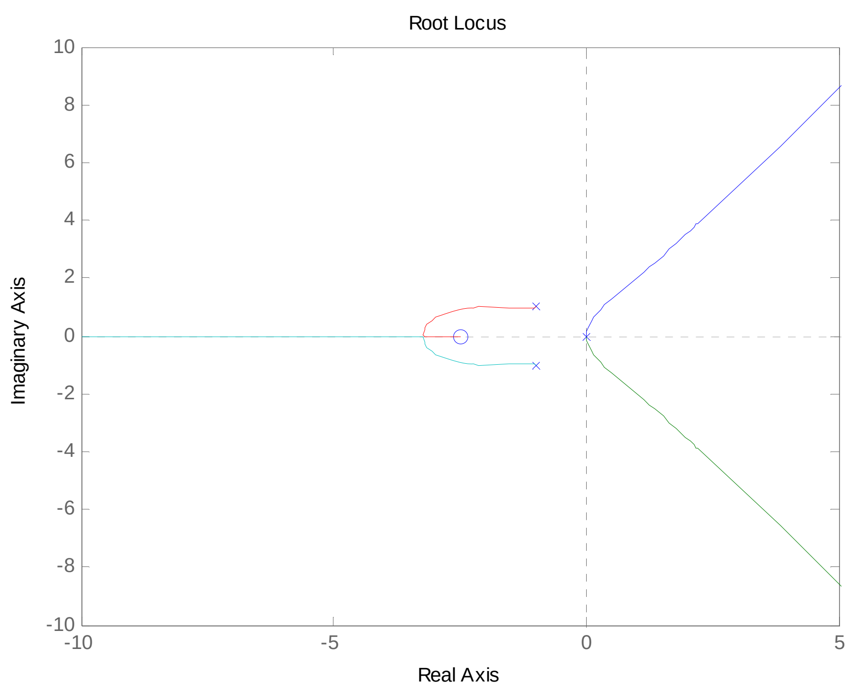
Asymptotes angles: $\theta_i = \frac{2i+1}{n-m} \times 180 = \frac{2i+1}{3} \times 180$

or θ = $60^o, 180^o, 300^o$

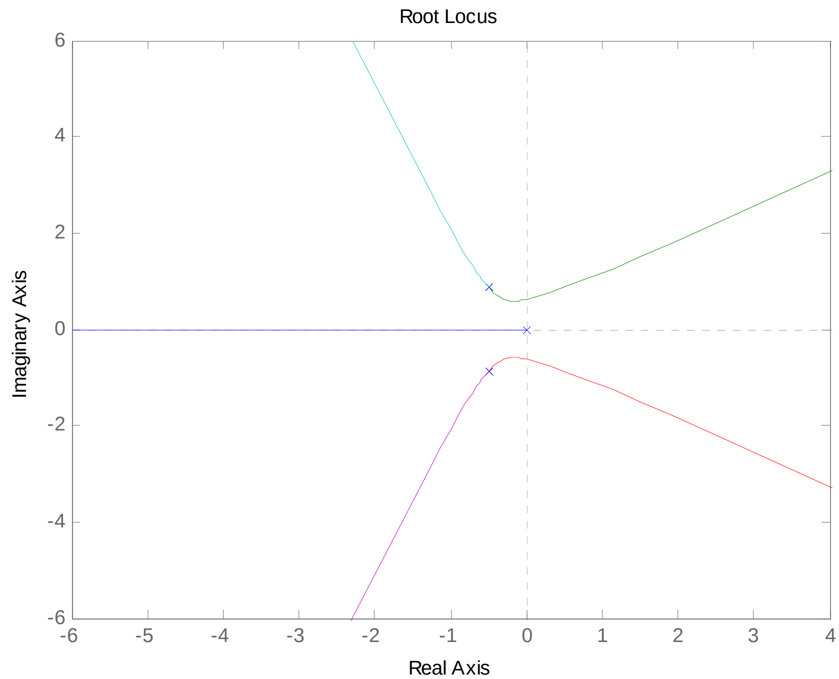
$\sigma_1 = \frac{-2 - j - 2 + j}{3} = -\frac{4}{3}$



Root Locus diagram – 7-15(j):



Root Locus diagram – 7-15(k):



7-16) (a) **Asymptotes: *K* > 0:** $45^o, \quad 135^o, \quad 225^o, \quad 315^o$

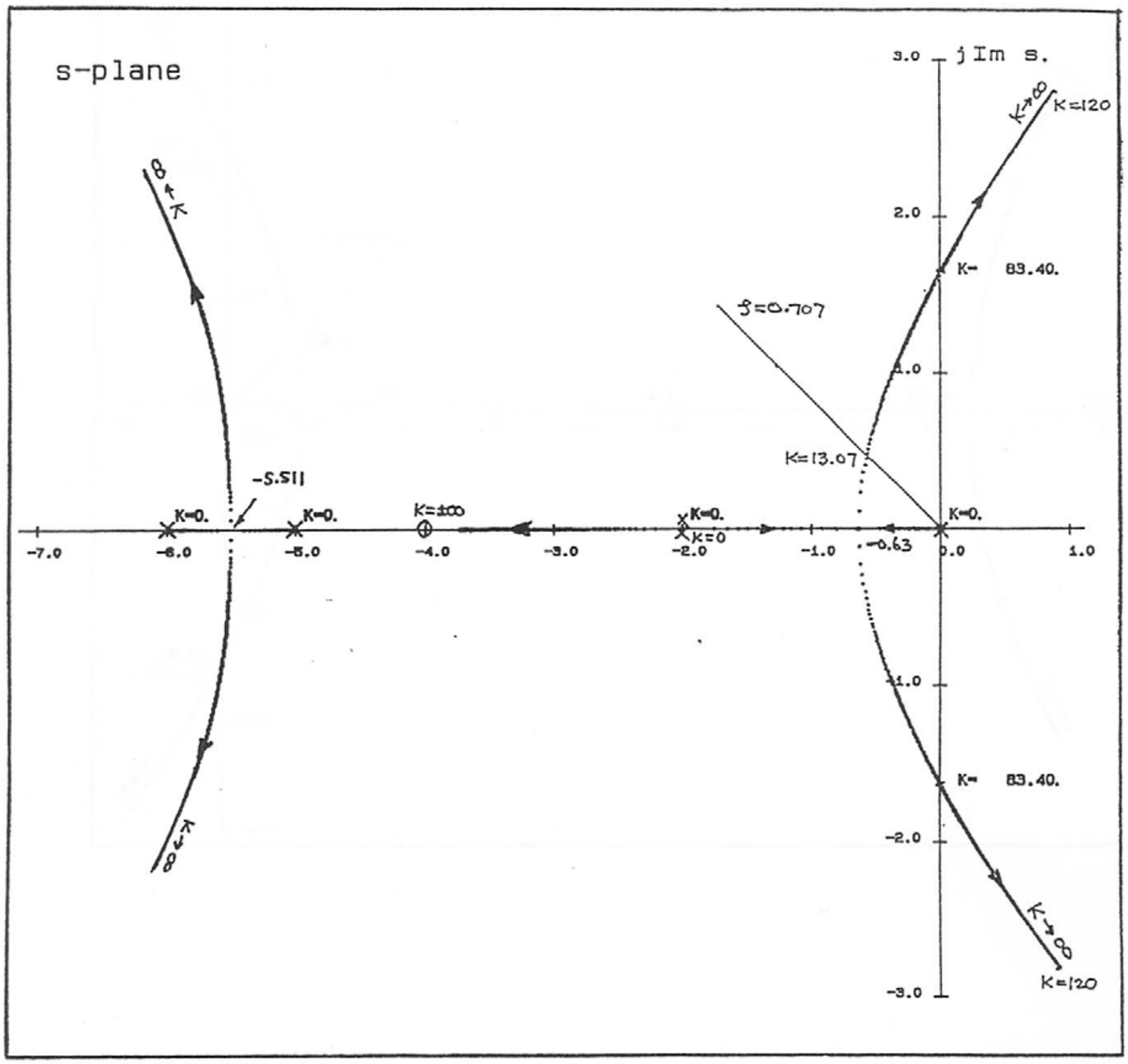
Intersect of Asymptotes:

$$\sigma_1 = \frac{-2-2-5-6-(-4)}{5-1} = -2.75$$

Breakaway-point Equation: $4s^5 + 65s^4 + 396s^3 + 1100s^2 + 1312s + 480 = 0$

Breakaway Points: –0.6325, –5.511 (on the RL)

When $\zeta = 0.707,$ ***K* = 13.07**



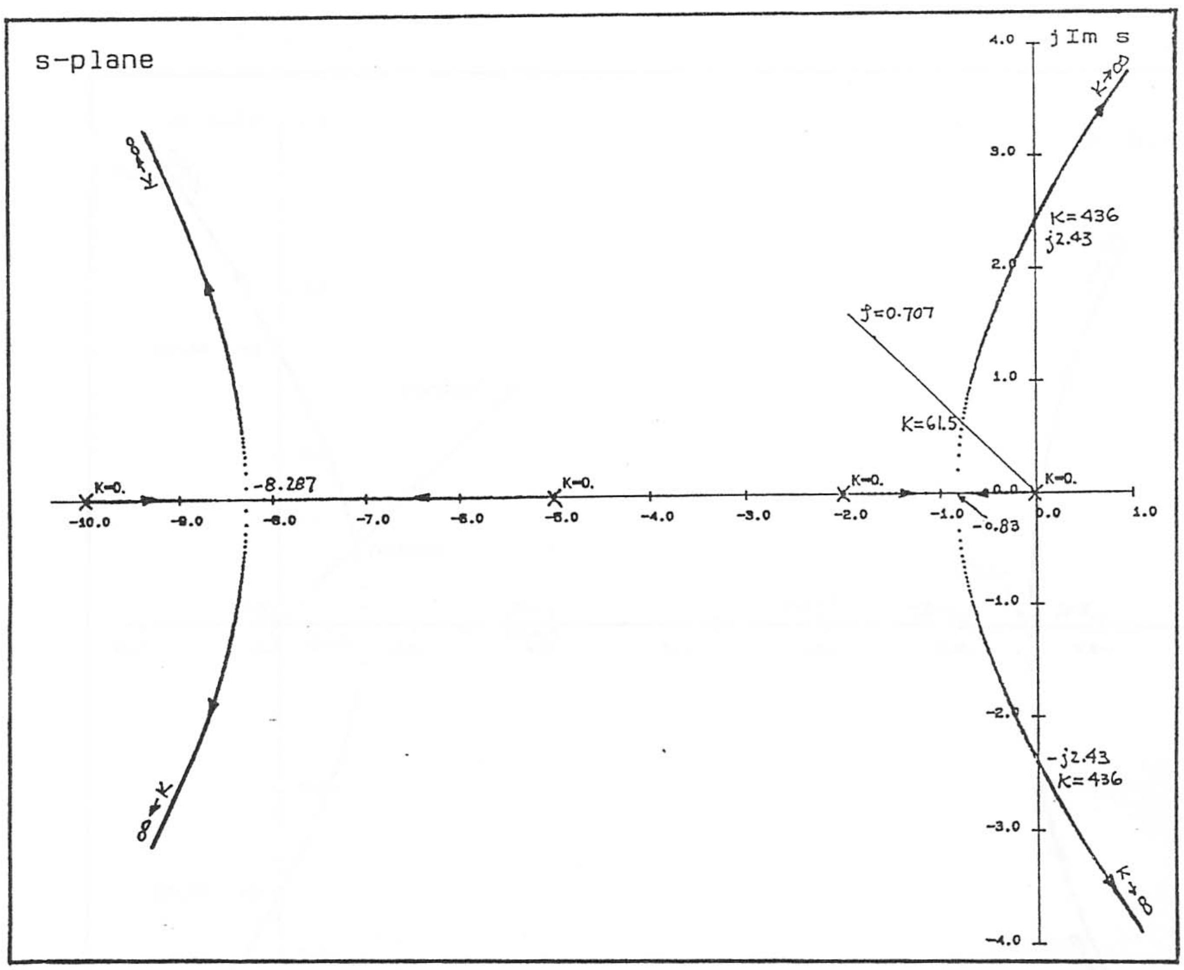
7-16 (b) **Asymptotes:** ***K*** **> 0:** 45^o, 135^o, 225^o, 315^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-2-5-10}{4} = -4.25$$

Breakaway-point Equation: $4s^3 + 51s^2 + 160s + 100 = 0$

When $\zeta = 0.707$, ***K* = 61.5**

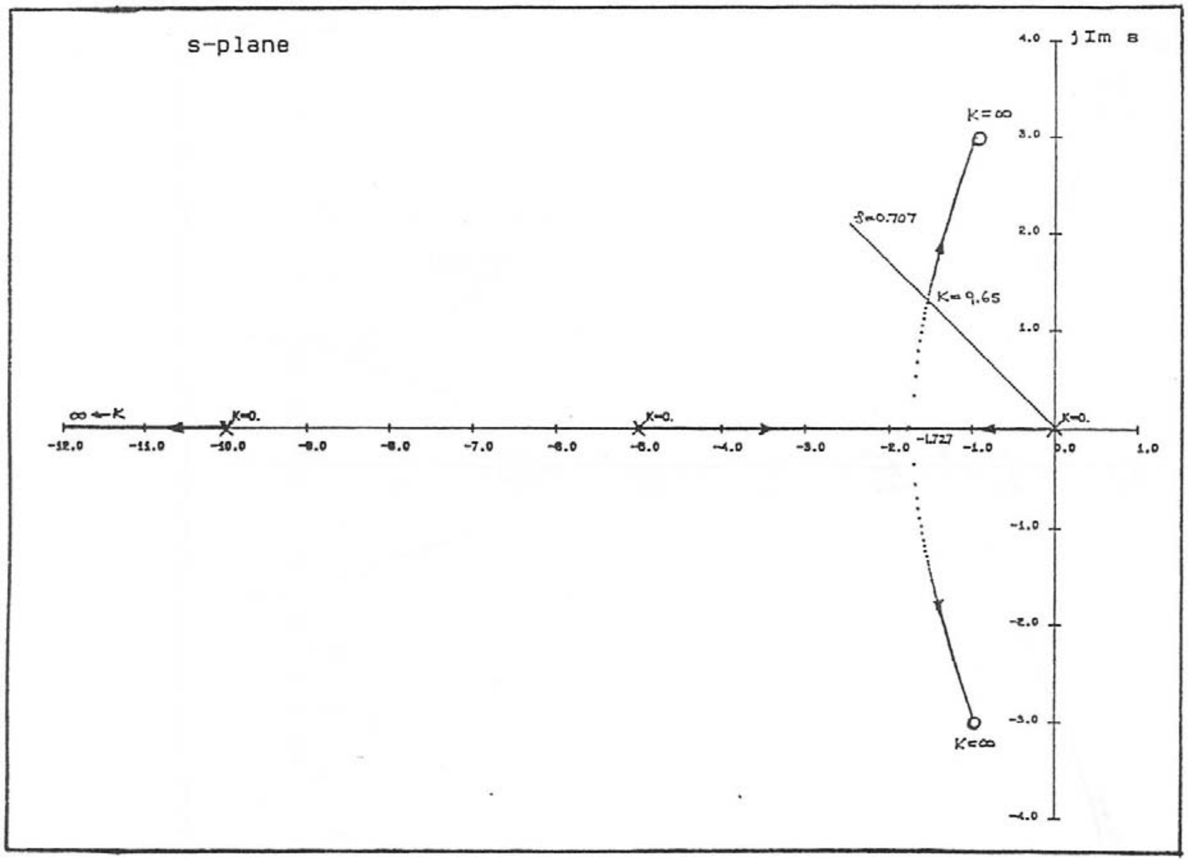


7-16 (c) **Asymptotes:** ***K* > 0:** 180^o

Breakaway-point Equation: $s^4 + 4s^3 + 10s^2 + 300s + 500 = 0$

Breakaway Points: –1.727 (on the RL)

When $\zeta = 0.707$, ***K* = 9.65**

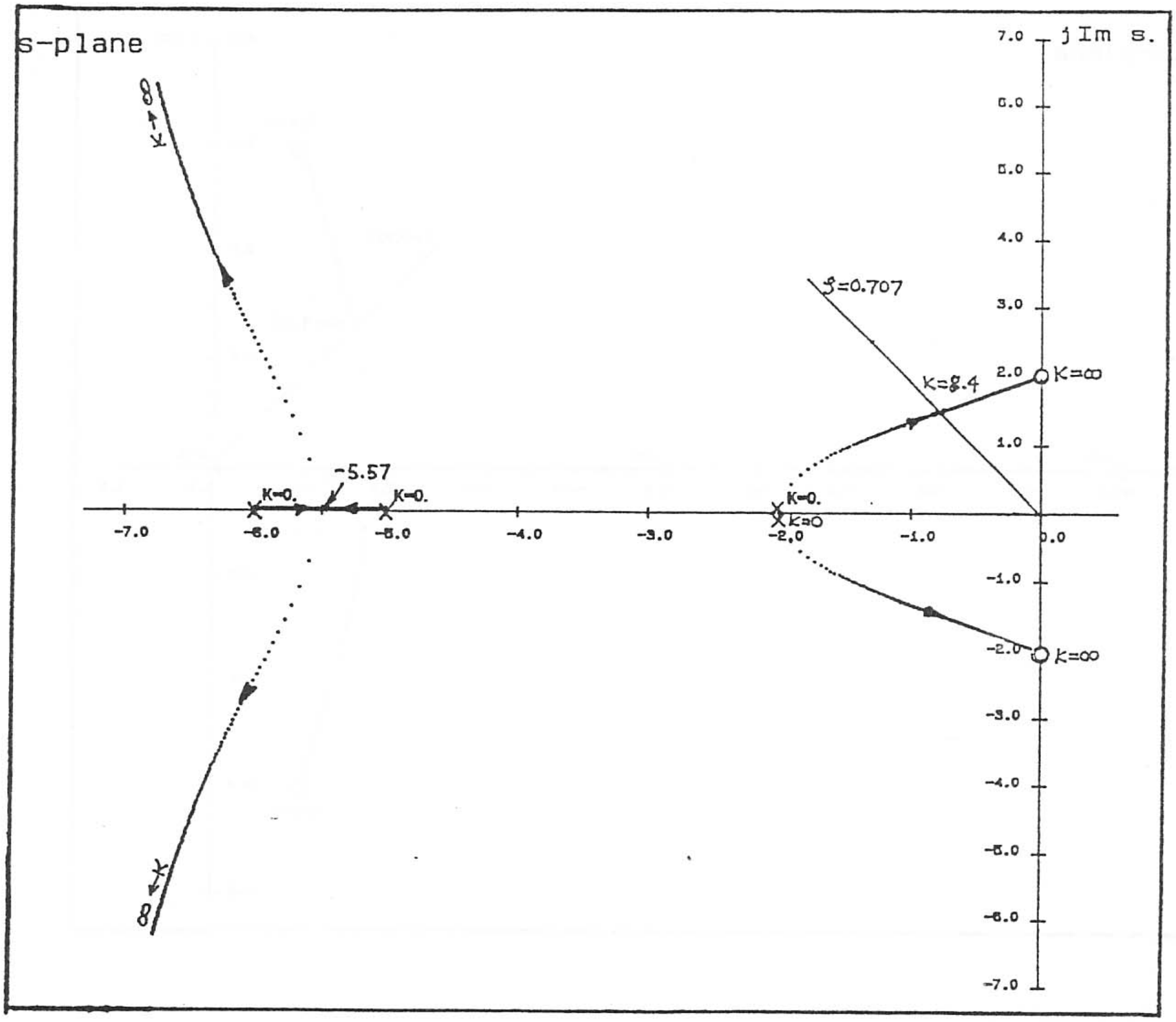

s-plane
jIm s
K=∞
ζ=0.707
K=9.65
∞←K
K=0.
K=0.
K=0.
-1.727
K=∞

7-16 (d) ***K* > 0:** $90^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-2-2-5-6}{4-2} = -7.5$$

When $\zeta = 0.707,$ ***K* = 8.4**



7-17) MATLAB code:

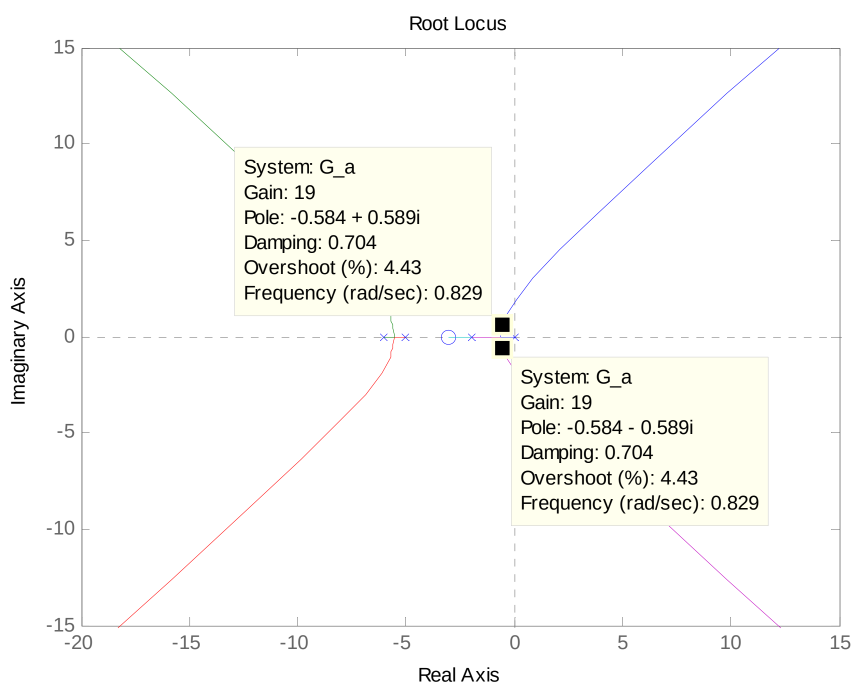
```
clear all;
close all;
s = tf('s')

%a)
num_G_a=(s+3);
den_G_a=s*(s^2+4*s+4)*(s+5)*(s+6);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
```

```
num_G_b= 1;
den_G_b=s*(s+2)*(s+4)*(s+10);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
num_G_c=(s^2+2*s+8);
den_G_c=s*(s+5)*(s+10);
G_c=num_G_c/den_G_c;
figure(3);
rlocus(G_c)

%d)
num_G_d=(s^2+4);
den_G_d=(s+2)^2*(s+5)*(s+6);
G_d=num_G_d/den_G_d;
figure(4);
rlocus(G_d)

%e)
num_G_e=(s+10);
den_G_e=s^2*(s+2.5)*(s^2+2*s+2);
G_e=num_G_e/den_G_e;
figure(5);
rlocus(G_e)

%f)
num_G_f=1;
den_G_f=(s+1)*(s^2+4*s+5);
G_f=num_G_f/den_G_f;
figure(6);
rlocus(G_f)

%g)
num_G_g=(s+2);
den_G_g=(s+1)*(s^2+6*s+10);
G_g=num_G_g/den_G_g;
figure(7);
rlocus(G_g)

%h)
num_G_h=(s+3)*(s+2);
den_G_h=s*(s+1);
G_h=num_G_h/den_G_h;
figure(8);
rlocus(G_h)
```

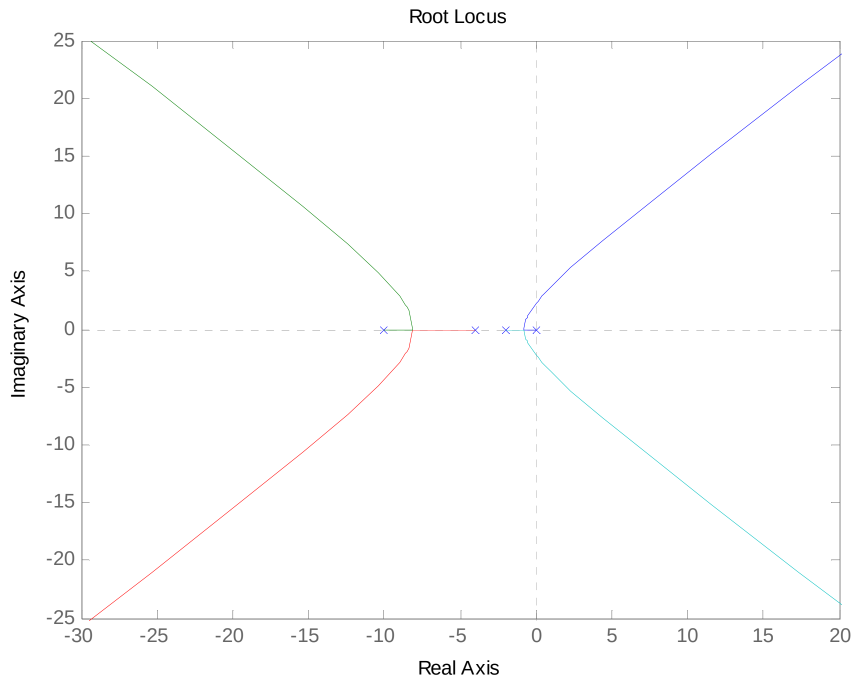
```
%i)
num_G_i=1;
den_G_i=s*(s^2+4*s+5);
G_i=num_G_i/den_G_i;
figure(9);
rlocus(G_i)
```

Root Locus diagram – 7-17(a):

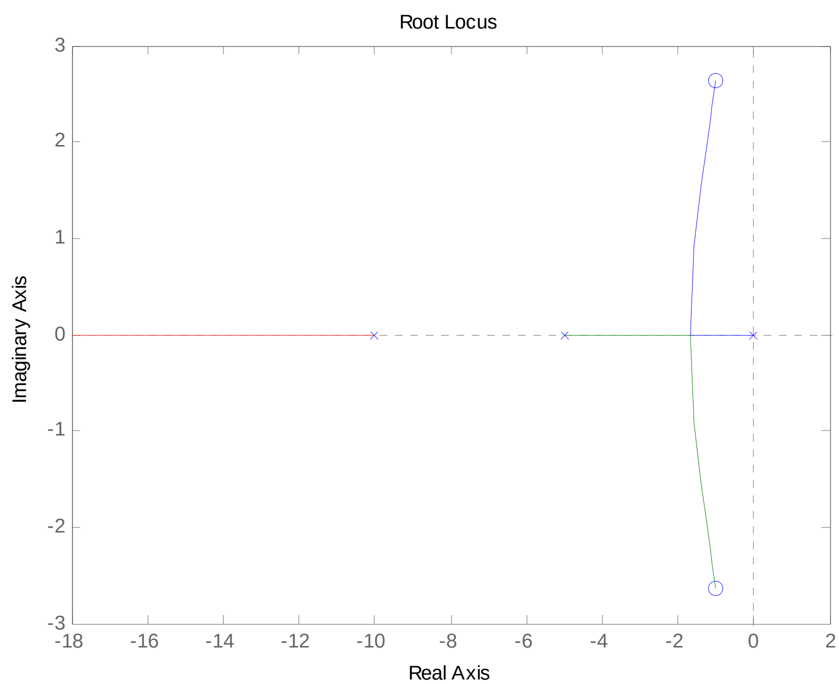
By using "Data Cursor" tab on the figure window and clicking on the root locus diagram, gain and damping values can be observed. Damping of ~0.707 can be observed on intersection of the root locus diagram with two lines originating from (0,0) by angles of ArcCos(0.707) from the real axis. These intersection points are shown for part (a) where the corresponding gain is 19. In the other figures for section (b) to (i), similar points have been picked by the "Data Cursor", and the gains are reported here.



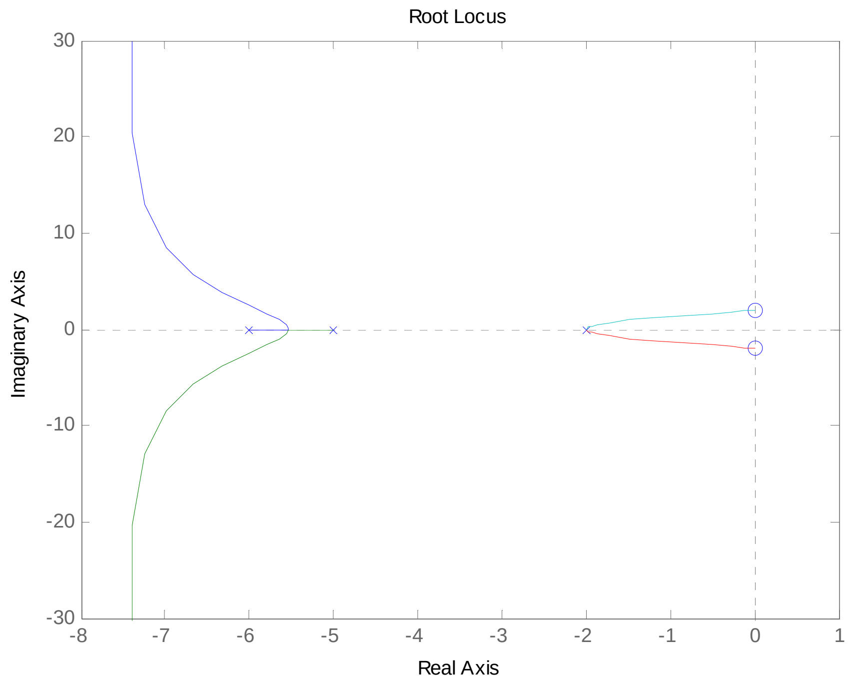
Root Locus diagram – 7-17(b): ($K = 45.5$ @ damping = ~0.0707)



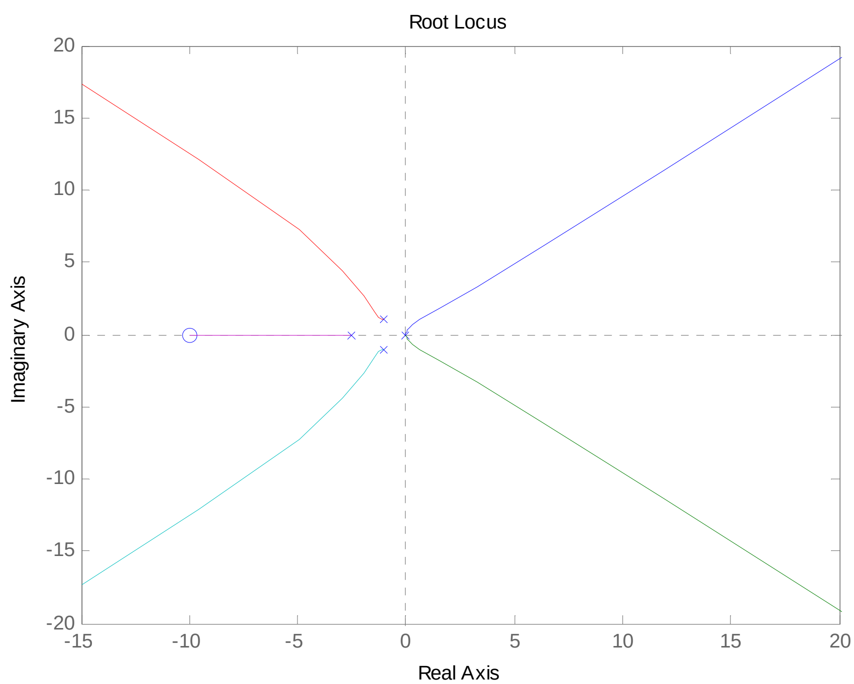
Root Locus diagram – 7-17(c): (K = 12.8 @ damping = ~0.0707)



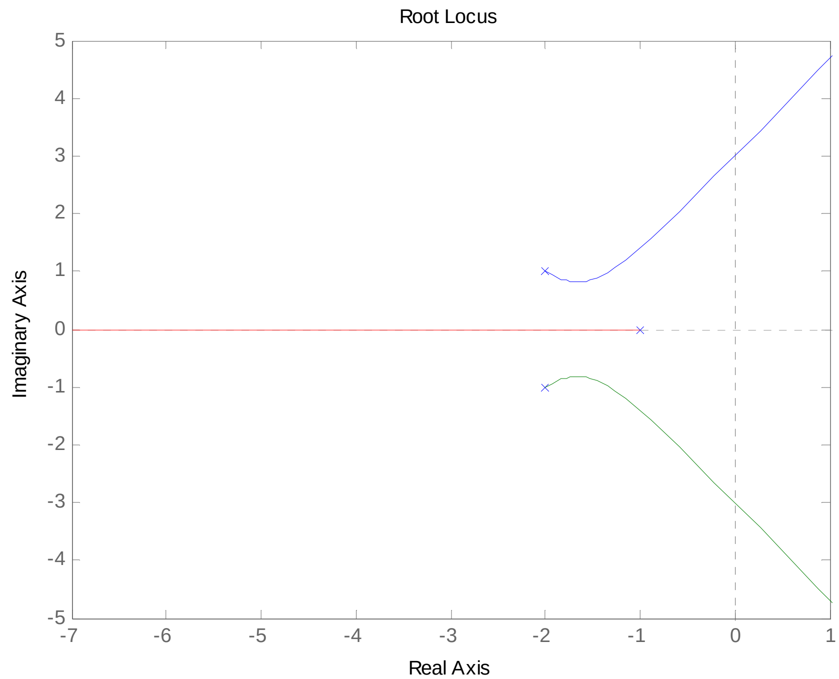
Root Locus diagram – 7-17(d): (K = 8.3 @ damping = ~0.0707)



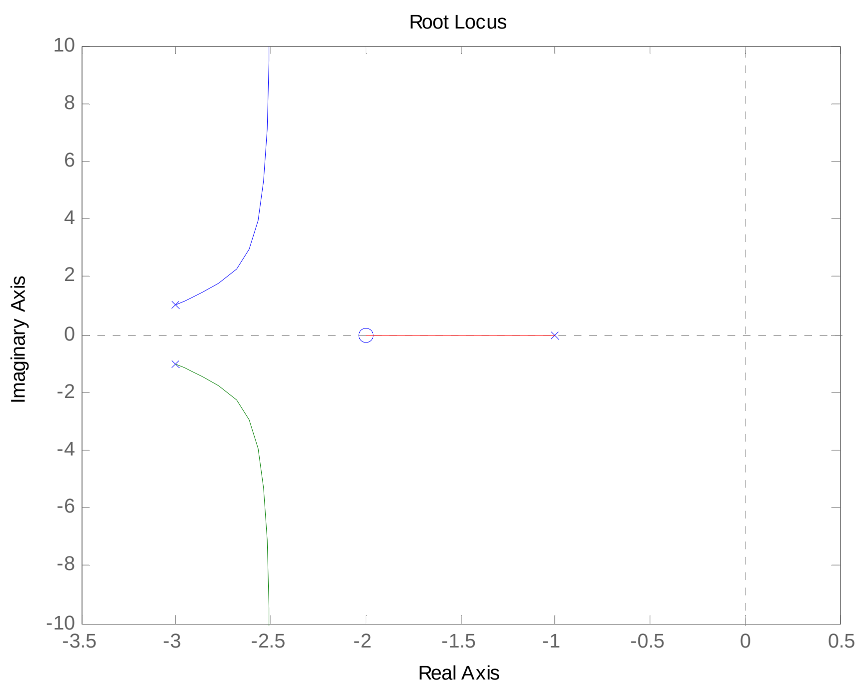
Root Locus diagram – 7-17(e): ($K = 0$ @ damping = 0.0707)



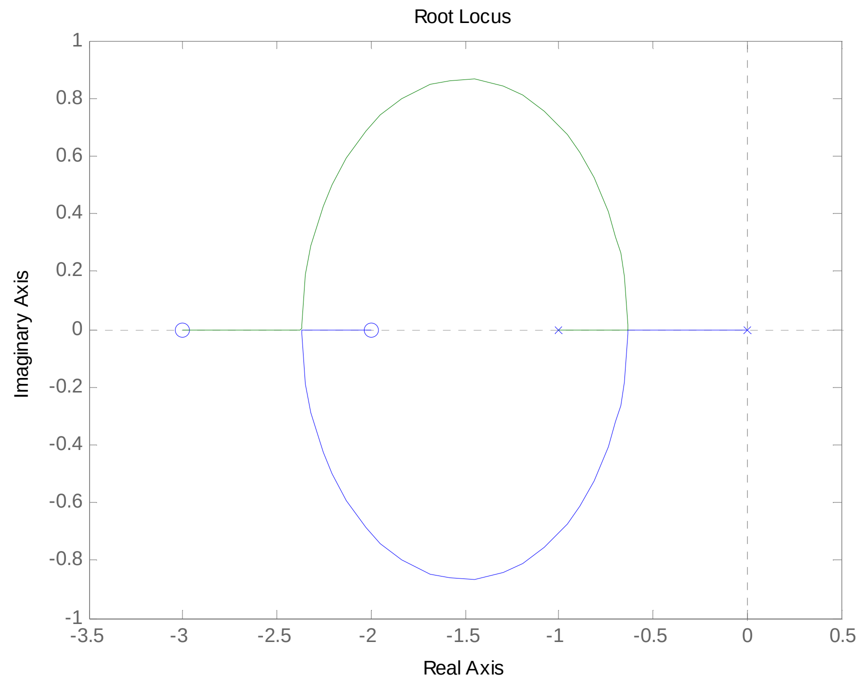
Root Locus diagram – 7-17(f): ($K = 2.33$ @ damping = ~0.0707)



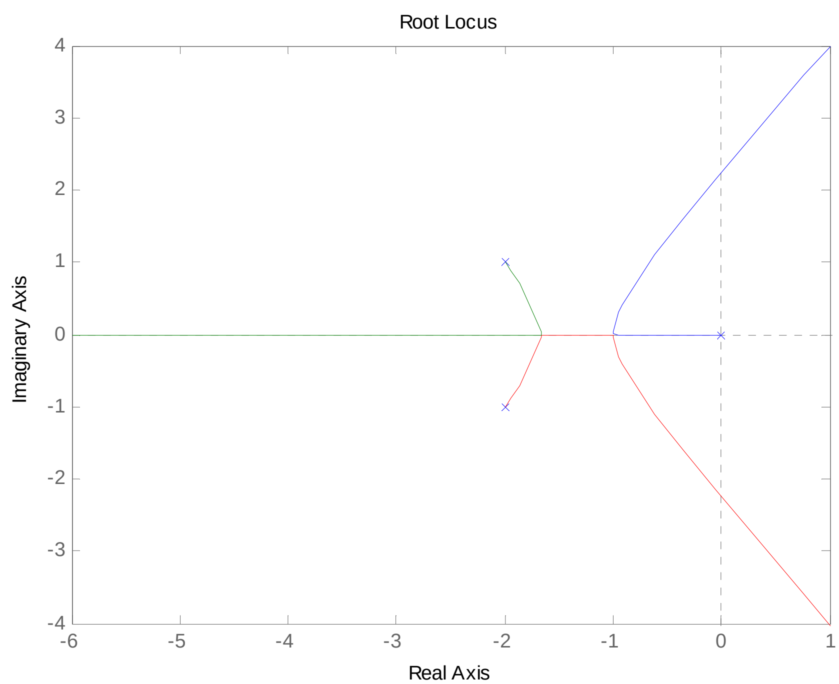
Root Locus diagram – 7-17(g): (K = 7.03 @ damping = ~0.0707)



Root Locus diagram – 7-17(h): (no solution exists for damping =0.0707)



Root Locus diagram – 7-17(i): ($K = 2.93$ @ damping = ~0.0707)

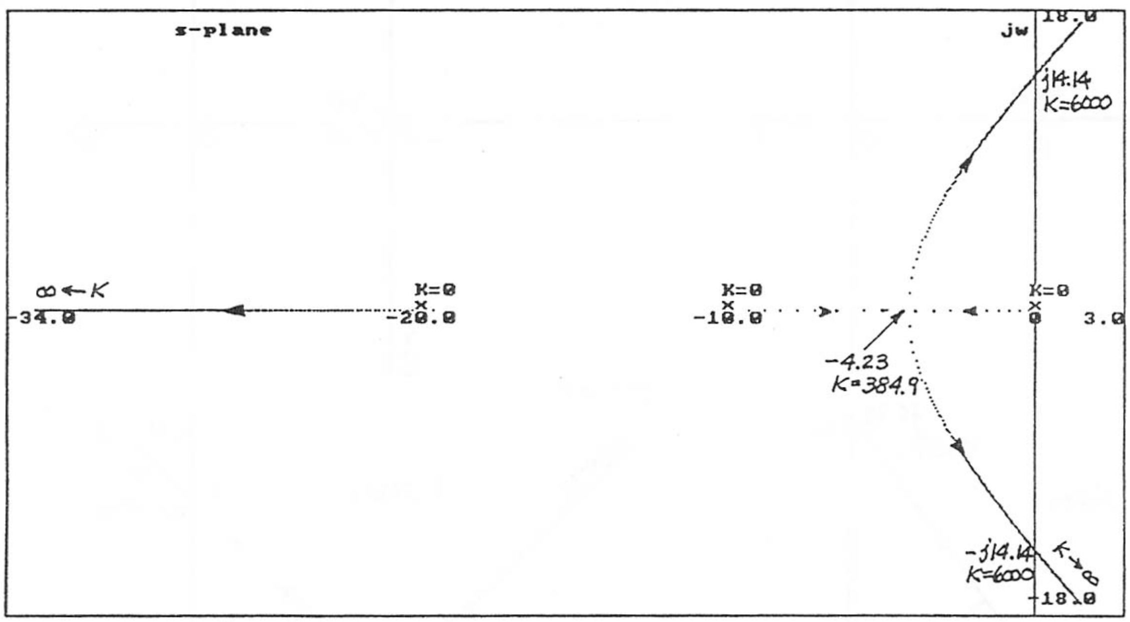


7-18) **(a)** **Asymptotes:** ***K* > 0:** 60^o, 180^o, 300^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-10-20}{3} = -10$$

Breakaway-point Equation: $3s^2 + 60s + 200 = 0$ **Breakaway Point: (RL)** −4.2265, ***K* = 384.9**



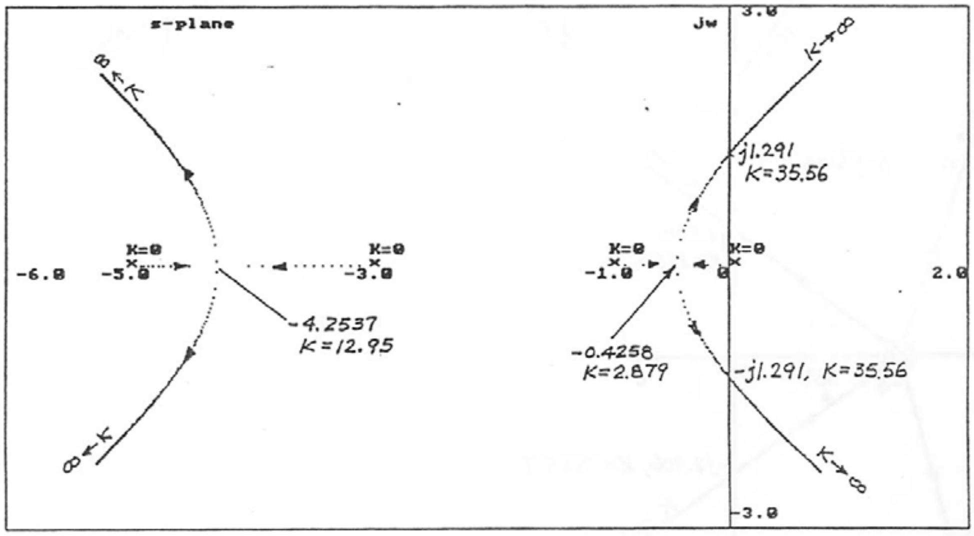
(b) Asymptotes: ***K* > 0:** 45^o, 135^o, 225^o, 315^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-3-5}{4} = -2.25$$

Breakaway-point Equation: $4s^3 + 27s^2 + 46s + 15 = 0$

Breakaway Points: (RL) −0.4258 ***K* = 2.879**, −4.2537 ***K* = 12.95**



c) Zeros: $s = 0.5$ and poles: $s = 1$

Angle of asymptotes: $\theta = (2i + 1)180 = 180$

The breakaway points: $\frac{1}{(s+1)^2} = \frac{1}{s+0.5} \rightarrow s^2 + s + 0.5 = 0$

Then $s = -0.5 - 0.5j, -0.5 + 0.5j$ and $\sigma_1 = \frac{+1 - 0.5}{1} = 0.5$

d) Poles: $s = -0.5, 4.5$

Angle of asymptotes: $\theta_i = \frac{2i + 1}{2} \times 180 = 90, 270$

breakaway points:

$s^2 + s + 0.75 = 0 \rightarrow s = -1 - \sqrt{2}j, -1 + \sqrt{2}j$

$\sigma_1 = \frac{-0.5 + 1.5}{2} = 0.5$

e) Zeros: $s = -\frac{1}{3}, -1$ and poles: $s = 0, 0.5, 1$

Angle of asymptotes: $\theta_i = \frac{2i + 1}{3 - 2}180 = 180$

breakaway points: $\frac{1}{s}+\frac{1}{s+\frac{1}{2}}+\frac{1}{s-1}=\frac{1}{s+\frac{1}{3}}+\frac{1}{s+1} \rightarrow s=0.383,-2.22$

$$\sigma=-\frac{1-0.5+\frac{1}{3}+1}{1}=-\frac{11}{6}$$

f) Poles: $s=0,-3+4j,-3-4j$

Angles of asymptotes: $\theta_i=\frac{2i+1}{3}\times 180=60,180,300$

$$\sigma_1=-\frac{0+3-4j+3+4j}{3}=2$$

breakaway point: $-\frac{d}{ds}[s(s^2+6s+25)]=0$

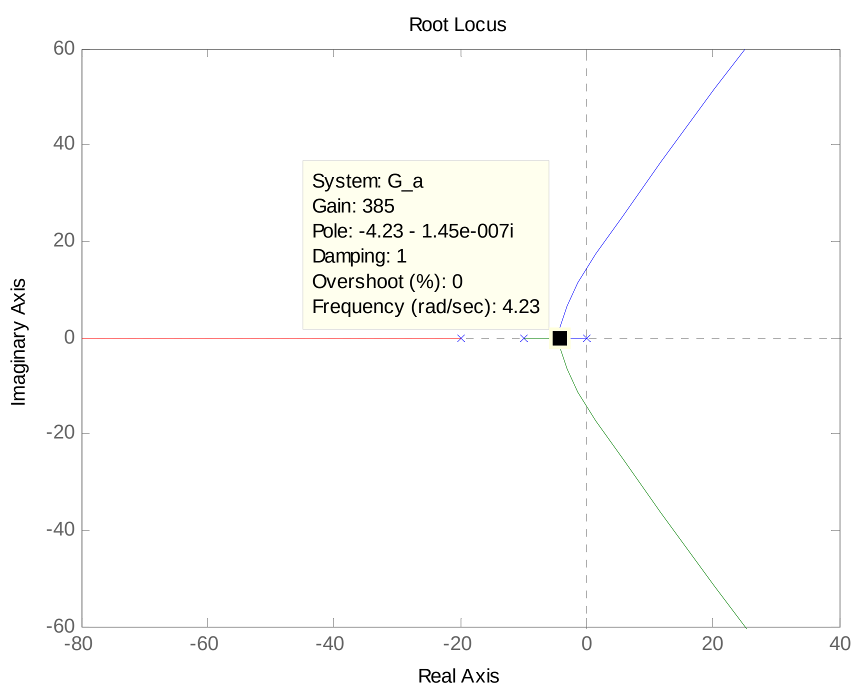
$3s^2+12s+25=0 \rightarrow s\approx -2+2.1j,-2-2.1j$

7-19) MATLAB code:

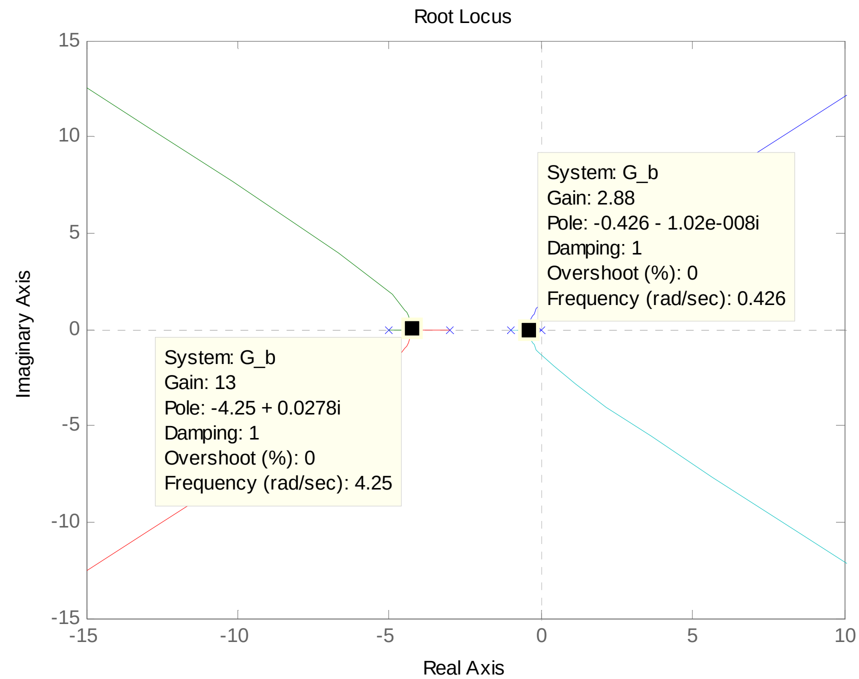
```
clear all;
close all;
s = tf('s')

%a)
num_G_a=1;
den_G_a=s*(s+10)*(s+20);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
num_G_b= 1;
den_G_b=s*(s+1)*(s+3)*(s+5);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
num_G_c=(s-0.5);
den_G_c=(s-1)^2;
G_c=num_G_c/den_G_c;
figure(3);
rlocus(G_c)

%d)
```

```
num_G_d=1;
den_G_d=(s+0.5)*(s-1.5);
G_d=num_G_d/den_G_d;
figure(4);
rlocus(G_d)

%e)
num_G_e=(s+1/3)*(s+1);
den_G_e=s*(s+1/2)*(s-1);
G_e=num_G_e/den_G_e;
figure(5);
rlocus(G_e)

%f)
num_G_f=1;
den_G_f=s*(s^2+6*s+25);
G_f=num_G_f/den_G_f;
figure(6);
rlocus(G_f)
```

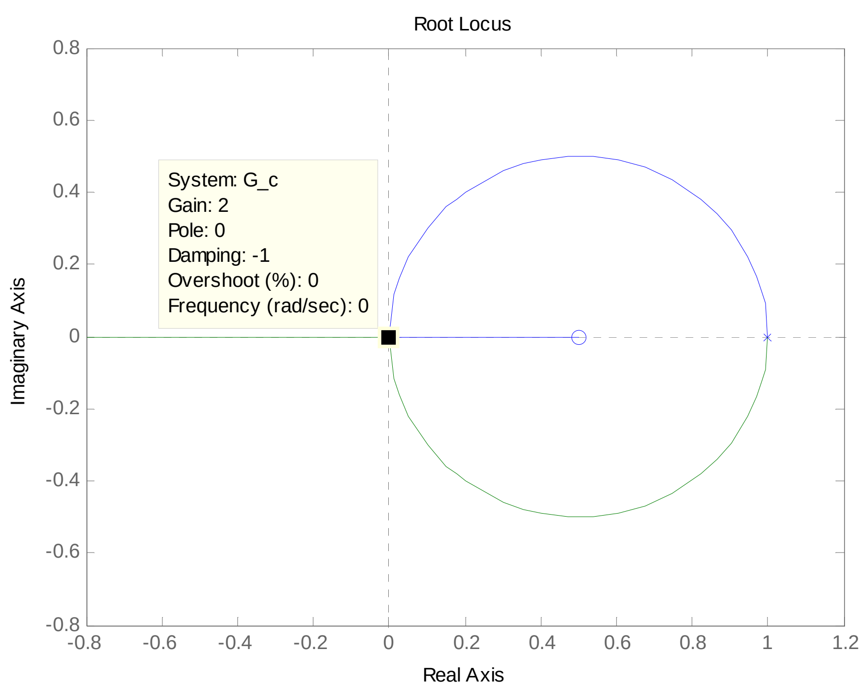
Root Locus diagram – 7-19(a):



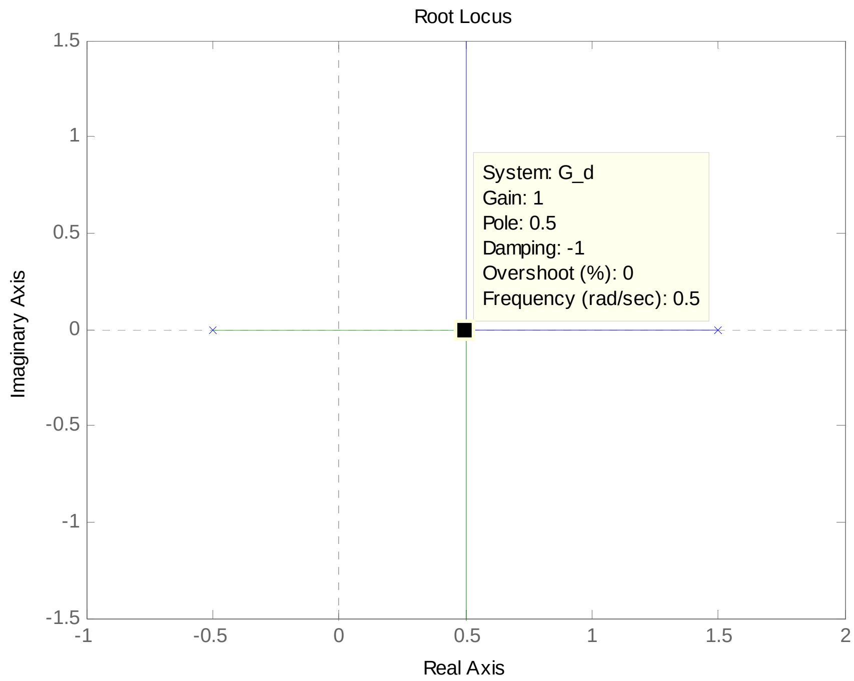
Root Locus diagram – 7-19(b):



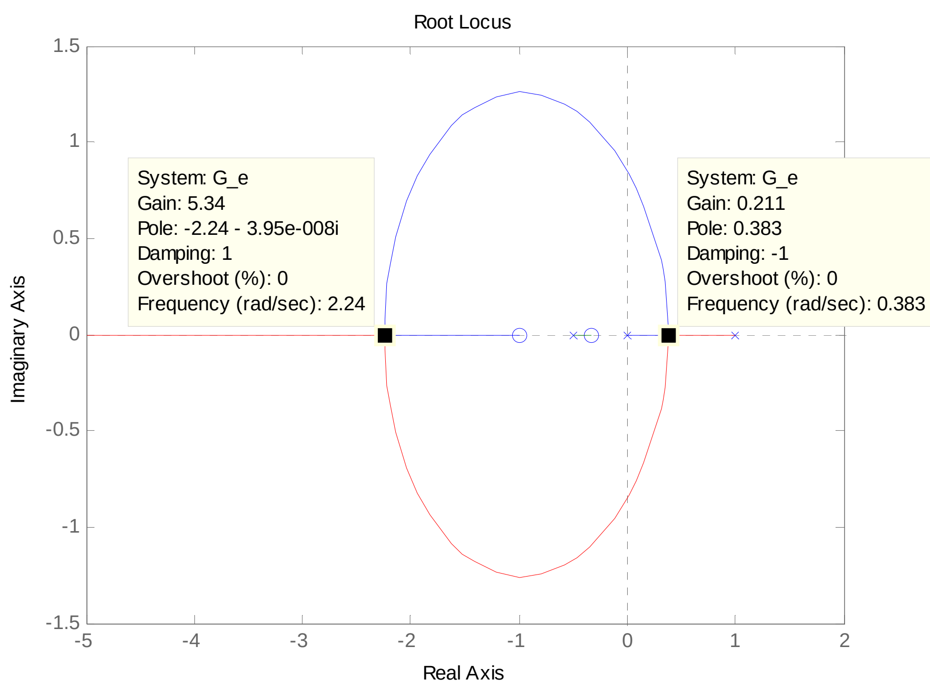
Root Locus diagram – 7-19(c):



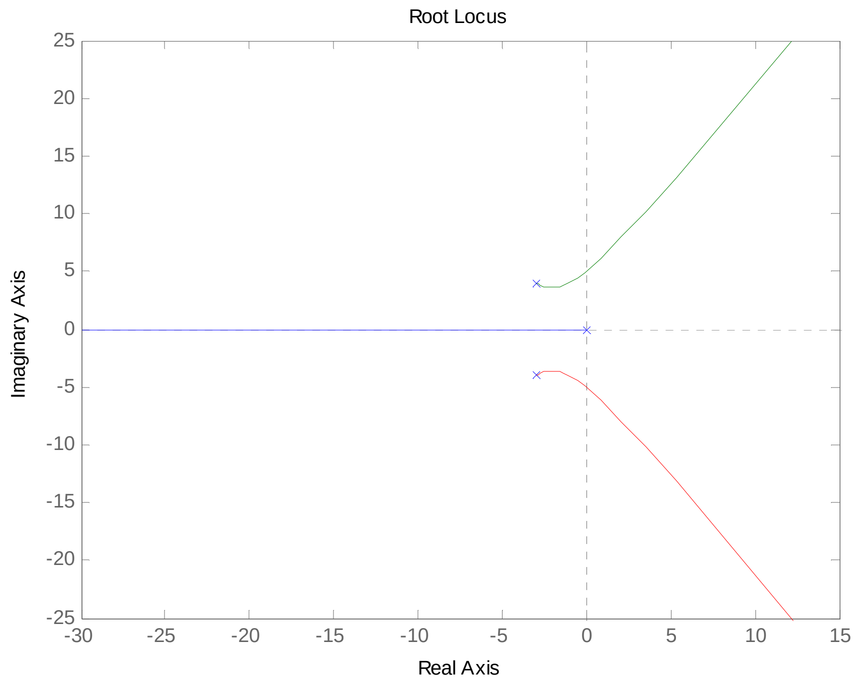
Root Locus diagram – 7-19(d):



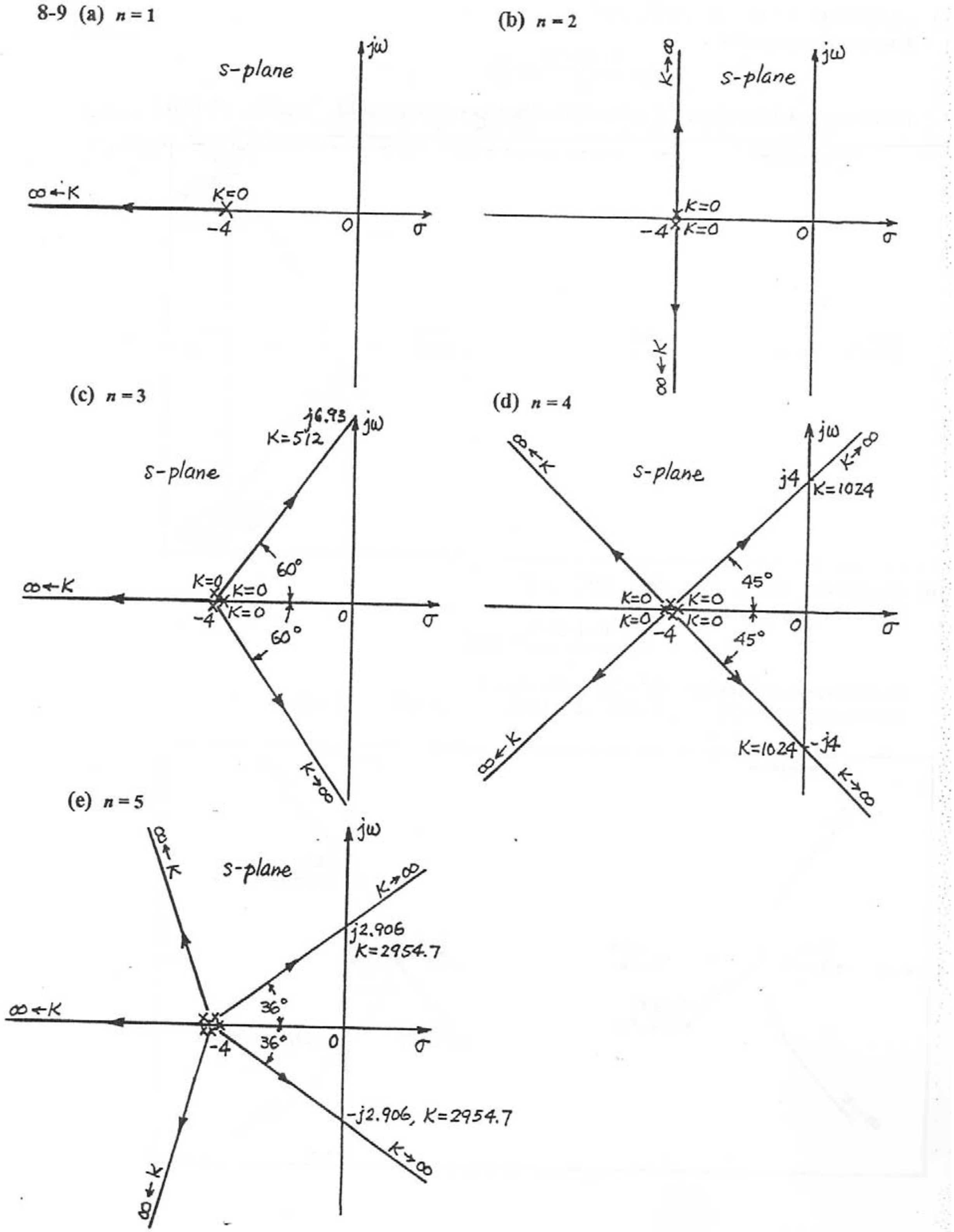
Root Locus diagram – 7-19(e):



Root Locus diagram – 7-19(f): (No breakaway points)


Root Locus
Imaginary Axis
Real Axis
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-25
-20
-15
-10
-5
0
5
10
15

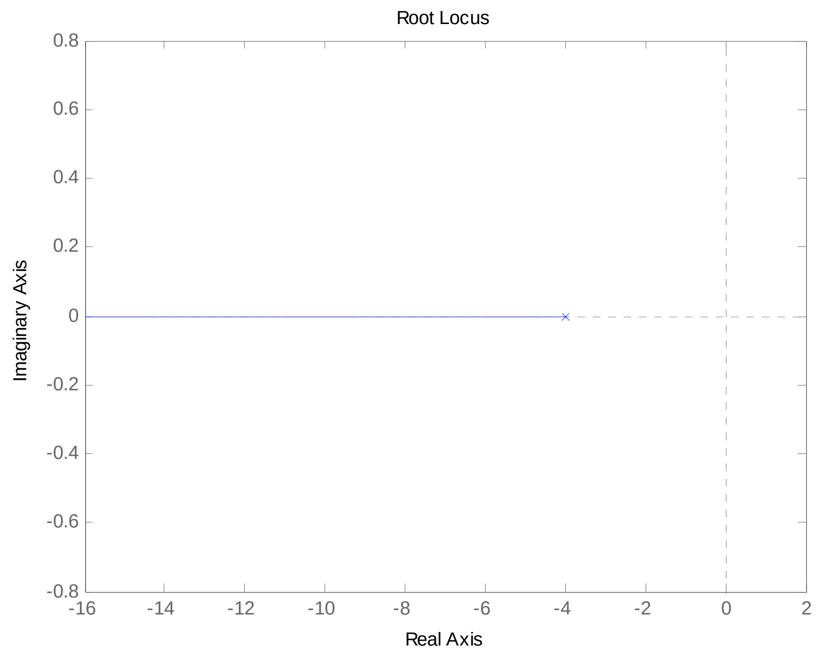
7-20)


8-9 (a) n = 1
s-plane
jω
∞ ← K
K=0
-4
0
σ
(b) n = 2
K → ∞
s-plane
jω
K=0
-4
K=0
0
σ
∞ ← K
(c) n = 3
j6.93
jω
K=512
s-plane
60°
∞ ← K
K=0
K=0
K=0
-4
0
σ
60°
K → ∞
(d) n = 4
jω
∞ ← K
K → ∞
s-plane
j4
K=1024
45°
K=0
K=0
K=0
K=0
-4
0
σ
45°
∞ ← K
K=1024
-j4
K → ∞
(e) n = 5
∞ ← K
jω
s-plane
K → ∞
j2.906
K=2954.7
36°
∞ ← K
-4
36°
0
σ
-j2.906, K=2954.7
K → ∞
∞ ← K

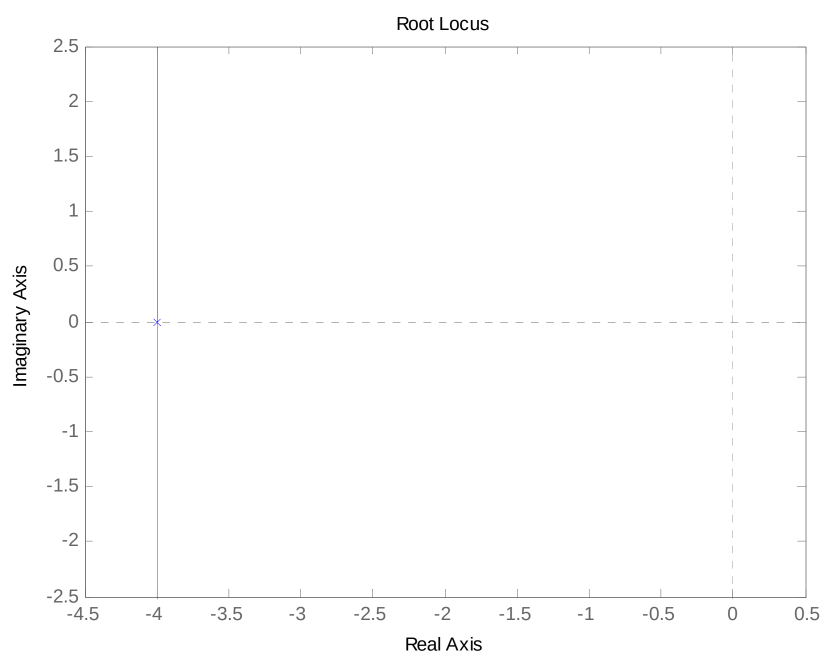
7-21) MATLAB code:

```
clear all;
close all;
s = tf('s')

%a)
n=1;
num_G_a= 1;
den_G_a=(s+4)^n;
G_a=num_G_a/den_G_a;
figure(n);
rlocus(G_a)

%b)
n=2;
num_G_b= 1;
den_G_b=(s+4)^n;
G_b=num_G_b/den_G_b;
figure(n);
rlocus(G_b)

%c)
n=3;
num_G_c= 1;
den_G_c=(s+4)^n;
G_c=num_G_c/den_G_c;
figure(n);
rlocus(G_c)

%d)
n=4;
num_G_d= 1;
den_G_d=(s+4)^n;
G_d=num_G_d/den_G_d;
figure(n);
rlocus(G_d)

%e)
n=5;
num_G_e= 1;
den_G_e=(s+4)^n;
G_e=num_G_e/den_G_e;
figure(n);
rlocus(G_e)
```

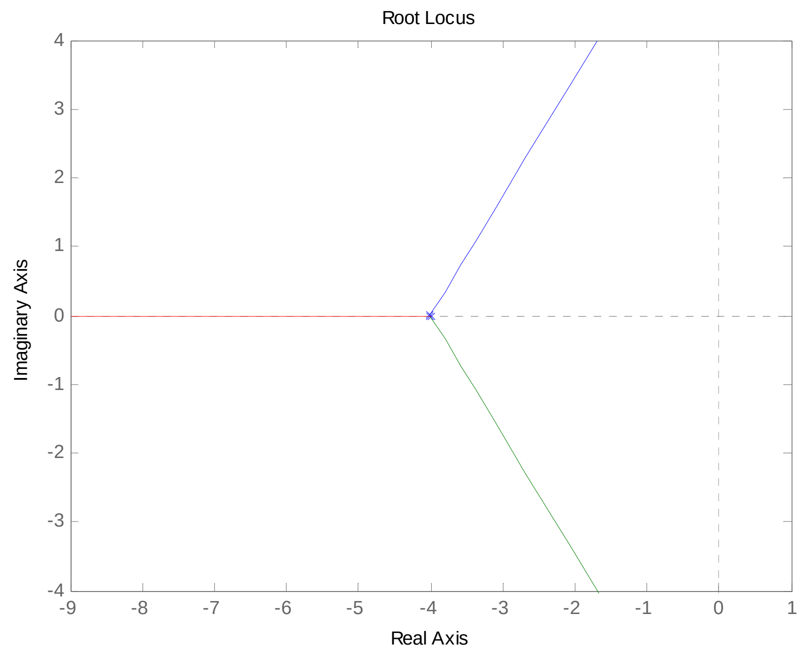
Root Locus diagram – 7-21(a):



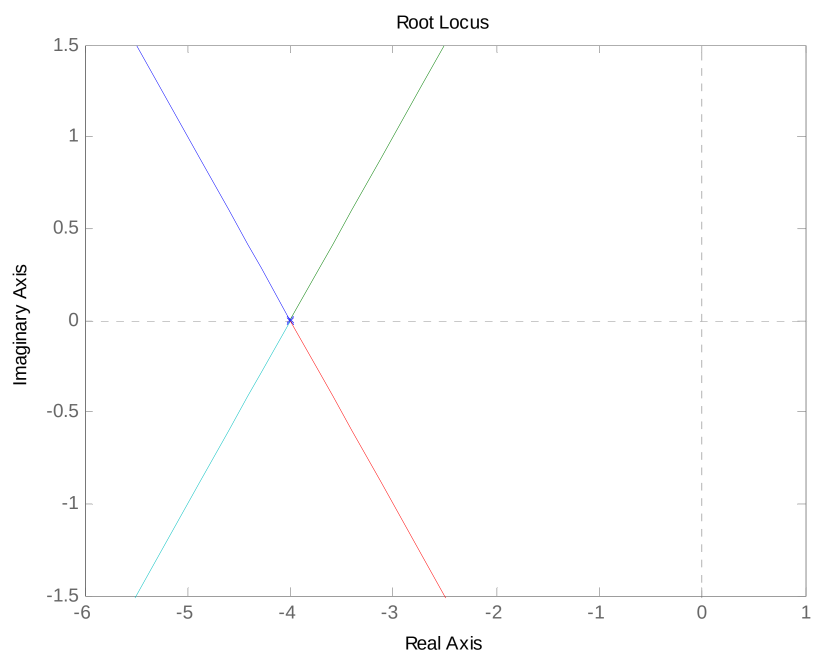
Root Locus diagram – 7-21(b):



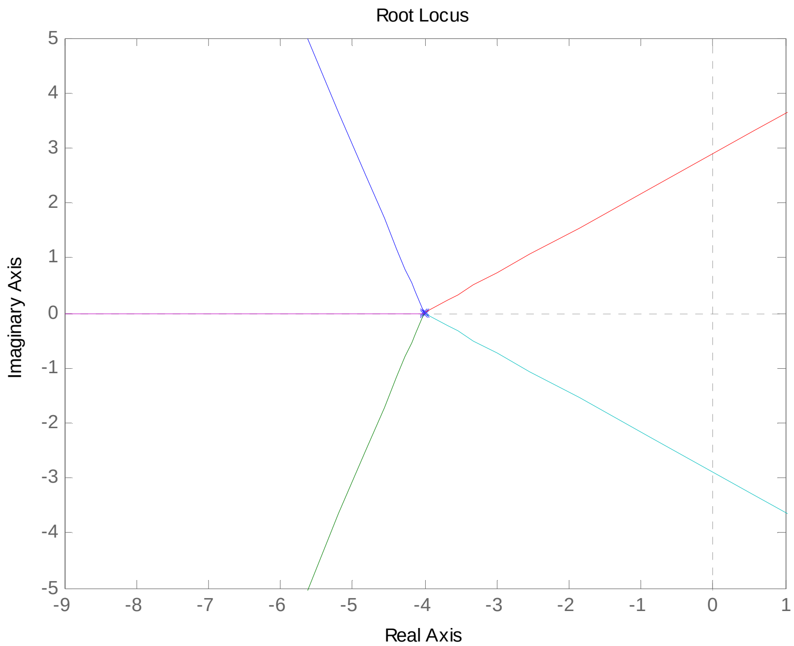
Root Locus diagram – 7-21(c):



Root Locus diagram – 7-21(d):



Root Locus diagram – 7-21(e):



7-22) $P(s) = s^3 + 25s^2 + 2s + 100 \qquad Q(s) = 100s$

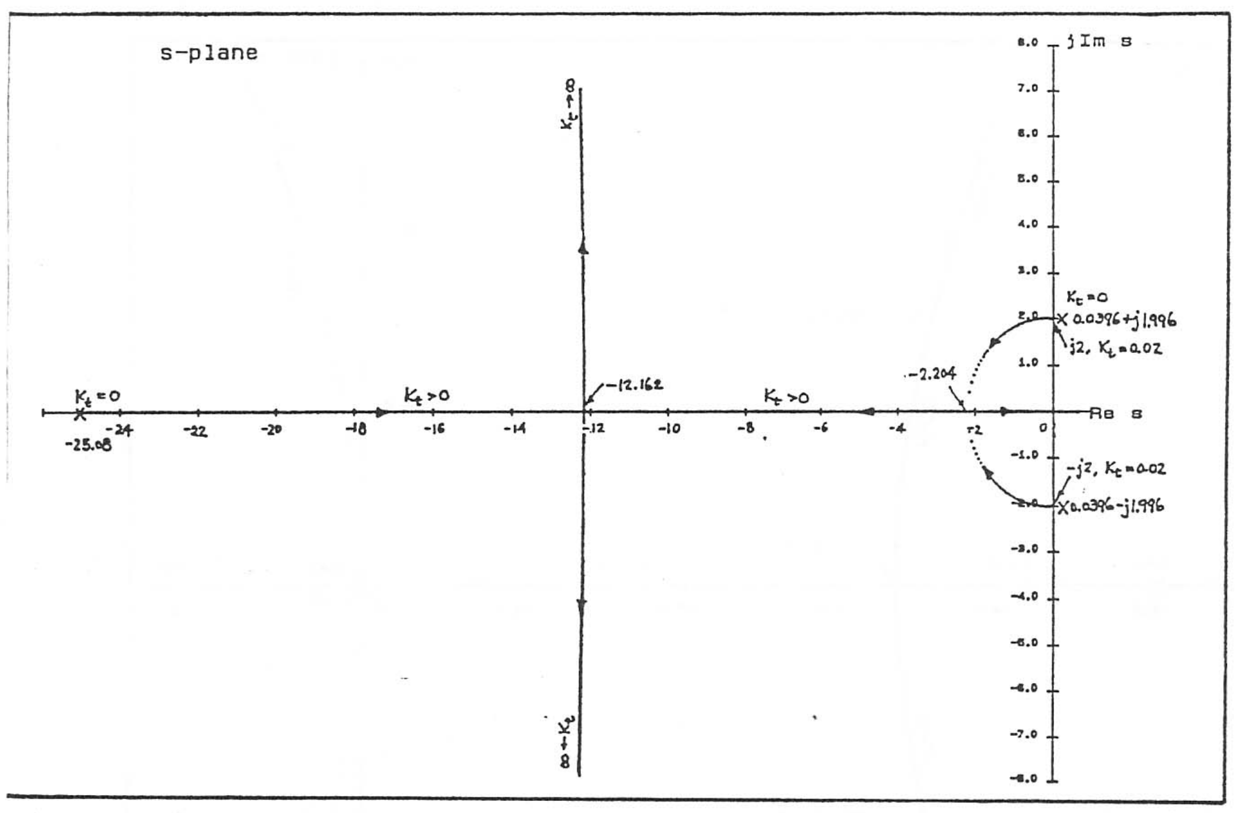
Asymptotes: $K_t > 0$: 90^o, 270^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{-25-0}{3-1} = -12.5$$

Breakaway-point Equation: $s^3 + 12.5s^2 - 50 = 0$

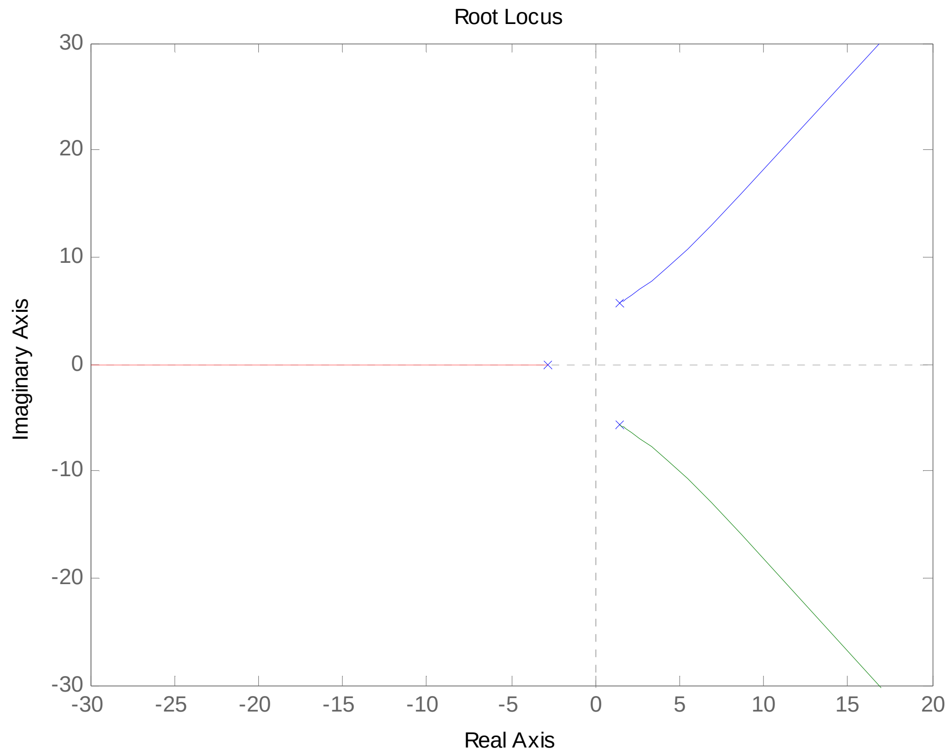
Breakaway Points: **(RL)** −2.2037, −12.162



7-23) MATLAB code:

```
s = tf('s')
num_G= 100;
den_G=s^3+25*s+2*s+100;
G=num_G/den_G;
figure(1);
rlocus(G)
```

Root Locus diagram – 7-23:



7-24) **Characteristic equation:** $s^3 + 5s^2 + K_t s + K = 0$

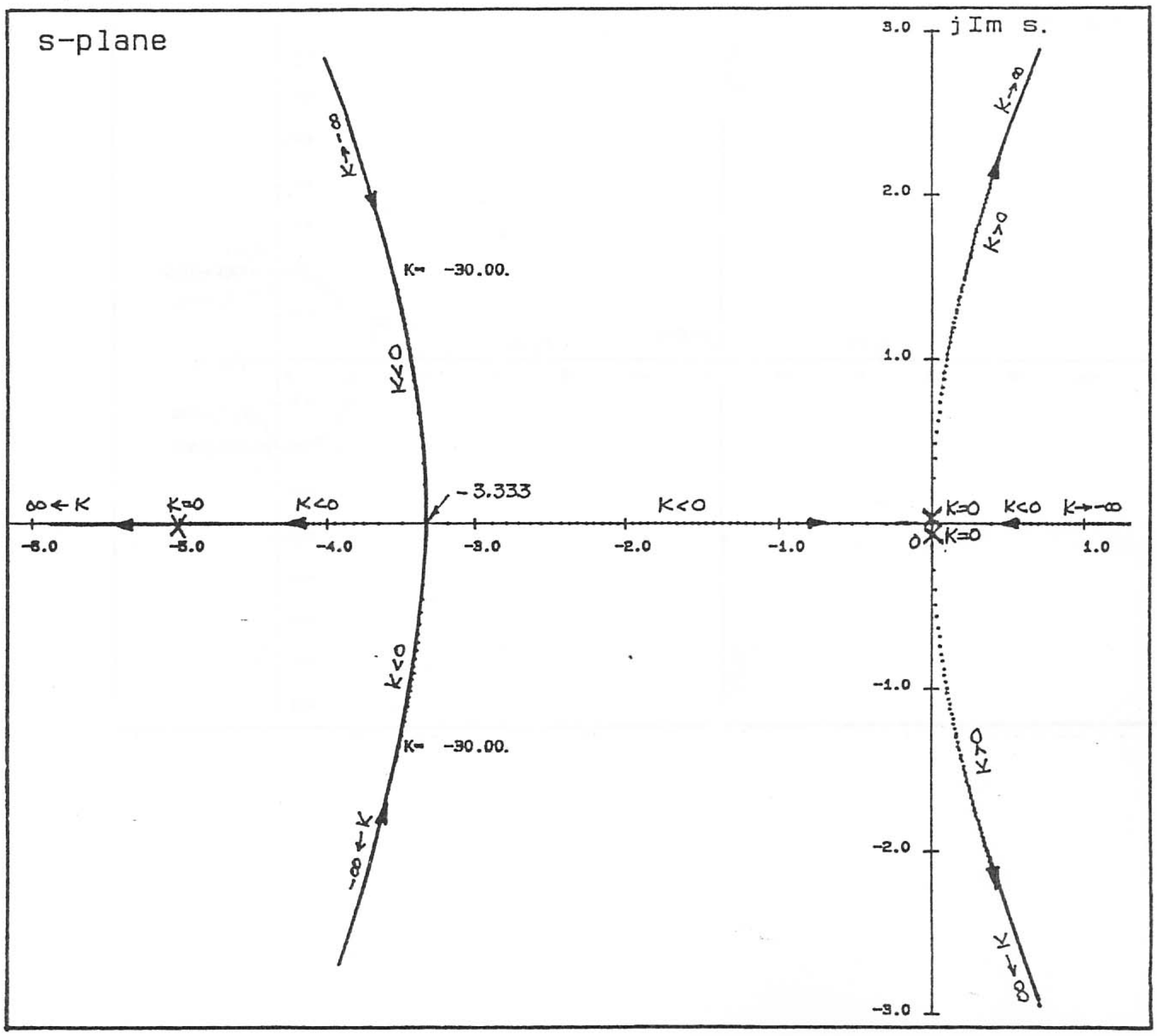
(a) $K_t = 0$: $P(s) = s^2\left(s+5\right)$ $Q(s) = 1$

Asymptotes: ***K* > 0:** 60^o, 180^o, 300^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{-5-0}{3} = -1.667$$

Breakaway-point Equation: $3s^2 + 10s = 0$ **Breakaway Points:** 0, -3.333



7-24 (b) $P(s) = s^3 + 5s^2 + 10 = 0 \qquad Q(s) = s$

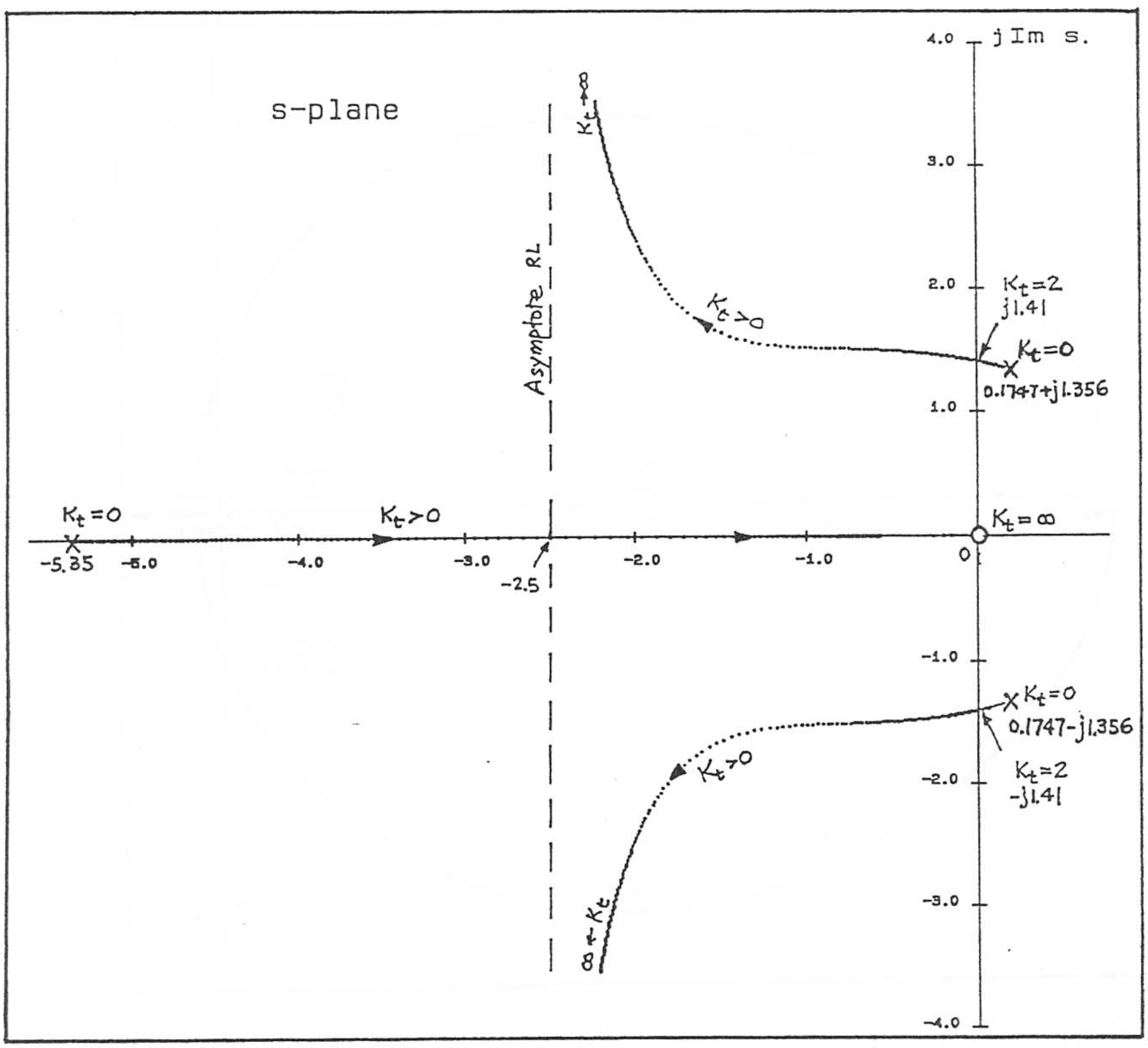
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-5-0}{2-1} = 0$$

Breakaway-point Equation: $2s^3 + 5s - 10 = 0$

There are no breakaway points on RL.



7-25)

By collapsing the two loops, and finding the overall close loop transfer function, the characteristic equation (denominator of closed loop transfer function) can be found as:

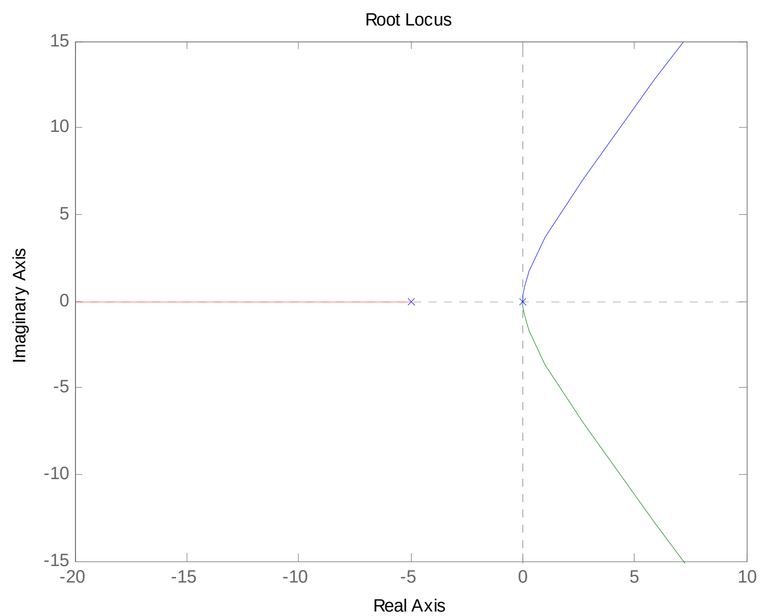
$$1+GH=\frac{s^3+5s^2+K_t s+K}{s^2(s+5)+K_t s}$$

For part (a):

K_t =0. Therefore, assuming

Den(GH)= s^3+5s^2 and Num(GH) =1, we can use rlocus command to construct the root locus diagram.

Root locus diagram, part (a):

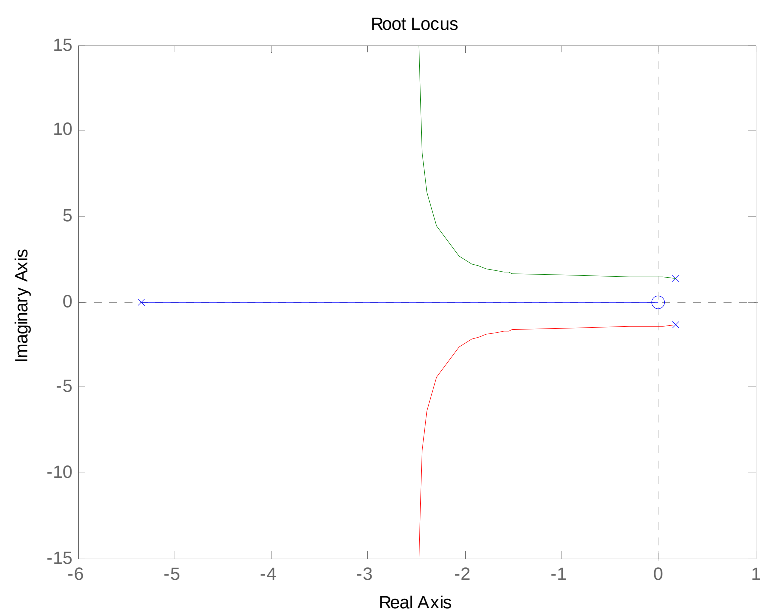


For part (b):

K =10. Therefore, assuming

Den(GH)= s^3+5s^2+10 and Num(GH) = s , we can use rlocus command to construct the root locus diagram.

Root locus diagram, part (b):



MATLAB code (7-25):

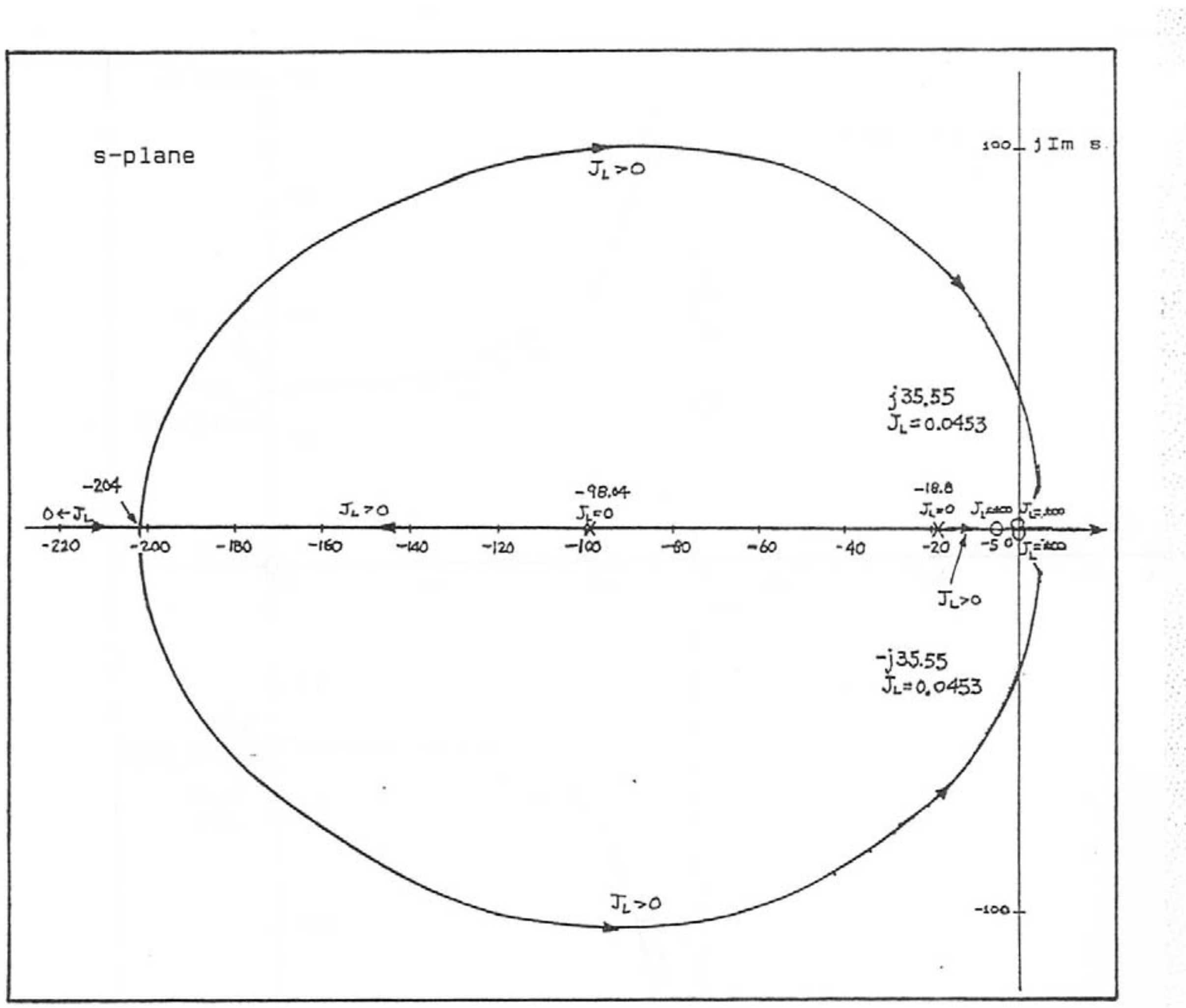
```
s = tf('s')
%a)
num_G_a= 1;
den_G_a=s^3+5*s^2;
GH_a=num_G_a/den_G_a;
figure(1);
rlocus(GH_a)

%b)
num_G_b= s;
den_G_b=s^3+5*s^2+10;
GH_b=num_G_b/den_G_b;
figure(2);
rlocus(GH_b)
```

7-26) $P(s) = s^2 + 116.84s + 1843 \qquad Q(s) = 2.05s^2(s+5)$

Asymptotes: $J_L = 0{:}\quad 180^o$

Breakaway-point Equation: $-2.05s^4 - 479s^3 - 12532s^2 - 37782s = 0$

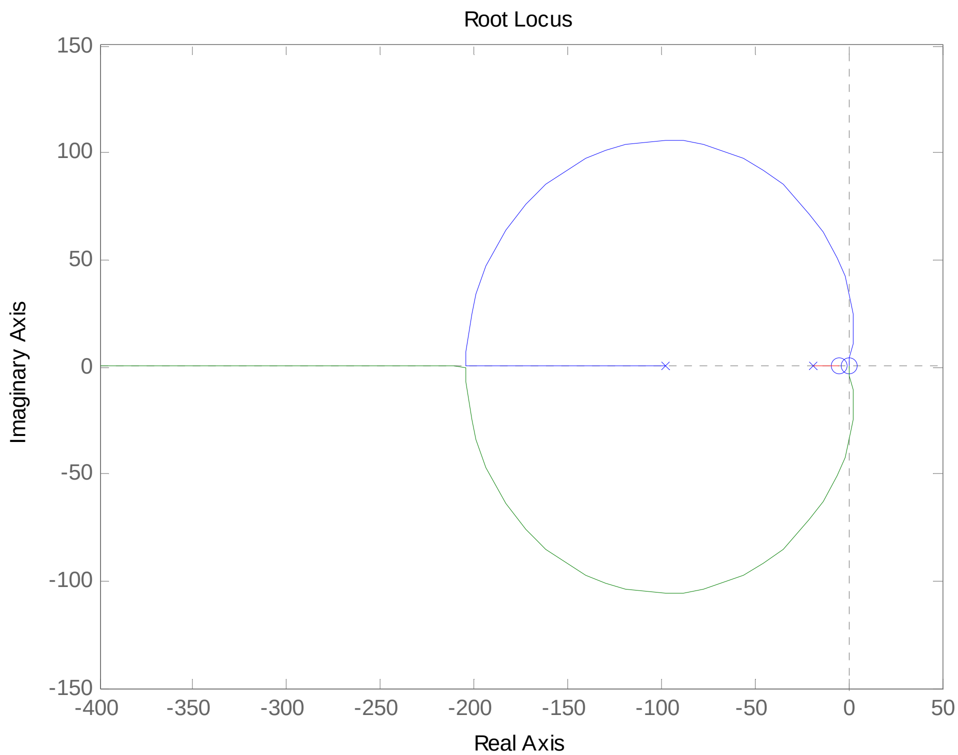
Breakaway Points: **(RL)** 0, −204.18



7-27) MATLAB code:

```
s = tf('s')
num_G = (2.05*s^3 + 10.25*s^2);
den_G = (s^2 + 116.84*s + 1843);
G = num_G/den_G;
figure(1);
rlocus(G)
```

Root locus diagram:

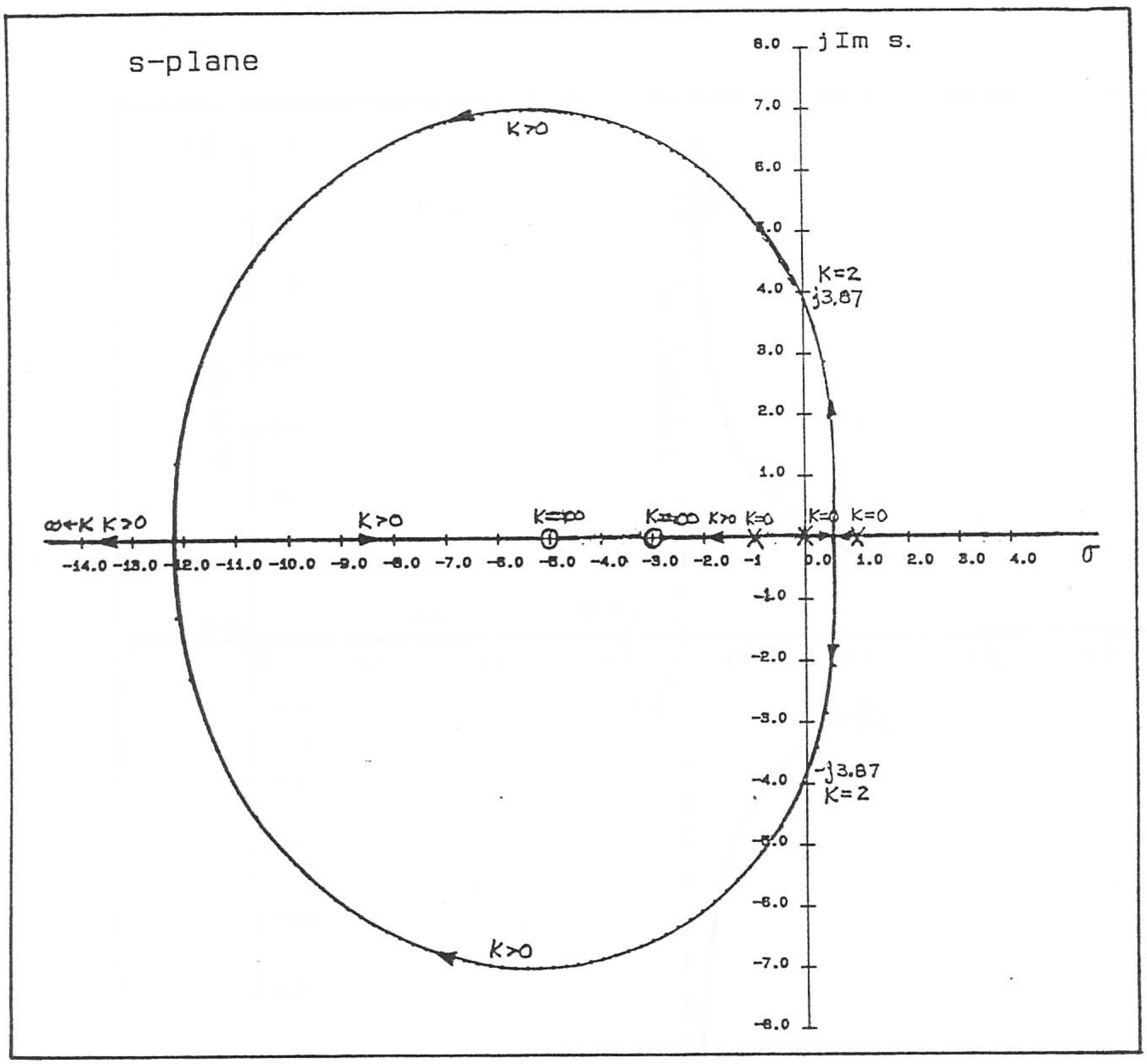


7-28) (a) $P(s) = s\left(s^2 - 1\right)$ $Q(s) = (s+5)(s+3)$

Asymptotes: ***K* > 0:** 180^o

Breakaway-point Equation: $s^4 + 16s^3 + 46s^2 - 15 = 0$

Breakaway Points: **(RL)** 0.5239, –12.254



7-28 (b) $P(s) = s\left(s^2 + 10s + 29\right) \qquad Q(s) = 10(s+3)$

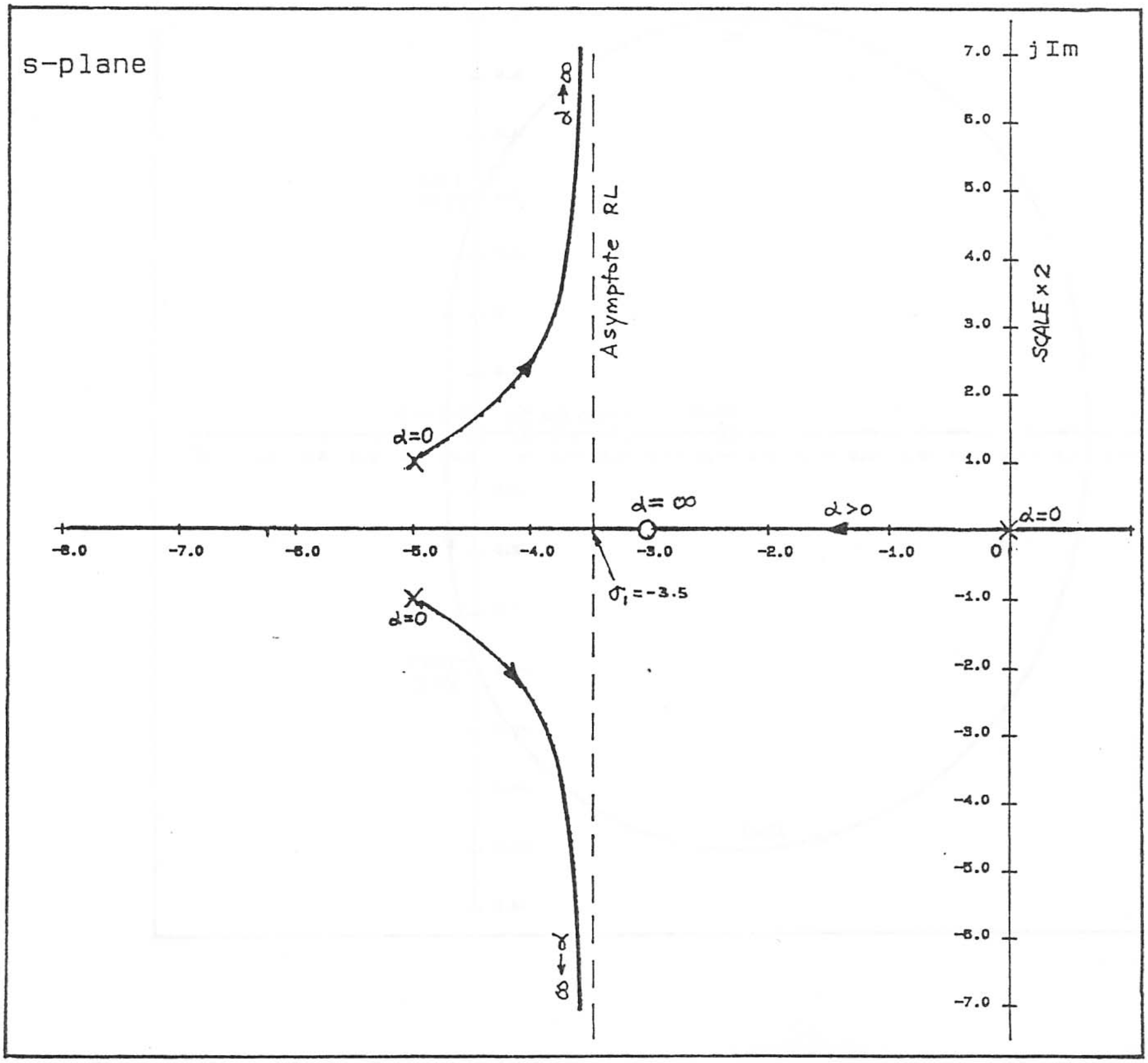
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 10 - (-3)}{3 - 1} = -3.5$$

Breakaway-point Equation: $20s^3 + 190s^2 + 600s + 870 = 0$

There are no breakaway points on the RL.

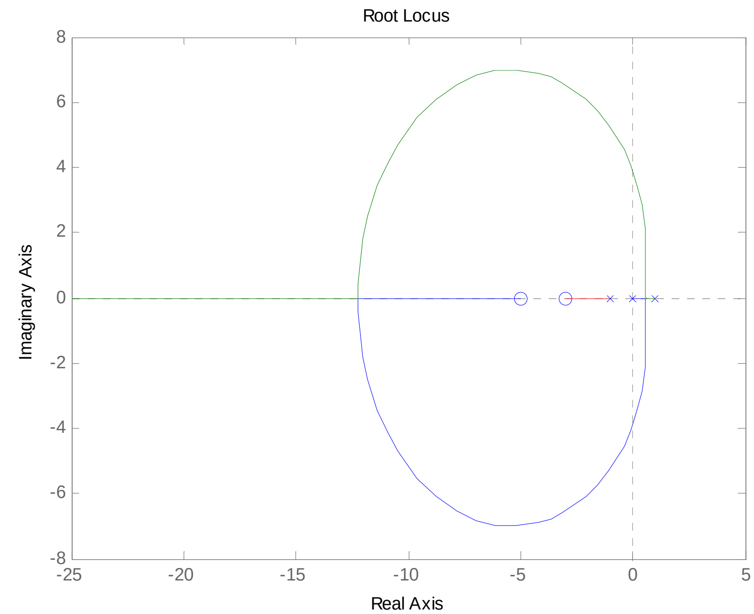


7-29)

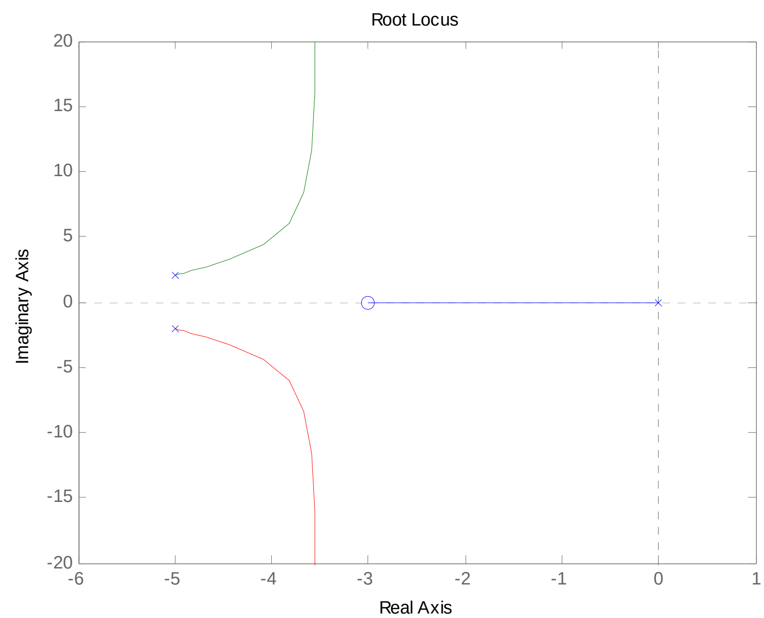
MATLAB code (7-29):	**Root locus diagram, part (a):**

s = tf('s')
%a)
num_G_a = (s+5)*(s+3);
den_G_a = s*(s^2 - 1);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)	

```
K=10;
%b)
num_G_b  = (3*K+K*s);
den_G_b =
(s^3+K*s^2+K*3*s-s);
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)
```



Root locus diagram, part (b):



7-30) Poles: $s = 0, -3.6$ zeros: $s = -0.4$

Angles of asymptotes: $\theta_i = \frac{2i+1}{3-1} \times 180 = 90^\circ, 270^\circ$

$$\sigma = -\frac{3.6 - 0.4}{3 - 1} = -1.6$$

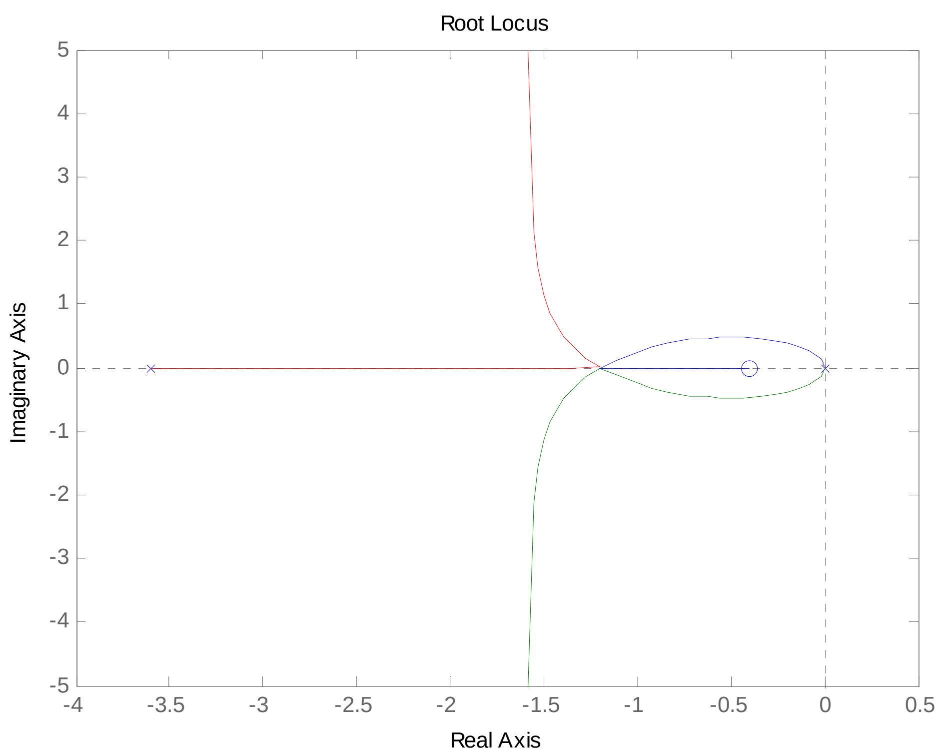
breakaway points: $\frac{1}{s^2} + \frac{1}{s+3.6} = \frac{1}{s+0.4}$

$\Rightarrow \; s^3 + 2.4s^2 + 1.44s = 0 \;\rightarrow\; s = 0, -1.2$

MATLAB code:

```
s = tf('s')
num_G=(s+0.4);
den_G=s^2*(s+3.6);
G=num_G/den_G;
figure(1);
rlocus(G)
```

Root locus diagram:



7-31 (a) $P(s) = s(s+12.5)(s+1) \qquad Q(s) = 83.333$

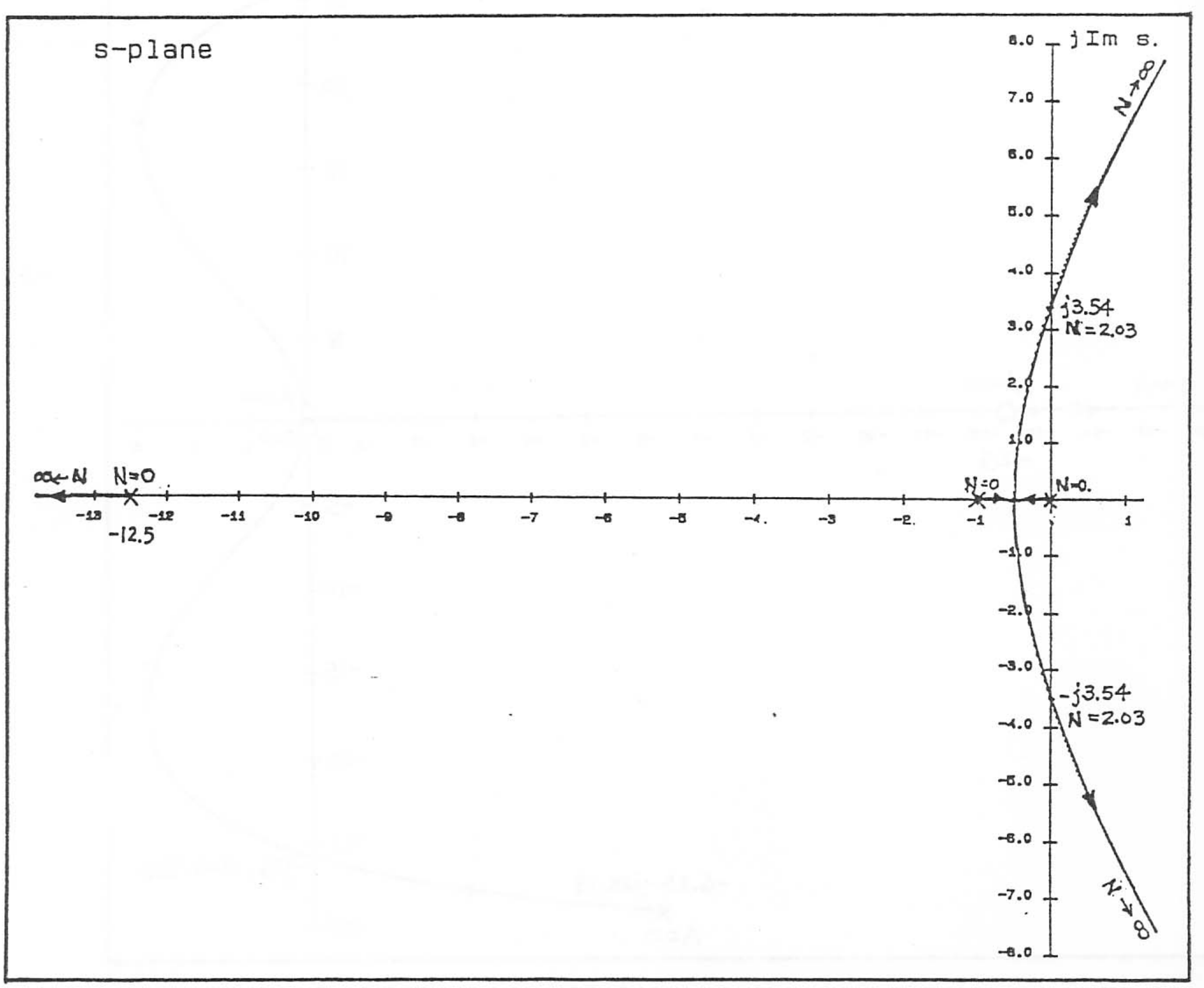
Asymptotes: ***N* > 0:** $60^o, \quad 180^o, \quad 300^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-12.5-1}{3} = -4.5$$

Breakaway-point Equation: $3s^2 + 27s\_12.5 = 0$

Breakaway Point: (RL) –0.4896

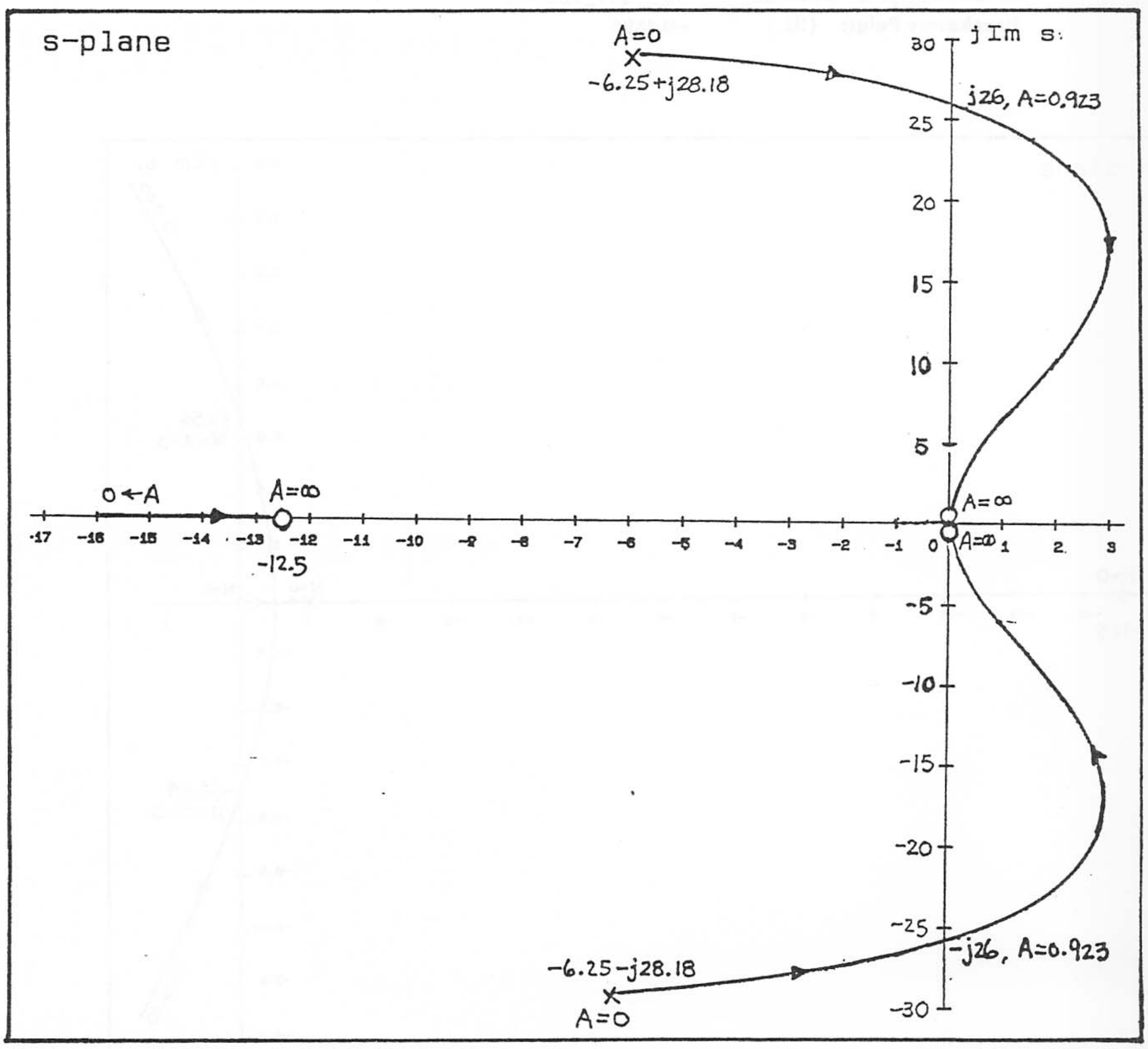


7-31 (b) $P(s) = s^2 + 12.5s + 833.333 \qquad Q(s) = 0.02s^2(s+12.5)$

A > 0: 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 53.125s^2 + 416.67s = 0$

Breakaway Points: **(RL)** 0



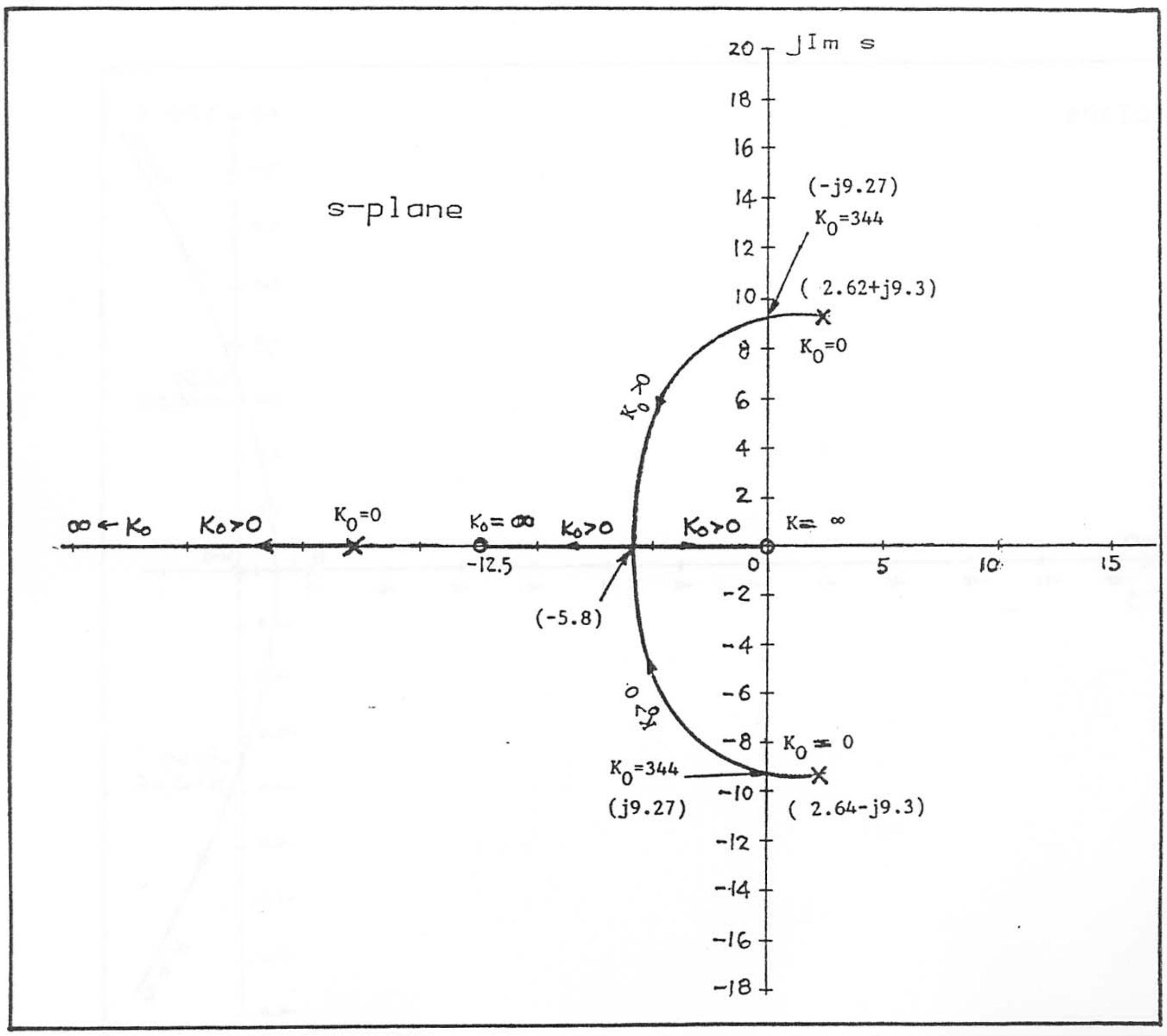
7-31 c) $P(s) = s^3 + 12.5s^2 + 1666.67 = (s + 17.78)(s - 2.64 + j9.3)(s - 2.64 - j9.3)$

$Q(s) = 0.02s(s + 12.5)$

Asymptotes: $K_o > 0$: 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 3.125s^2 - 66.67s - 416.67 = 0$

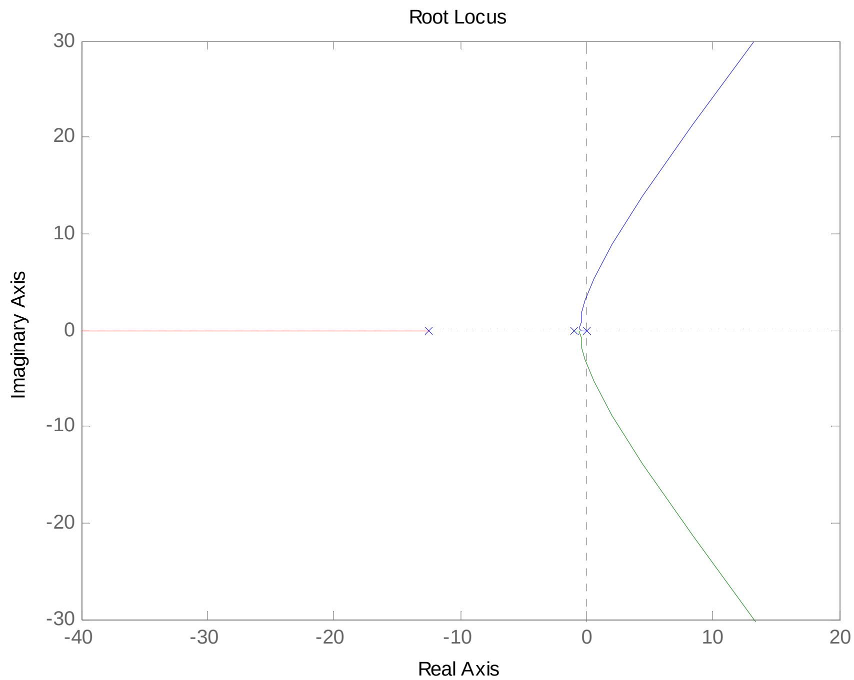
Breakaway Point: (RL) -5.797



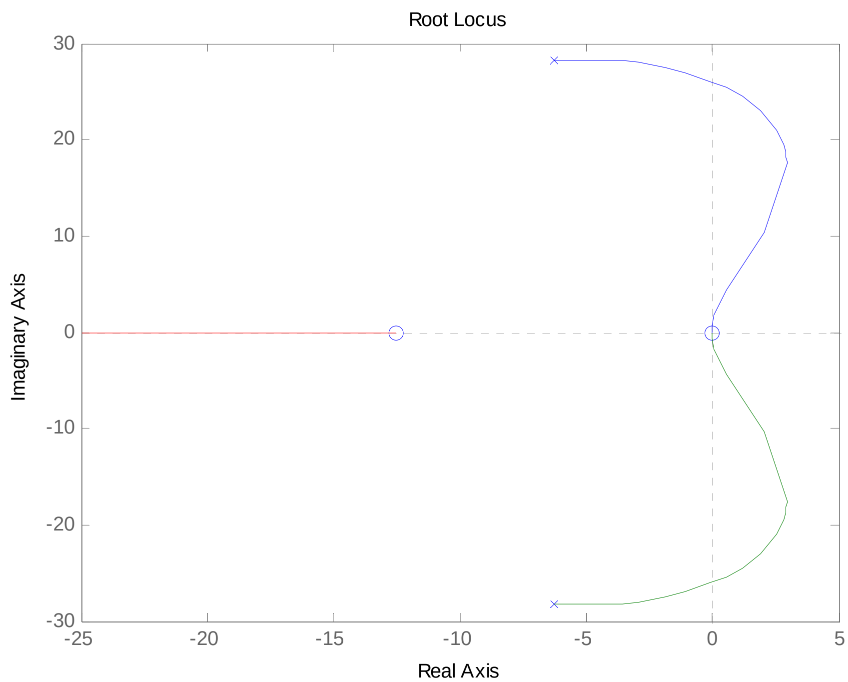
7-32) MATLAB code:

```
s = tf('s')
%a)
A=50;
K0=50;
num_G_a = 250;
den_G_a = 0.06*s*(s + 12.5)*(A*s+K0);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
N=10;
K0=50;
num_G_b = 0.06*s*(s+12.5)*s
den_G_b = K0*(0.06*s*(s+12.5))+250*N;
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
A=50;
N=20;
num_G_c = 0.06*s*(s+12.5);
den_G_c = 0.06*s*(s+12.5)*A*s+250*N;
G_c = num_G_c/den_G_c;
figure(3);
rlocus(G_c)
```

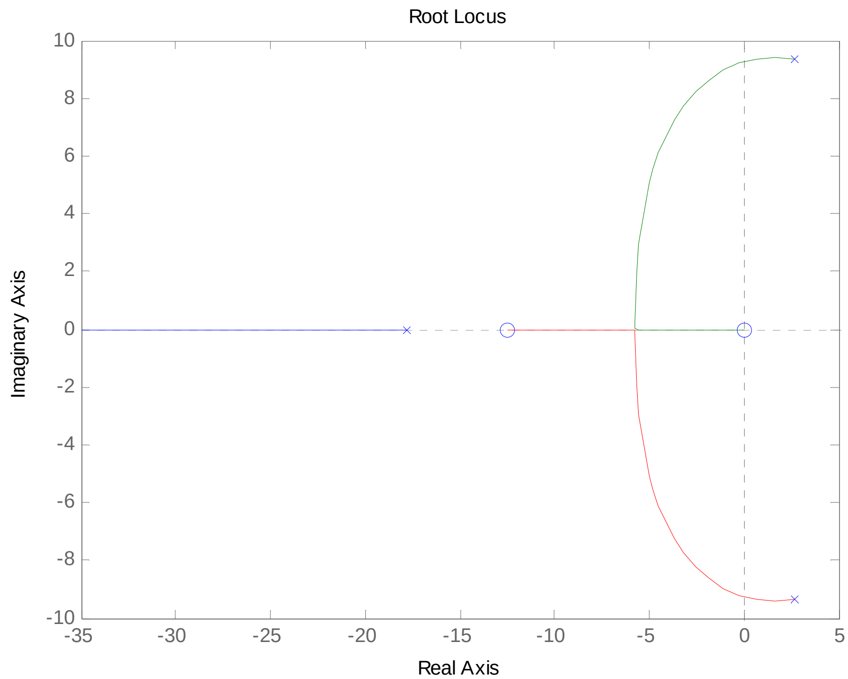
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c):



7-33) (a) $A = K_o = 100$: $P(s) = s(s+12.5)(s+1)$ $Q(s) = 41.67$

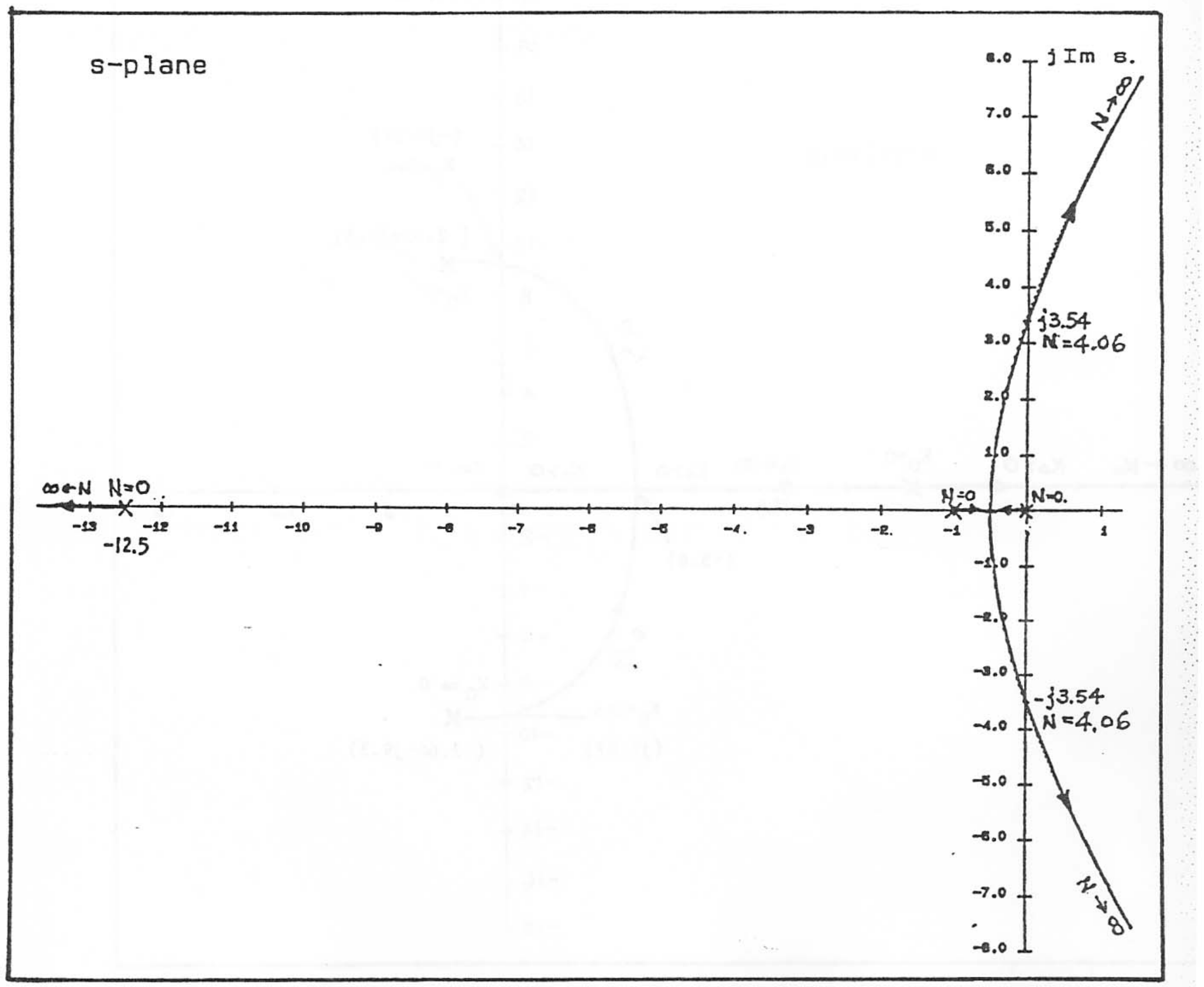
Asymptotes: ***N* > 0:** 60^o 180^o 300^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-12.5}{3} = -4.5$$

Breakaway-point Equation: $3s^2 + 27s + 12.5 = 0$

Breakaway Points: (RL) –0.4896



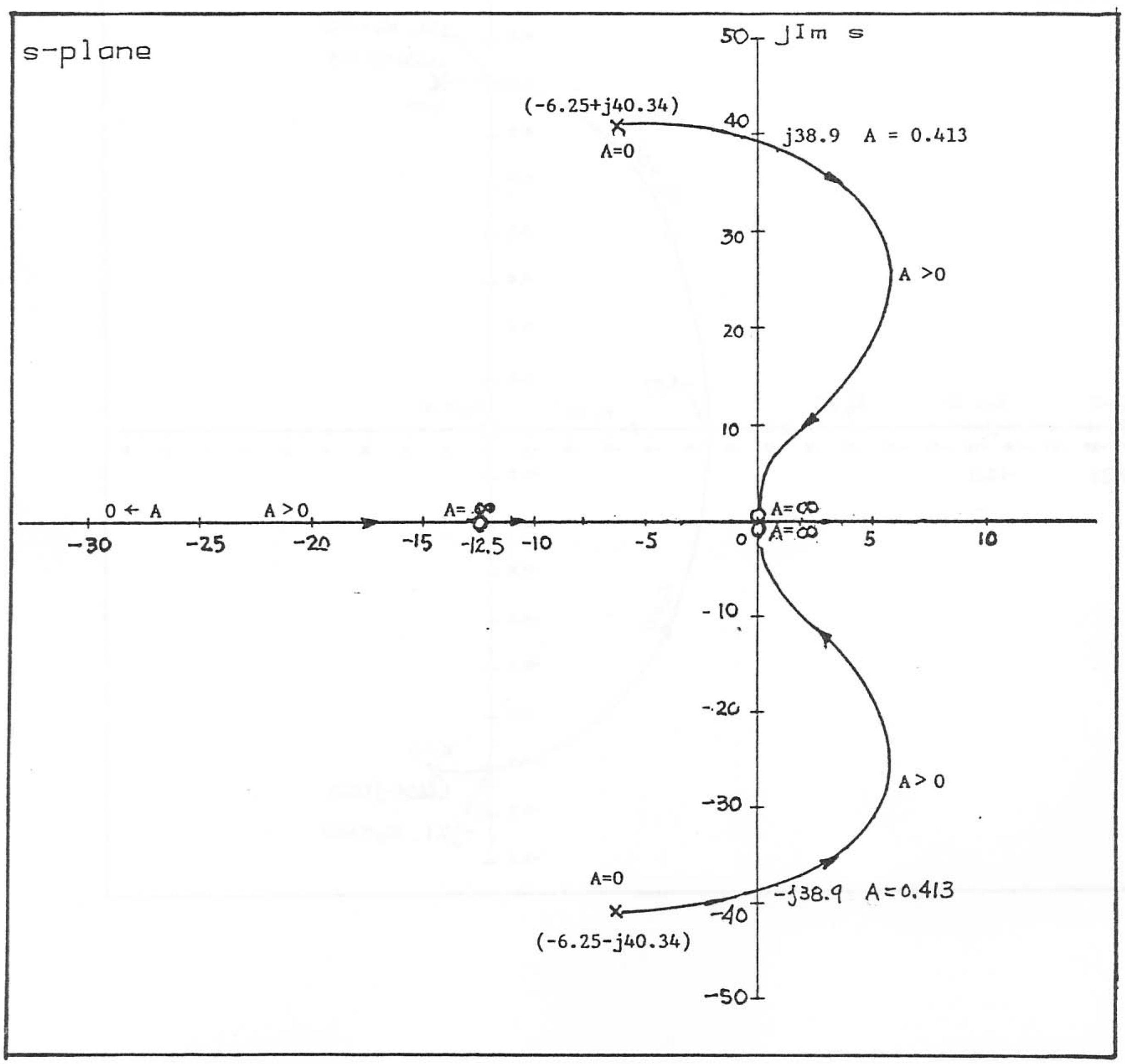
7-33 (b) $P(s) = s^2 + 12.5 + 1666.67 = (s + 6.25 + j40.34)(s + 6.25 - j40.34)$

$Q(s) = 0.02s^2(s + 12.5)$

Asymptotes: ***A* > 0:** 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 103.13s^2 + 833.33s = 0$

Breakaway Points: (RL) 0



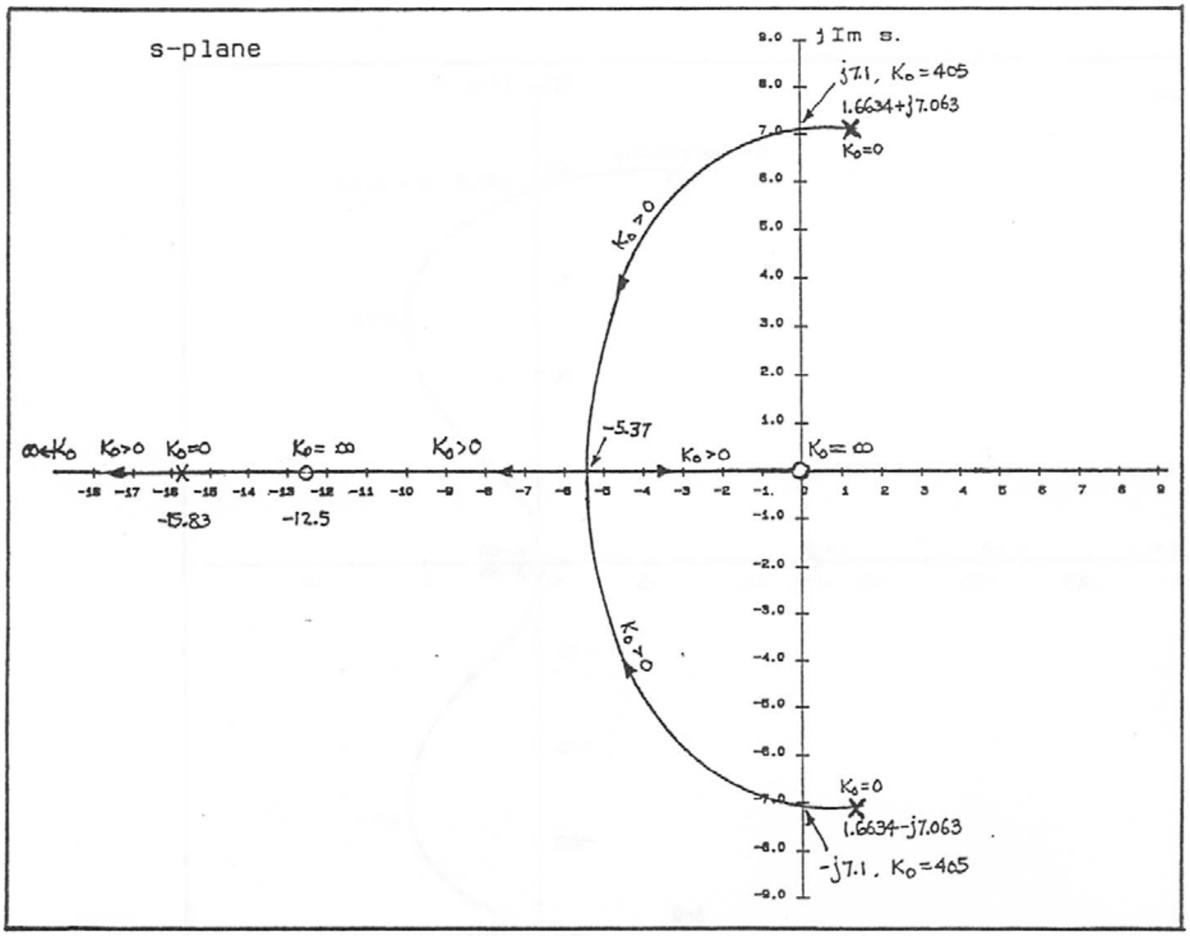
7-33 (c) $P(s) = s^3 + 12.5s^2 + 833.33 = (s + 15.83)(s - 1.663 + j7.063)(s - 1.663 - j7.063)$

$Q(s) = 0.01s(s + 12.5)$

Asymptotes: $K_o > 0$: 180^o

Breakaway-point Equation: $0.01s^4 + 0.15s^3 + 1.5625s^2 - 16.67s - 104.17 = 0$

Breakaway Point: **(RL)** –5.37

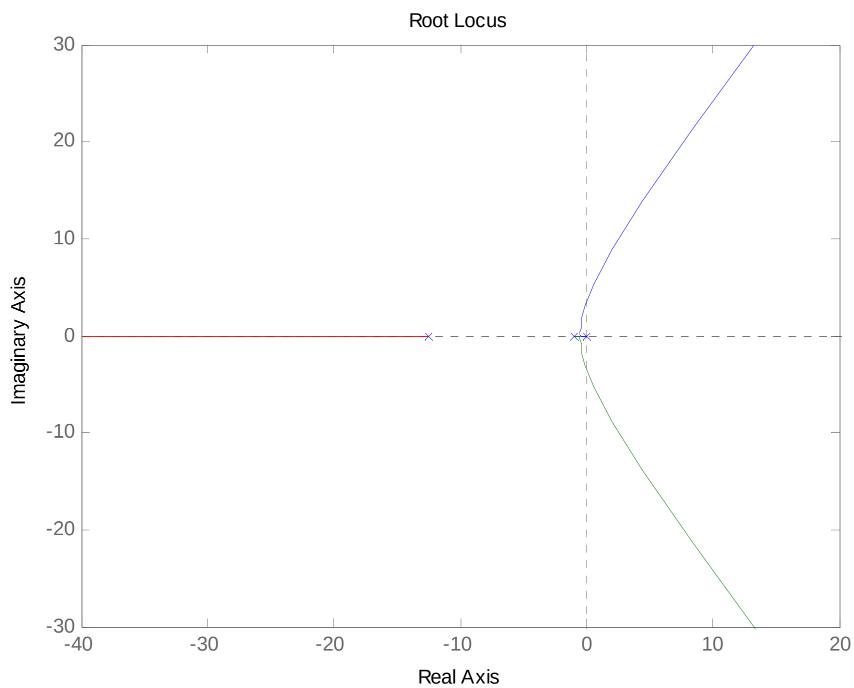


7-34) MATLAB code:

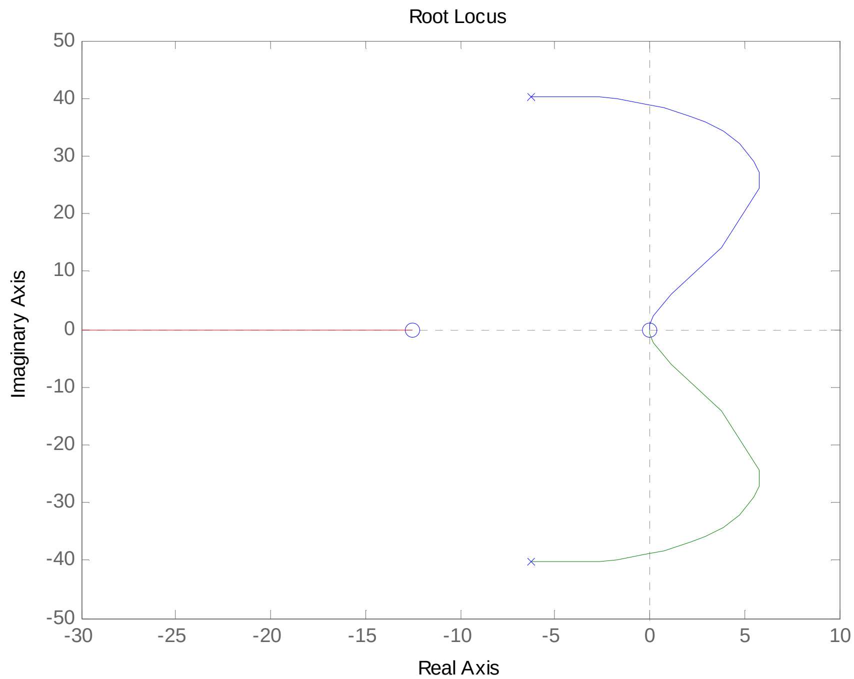
```
s = tf('s')
%a)
A=100;
K0=100;
num_G_a = 250;
den_G_a = 0.06*s*(s + 12.5)*(A*s+K0);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
N=20;
K0=50;
```

```
num_G_b = 0.06*s*(s+12.5)*s
den_G_b = K0*(0.06*s*(s+12.5))+250*N;
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
A=100;
N=20;
num_G_c = 0.06*s*(s+12.5);
den_G_c = 0.06*s*(s+12.5)*A*s+250*N;
G_c = num_G_c/den_G_c;
figure(3);
rlocus(G_c)
```

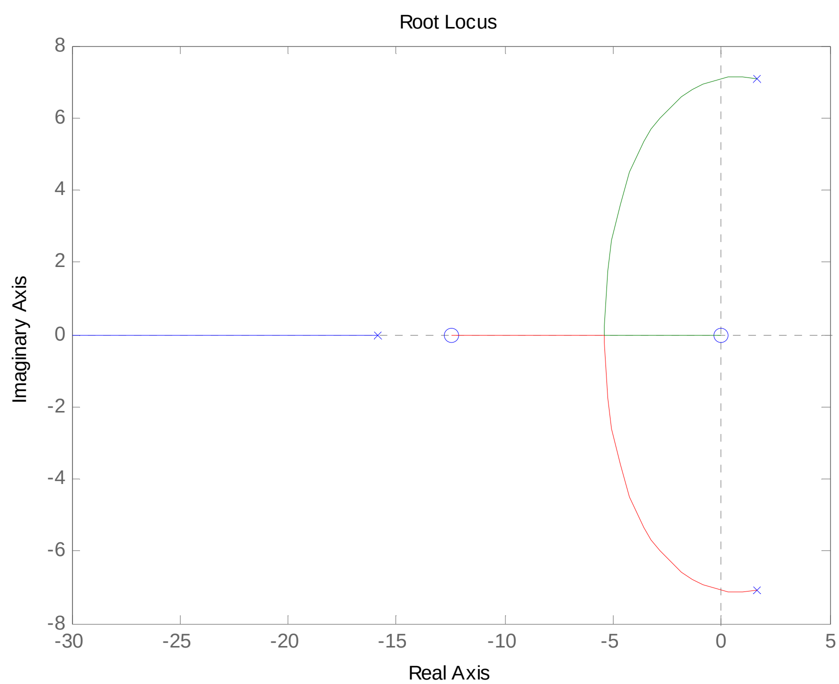
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c):



7-35) a) zeros: $s = -2$, poles: $s = -2j, +2j, -5$

Angle of asymptotes: $\theta_i = \dfrac{2i+1}{4-2} \times 180 = 90{,}270$

$\sigma = -3$

Breakaway points: $\dfrac{1}{s^2+4} + \dfrac{1}{(s+5)^2} = \dfrac{1}{(s+2)^2}$

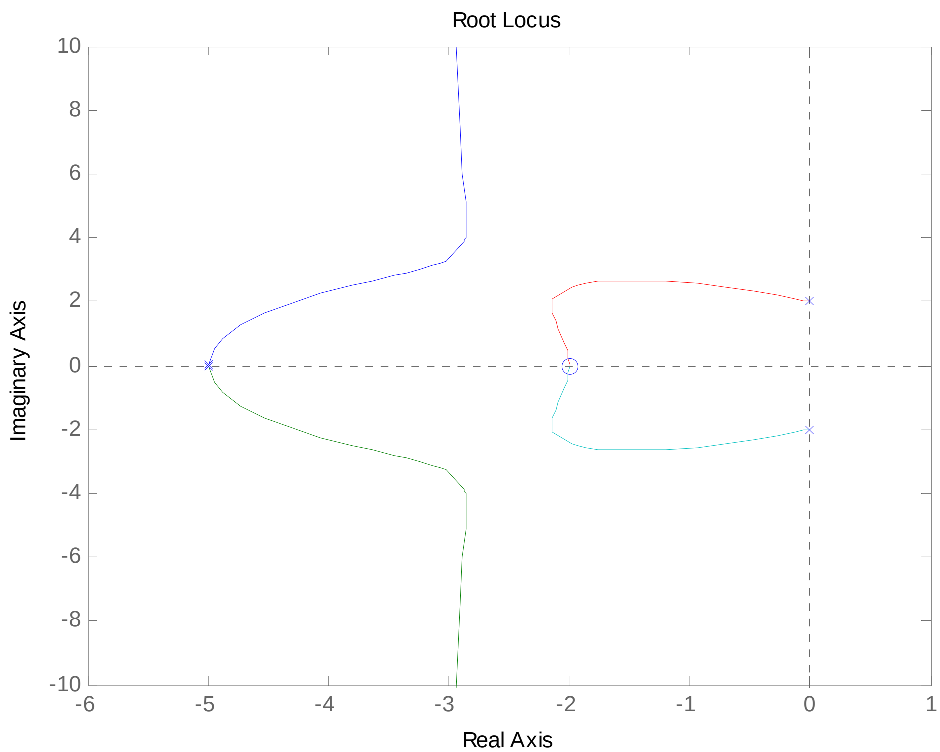
$\Rightarrow \; s^2 + 6s + 25 = 0 \;\; \rightarrow s = -1.5 + 2j, 1.5 - 2j$

b) There is no closed loop pole in the right half s-plane; therefore the system is stable for all K>0

c) MATLAB code:

```
num_G=25*(s+2)^2;
den_G=(s^2+4)*(s+5)^2;
G_a=num_G/den_G;
figure(1);
rlocus(G_a)
```

Root locus diagram:



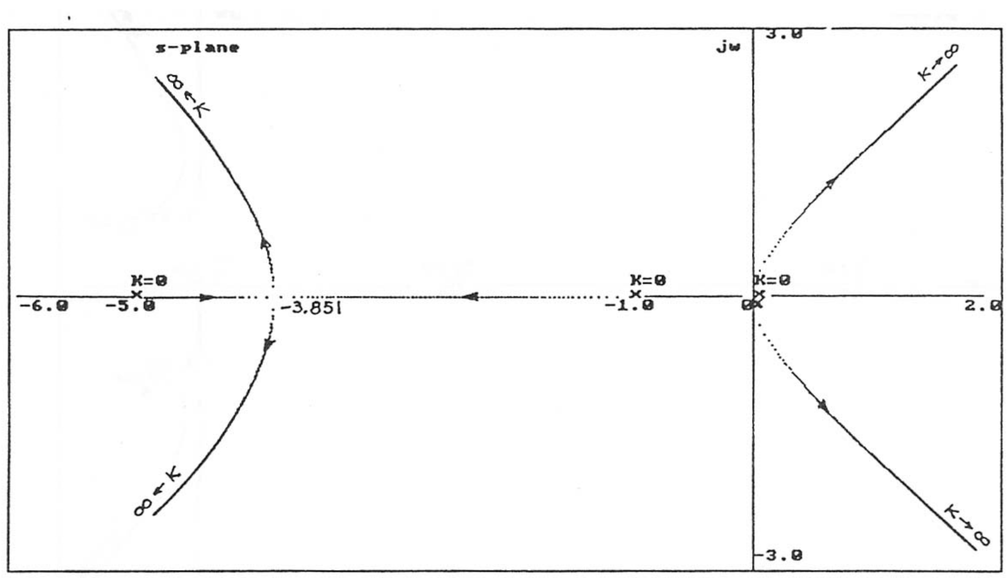
7-36) (a) $P(s) = s^2(s+1)(s+5) \qquad Q(s) = 1$

Asymptotes: *K* > 0: $45^o, \quad 135^o, \quad 225^o, \quad 315^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-1-5}{4} = -1.5$$

Breakaway-point Equation: $4s^3 + 18s^2 + 10s = 0$ **Breakaway point: (RL)** 0, −3.851



(b) $P(s) = s^2(s+1)(s+5) \qquad Q(s) = 5s+1$

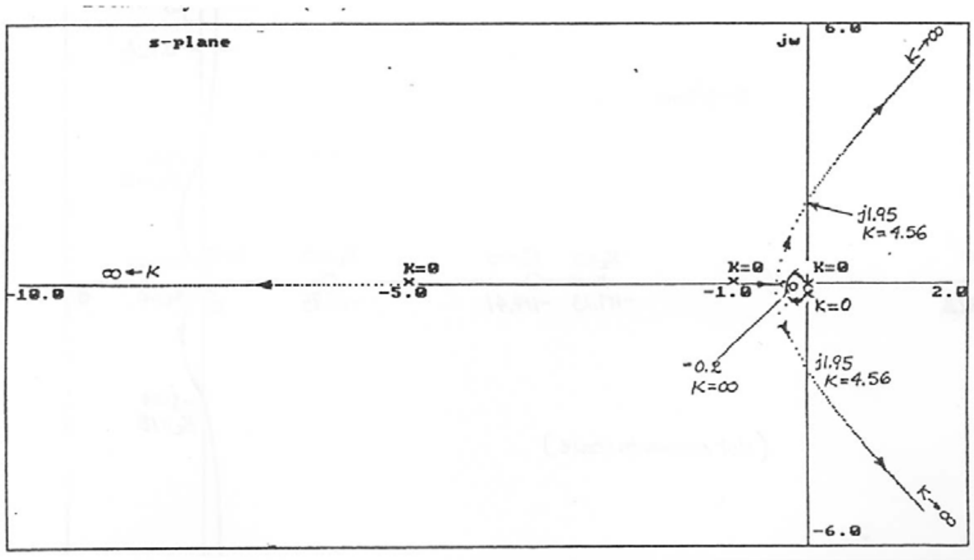
Asymptotes: *K* > 0: $60^o, \quad 180^o, \quad 300^o$

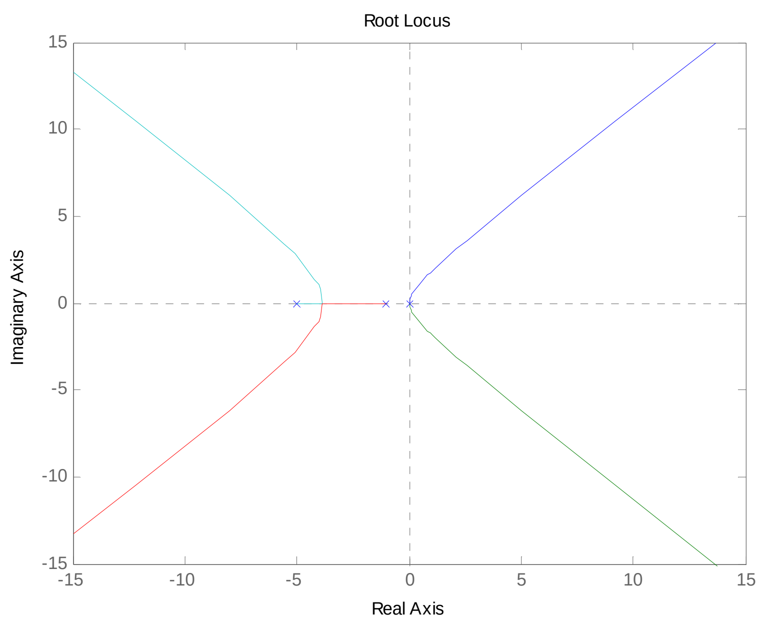
Intersect of Asymptotes:

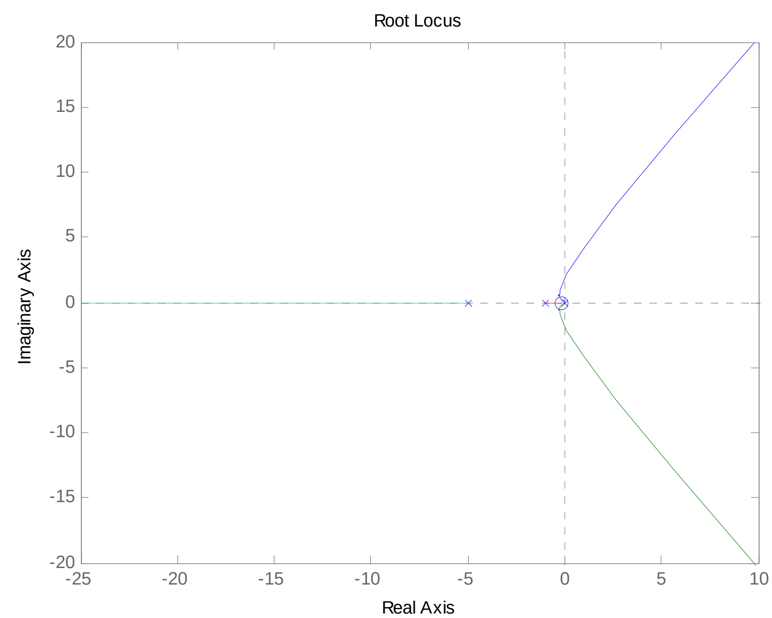
$$\sigma_1 = \frac{0+0-1-5-(-0.2)}{4-1} = -\frac{5.8}{3} = -1.93$$

Breakaway-point Equation: $15s^4 + 64s^3 + 43s^2 + 10s = 0$

Breakaway Points: (RL) −3.5026



7-37)
MATLAB code (7-37):
`s = tf('s')`
`%a)`
`num_GH_a= 1;`
`den_GH_a=s^2*(s+1)*(s+5);`
`GH_a=num_GH_a/den_GH_a;`
`figure(1);`
`rlocus(GH_a)`
 	
`%b)`
`num_GH_b= (5*s+1);`
`den_GH_b=s^2*(s+1)*(s+5);`
`GH_b=num_GH_b/den_GH_b;`
`figure(2);`
`rlocus(GH_b)`
	Root locus diagram, part (a):
	
	Root locus diagram, part (b):



7-38) a) e^{-s} can be approximated by (easy way to verify is to compare both funtions' Taylor series expansions)

$$e^{-s} \approx \frac{2-s}{2+s}$$

Therefore:

$$G(s) = -\frac{K(s-2)}{(s+1)(s+2)}$$

Zeros: $s = 2$ and poles: $s = -1, -2$

Angle of asymptotes : $\theta_i = (2i+1)180 = 180$

$$\sigma_0 = -(1+2-2) = -1$$

Breakaway points: $\frac{1}{s+1} + \frac{1}{s+2} = \frac{1}{2-s}$

Which means: $s^2 + 4s = 0$ → $s = 0, s = \pm 2$

b) $s+1+K\frac{2 \quad s}{s+2} = 0$ → $s^2 + 3s + 2 - Ks + 2K = 0$

S^2	1	2+2k

S	3-k	0
S^0	(3-k)(2+2k)	

As a result:

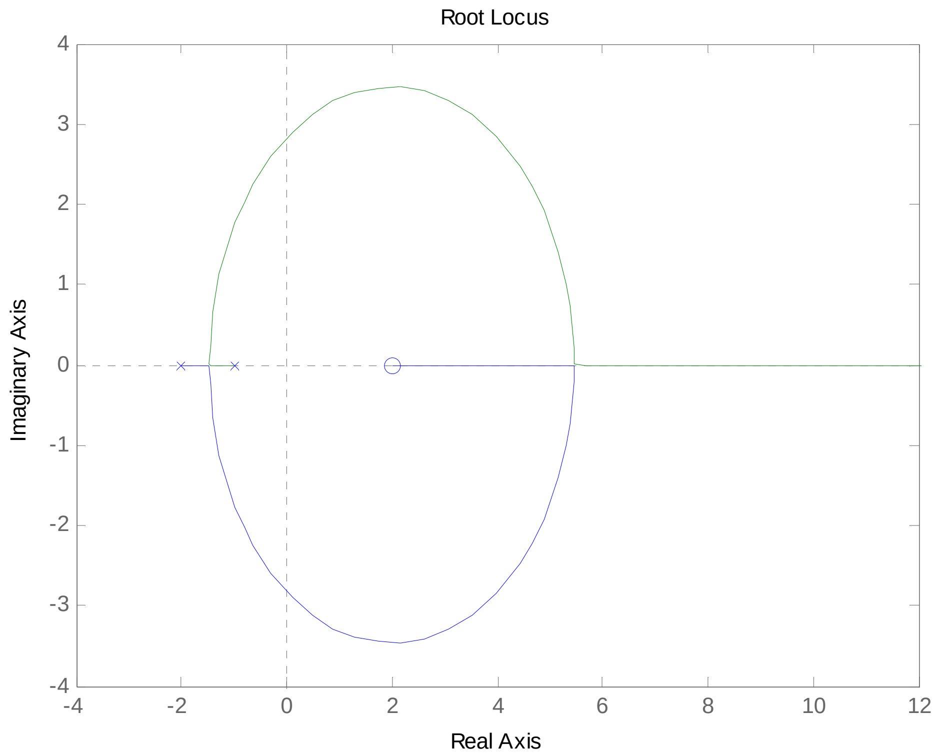
$$\begin{cases} 3-K>0 \rightarrow K<3 \\ (3-K)(2+2K)>0 \rightarrow 2+2k>0 \rightarrow K>-1 \end{cases}$$

Since K must be positive, the range of stability is then $0<k<3$

c) In this problem, e^{-Ts} term is a time delay. Therefore, MATLB PADE command is used for pade approximation, where brings e^{-Ts} term to the polynomial form of degree N.

```
s = tf('s')
T=1
N=1;
num_GH= pade(exp(-1*T*s),N);
den_GH=(s+1);
GH=num_GH/den_GH;
figure(5);
rlocus(GH)
```

Root locus diagram:



7-39)

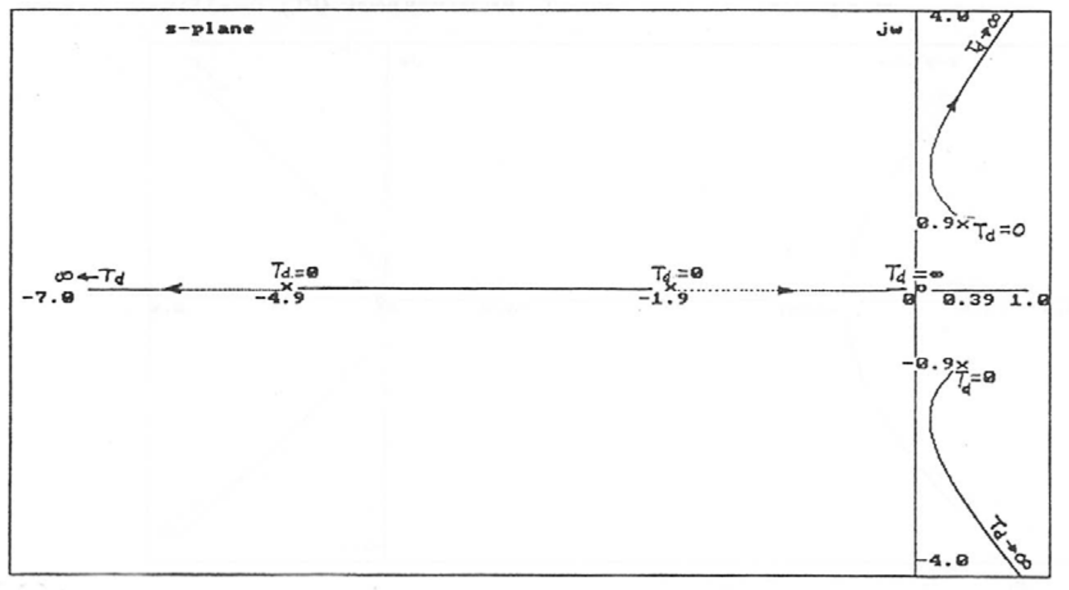
(a) $P(s) = s^2(s+1)(s+5)+10 = (s+4.893)(s+1.896)(s-0.394+j0.96)(s-0.394+j0.96)$

$Q(s) = 10s$

Asymptotes: $T_d > 0$: 60^o, 180^o, 300^o

Intersection of Asymptotes: $\sigma_1 = \dfrac{-4.893-1.896+0.3944+0.3944}{4-1} = -2$

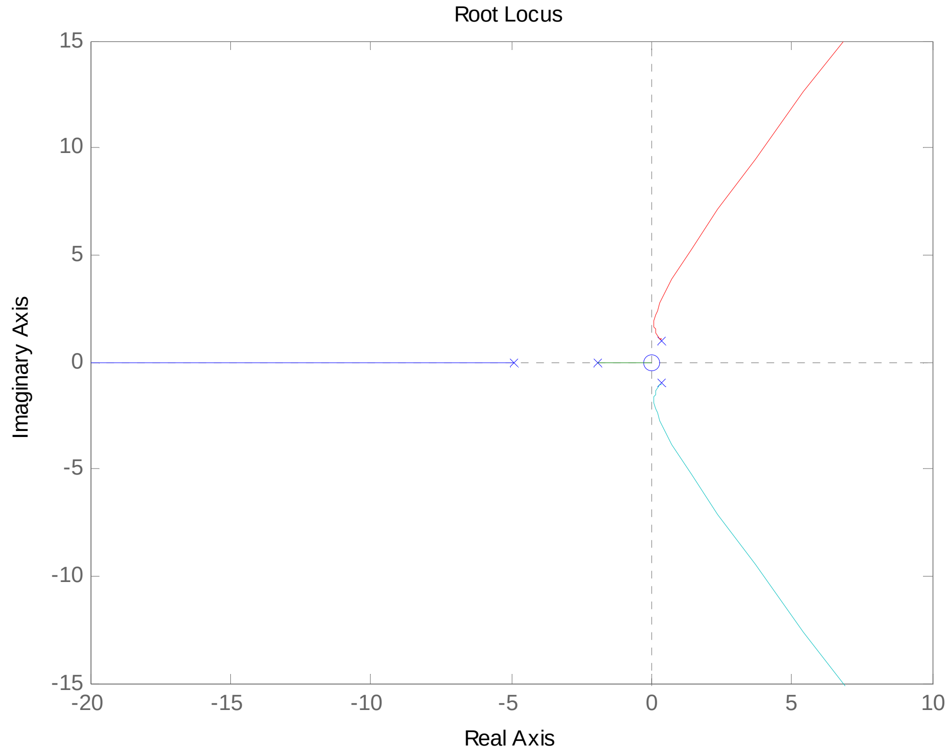
There are no breakaway points on the RL.



(b) MATLAB code:

```
s = tf('s')
num_GH= 10*s;
den_GH=s^2*(s+1)*(s+5)+10;
GH=num_GH/den_GH;
figure(1);
rlocus(GH)
```

Root locus diagram:



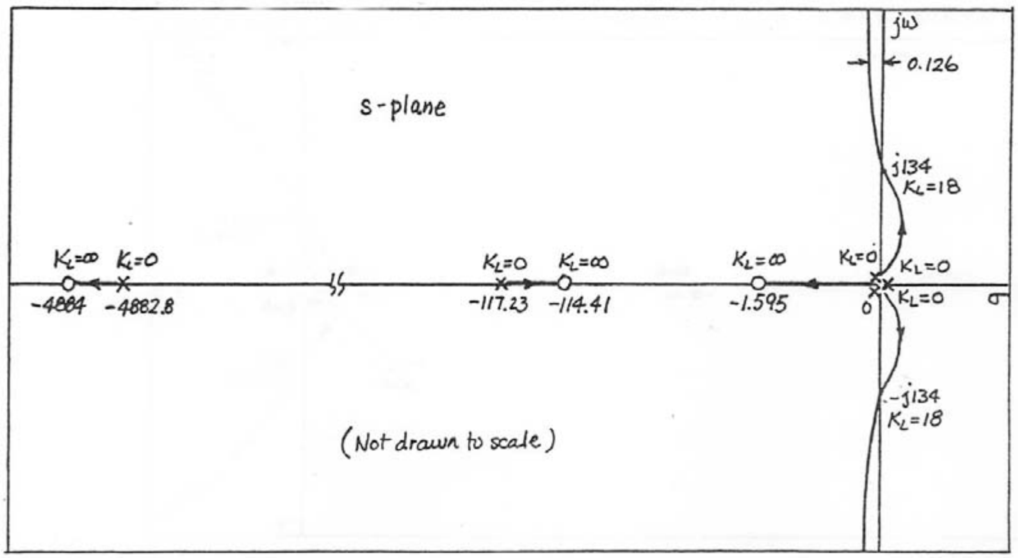
7-40) (a) ***K* = 1:** $P(s) = s^3(s+117.23)(s+4882.8) \qquad Q(s) = 1010(s+1.5948)(s+114.41)(s+4884)$

Asymptotes: $K_L > 0: \quad 90^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-117.23 - 4882.8 + 1.5948 + 114.41 + 4884}{5-3} = -0.126$$

Breakaway Point: (RL) 0



7-40 (b) ***K* = 1000:** $P(s) = s^3(s+117.23)(s+4882.8)$

$$Q(s) = 1010(s^3 + 5000s^2 + 5.6673\times10^5 s + 891089110)$$
$$= 1010(s+4921.6)(s+39.18+j423.7)(s+39.18-423.7)$$

Asymptotes: $K_L > 0$: 90^o, 270^o

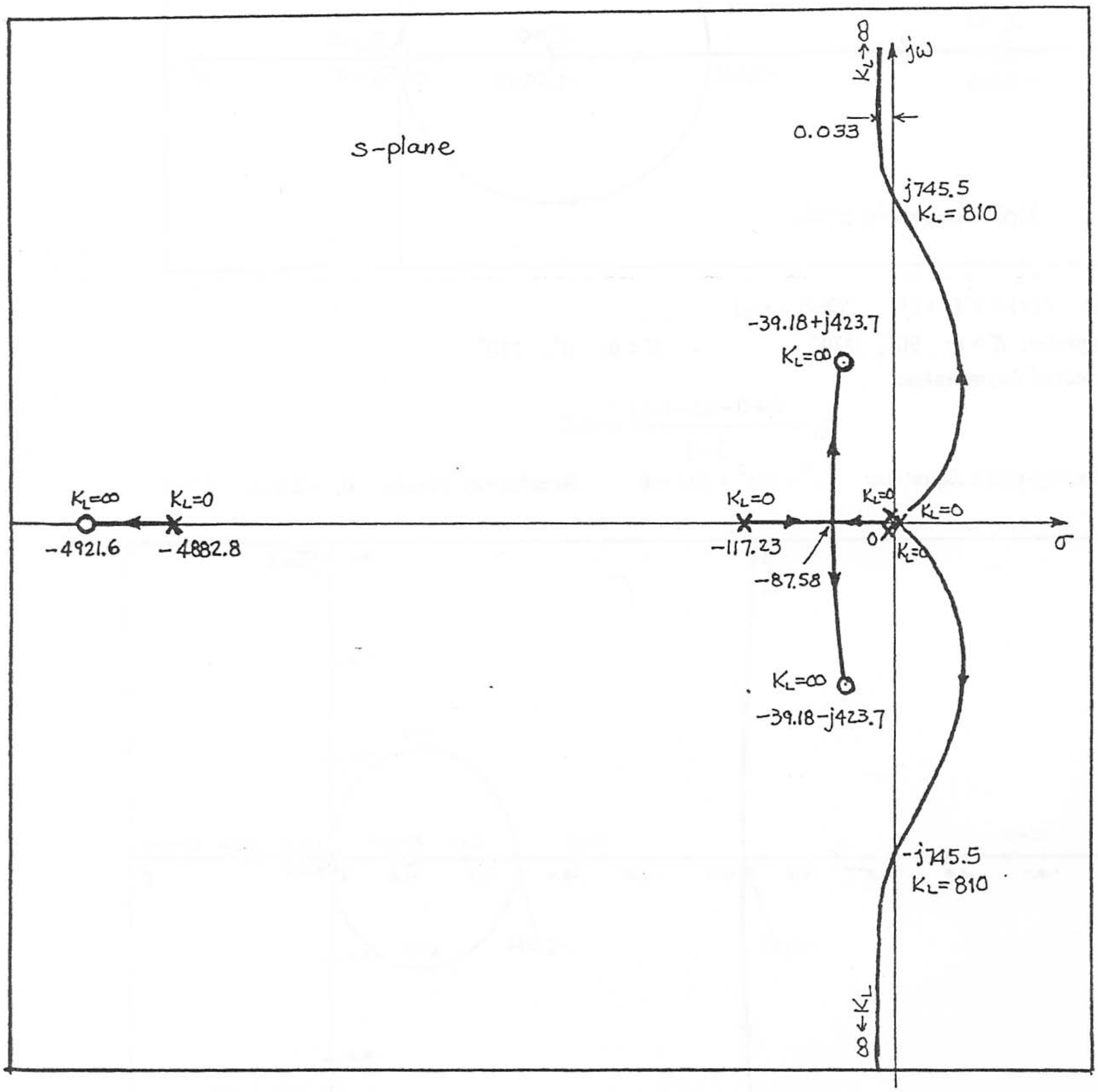
Intersect of Asymptotes:

$$\sigma_1 = \frac{-117.23 - 4882.8 + 4921.6 + 39.18 + 39.18}{5-3} = -0.033$$

Breakaway-point Equation:

$$2020s^7 + 2.02\times10^7 s^6 + 5.279\times10^{10} s^5 + 1.5977\times10^{13} s^4 + 1.8655\times10^{16} s^3 + 1.54455\times10^{18} s^2 = 0$$

Breakaway points: (RL) 0, −87.576

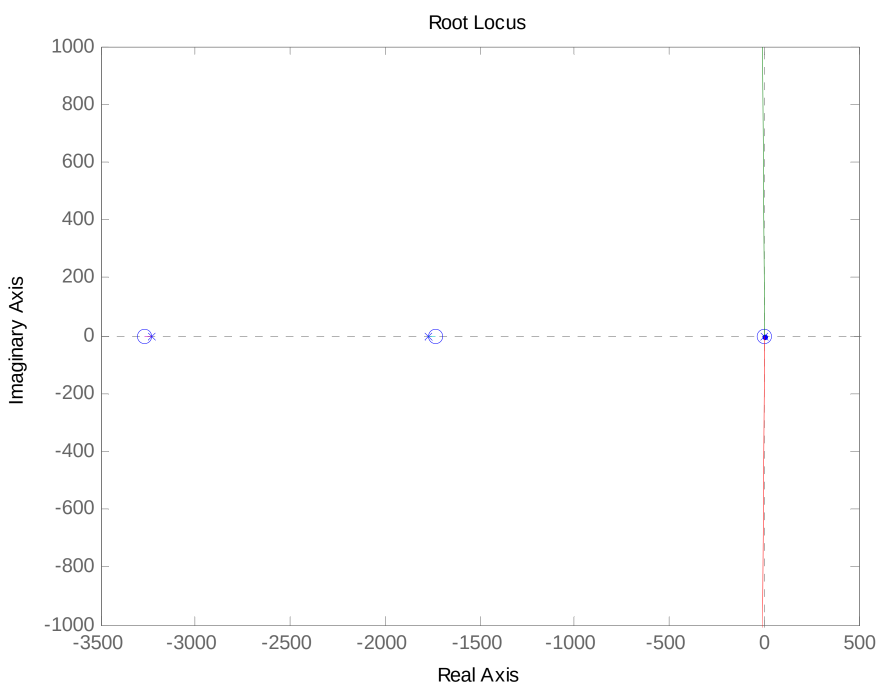


7-41) MATLAB code:

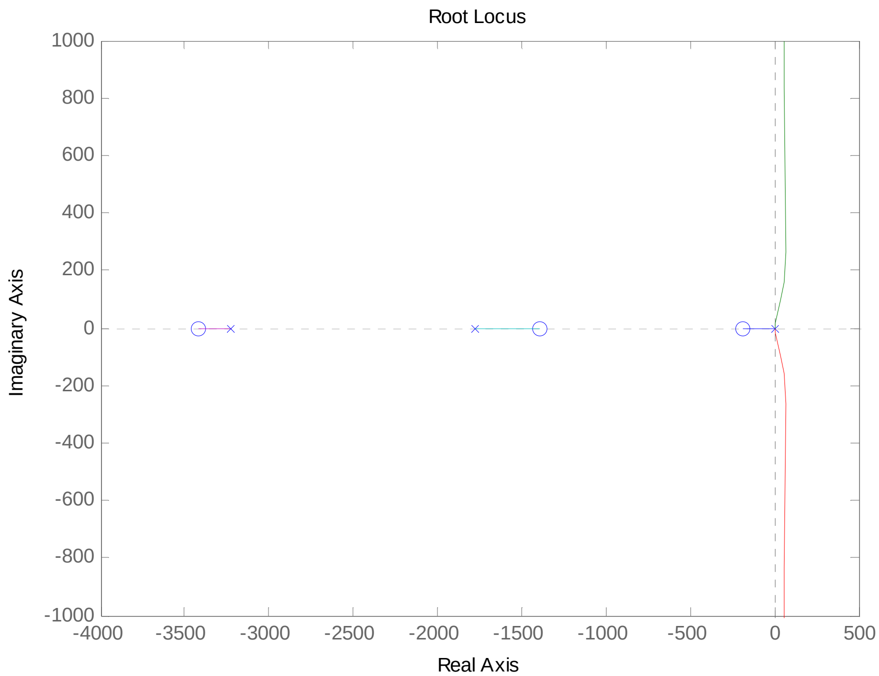
```
s = tf('s')
Ki=9;
Kb=0.636;
Ra=5;
La=.001;
Ks=1;
n=.1;
Jm=0.001;
Jl=0.001;
Bm=0;
%a)
```

```
K=1;
num_G_a=((n^2*La*Jl+La*Jm)*s^3+(n^2*Ra*Jl+Ra*Jm+Bm*La)*s^2+Ra*Bm*s+Ki
*Kb*s+n*Ks*K*Ki);
den_G_a=((La*Jm*Jl)*s^5+(Jl*Ra*Jm+Jl*Bm*La)*s^4+(Ki*Kb*Jl+Ra*Bm*Jl)*s
^3);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)
%b)
K=1000;
num_G_b=((n^2*La*Jl+La*Jm)*s^3+(n^2*Ra*Jl+Ra*Jm+Bm*La)*s^2+Ra*Bm*s+Ki
*Kb*s+n*Ks*K*Ki);
den_G_b=((La*Jm*Jl)*s^5+(Jl*Ra*Jm+Jl*Bm*La)*s^4+(Ki*Kb*Jl+Ra*Bm*Jl)*s
^3);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)
```

Root locus diagram, part (a):



Root locus diagram, part (b):



7-42

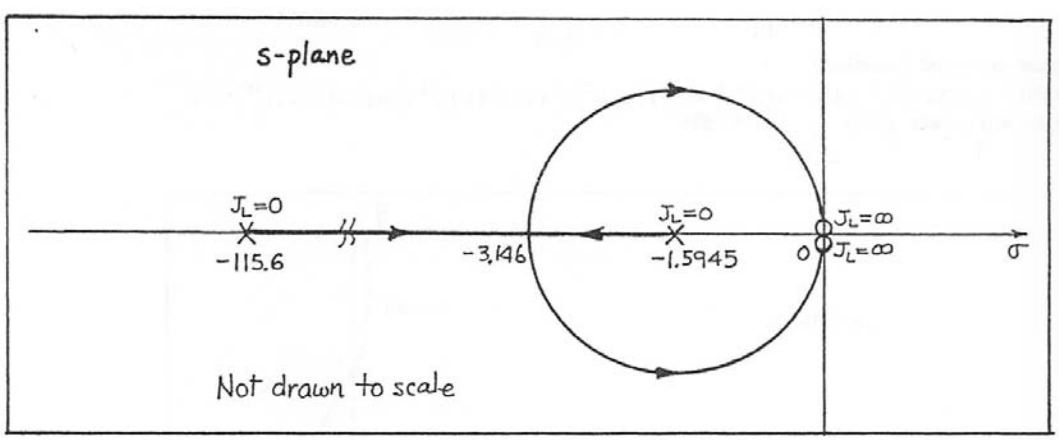
(a) Characteristic Equation: $s^3 + 5000s^2 + 572{,}400s + 900{,}000 + J_L\left(10s^3 + 50{,}000s^2\right) = 0$

$P(s) = s^3 + 5000s^2 + 572{,}400s + 900{,}000 = (s + 1.5945)(s + 115.6)(s + 4882.8) \qquad Q(s) = 10s^2(s + 5000)$

Since the pole at –5000 is very close to the zero at –4882.8, $P(s)$ and $Q(s)$ can be approximated as:

$P(s) \cong (s + 1.5945)(s + 115.6) \qquad Q(s) \cong 10.24s^2$

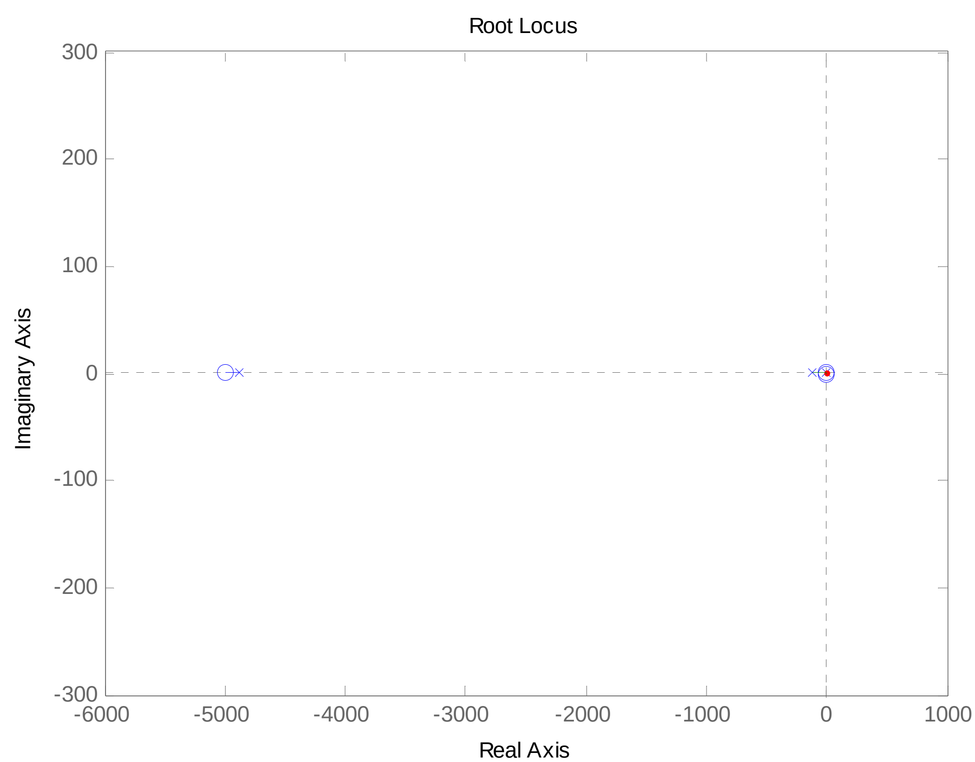
Breakaway-point Equation: $1200s^2 + 3775s = 0$ **Breakaway Points: (RL):** 0, –3.146



(b) MATLAB code:

```
s = tf('s')
K=1;
Jm=0.001;
La=0.001;
n=0.1;
Ra=5;
Ki=9;
Bm=0;
Kb=0.0636;
Ks=1;

num_G_a = (n^2*La*s^3+n^2*Ra*s^2);
den_G_a = (La*Jm*s^3+(Ra*Jm+Bm*La)*s^2+(Ra*Bm+Ki*Kb)*s+n*Ki*Ks*K);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)
```

Root locus diagram:



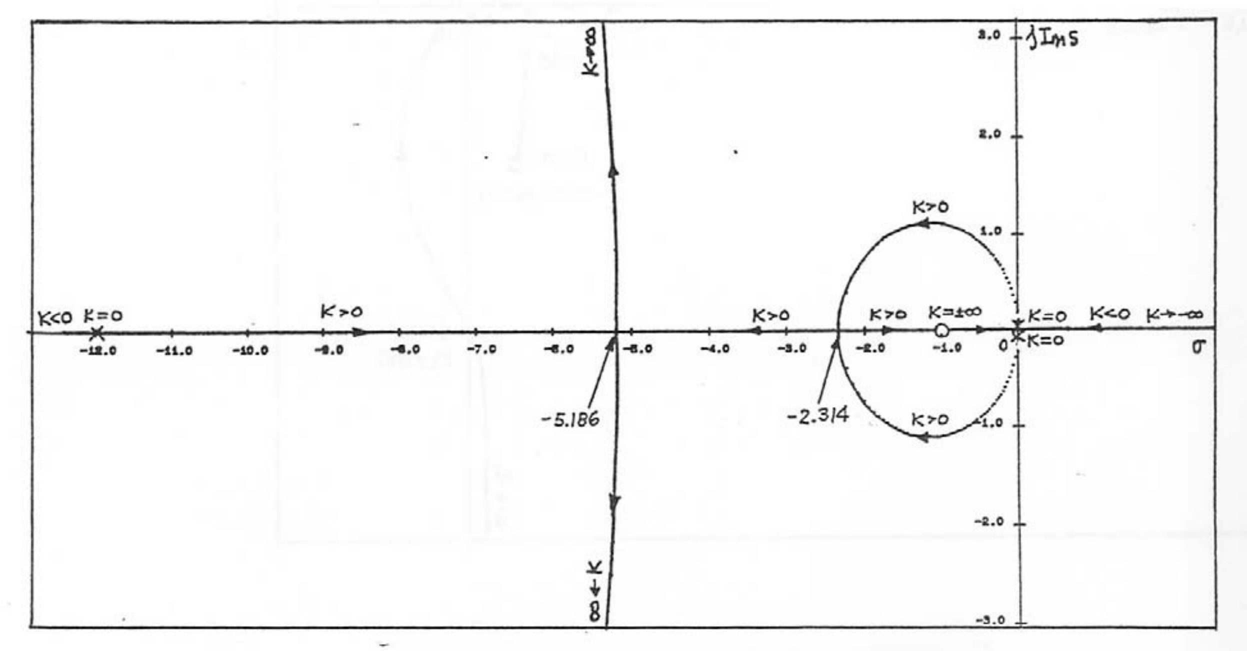
7-43) (a) $\alpha = 12$: $P(s) = s^2(s+12)$ $Q(s) = s+1$

Asymptotes: *K* > 0: 90^o, 270^o ***K* < 0:** 0^o, 180^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-12-(-1)}{3-1} = -5.5$$

Breakaway-point Equation: $2s^3 + 15s^2 + 24s = 0$ **Breakaway Points:** 0, −2.314, −5.186



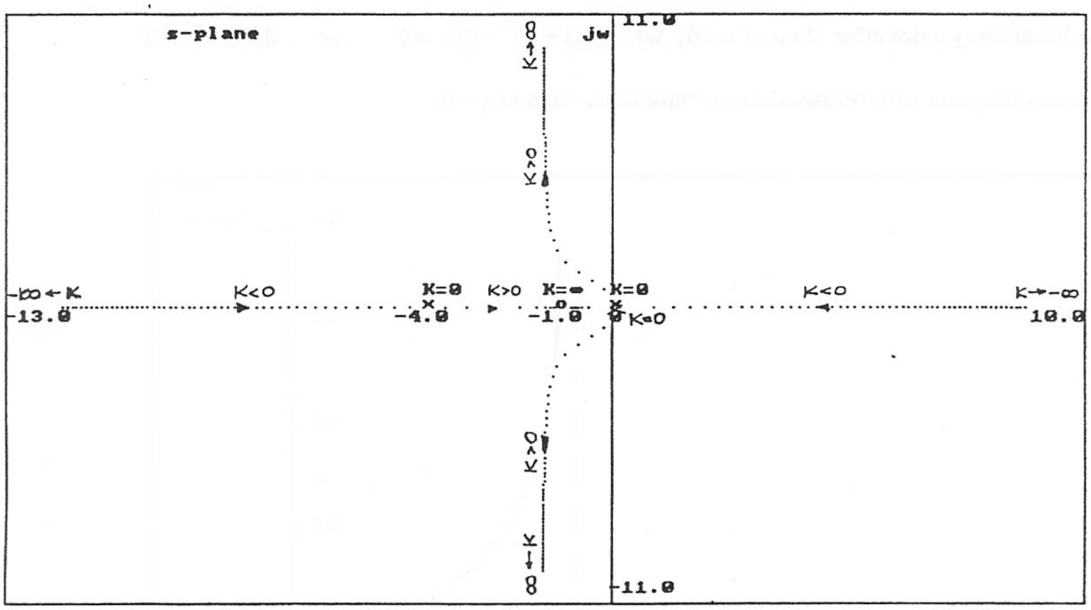
7-43 (b) $\alpha = 4$: $P(s) = s^2(s+4)$ $Q(s) = s+1$

Asymptotes: *K* > 0: 90^o, 270^o ***K* < 0:** 0^o, 180^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-4-(-10}{3-1} = -1.5$$

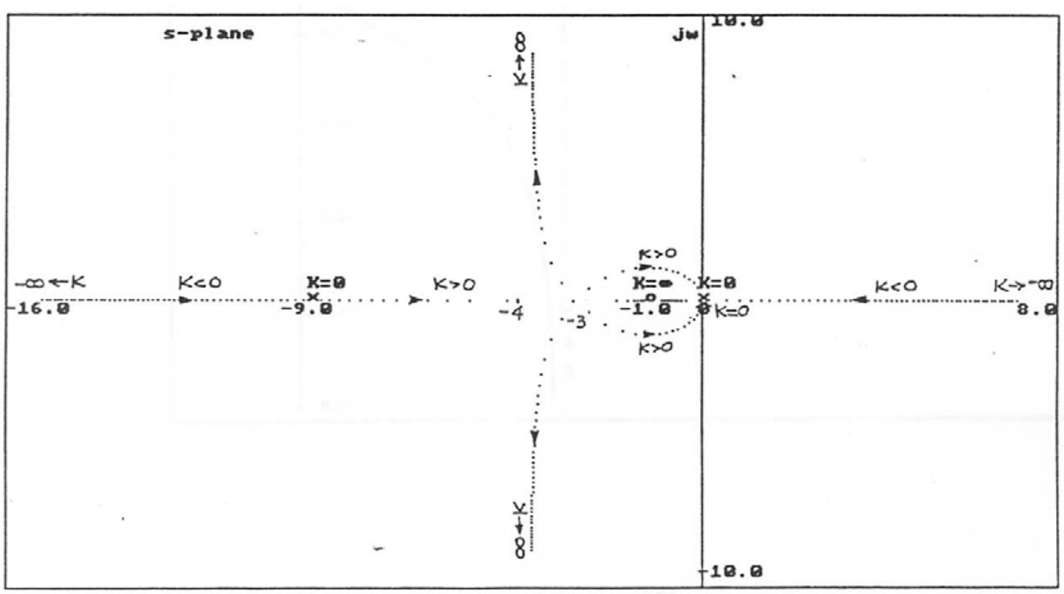
Breakaway-point Equation: $2s^3 + 7s^2 + 8s = 0$ **Breakaway Points: *K* > 0** 0. None for ***K* < 0.**



(c) **Breakaway-point Equation:** $2s^2 + (\alpha + 3)s + 2s = 0$ **Solutions:** $s = -\frac{\alpha + 3}{4} \pm \frac{\sqrt{(\alpha + 3)^2 - 16\alpha}}{4}$, $s = 0$

For one nonzero breakaway point, the quantity under the square-root sign must equal zero.

Thus, $\alpha^2 - 10\alpha + 9 = 0$, $\alpha = 1$ or $\alpha = 9$. The answer is $\alpha = 9$. The $\alpha = 1$ solution represents pole-zero cancellation in the equivalent $G(s)$. When $\alpha = 9$, the nonzero breakaway point is at $s = -3$. $\sigma_1 = -4$.



7-44)

For part (c), after finding the expression for:

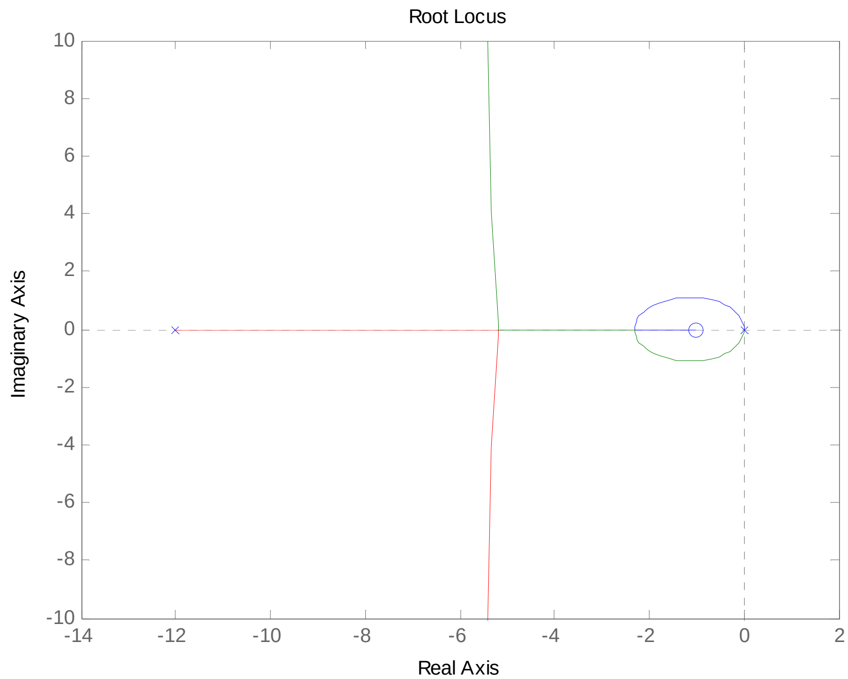
$$\frac{dk}{ds} = \frac{-3-\alpha \pm \sqrt{(\alpha-1)(\alpha-9)}}{4},$$

there is one acceptable value of alpha that makes the square root zero ($\alpha = 9$). Zero square root means one answer to the breakaway point instead of 2 answers as a result of $\pm$ sign. $\alpha = 1$ is not acceptable since it results in $s = -1 @ \frac{dk}{ds} = 0$ and then $k = \frac{0}{0}$.

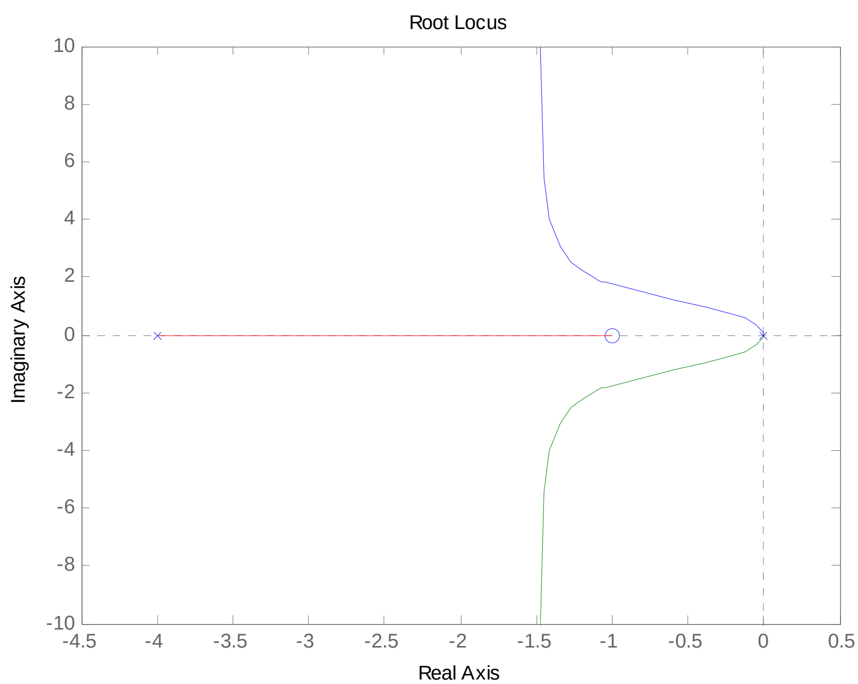
MATLAB code:

```
s = tf('s')

%(a)
alpha=12
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(1);
rlocus(GH)

%(b)
alpha=4
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(2);
rlocus(GH)

%(c)
alpha=9
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(3);
rlocus(GH)
```

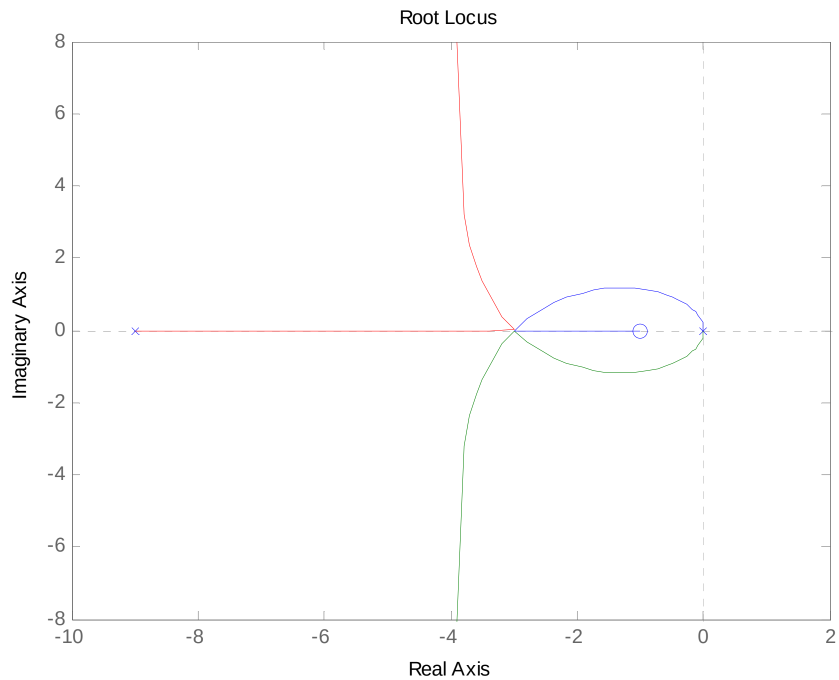
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c): (`alpha=9 resulting in 1 breakaway point`)



7-45) (a) $P(s) = s^2(s+3) \qquad Q(s) = s + \alpha$

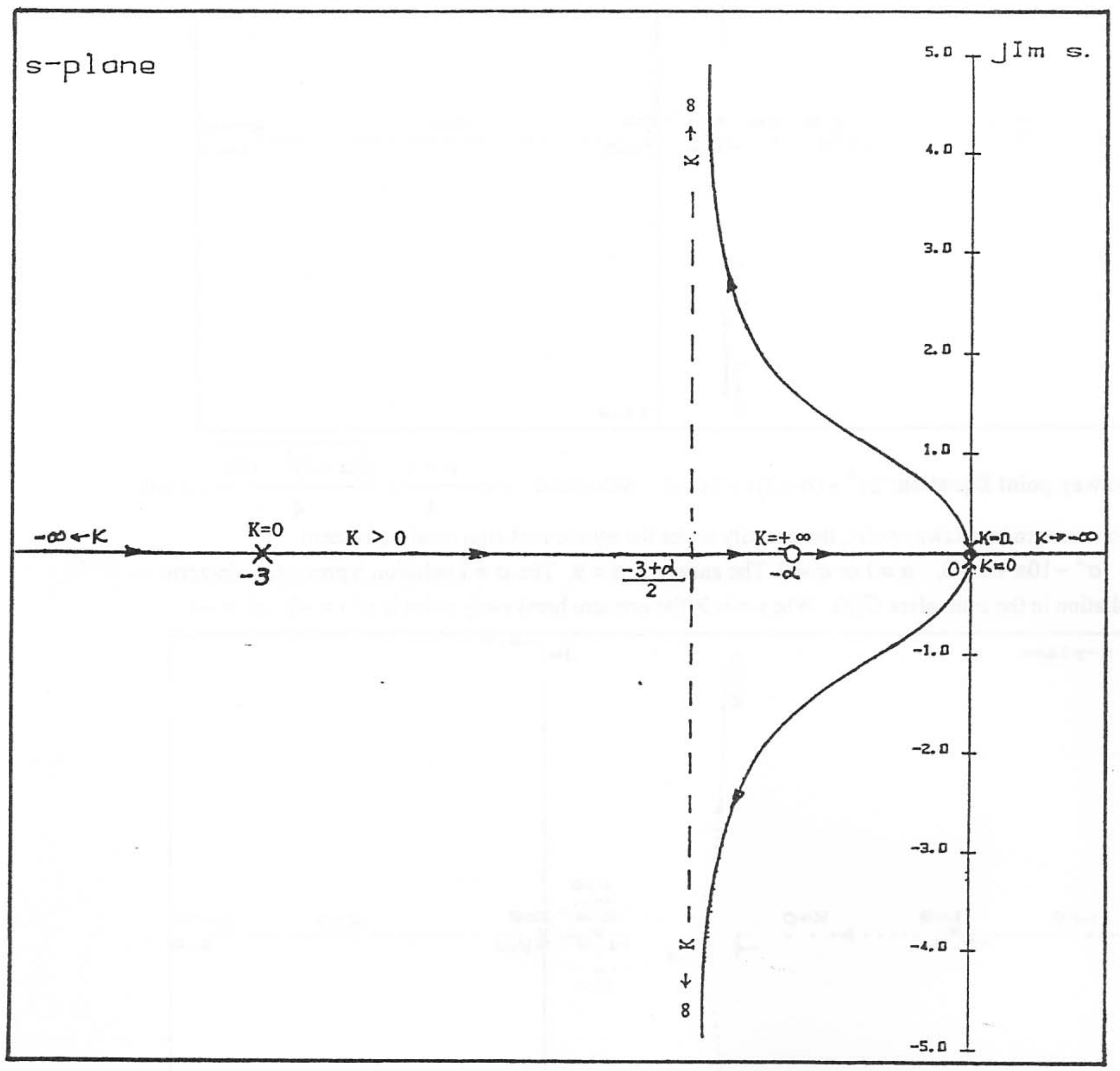
Breakaway-point Equation: $2s^3 + 3(1+\alpha)s + 6\alpha = 0$

The roots of the breakaway-point equation are:

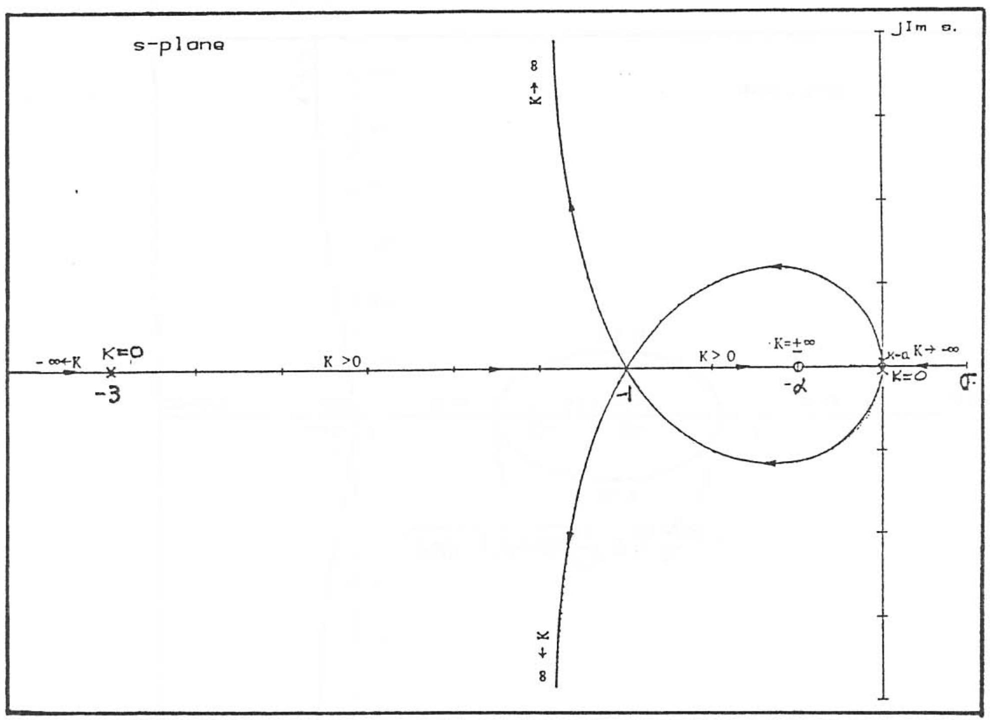
$$s = \frac{-3(1+\alpha)}{4} \pm \frac{\sqrt{9(1+\alpha)^2 - 48\alpha}}{4}$$

For no breakaway point other than at $s = 0$**, set** $9(1+\alpha)^2 - 48\alpha < 0$ **or** $0.333 < \alpha < 3$

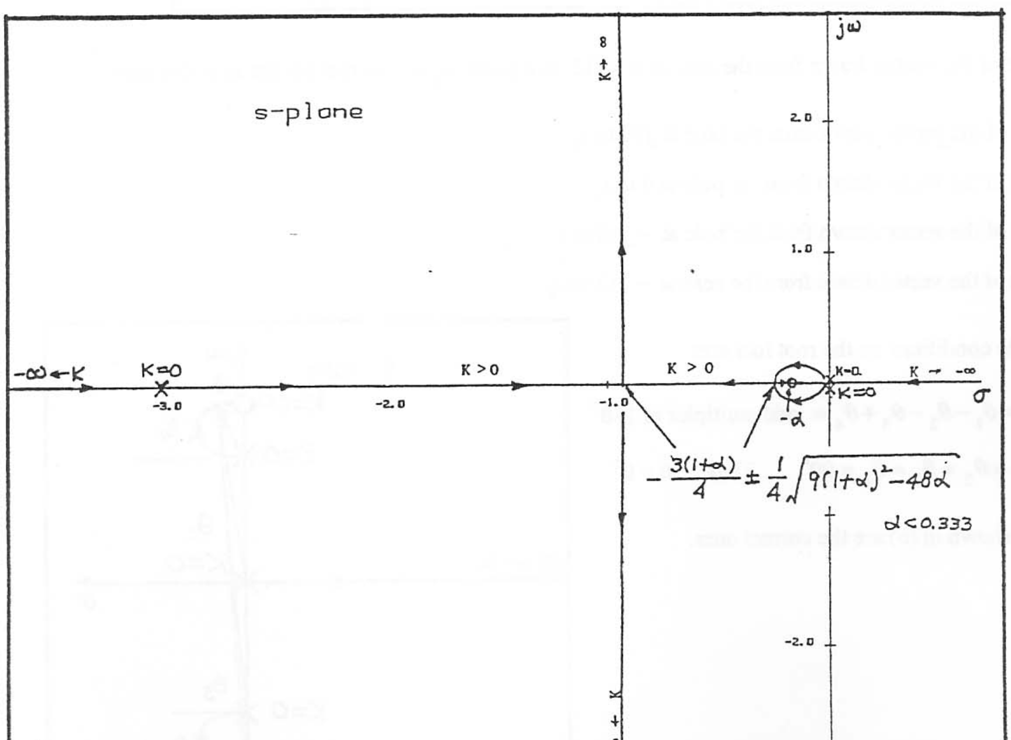
Root Locus Diagram with No Breakaway Point other than at *s* = 0.



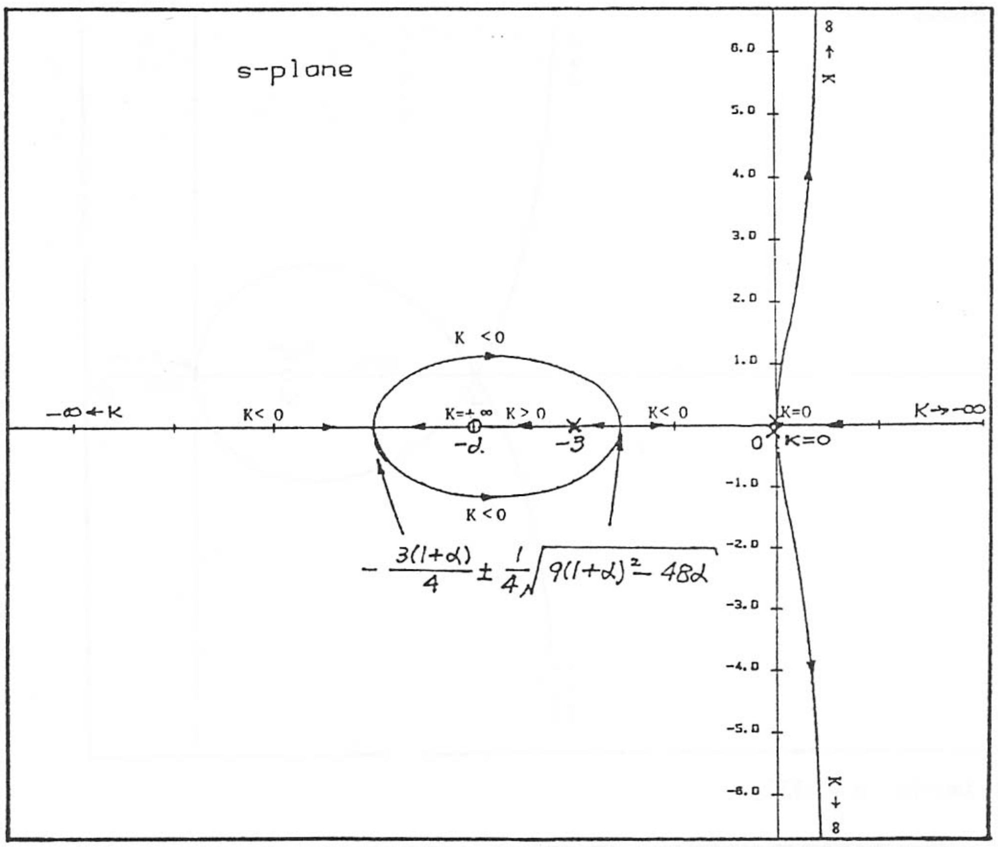
7-45 (b) One breakaway point other than at *s* = 0: $\alpha = 0.333$, Breakaway point at $s = -1$.



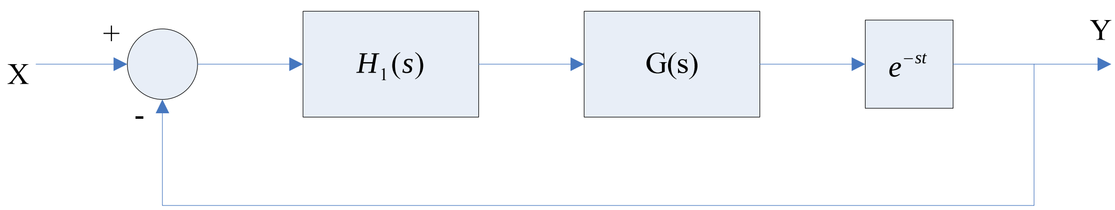
(c) Two breakaway points: $\alpha < 0.333$:



7-45 (d) Two breakaway points: $\alpha > 3$:



7-46) First we can rearrange the system as:



where

$$H_1(s) = \frac{H}{1 + (1 - e^{-st})GH}$$

Now designing a controller is similar to the designing a controller for any unity feedback system.

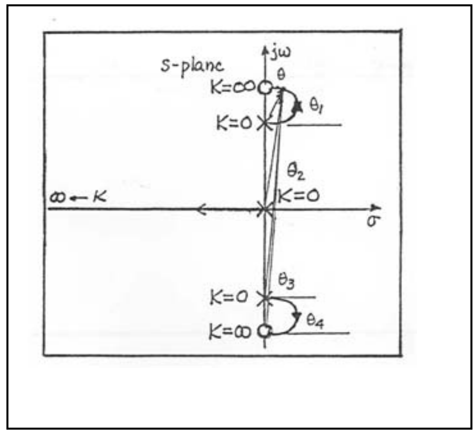
7-47) Let the angle of the vector drawn from the zero at $s = j12$ to a point s_1 on the root locuss near the zero be θ. Let

$\theta_1 =$ angle of the vector drawn from the pole at $j10$ to s_1.

$\theta_2 =$ angle of the vector drawn from the pole at 0 to s_1.

$\theta_3 =$ angle of the vector drawn from the pole at $-j10$ to s_1.

$\theta_4 =$ angle of the vector drawn from the zero at $-j12$ to s_1.



Then the angle conditions on the root loci are:

$$\theta = \theta_1 - \theta_2 - \theta_3 + \theta_4 = \text{ odd multiples of } 180^o$$

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = 90^o \qquad \text{Thus,} \quad \theta = 0^o$$

The root loci shown in (b) are the correct ones.

Answers to True and False Review Questions:

6. (F) 7. (T) 8. (T) 9. (F) 10. (F) 11. (T) 12. (T) 13. (T) 14. (T)

Chapter 8

8-1 (a) ***K* = 5** $\omega_n = \sqrt{5} = 2.24 \text{ rad/sec}$ $\zeta = \frac{6.54}{4.48} = 1.46$ $M_r = 1$ $\omega_r = 0 \text{ rad/sec}$

(b) ***K* = 21.39** $\omega_n = \sqrt{21.39} = 4.62 \text{ rad/sec}$ $\zeta = \frac{6.54}{9.24} = 0.707$ $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1$

$$\omega_r = \omega_n\sqrt{1-\zeta^2} = 3.27 \text{ rad/sec}$$

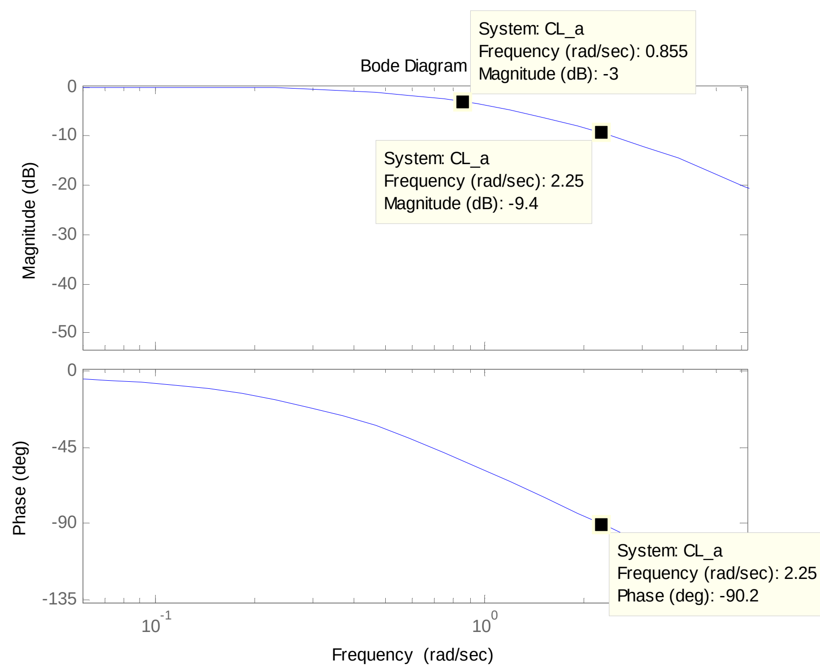
(c) ***K* = 100** $\omega_n = 10 \text{ rad/sec}$ $\zeta = \frac{6.54}{20} = 0.327$ $M_r = 1.618$ $\omega_r = 9.45 \text{ rad/sec}$

8-2

MATLAB code:

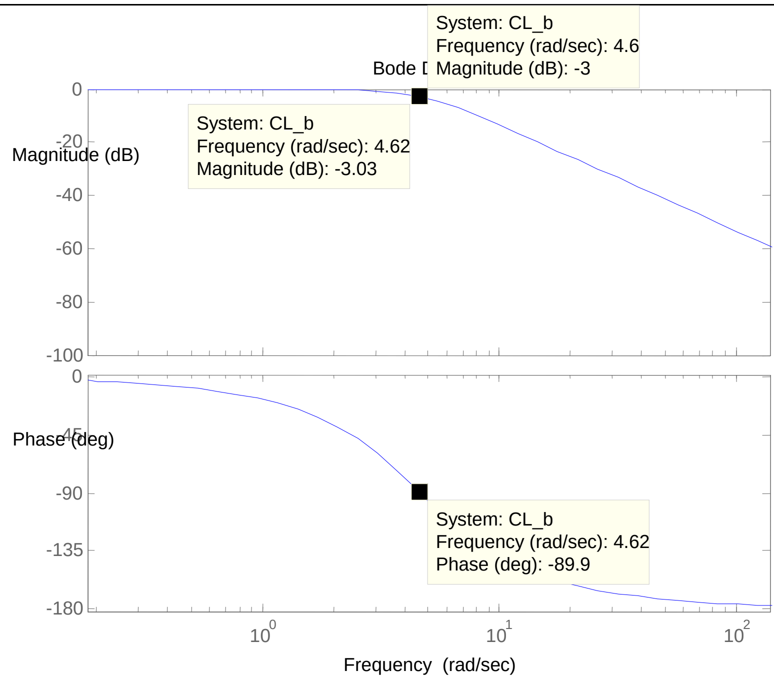
```
% Question 8-2,
clear all;
close all;
s = tf('s')

%a)
num_G_a= 5;
den_G_a=s*(s+6.54);
G_a=num_G_a/den_G_a;
CL_a=G_a/(1+G_a)
BW = bandwidth(CL_a)
bode(CL_a)

%b)
figure(2);
num_G_b=21.38;
den_G_b=s*(s+6.54);
G_b=num_G_b/den_G_b;
CL_b=G_b/(1+G_b)
BW = bandwidth(CL_b)
bode(CL_b)

%c)
figure(3);
num_G_c=100;
den_G_c=s*(s+6.54);
G_c=num_G_c/den_G_c;
```

Bode diagram (a) – k=5: data points from top to bottom indicate bandwidth BW, resonance peak M_r, and resonant frequency ω_r.

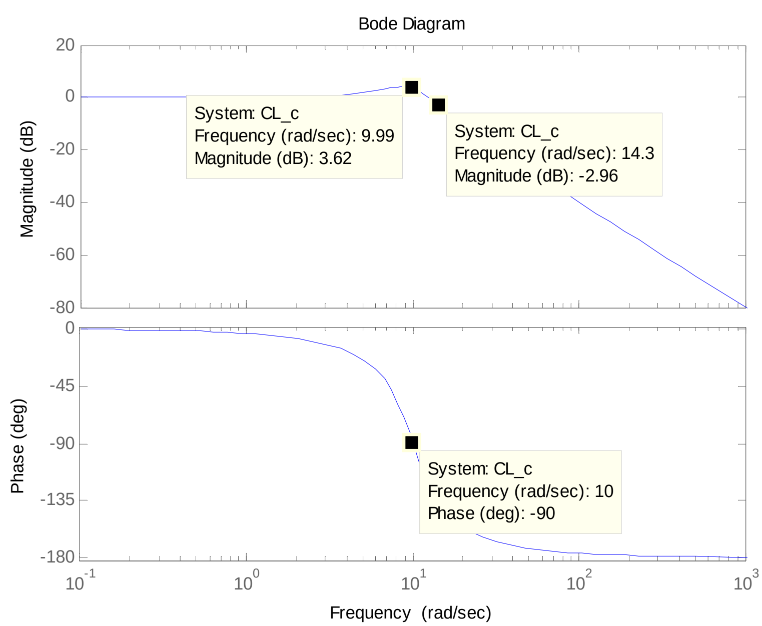


Bode diagram (b) – k=21.38: data points from top to bottom indicate bandwidth BW, resonance peak M_r, and resonant frequency ω_r.

```
CL_c=G_c/(1+G_c)
BW = bandwidth(CL_c)
bode(CL_c)
```



Bode diagram (c) – k=100: data points from top to bottom indicate resonance peak M_r, bandwidth BW, and resonant frequency ω_r.



8-3) If $u(t) = U\sin(\omega t)$ is the input, then

$$G(j\omega) = \frac{j\omega + \frac{1}{A_1}}{j\omega + \frac{1}{A_2}} = \frac{A_2(1 + A_1 j\omega)}{A_1(1 + A_2 j\omega)}$$

where

$$|G(j\omega)| = \frac{A_2}{A_1}\sqrt{\frac{1 + A_1^2\omega^2}{1 + A_2^2\omega^2}}$$

and

$$\angle G(j\omega) = \tan^{-1} A_1\omega - \tan^{-1} T_2\omega = \Phi$$

Therefore:

$$y(t) = \frac{UA_2}{A_1}\sqrt{\frac{1 + A_1^2\omega^2}{1 + A_2^2\omega^2}}\sin(\omega t + \tan^{-1} A_1\omega - \tan^{-1} A_2\omega)$$

As a result:

$$\begin{cases} if\ A_1 > A_2 \rightarrow \Phi > 0 \rightarrow the\ network\ is\ a\ lead\ network \\ if\ A_1 < A_2 \rightarrow \Phi < 0 \rightarrow the\ network\ is\ a\ lag\ network \end{cases}$$

8-4 (a) $M_r = 2.944\ (9.38\ \text{dB})$ $\omega_r = 3\ \text{rad/sec}$ $\text{BW} = 4.495\ \text{rad/sec}$

(b) $M_r = 15.34\ (23.71\ \text{dB})$ $\omega_r = 4\ \text{rad/sec}$ $\text{BW} = 6.223\ \text{rad/sec}$

(c) $M_r = 4.17\ (12.4\ \text{dB})$ $\omega_r = 6.25\ \text{rad/sec}$ $\text{BW} = 9.18\ \text{rad/sec}$

(d) $M_r = 1\ (0\ \text{dB})$ $\omega_r = 0\ \text{rad/sec}$ $\text{BW} = 0.46\ \text{rad/sec}$

(e) $M_r = 1.57\ (3.918\ \text{dB})$ $\omega_r = 0.82\ \text{rad/sec}$ $\text{BW} = 1.12\ \text{rad/sec}$

(f) $M_r = \infty$ (unstable) $\omega_r = 1.5$ rad / sec BW = 2.44 rad / sec

(g) $M_r = 3.09$ (9.8 dB) $\omega_r = 1.25$ rad / sec BW = 2.07 rad / sec

(h) $M_r = 4.12$ (12.3 dB) $\omega_r = 3.5$ rad / sec BW = 5.16 rad / sec

8-5)

Maximum overshoot = 0.1 Thus, $\zeta = 0.59$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.05 \qquad t_r = \frac{1-0.416\zeta+2.917\zeta^2}{\omega_n} = 0.1 \text{ sec}$$

Thus, minimum $\omega_n = 17.7$ rad / sec Maximum $M_r = 1.05$

Minimum BW = $\omega_n\left(\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}\right)^{1/2} = 20.56$ rad/sec

8-6)

Maximum overshoot = 0.2 Thus, $0.2 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$ $\zeta = 0.456$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.232 \qquad t_r = \frac{1-0.416\zeta+2.917\zeta^2}{\omega_n} = 0.2$$

Thus, minimum $\omega_n = 14.168$ rad/sec

Maximum $M_r = 1.232$ Minimum BW = $\left(\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}\right)^{1/2} = 18.7$ rad/sec

8-7) Maximum overshoot = 0.3 Thus, $0.3 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$ $\zeta = 0.358$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.496 \qquad t_r = \frac{1-0.416\zeta+2.917\zeta^2}{\omega_n} = 0.2$$

Thus, minimum $\omega_n = 6.1246$ rad/sec

Maximum M_r = 1.496 Minimum BW = $\left(\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}\right)^{1/2}=1.4106$ rad/sec

8-8) (a)

$$G(j\omega)=\frac{0.5K}{-0.375\,\omega^2+j(\omega-0.25\,\omega^3)}$$

At the gain crossover:

$$|G(j\omega)|=\frac{0.5K}{\sqrt{(0.375^2\omega^4)+(\omega^2-0.25\,\omega^3)^2}}=1$$

Therefore:

$$(0.375)^2\omega^4+(\omega-0.25\,\omega^3)^2-0.25K^2=0$$

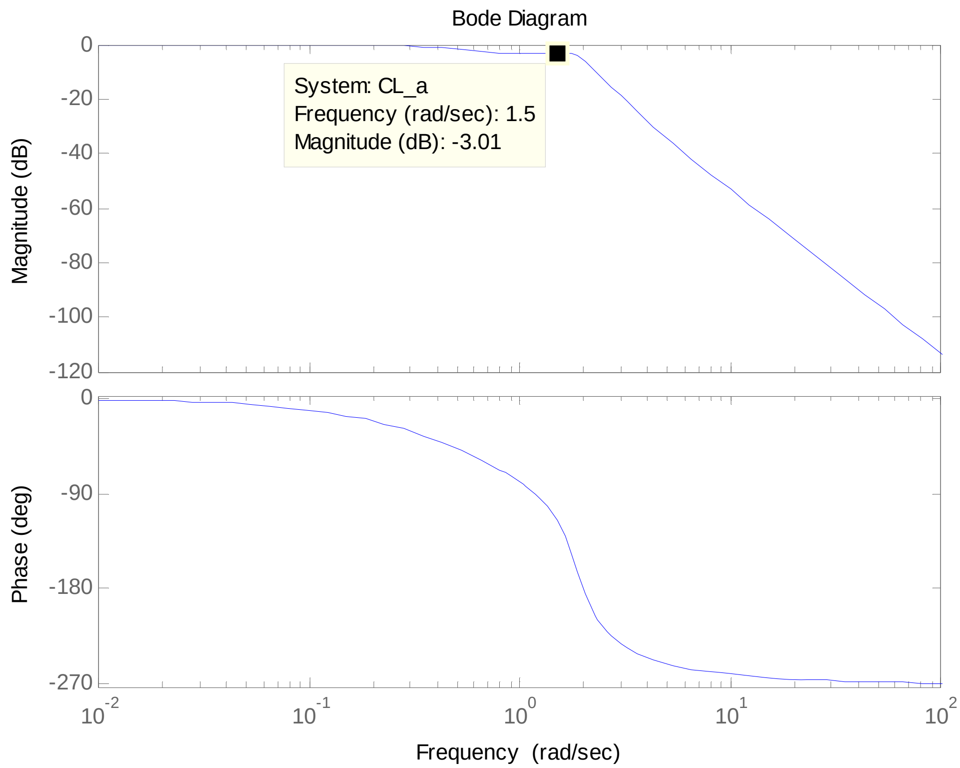
at ω = 1.5 ⇨ $K=2.138$

(b)

MATLAB code:

```
%solving for k:
syms kc
omega=1.5
sol=eval(solve('0.25*kc^2=0.7079^2*((-0.25*omega^3+omega)^2+(-
0.375*omega^2+0.5*kc)^2)',kc))
%ploting bode with K=1.0370
s = tf('s')
K=1.0370;
num_G_a= 0.5*K;
den_G_a=s*(0.25*s^2+0.375*s+1);
G_a=num_G_a/den_G_a;
CL_a = G_a/(1+G_a)
BW = bandwidth(CL_a)
bode(CL_a);
```

Bode diagram: data point shows -3dB point at 1.5 rad/sec frequency which is the closed loop bandwidth



8-9) $\theta = \sin^{-1}\left(\frac{1}{Mp}\right) - \sin^{-1}\left(\frac{1}{2.2}\right) \approx 27^o$

$\alpha = 90 - \theta = 63^o$

$$OA = -\frac{M^2}{M^2-1} = -1.26$$

Therefore:

$$\begin{aligned} AB &= \frac{M}{M^2-1}\cos\alpha \\ &= \left(\frac{M}{M^2-1}\right)\cos(90-\theta) \\ &= \frac{M}{M^2-1}\sin\alpha \\ &= \frac{M}{M^2-1}\left(\frac{1}{M}\right) = \frac{1}{M^2-1} \end{aligned}$$

As a result:

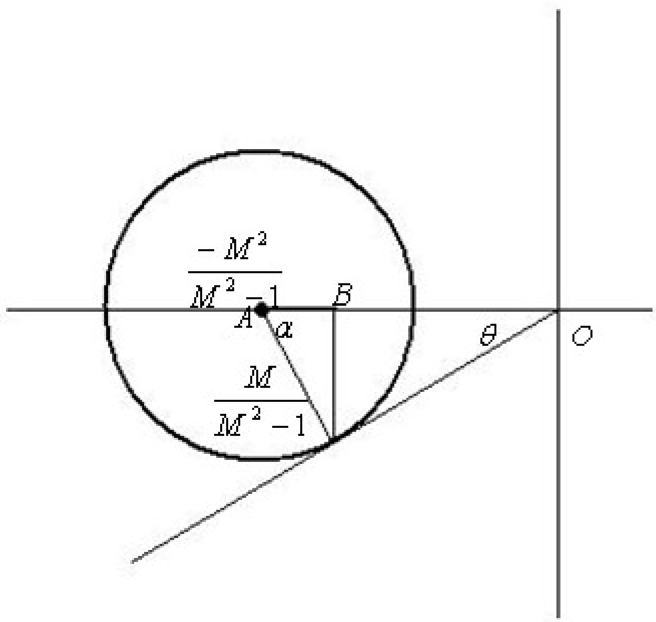
$$OB = -\frac{M^2}{M^2-1} - \frac{1}{M^2-1} = -1$$

Therefore:

$$|G(j\omega)|_{\substack{s=-1\\k=1}} = -0.54$$

To change the crossover frequency requires adding gain as:

$$K = -\frac{1}{-0.54} = 1.85$$

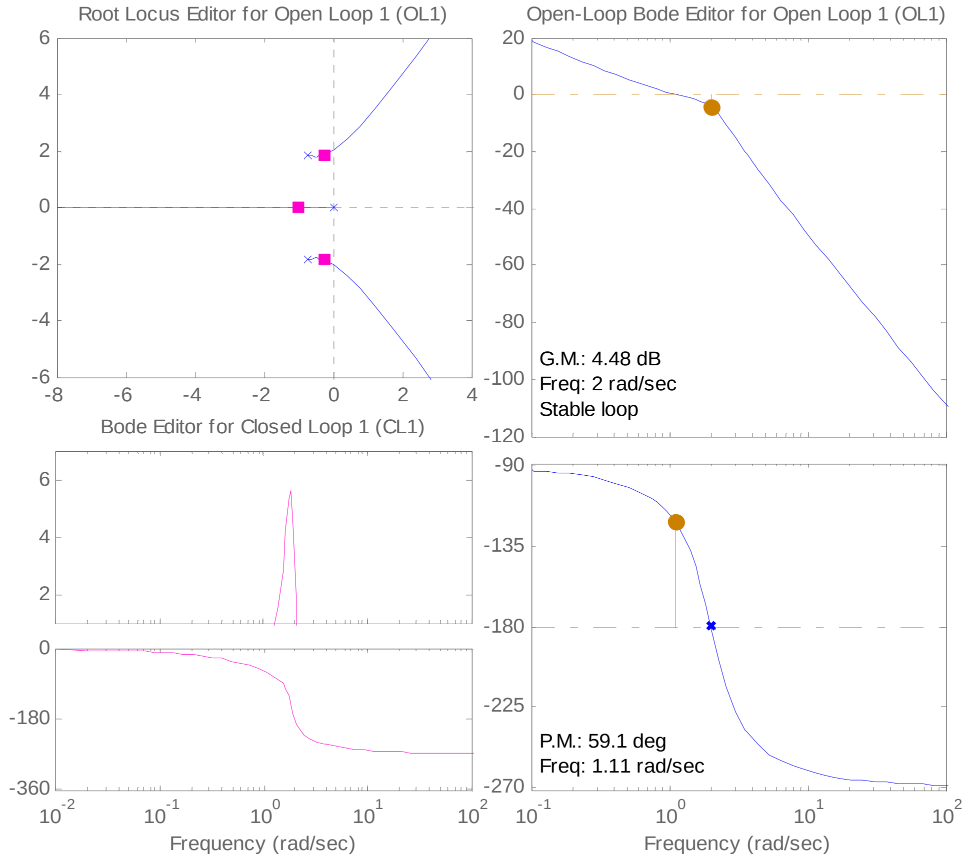


(b) MATLAB code:

```
s = tf('s')
%(b)
K = 0.95*2;
num_G_a = 0.5*K;
den_G_a = s*(0.25*s^2+0.375*s+1);
G_a = num_G_a/den_G_a;
CL_a = G_a/(1+G_a)
bode(CL_a);
figure(2);
sisotool
```

Peak mag = 2.22 can be converted to dB units by: 20*Log(2.22)= 6.9271 dB

By using sisotool and importing the loop transfer function, the overall gain (0.5K) was changed until the magnitude of the resonance in Bode was about 6.9 dB. At 0.5K=~0.95 or K=1.9, this resonance peak was achieved as can be seen in the BODE diagram of the following figure:



8-10)

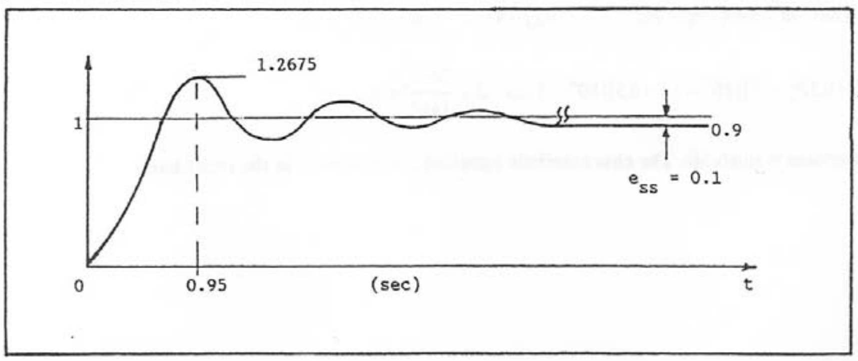
$$M_r = 1.4 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{Thus, } \zeta = 0.387 \qquad \text{Maximum overshoot} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.2675\ (26.75\%)$$

$$\omega_r = 3 \text{ rad/sec} = \omega_n\sqrt{1-2\zeta^2} = 0.8367\omega_n \text{ rad/sec} \qquad \omega_n = \frac{3}{0.8367} = 3.586 \text{ rad/sec}$$

$$t_{\max} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{3.586\sqrt{1-(0.387)^2}} = 0.95 \text{ sec} \qquad \text{At } \omega = 0, \quad |M| = 0.9.$$

This indicates that the steady-state value of the unit-step response is 0.9.

Unit-step Response:



8-11) a) The closed loop transfer function is:

$$\frac{Y(s)}{X(s)} = \frac{GH}{1+GH} = \frac{K}{10s^2+s+K} = \frac{K}{s^2+0.1s+0.1K}$$

as $\frac{1}{2\xi\sqrt{1-\xi^2}} = 1.4$,which means $\xi = 0.387$

According to the transfer function: $\xi\omega_n = 0.1 \Rightarrow \omega_n = 0.129$ rad/s
As $\omega_n^2 = 0.1K$; then, $K = 10\,\omega_n^2 = 0.1669$

b) $$\omega_R = \omega_0\sqrt{1 \quad 2\xi^2} = 0.100\frac{rad}{s}$$

$$M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.268$$

$$PM = \angle GH(j\omega_g) - 180^o$$

$$|GH(j\omega)|_{\omega=\omega_g} = 1 \Rightarrow \frac{K}{|(j\omega)(10/\omega+1)|} = \frac{K}{\sqrt{100\,\omega_g^4 + \omega_g^2}} = 1$$

As K = 0.1664, then $100\,\omega_g^4 + \omega_g^4 = 0.0277$
which means $\omega_g = 0.1104$
Accordingly PM = 42^o

As $M(\omega) = \left|\frac{Y(j\omega)}{X(j\omega)}\right| > \left(\frac{\sqrt{2}}{2}\right)K$, then $\omega_b = 0.179\frac{rad}{s}$

8-12)

T	BW (rad/sec)	M_r
0	1.14	1.54
0.5	1.17	1.09
1.0	1.26	1.00
2.0	1.63	1.09
3.0	1.96	1.29
4.0	2.26	1.46
5.0	2.52	1.63

8-13)

T	BW (rad/sec)	M_r
0	1.14	1.54
0.5	1.00	2.32
1.0	0.90	2.65
2.0	0.74	2.91
3.0	0.63	3.18
4.0	0.55	3.37
5.0	0.50	3.62

8-14) The Routh array is:

S^3	0.25	1
S^2	0.375	0.5K
S^1	1-1/3	0
S^0	0.5K	

Therefore:

$$GH \approx \frac{0.5K}{0.375s^2 + \left(1 - \frac{K}{3}\right)s + 0.5K}$$

As $H(\omega) = |GH(j\omega_b)| \approx \frac{\sqrt{2}}{k}K$, if GH is rearranged as:

$$GH \approx \frac{1}{\frac{0.75}{k}s^2 + 2\left(\frac{1}{k} - \frac{1}{3}\right)s + 1}$$

then

$$\frac{\omega_b^2}{\omega_n^2} = (1 - \xi^2) + \sqrt{2 - 4\xi^2(1 - \xi^2)}$$

which gives

$$\omega_b^2 = \omega_n^2\left[(1 - 2\xi^2) + \sqrt{2 - 4\xi^2(1 - \xi^2)}\right] = (1.5)^2$$

where $\omega_n^2 = \frac{K}{0.75}$ and $\xi^2 = \frac{1}{0.75k}$

therefore, $K = 2.146, \omega_n = 1.692$ and $\xi = 0.6213$

(c)

MATLAB code:

```
s = tf('s')
%c)
K = 1.03697;
num_G_a = 0.5*K;
den_G_a =
s*(0.25*s^2+0.375*s+1
);
%create closed-loop
system
G_a =
num_G_a/den_G_a;
```

Bode diagram:

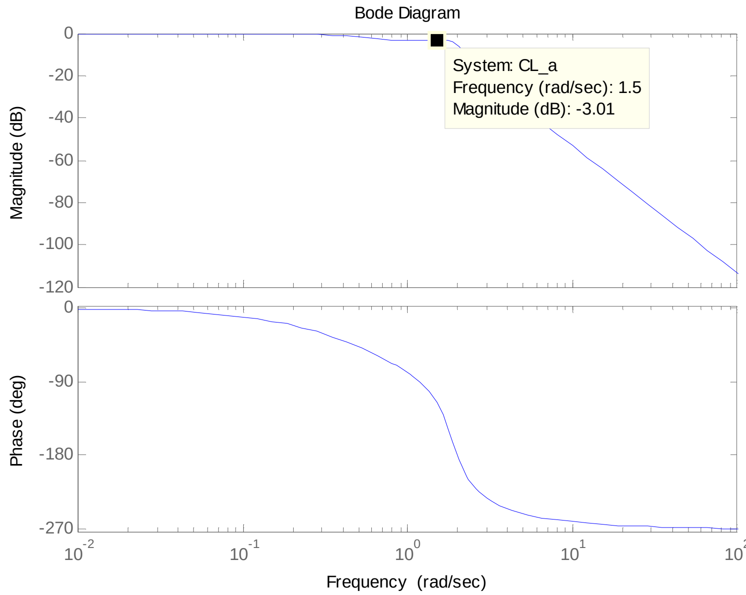
```
CL_a = G_a/(1+G_a)
bode(CL_a);
```

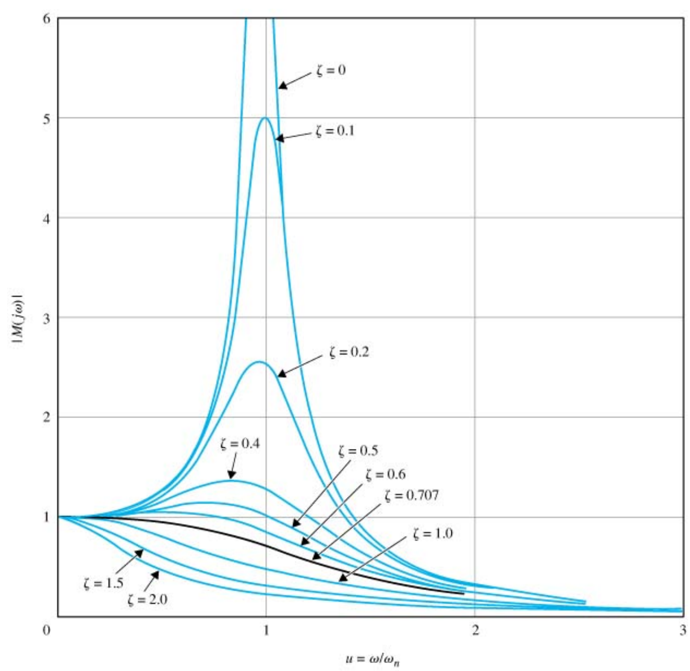
Notes:

1 - BW is verified by finding -3dB point at Freq = 1.5 rad/sec in the Bode graph at calculated k.

2- By comparison to diagram of typical 2nd order poles with different damping ratios, damping ratio is approximated as:

$\xi = \sim .707$





8-15 (a)

$$L(s) = \frac{20}{s(1+0.1s)(1+0.5s)} \qquad P_\omega = 1, \quad P = 0$$

When $\omega = 0$: $\angle L(j\omega) = -90^o$ $|L(j\omega)| = \infty$ When $\omega = \infty$: $\angle L(j\omega) = -270^o$ $|L(j\omega)| = 0$

$$L(j\omega) = \frac{20}{-0.6\omega^2 + j\omega\left(1-0.05\omega^2\right)} = \frac{20\left[-0.6\omega^2 - j\omega\left(1-0.05\omega^2\right)\right]}{0.36\omega^4 + \omega^2\left(1-0.05\omega^2\right)^2}$$

Setting $\text{Im}\left[L(j\omega)\right] = 0$

$1-0.05\omega^2 = 0$ Thus, $\omega = \pm 4.47$ rad / sec $L(j4.47) = -1.667$

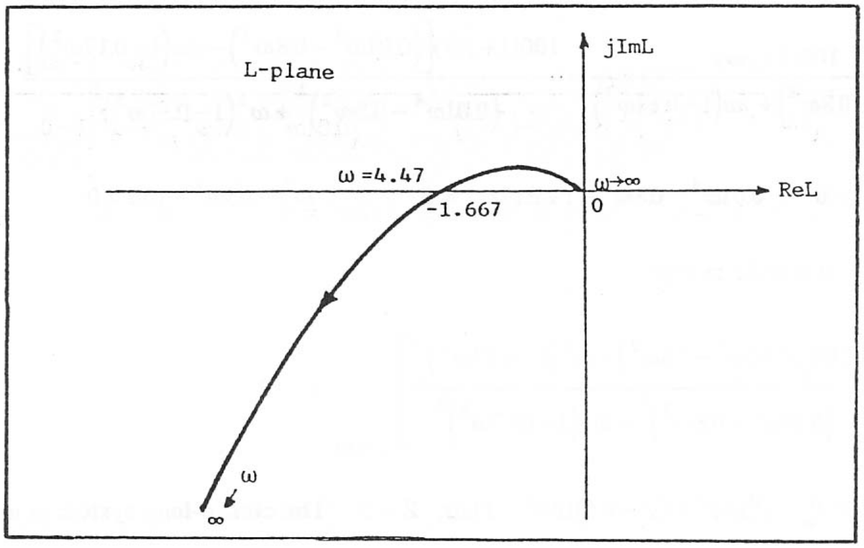
$$\Phi_{11} = 270^o = \left(Z - 0.5P_\omega - P\right)180^o = \left(Z-0.5\right)180^o \quad \text{Thus, } Z = \frac{360^o}{180^o} = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

MATLAB code:

```
s = tf('s')
%a)
figure(1);
num_G_a= 20;
den_G_a=s*(0.1*s+1)*(0.5*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



(b)

$$L(s) = \frac{10}{s(1+0.1s)(1+0.5s)}$$

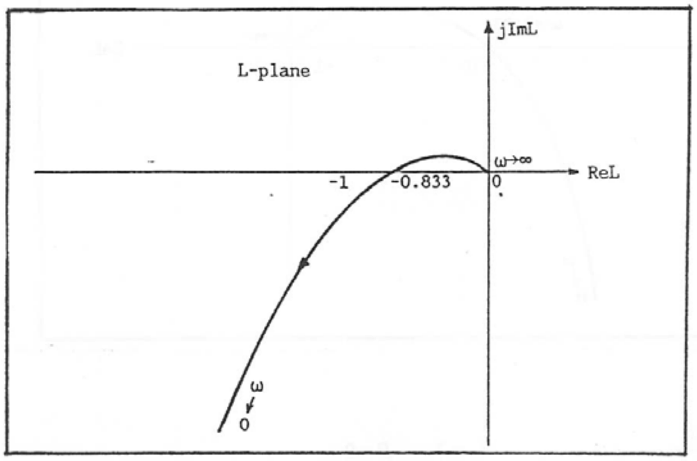
Based on the analysis conducted in part (a), the intersect of the negative real axis by the $L(j\omega)$ plot is at –0.8333, and the corresponding ω is 4.47 rad/sec.

$\Phi_{11} = -90^o = (Z - 0.5P_\omega - P)180^o = 180Z - 90^o$ Thus, $Z = 0$. **The closed-loop system is stable.**

MATLAB code:

```
s = tf('s')
%b)
figure(1);
num_G_a= 10;
den_G_a=s*(0.1*s+1)*(0.5*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



(c)

$$L(s) = \frac{100(1+s)}{s(1+0.1s)(1+0.2s)(1+0.5s)} \qquad P_\omega = 1, \quad P = 0.$$

When $\omega = 0$: $\angle L(j0) = -90^o$ $|L(j0)| = \infty$ When $\omega = \infty$: $\angle L(j\infty) = -270^o$ $|L(j\infty)| = 0$

When $\omega = \infty$: $\angle L(j\omega) = -270^o$ $|L(j\omega)| = 0$ When $\omega = \infty$: $\angle L(j\omega) = -270^o$ $|L(j\omega)| = 0$

$$L(j\omega) = \frac{100(1+j\omega)}{\left(0.01\omega^4 - 0.8\omega^2\right) + j\omega\left(1-0.17\omega^2\right)} = \frac{100(1+j\omega)\left[\left(0.01\omega^4 - 0.8\omega^2\right) - j\omega\left(1-0.17\omega^2\right)\right]}{\left(0.01\omega^4 - 0.8\omega^2\right)^2 + \omega^2\left(1-0.17\omega^2\right)^2}$$

Setting $\text{Im}\left[L(j\omega)\right] = 0$ $\quad 0.01\omega^4 - 0.8\omega^2 - 1 + 0.17\omega^2 = 0 \qquad \omega^4 - 63\omega^2 - 100 = 0$

Thus, $\omega^2 = 64.55 \quad \omega = \pm 8.03$ rad/sec

$$L(j8.03) = \left|\left(\frac{100\left[\left(0.01\omega^4 - 0.8\omega^2\right) + \omega^2\left(1-0.17\omega^2\right)\right]}{\left(0.01\omega^2 - 0.8\omega^2\right)^2 + \omega^2\left(1-0.17\omega^2\right)^2}\right)\right|_{\omega=8.03} = -10$$

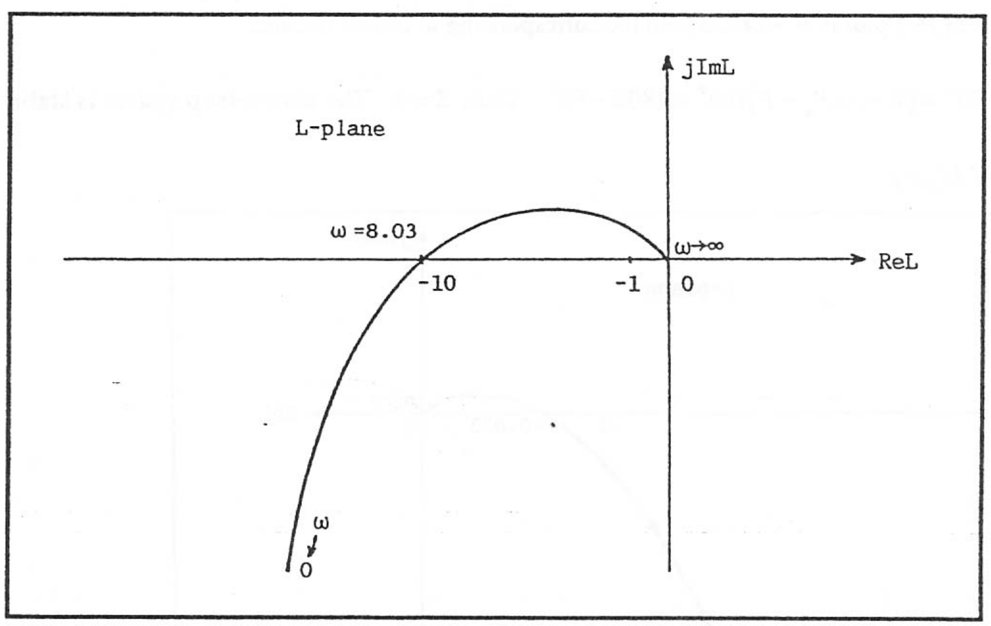
$\Phi_{11} = 270^\circ = (Z - 0.5P_\omega - P)180^\circ = (Z - 0.5)180^\circ$ Thus, $Z = 2$ **The closed-loop system is unstable.**

The characteristic equation has two roots in the right-half *s*-plane.

MATLAB code:

```
s = tf('s')
%c)
figure(1);
num_G_a= 100*(s+1);
den_G_a=s*(0.1*s+1)*(0.2*s+1)*(0.5*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



(d)

$$L(s) = \frac{10}{s^2(1+0.2s)(1+0.5s)} \qquad P_\omega = 2 \qquad P = 0$$

When $\omega = 0$: $\angle L(j\omega) = -180^o$ $\quad |L(j\omega)| = \infty$ $\qquad$ When $\omega = \infty$: $\angle L(j\omega) = -360^o$ $\quad |L(j\omega)| = 0$

$$L(j\omega) = \frac{10}{\left(0.1\omega^4 - \omega^2\right) - j0.7\omega^3} = \frac{10\left(0.1\omega^4 - \omega^2 + j0.7\omega^3\right)}{\left(0.1\omega^4 - \omega^2\right)^2 + 0.49\omega^6}$$

Setting $\text{Im}\left[L(j\omega)\right] = 0$, $\omega = \infty$. The Nyquist plot of $L(j\omega)$ does not intersect the real axis except at the origin where $\omega = \infty$.

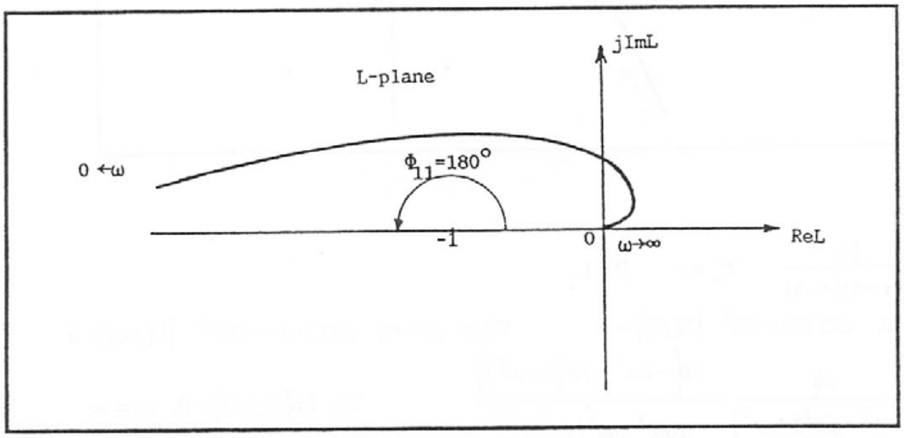
$$\Phi_{11} = \left(Z - 0.5P_\omega - P\right)180^o = \left(Z - 1\right)180^o \quad \text{Thus, } Z = 2.$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

MATLAB code:

```
s = tf('s')
%d)
figure(1);
num_G_a= 10;
den_G_a=s^2* (0.2*s+1)*(0.5*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



8-15 (e)

$$L(s)=\frac{3(s+2)}{s\left(s^3+3s+1\right)} \qquad P_\omega = 1 \qquad P=2$$

When $\omega = 0$: $\angle L(j0) = -90^o$ $\left|L(j0)\right| = \infty$ When $\omega=\infty$: $\angle L(j\infty) = -270^o$ $\left|L(j\infty)\right| = 0$

$$L(j\omega)=\frac{3(j\omega+2)}{\left(\omega^4-3\omega^2\right)+j\omega}=\frac{3(j\omega+2)\left[\left(\omega^4-3\omega^2\right)-j\omega\right]}{\left(4^4-3\omega^2\right)^2+\omega^2}$$

Setting $\text{Im}\left[L(j\omega)\right]=0$,

$\omega^4 - 3\omega^2 - 2 = 0$ or $\omega^2 = 3.56$ $\omega = \pm 1.89$ rad/sec. $L(j1.89) = 3$

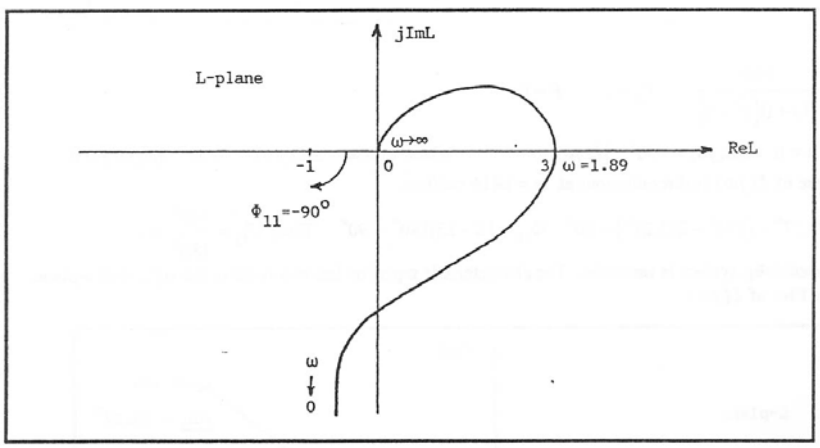
$$\Phi_{11} = \left(Z - 0.5P_\omega - P\right)180^o = \left(Z - 2.5\right)180^o = -90^o \qquad \text{Thus, } Z = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

MATLAB code:

```
s = tf('s')
%e)
figure(1);
num_G_a= 3*(s+2);
den_G_a=s*(s^3+3*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



8-15 (f)

$$L(s) = \frac{0.1}{s(s+1)(s^2+s+1)} \qquad P_\omega = 1 \qquad P = 0$$

When $\omega = 0$: $\angle L(j0) = -90^o$ $\quad |L(j0)| = \infty$ When $\omega = \infty$: $\angle L(j\infty) = -360^o$ $\quad |L(j\infty)| = 0$

$$L(j\omega) = \frac{0.1}{(\omega^4 - 2\omega^2) + j\omega(1-2\omega^2)} = \frac{0.1\left[(\omega^4 - 2\omega^2) - j\omega(1-2\omega^2)\right]}{(\omega^4 - 2\omega^2)^2 + \omega^2(1-2\omega^2)^2}$$ Settiing $\operatorname{Im}[L(j\omega)] = 0$

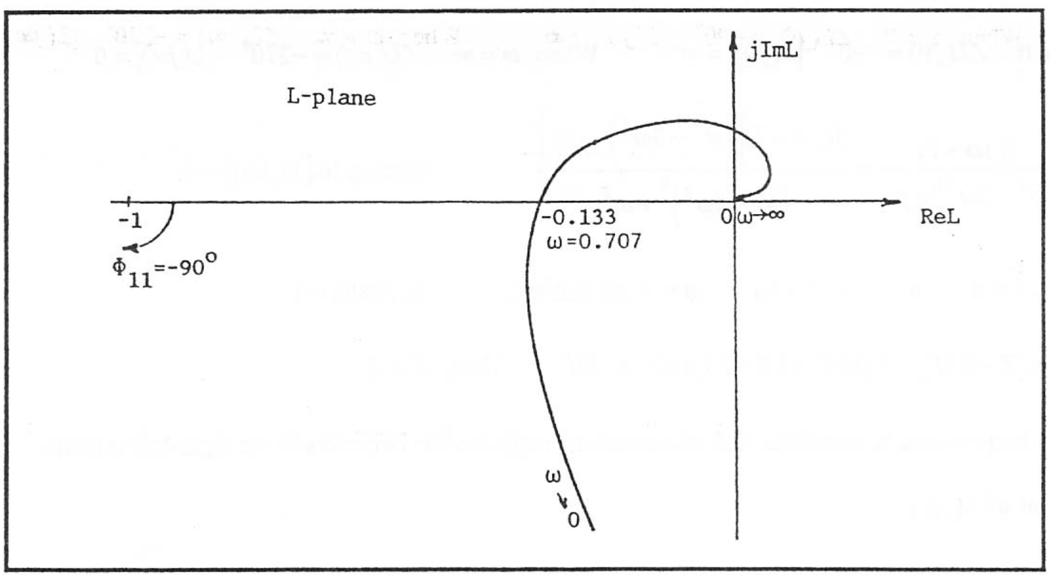
$\omega = \infty$ or $\omega^2 = 0.5$ $\quad \omega = \pm 0.707$ rad / sec $\quad L(j0.707) = -0.1333$

$\Phi_{11} = (Z - 0.5P_\omega - P)180^o = (Z - 0.5)180^o = -90^o$ Thus, $Z = 0$ **The closed-loop system is stable.**

MATLAB code:

```
s = tf('s')
%f)
figure(1);
num_G_a= 0.1;
den_G_a=s*(s+1)*(s^2+s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



8-15 (g)

$$L(s) = \frac{100}{s(s+1)(s^2+2)} \qquad P_\omega = 3 \qquad P = 0$$

When $\omega = 0$: $\angle L(j0) = -90^o$ $|L(j0)| = \infty$ When $\omega = \infty$: $\angle L(j\infty) = -360^o$ $|L(j\infty)| = 0$

The phase of $L(j\omega)$ is discontinuous at $\omega = 1.414$ rad/sec.

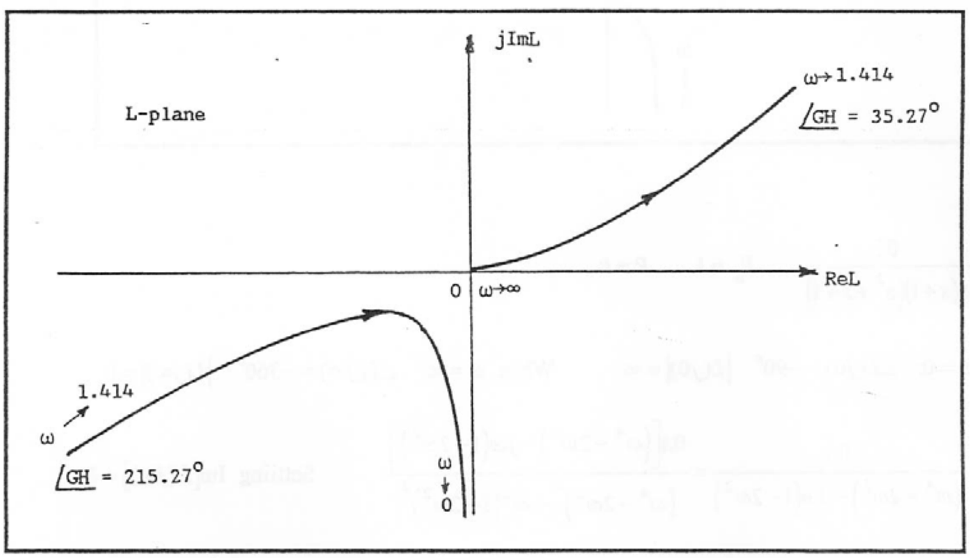
$$\Phi_{11} = 35.27^o + \left(270^o - 215.27^o\right) = 90^o \quad \Phi_{11} = (Z - 1.5)180^o = 90^o \quad \text{Thus, } P_{11} = \frac{360^o}{180^o} = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half *s*-plane.

MATLAB code:

```
s = tf('s')
%g)
figure(1);
num_G_a= 100;
den_G_a=s*(s+1)*(s^2+2);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



8-15 (h)

$$L(s) = \frac{10(s+10)}{s(s+1)(s+100)} \qquad P_\omega = 1 \qquad P = 0$$

When $\omega = 0$: $\angle L(j0) = -90^o$ $|L(j0)| = \infty$ When $\omega = \infty$: $\angle L(j\infty) = -180^o$ $|L(j\infty)| = 0$

$$L(j\omega) = \frac{10(j\omega+10)}{-101\omega^2 + j\omega(100-\omega^2)} = \frac{10(j\omega+10)\left[-101\omega^2 - j\omega\left(100-\omega^2\right)\right]}{10201\omega^4 + \omega^2\left(100-\omega^2\right)^2}$$

Setting $\mathrm{Im}\left[L(j\omega)\right]=0, \quad \omega=0$ is the only solution. Thus, the Nyquist plot of $L(j\omega)$ does not intersect the real axis, except at the origin.

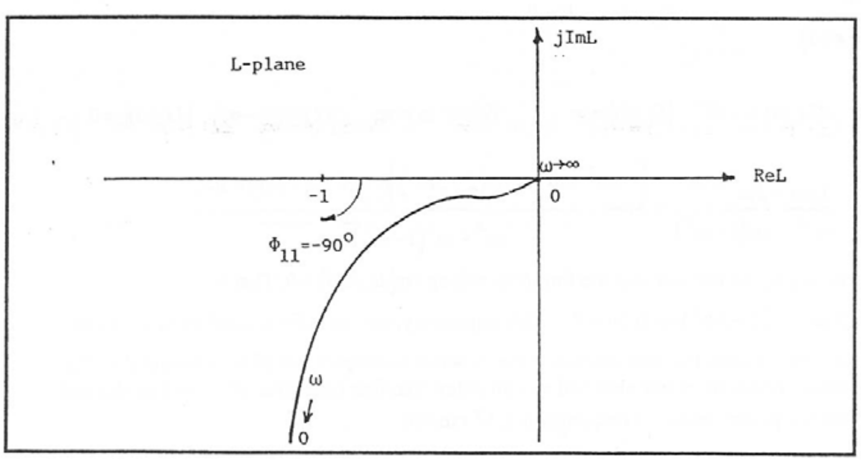
$$\Phi_{11}=\left(Z-0.5P_{\omega}-P\right)180^{o}=\left(Z-0.5\right)180^{o}=-90^{o} \qquad \text{Thus, } Z=0.$$

The closed-loop system is stable.

MATLAB code:

```
s = tf('s')
%h)
figure(1);
num_G_a= 10*(s+10);
den_G_a=s*(s+1)*(s+100);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



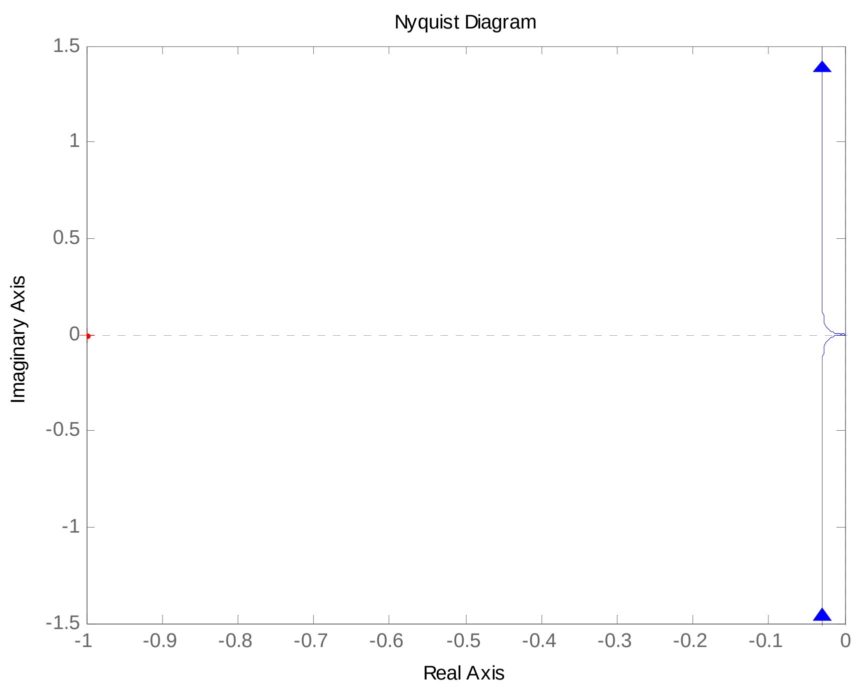
8-16

MATLAB code:

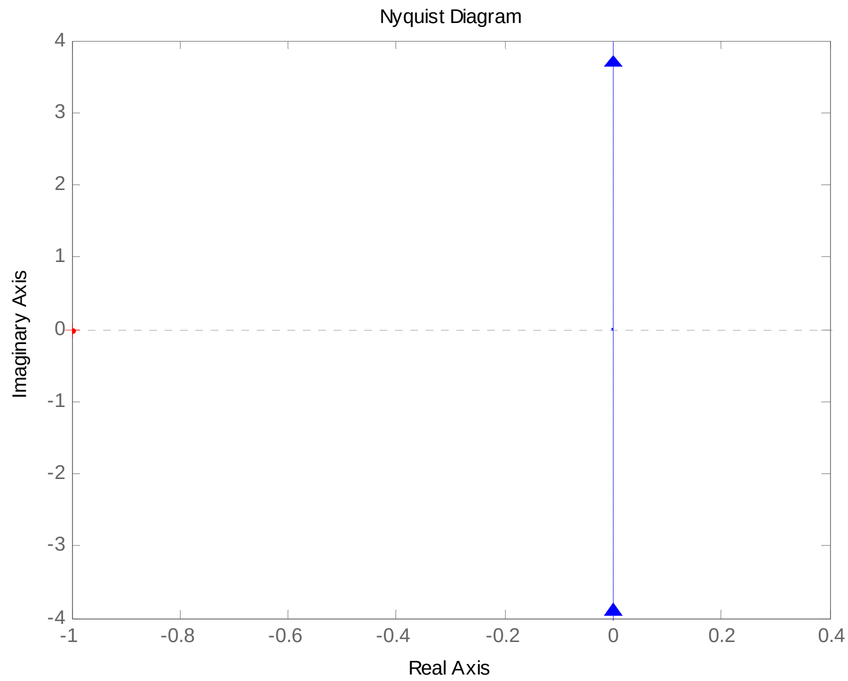
```
s = tf('s')
%a)
figure(1);
num_G_a= 1;
den_G_a=s*(s+2)*(s+10);
G_a=num_G_a/den_G_a;
nyquist(G_a)

%b)
figure(2);
num_G_b= 1*(s+1);
den_G_b=s*(s+2)*(s+5)*(s+15);
G_b=num_G_b/den_G_b;
nyquist(G_b)

%c)
figure(3);
num_G_c= 1;
den_G_c=s^2*(s+2)*(s+10);
G_c=num_G_c/den_G_c;
nyquist(G_c)

%d)
figure(4);
num_G_d= 1;
den_G_d=(s+2)^2*(s+5);
G_d=num_G_d/den_G_d;
```

```
nyquist(G_d)

%e)
figure(5);
num_G_e= 1*(s+5)*(s+1);
den_G_e=(s+50)*(s+2)^3;
G_e=num_G_e/den_G_e;
nyquist(G_e)
```

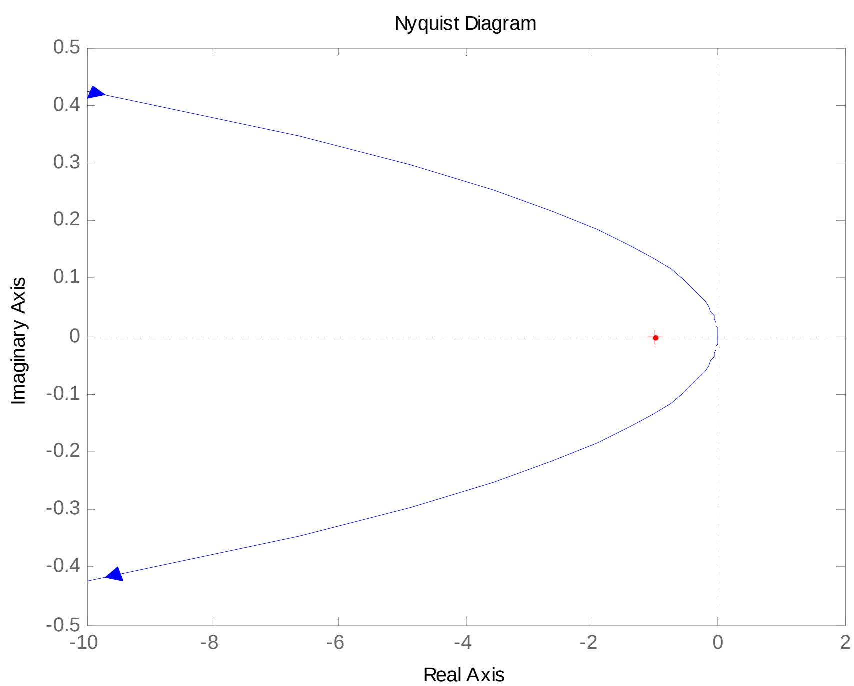
Nyquist graph, part(a):



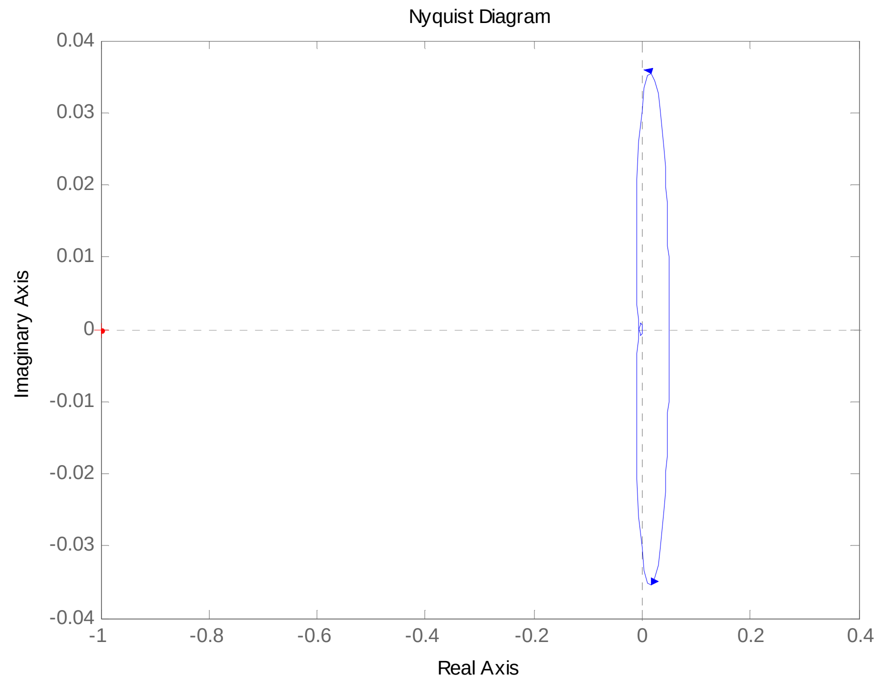
Nyquist graph, part(b):



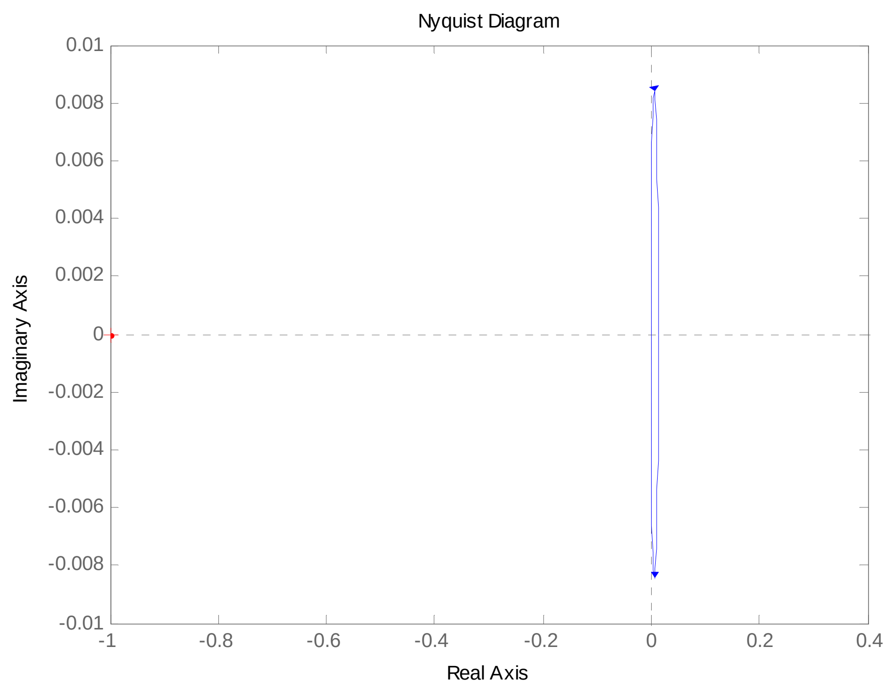
Nyquist graph, part(c):



Nyquist graph, part(d):



Nyquist graph, part (e):



8-17 (a)

$$G(s)=\frac{K}{(s+5)^2} \qquad P_\omega = 0 \qquad P = 0$$

$$\angle G(j0)=0^o \quad (K>0) \qquad \angle G(0)=180^o \quad (K<0) \quad |G(j0)|=\frac{K}{25}$$

$$G(j\infty)=-180^o \quad (K>0) \qquad \angle G(j\infty)=0^o \quad (K<0) \quad |G(j\infty)|=0$$

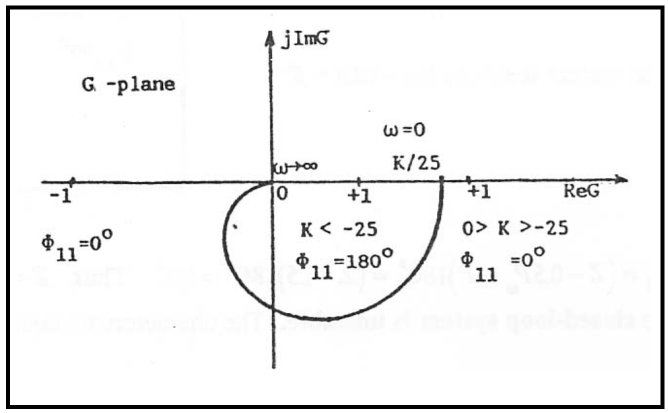
For stability, $Z = 0$.

$\Phi_{11} = -(0.5P_\omega + P)180^\circ = 0^\circ$

$0<K<\infty \quad \Phi_{11}=0^o$ **Stable**

$K<-25 \quad \Phi_{11}=180^o$ **Unstable**

$-25<K<0 \quad \Phi_{11}=0^o$ **Stable**



The system is stable for $-25 < K < \infty$.

8-17 (b)

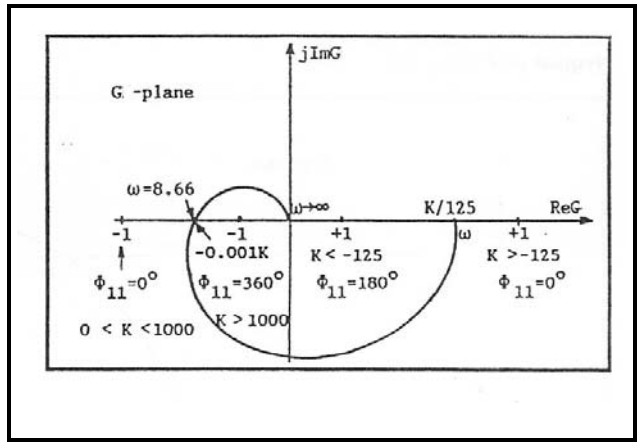
$$G(s)=\frac{K}{(s+5)^3} \qquad P_\omega = 0 \qquad P = 0$$

$$\angle G(j0)=0^o \quad (K>0) \qquad \angle G(0)=180^o \quad (K<0) \quad |G(j0)|=\frac{K}{125}$$

$$G(j\infty)=-270^o \quad (K>0) \qquad \angle G(j\infty)=270^o \quad (K<0) \quad |G(j\infty)|=0$$

For stability, $Z = 0$.

$$\Phi_{11} = -\left(0.5P_{\omega} + P\right)180^{o} = 0^{o}$$



$0 < K < 1000$	$\Phi_{11} = 0^{o}$	**Stable**
$K > 1000$	$\Phi_{11} = 360^{o}$	**Unstable**
$K < -125$	$\Phi_{11} = 180^{o}$	**Unstable**
$-125 < K < 0$	$\Phi_{11} = 0^{o}$	**Stable**

The system is stable for $-125 < K < 0$.

8-17 (c)

$$G(s) = \frac{K}{(s+5)^4} \qquad P_{\omega} = P = 0$$

$$\angle G(j0) = 0^{o} \quad (K > 0) \qquad \angle G(0) = 180^{o} \quad (K < 0) \qquad \left|G(j0)\right| = \frac{K}{625}$$

$$G(j\infty) = 0^{o} \quad (K > 0) \qquad \angle G(j\infty) = 180^{o} \quad (K < 0) \qquad \left|G(j\infty)\right| = 0$$

For stability, $Z = 0$.

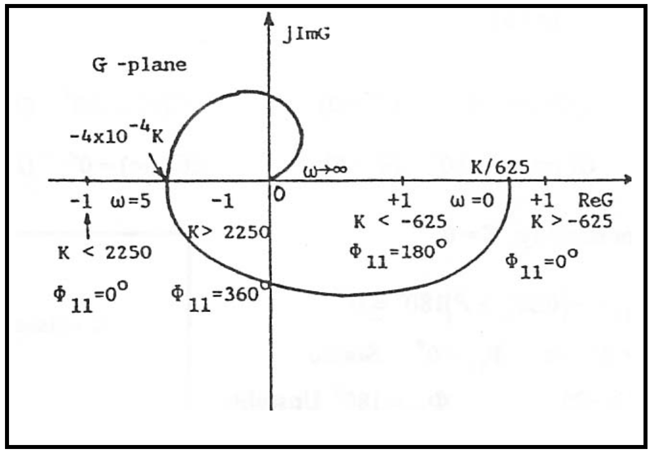
$\Phi_{11} = -(0.5P_\omega + P)180^\circ = 0^\circ$

$0 < K < 2500$ $\Phi_{11} = 0^o$ **Stable**

$K > 2500$ $\Phi_{11} = 360^o$ **Unstable**

$K < -625$ $\Phi_{11} = 180^o$ **Unstable**

$-625 < K < 0$ $\Phi_{11} = 0^o$ **Stable**



The system is stable for –625 < *K* < 2500.

8-18) The characteristic equation:

$$1 + \frac{K}{(s+1)(s^2+2s+2)} = 0$$

or

$s^3 + 3s^2 + 4s + K + 2 = 0$

if $K > 0$, the cross real axis at s = 0.1. ⇨ For stability $-10 < K < 10$

if K < 0, the Nyquist cross the real axis at $s = 0.5$ .So, for stability, $-2 < K < 2$

therefore, the rage of stability for the system is $-2 < K < 10$

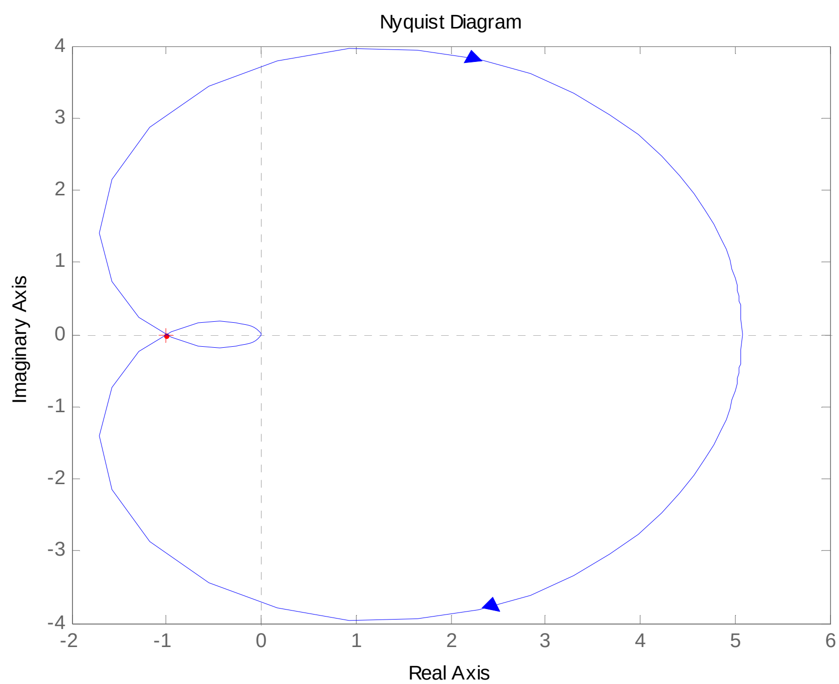
MATLAB code:

```
s = tf('s')

K=1
G= K/(s^2+2*s+2);
H=1/(s+1);
GH=G*H;

sisotool
```

```
K=10.15
G2= K/(s^2+2*s+2);
H2=1/(s+1);
GH2=G2*H2;
nyquist(GH2)
xlim[(-1.5,.5)]
ylim[(-1,1)]
```

After generating the feed-forward (G) and feedback (H) transfer functions in the MATLAB code, these transfer functions are imported to sisotool. Nyquist diagram is added to the results of sisotool. The overall gain of the transfer function is changed until Nyquist diagram passes through -1+0j point. Higher values of K resulted in unstable Nyquist diagram. Therefore K<10.15 determines the range of stability for the closed loop system.

Nyquist at margin of stability:



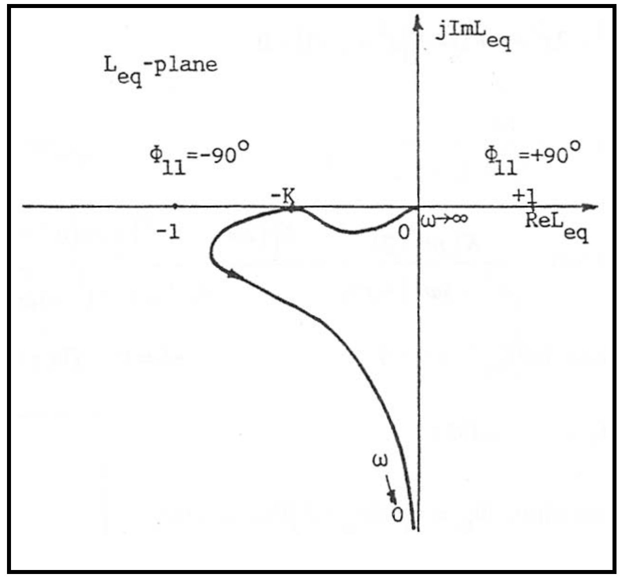
8-19)

$$s\left(s^3+2s^2+s+1\right)+K\left(s^2+s+1\right)=0$$

$$L_{eq}(s) = \frac{K\left(s^2+s+1\right)}{s\left(s^3+2s^2+s+1\right)} \qquad P_\omega = 1 \quad P = 0 \quad L_{eq}(j0) = \infty\angle -90^o \quad L_{eq}(j\infty) = 0\angle 180^o$$

$$L_{eq}(j\omega) = \frac{K\left[\left(1-\omega^2\right)+j\omega\right]}{\left(\omega^4-\omega^2\right)+j\omega\left(1-2\omega^2\right)} = \frac{K\left[-\left(\omega^6+\omega^4\right)-j\omega\left(\omega^4-2\omega^2+1\right)\right]}{\left(\omega^4-\omega^2\right)^2+\omega^2\left(1-2\omega^2\right)^2}$$

Setting $\mathrm{Im}\left[L_{eq}(j\omega)\right] = 0$



$$\omega^4 - 2\omega^2 + 1 = 0$$

Thus, $\omega = \pm 1$ rad/sec are the real solutions.

$$L_{eq}(j1) = -K$$

For stability,

$$\Phi_{11} = -(0.5P_\omega + P)180^o = -90^o$$

When K = 1 the system is marginally stable.

$K > 0$	$\Phi_{11} = -90^o$	**Stable**
$K < 0$	$\Phi_{11} = +90^o$	**Unstable**

Routh Tabulation

s^4	1	$K+1$	K
s^3	2	$K+1$	
s^2	$\frac{K+1}{2}$	K	$K > -1$
s^1	$\frac{K^2-2K+1}{K+1} = \frac{(K-1)^2}{K+1}$		
s^0	K		$K > 0$

When *K* = 1 the coefficients of the s^1 row are all zero. The auxiliary equation is $s^2+1=0$ The solutions are $\omega = \pm 1$ rad/sec. Thus the Nyquist plot of $L_{eq}(j\omega)$ intersects the –1 point when *K* = 1, when $\omega = \pm 1$ rad/sec. **The system is stable for 0 < *K* < ∝, except at *K* = 1.**

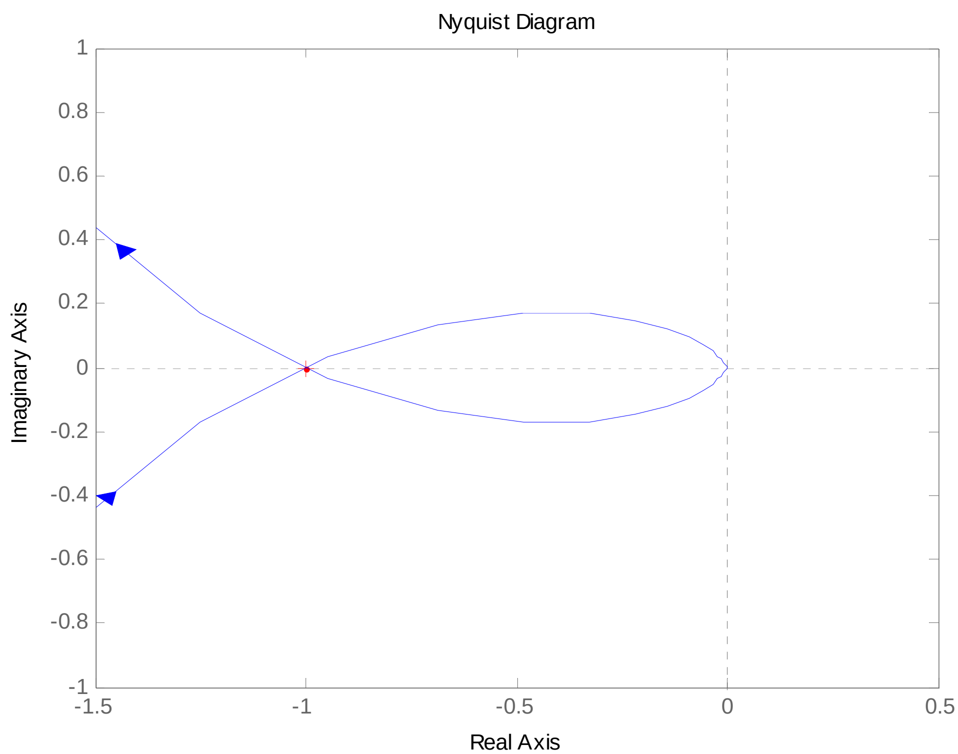
8-20) Solution is similar to the previous problem. Let's use Matlab as an alternative approach

MATLAB code:

```
s = tf('s')

figure(1);
K=8.09
num_GH= K;
den_GH=(s^3+3*s^2+3*s+1);
GH=num_GH/den_GH;
nyquist(GH)
xlim([-1.5,.5])
ylim([-1,1])

sisotool;
```

After generating the loop transfer function and analyzing Nyquist in MATLAB sisotool, it was found that for values of K higher than ~8.09, the closed loop system is unstable. Following is the Nyquist diagram at margin of stability.

Part(a), Nyquist at margin of stability:



Part(b), Verification by Routh-Hurwitz criterion:

Using Routh criterion, the coefficient table is as follows:

S^3	1	3
S^2	3	K+1
S^1	(8-K)/3	0
S^0	K+1	0

The system is stable if the content of the 1st column is positive:

$$(8-K)/3>0 \rightarrow K<8$$

$$K+1>0 \rightarrow K>-1$$

which is consistent with the results of the Nyquist diagrams.

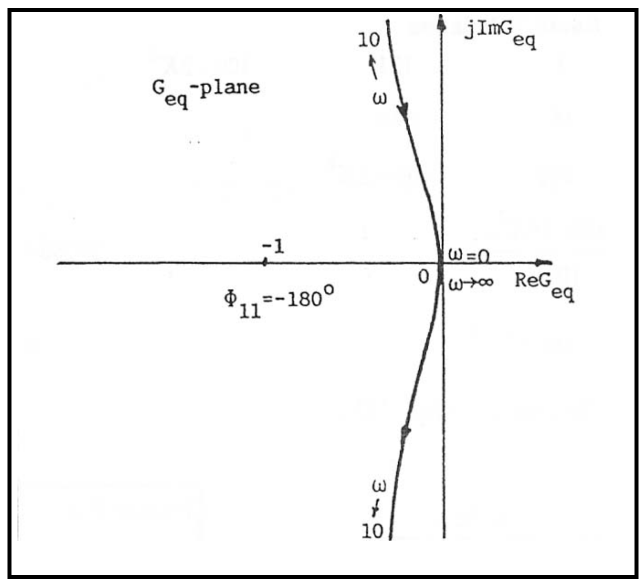
8-21)

Parabolic error constant $K_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} 10\left(K_P + K_D s\right) = 10K_P = 100$ Thus $K_P = 10$

Characteristic Equation: $s^2 + 10K_D s + 100 = 0$

$$G_{eq}(s) = \frac{10K_D s}{s^2+100} \qquad P_\omega = 2 \qquad P = 0$$

For stability,



$$\Phi_{11} = -(0.5P_\omega + P)180^o = -180^o$$

The system is stable for $0 < K_D < \infty$**.**

8-22 (a) The characteristic equation is $1+G(s)-G(s)-2[G(s)]^2 = 1-2[G(s)]^2 = 0$

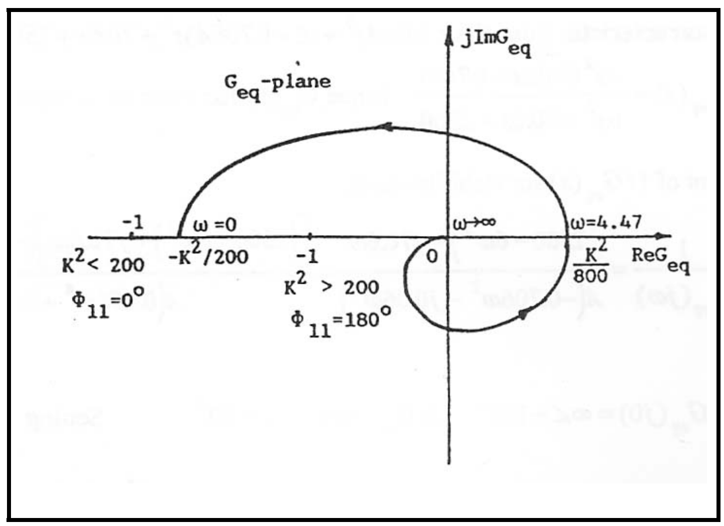
$$G_{eq}(s) = -2[G(s)]^2 = \frac{-2K^2}{(s+4)^2(s+5)^2} \qquad P_\omega = 0 \qquad P = 0$$

$$G_{eq}(j\omega) = \frac{-2K^2}{\left(400-120\omega^2+\omega^4\right)+j\omega\left(360-18\omega^2\right)} = \frac{-2K^2\left[\left(400-120\omega^2+\omega^2\right)-j\omega\left(360-18\omega^2\right)\right]}{\left(400-120\omega^2+\omega^2\right)+\omega^2\left(360-18\omega^2\right)^2}$$

$$G_{eq}(j0) = \frac{K^2}{200}\angle 180^o \qquad G_{eq}(j\infty) = 0\angle 180^o \qquad \text{Setting } \operatorname{Im}\left[G_{eq}(j\omega)\right] = 0$$

$$\omega = 0 \quad \text{and} \quad \omega = \pm 4.47 \text{ rad / sec} \qquad G_{eq}(j4.47) = \frac{K^2}{800}$$

For stability,



$$\Phi_{11} = -\left(0.5P_\omega + P\right)180^o = 0^o$$

The system is stable for $K^2 < 200$

or $|K| < \sqrt{200}$

8-22 (b)

Characteristic Equation: $s^4 + 18s^3 + 121s^2 + 360s + 400 - 2K^2 = 0$

Routh Tabulation

s^4	1	121	$400 - 2K^2$
s^3	18	360	
s^2	101	$400 - 2K^2$	
s^1	$\dfrac{29160 - 36K^2}{101}$		$29160 + 36K^2 > 0$
s^0	$400 - 2K^2$		$K^2 < 200$

Thus for stability, $|K| < \sqrt{200}$

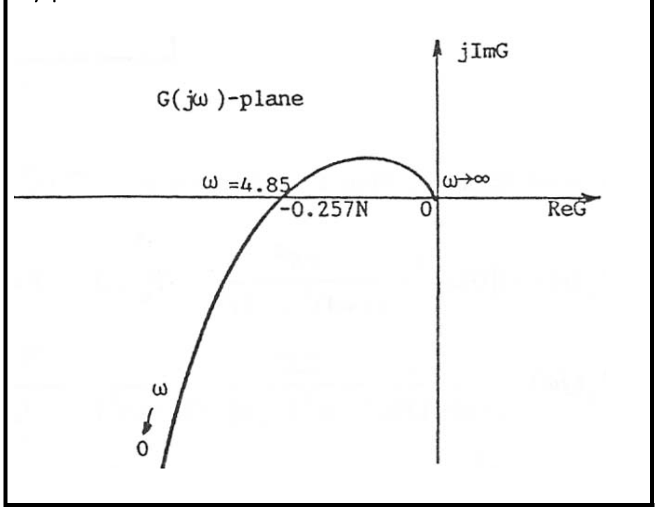
8-23 (a)

$$G(s) = \frac{83.33N}{s(s+2)(s+11.767)}$$

For stability, $N < 3.89$

Thus $N < 3$ since N must be an integer.

Nyquist Plot



(b)

$$G(s)=\frac{2500}{s(0.06s+0.706)(As+100)}$$

Characteristic Equation: $0.06As^3+(6+0.706A)s^2+70.6s+2500=0$

$G_{eq}(s)=\dfrac{As^2(0.06s+0.706)}{6s^2+70.6s+2500}$ Since $G_{eq}(s)$ has more zeros than poles, we should sketch the Nyquist

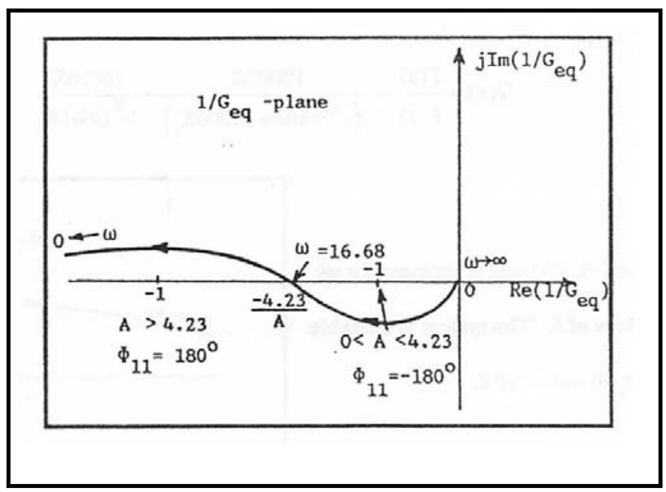
plot of $1/G_{eq}(s)$ for stability study.

$$\frac{1}{G_{eq}(j\omega)}=\frac{\left(2500-6\omega^2\right)+j70.6\omega}{A\left(-0.706\omega^2-j0.06\omega^3\right)}=\frac{\left[\left(2500-6\omega^2\right)+j70.6\omega\right]\left(-0.706\omega^2+j0.06\omega^3\right)}{A\left(0.498\omega^4+0.0036\omega^6\right)}$$

$1/G_{eq}(j0)=\infty\angle-180^o$ $1/G_{eq}(j\infty)=0\angle-90^o$ Setting $\text{Im}\left[\dfrac{1}{G_{eq}(j\omega)}\right]=0$

$100.156-0.36\omega^2=0$ $\omega=\pm16.68$ rad/sec $\dfrac{1}{G_{eq}(j16.68)}=\dfrac{-4.23}{A}$

For stability,



$$\Phi_{11}=-\left(0.5P_\omega+P\right)180^o=-180^o$$

For $A>4.23$ $\Phi_{11}=180^o$ **Unstable**

For $0<A<4.23$ $\Phi_{11}=-180^o$ **Stable**

The system is stable for 0 < *A* < 4.23.

(c)

$$G(s) = \frac{2500}{s(0.06s + 0.706)(50s + K_o)}$$

Characteristic Equation: $s(0.06s + 0.706)(50s + K_o) + 2500$ = 0

$$G_{eq}(s) = \frac{K_o s(0.06s + 0.706)}{3s^3 + 35.3s^2 + 2500} \qquad P_\omega = 0 \qquad P = 0 \qquad G_{eq}(j0) = 0\angle 90^o \qquad G_{eq}(j\infty) = 0\angle -90^o$$

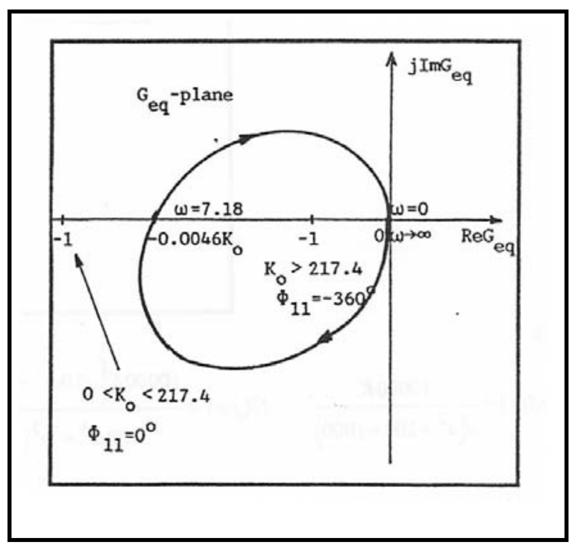
$$G_{eq}(j\omega) = \frac{K_o(-0.06\omega^3 + 0.706j\omega)}{\left(2500 - 35.3\omega^2\right) - j3\omega^3} = \frac{K_o\left(-0.06\omega^2 + 0.706j\omega\right)\left[\left(2500 - 35.5\omega^2\right) + j3\omega^3\right]}{\left(2500 - 35.3\omega^2\right)^2 + 9\omega^6}$$

Setting $\text{Im}\left[G_{eq}(j\omega)\right] = 0 \quad \omega^4 + 138.45\omega^2 - 9805.55 = 0 \qquad \omega^2 = 51.6 \qquad \omega = \pm 7.18$ rad/sec

$$G_{eq}(j7.18) = -0.004K_o$$

For stability, $\Phi_{11} = -\left(0.5P_\omega + P\right)180^o = 0^o$

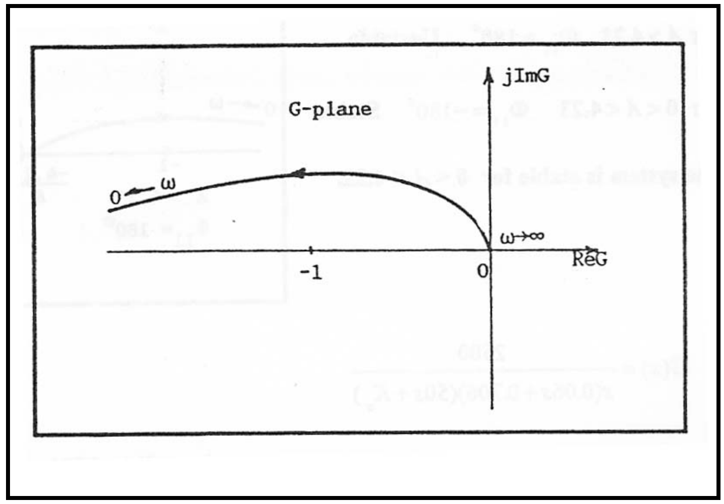
For stability, $0 < K_o < 217.4$



8-24 (a) $K_t = 0$:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{10000K}{s\left(s^2 + 10s + 10000K_t\right)} = \frac{10000K}{s^2(s+10)}$$

The (–1, *j*0) point is enclosed for all values of *K*. **The system is unstable for all values of *K*.**



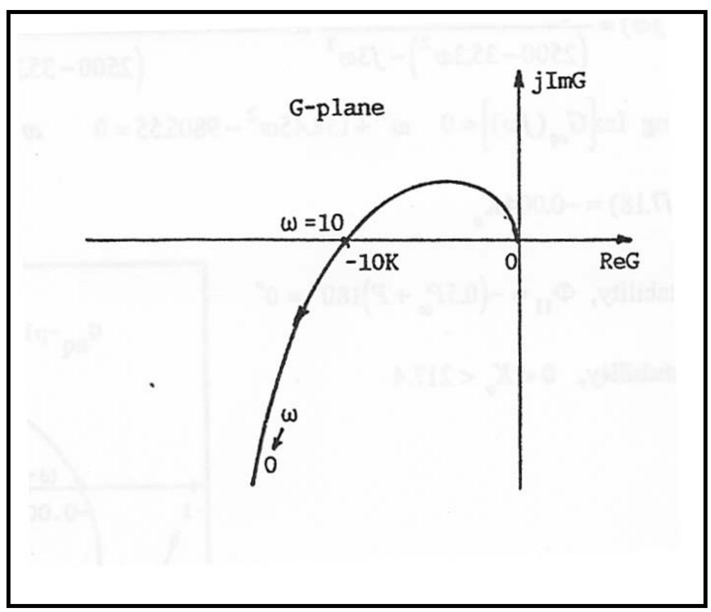
(b) $K_t = 0.01$:

$$G(s) = \frac{10000K}{s\left(s^2 + 10s + 100\right)} \qquad G(j\omega) = \frac{10000K\left[-10\omega^2 - j\omega\left(100 - \omega^2\right)\right]}{100\omega^4 + \omega^2\left(100 - \omega^2\right)^2}$$

Setting $\text{Im}\left[G(j\omega)\right] = 0 \quad \omega^2 = 100$

$\omega = \pm 10 \text{ rad/sec} \quad G(j10) = -10K$

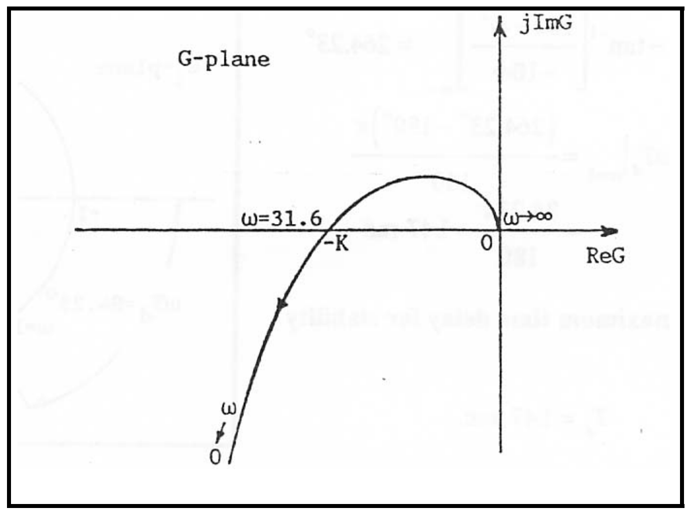
The system is stable for 0 < *K* < 0.1



(c) $K_t = 0.1$:

$$G(s) = \frac{10000K}{s\left(s^2 + 10s + 1000\right)} \qquad G(j\omega) = \frac{10000K\left[-10\omega^2 - j\omega\left(1000 - \omega^2\right)\right]}{100\omega^4 + \omega^2\left(1000 - \omega^2\right)^2}$$

Setting $\mathrm{Im}\left[G(j\omega)\right] = 0 \quad \omega^2 = 100 \qquad \omega = \pm 31.6$ rad/sec $\qquad G(j31.6) = -K$



For stability, 0 < *K* < 1

8-25) The characteristic equation for *K* = 10 is:

$$s^3 + 10s^2 + 10{,}000K_t s + 100{,}000 = 0$$

$$G_{eq}(s) = \frac{10{,}000K_t s}{s^3 + 10s^2 + 100{,}000} \qquad P_\omega = 0 \qquad P = 2$$

$$G_{eq}(j\omega) = \frac{10{,}000K_t j\omega}{100{,}000 - 10\omega^2 - j\omega^3} = \frac{10{,}000K_t\left[-\omega^4 + j\omega\left(10{,}000 - 10\omega^2\right)\right]}{\left(10{,}000 - 10\omega^2\right)^2 + \omega^6}$$

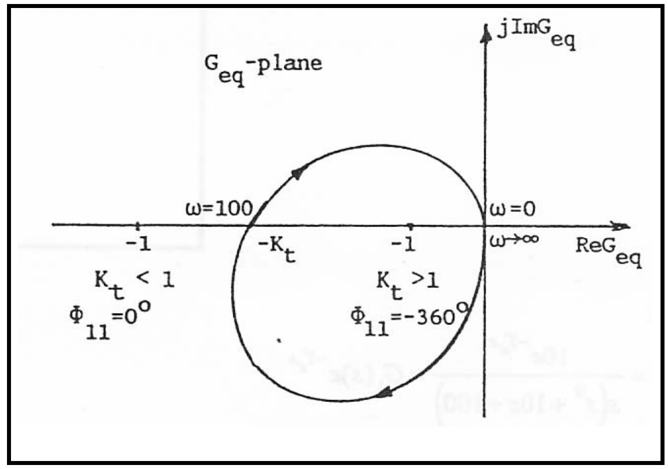
Setting $\mathrm{Im}\left[G_{eq}(j\omega)\right] = 0$

$\omega = 0, \quad \omega^2 = 10{,}000$

$\omega = \pm 100 \text{ rad / sec} \quad G_{eq}(j100) = -K_t$

For stability,

$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -360^\circ$



The system is stable for $K_t > 0$.

8-26)

$$\frac{Y(s)}{X(s)} = \frac{KK_f}{Js^2 + (a + KK_f)s + KK_f}$$

a) $K_f = 0 \Rightarrow \dfrac{Y(s)}{X(s)} = \dfrac{K}{Js^2 + as + K} = \dfrac{K}{s^2 + s + K}$

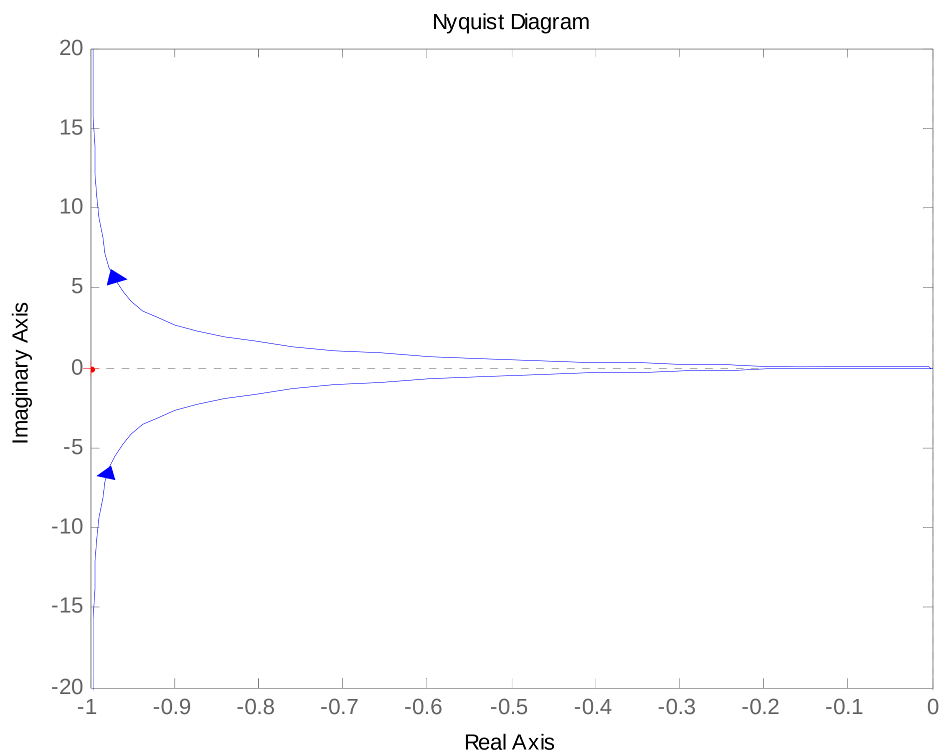
b) $K_f = 0.1 \Rightarrow \dfrac{Y(s)}{X(s)} = \dfrac{0.1K}{Js^2 + (a + 0.1K)s + 0.1K} = \dfrac{0.1K}{s^2 + (1 + 0.1K)s + 0.1K}$

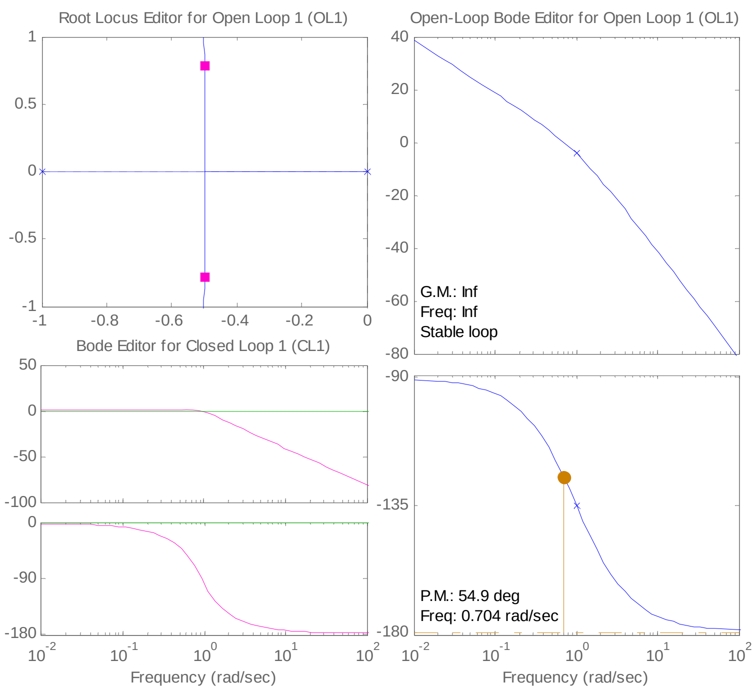
c) $K_f = 0.2 \Rightarrow \dfrac{Y(s)}{X(s)} = \dfrac{0.2K}{s^2 + (-1 + 0.2K)s + 0.2K}$

MATLAB code:

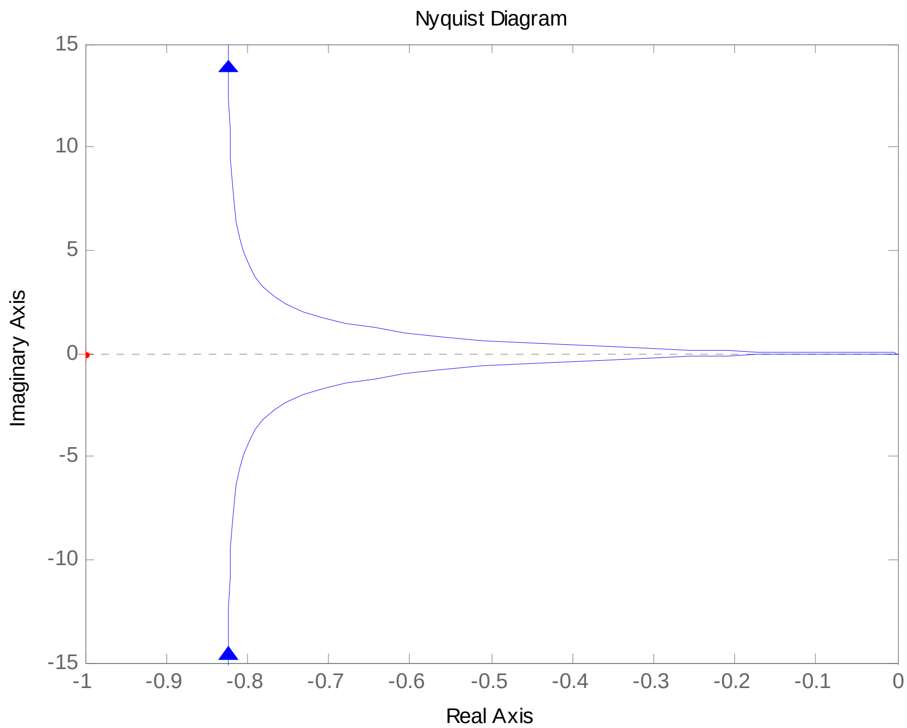
```
s = tf('s')
figure(1);
J=1;
B=1;
K=1;
Kf=0
G1= K/(J*s+B);
CL1=G1/(1+G1*Kf);
H2 = 1;
G1G2 = CL1/s;
L_TF=G1G2*H2;
nyquist(L_TF)

sisotool
```

Part (a), `Kf=0`**:** by plotting the Nyquist diagram in sisotool and varying the gain, it was observed that all values of gain (K) will result in a stable system. Location of poles in root locus diagram of the second figure will also verify that.

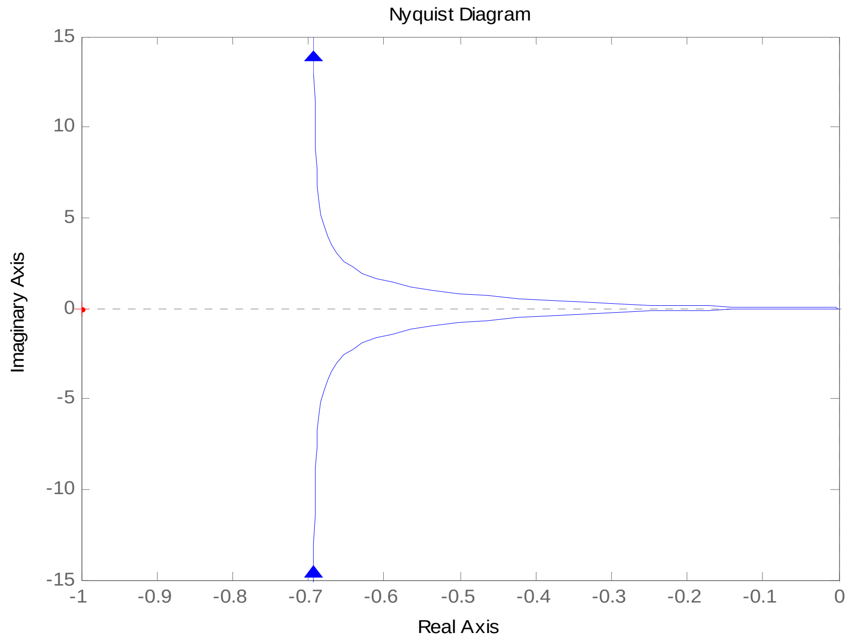




Part (b), `Kf=0.1`**:** The result and approach is similar to part (a), a sample of Nyquist diagram is presented for his case as follows:



Part (c), `Kf=0.2`**:** The result and approach is similar to part (a), a sample of Nyquist diagram is presented for his case as follows:



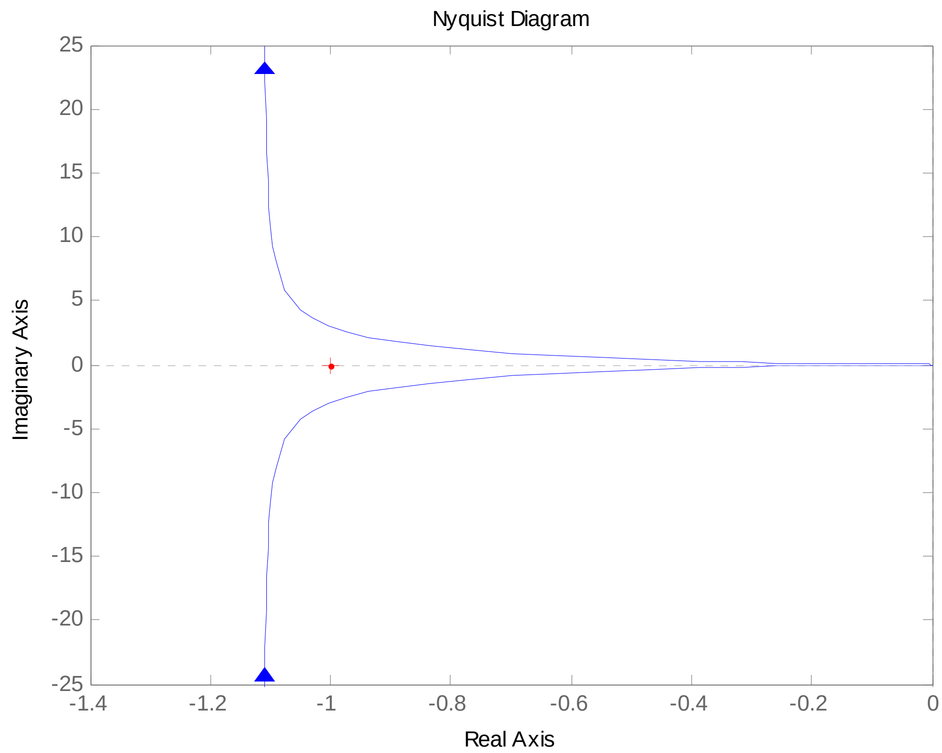
8-27)

$$\frac{Y(s)}{X(s)} = \frac{10K_f}{s^2 + (1 + 10K_f)s + 10K_f}$$

MATLAB code:

```
s = tf('s')
figure(1);
J=1;
B=1;
K=10;
Kf=0.2
G1= K/(J*s+B);
CL1=G1/(1+G1*Kf);
H2 = 1;
G1G2 = CL1/s;
L_TF=G1G2*H2;
nyquist(L_TF)
```

After assigning K=10, different values of Kf has been used in the range of $0.01<K<10^4$. The Nyquist diagrams shows the stability of the closed loop system for all $0<K<\infty$. A sample of Nyquist diagram is plotted as follows:



8-28) a) K > 2 ⇨ system is stable

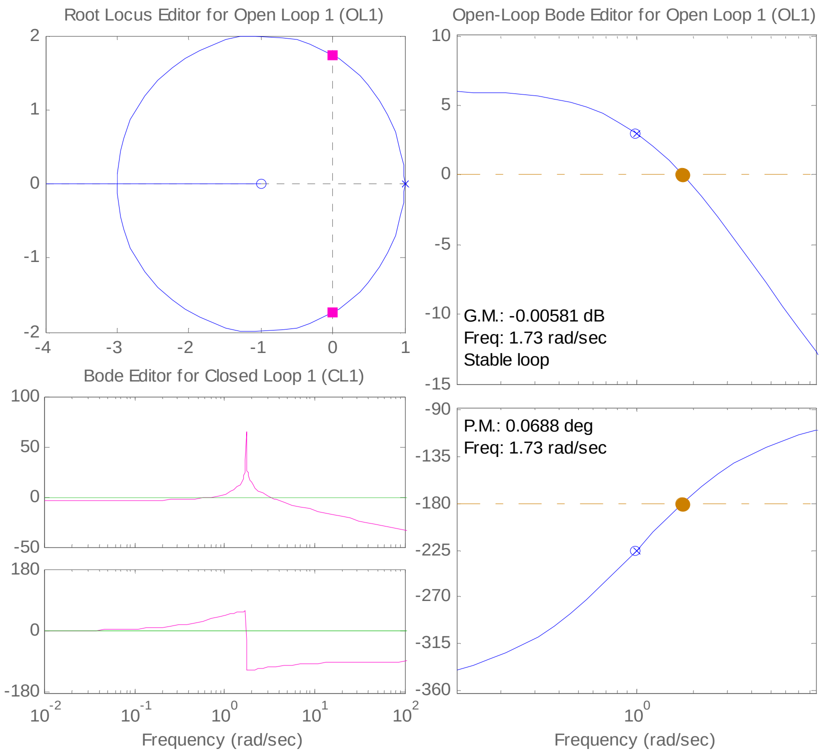
b) 0 < K < 1 and -2 < K < 0 ⇨ -2 < K < 1 ⇨system is stable

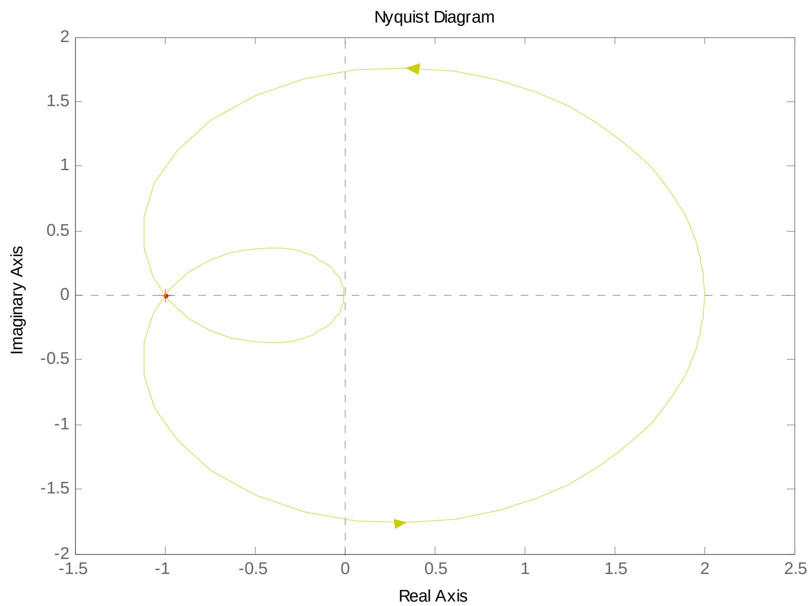
MATLAB code:

```
s = tf('s')
%a)
figure(1);
K=1
num_GH_a= K*(s+1);
den_GH_a=(s-1)^2;
GH_a=num_GH_a/den_GH_a;
nyquist(GH_a)
%b)
figure(2);
K=1
num_GH_b= K*(s-1);
den_GH_b=(s+1)^2;
GH_b=num_GH_b/den_GH_b;
nyquist(GH_b)
```

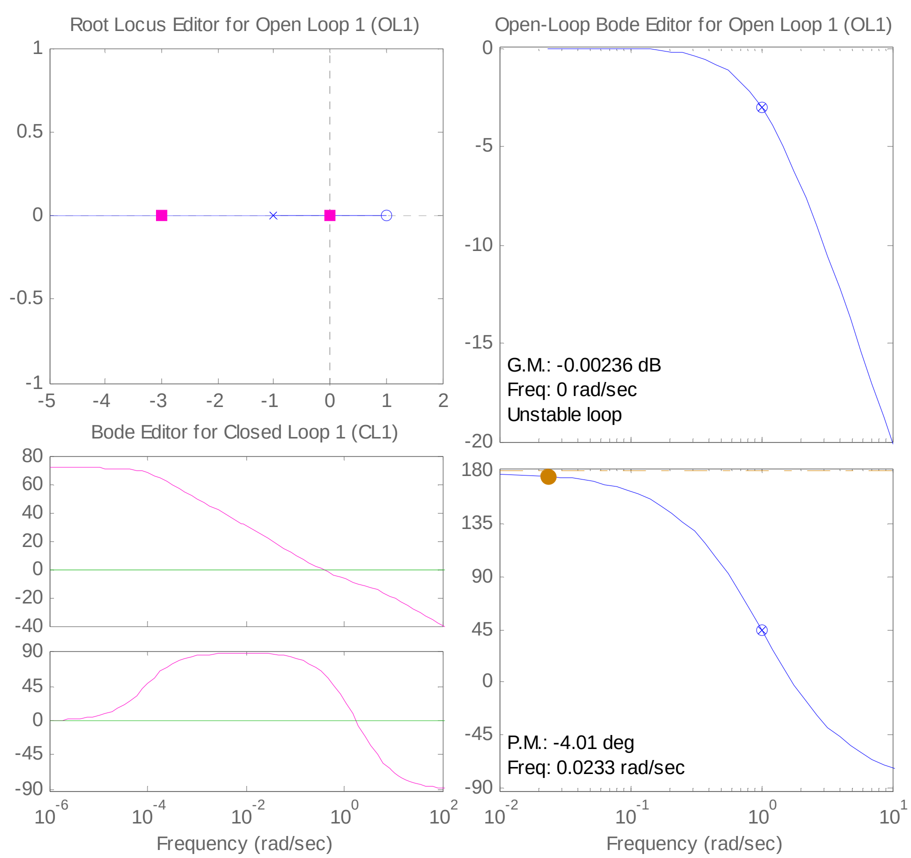
sisotool

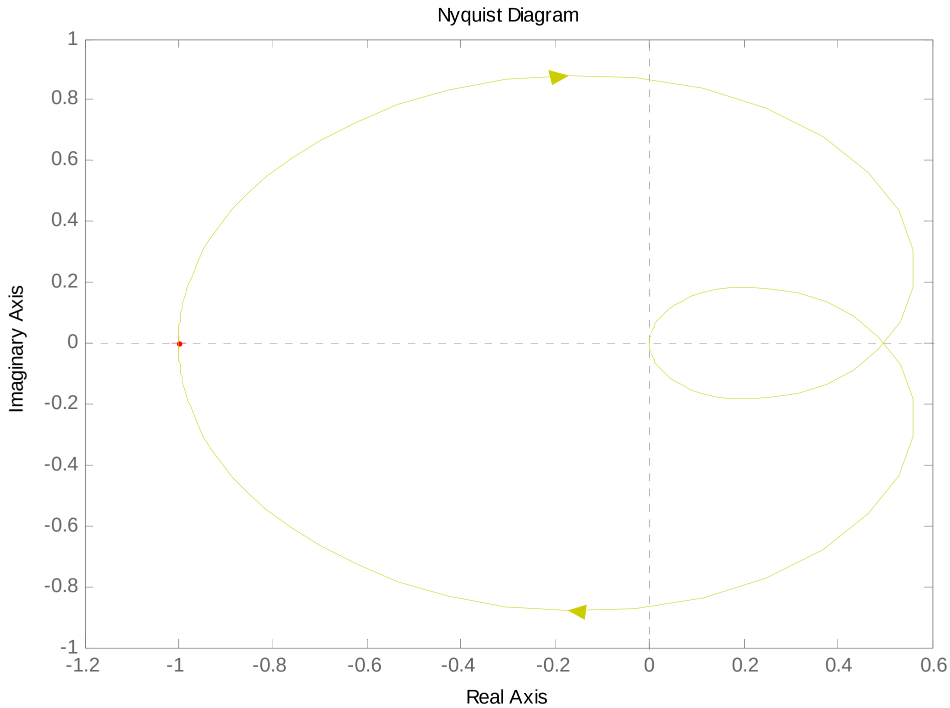
Part(a): Using MATLAB sisotool, the transfer function gain can be iteratively changed in order to obtain different phase margins. By changing the gain so that PM=0 (margin of stability), K>~2 resulted in stable Nyquist diagram for part(a). Following two figures illustrate the sisotool and Nyquist results at margin of stability for part (a).





Part(b): Similar methodology applied as in part (a). K<1 results in closed loop stability. Following are sisotool and Nyquist results at margin of stability (K=1):





8-29) (a)

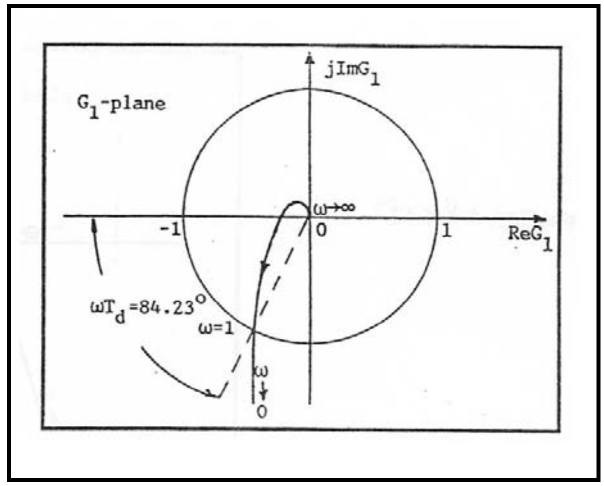
Let $G(s) = G_1(s)e^{-T_d s}$ Then $G_1(s) = \dfrac{100}{s\left(s^2 + 10s + 100\right)}$

Let $\left|\dfrac{100}{-10\omega^2 + j\omega\left(100 - \omega^2\right)}\right| = 1$ or $\dfrac{100}{\left[100\omega^4 + \omega^2\left(100 - \omega^2\right)^2\right]^{1/2}} = 1$

Thus $100\omega^4 + \omega^2\left(100 - \omega^2\right)^2 = 10,000$ $\omega^6 - 100\omega^4 + 10,000\omega^2 - 10,000 = 0$

The real solution for ω are $\omega = \pm 1$ rad/sec.

$$\angle G_1(j1) = -\tan^{-1}\left[\frac{100-\omega^2}{-10\omega}\right]\Bigg|_{\omega=1} = 264.23^o$$



Equating $\omega T_d\big|_{\omega=1} = \dfrac{\left(264.23^o - 180^o\right)\pi}{180}$

$$= \frac{84.23\pi}{180} = 1.47 \text{ rad}$$

Thus the maximum time delay for stability is

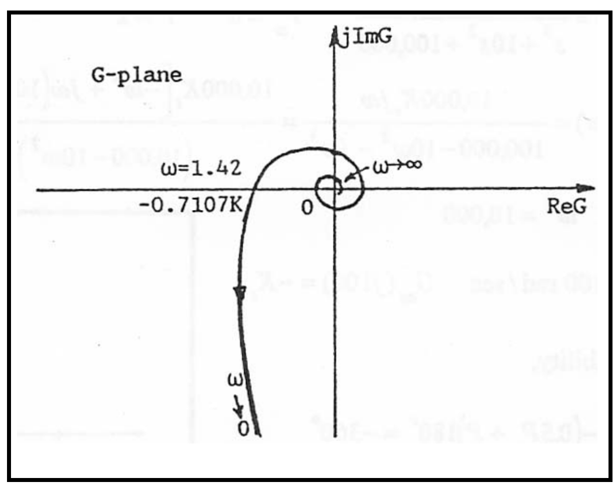
$T_d = 1.47$ **sec.**

(b) $T_d = 1$ sec.

$$G(s) = \frac{100Ke^{-s}}{s\left(s^2+10s+100\right)} \qquad G(j\omega) = \frac{100Ke^{-j\omega}}{-10\omega^2 + j\omega\left(100-\omega^2\right)}$$

At the intersect on the negative real axis, $\omega = 1.42$ rad/sec.

$G(j1.42) = -0.7107K.$



The system is stable for

0 < *K* < 1.407

8-30 (a) ***K*** **= 0.1**

$$G(s) = \frac{10e^{-T_d s}}{s\left(s^2 + 10s + 100\right)} = G_1(s)e^{-T_d s}$$

Let $\left|\frac{10}{-10\omega^2 + j\omega\left(100-\omega^2\right)}\right| = 1$ or $\frac{10}{\left[100\omega^4 + \omega^2\left(100-\omega^2\right)^2\right]^{1/2}} = 1$

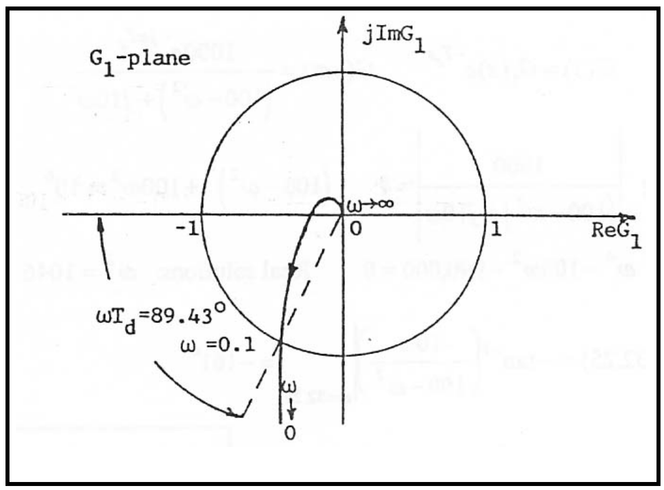
Thus $\omega^6 - 100\omega^4 + 10{,}000\omega^2 - 100 = 0$ The real solutions for ω is $\omega = \pm 0.1$ rad/sec.

$$\angle G_1(j0.1) = -\tan^{-1}\left[\frac{100-\omega^2}{-10\omega}\right]\Bigg|_{\omega=0.1} = 269.43^o$$

Equate $\omega T_d\big|_{\omega=0.1} = \frac{\left(269.43^o - 180^o\right)\pi}{180^o} = 1.56$ rad We have $T_d = 15.6$ sec.

We have the maximum time delay

for stability is 15.6 sec.



8-30 (b) $T_d = 0.1$ sec.

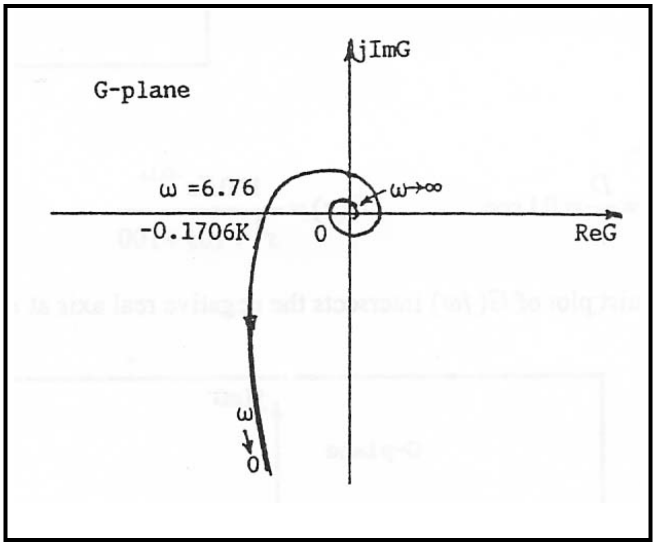
$$G(s) = \frac{100Ke^{-0.1s}}{s\left(s^2 + 10s + 100\right)} \qquad G(j\omega) = \frac{100Ke^{-0.1j\omega}}{-10\omega^2 + j\omega\left(100-\omega^2\right)}$$

At the intersect on the negative real axis,

$\omega = 6.76$ rad/sec. $G(j6.76) = -0.1706K$

The system is stable for

0 < *K* < 5.86

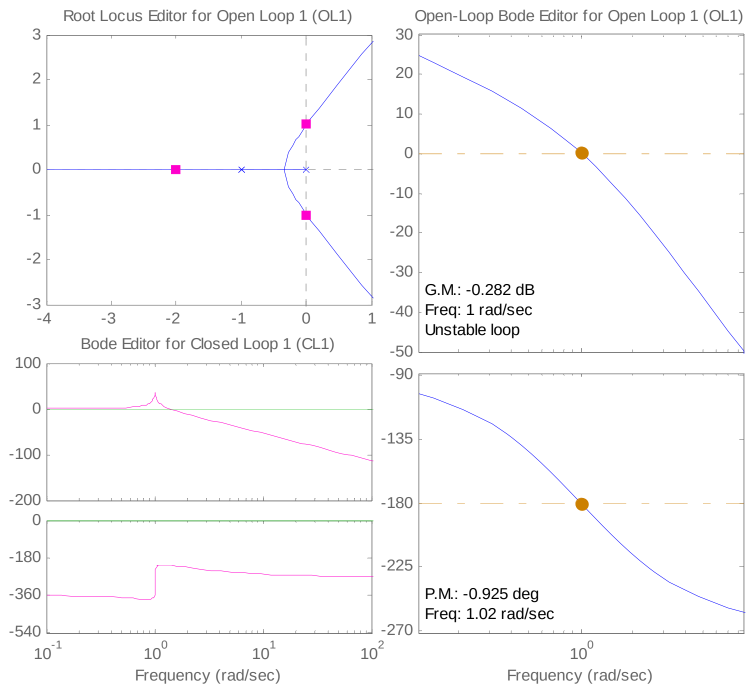


8-31)

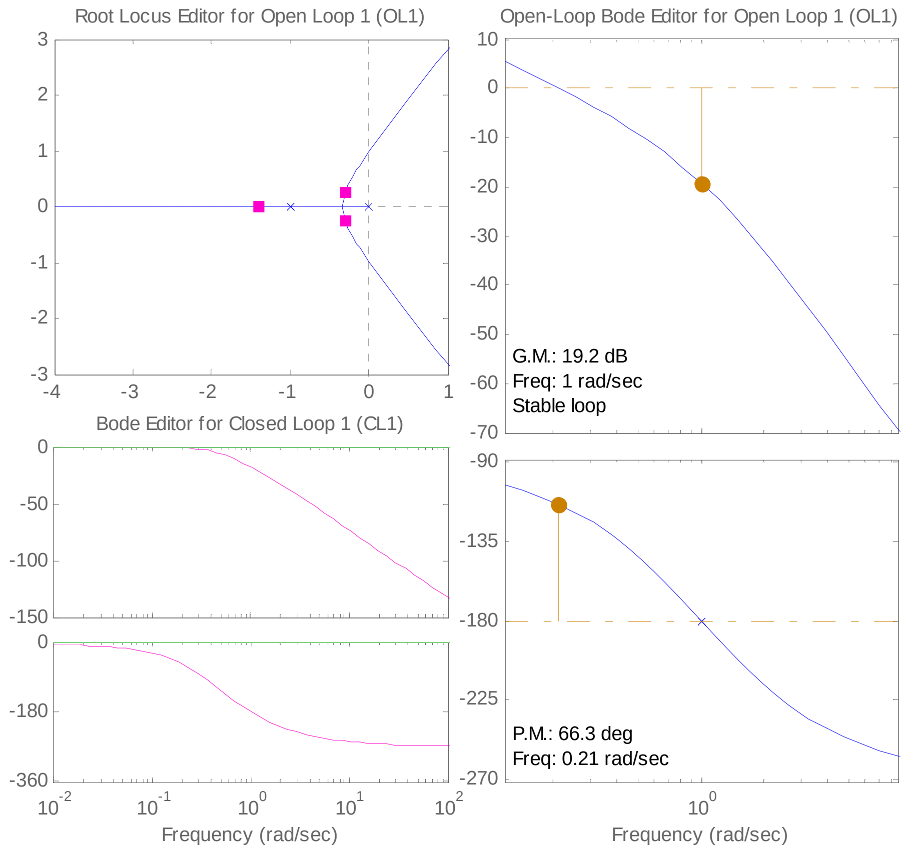
MATLAB code:

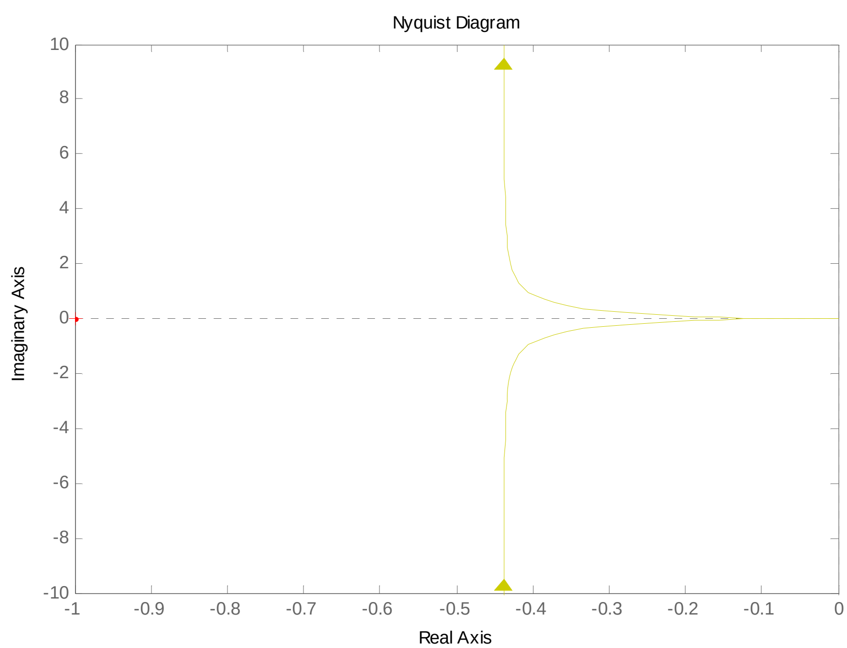
```
s = tf('s')
%a)

figure(1);
K=1
num_GH_a= K;
den_GH_a=s*(s+1)*(s+1);
GH_a=num_GH_a/den_GH_a;
nyquist(GH_a)

%b)
figure(2);
K=20
num_GH_b= K;
den_GH_b=s*(s+1)*(s+1);
GH_b=num_GH_b/den_GH_b;
nyquist(GH_b)

sisotool;
```

By using sisotool and importing the loop transfer function, different values of K has been tested which resulted in a stable system when K<2, and unstable system for K>2. Following diagrams correspond to margin of stability:

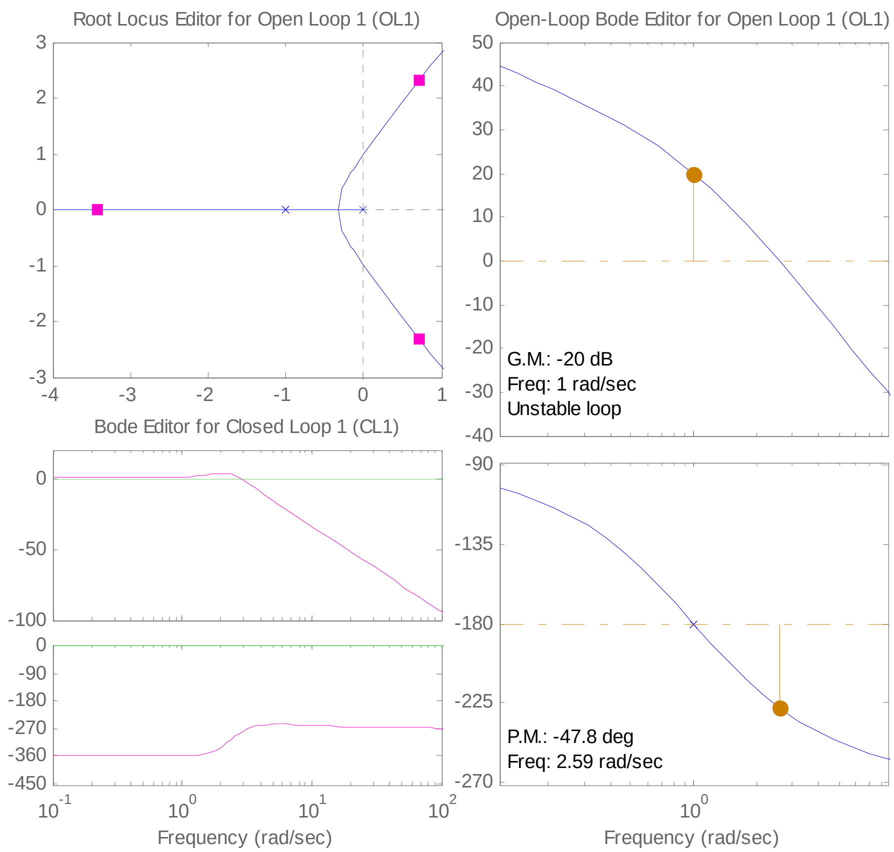


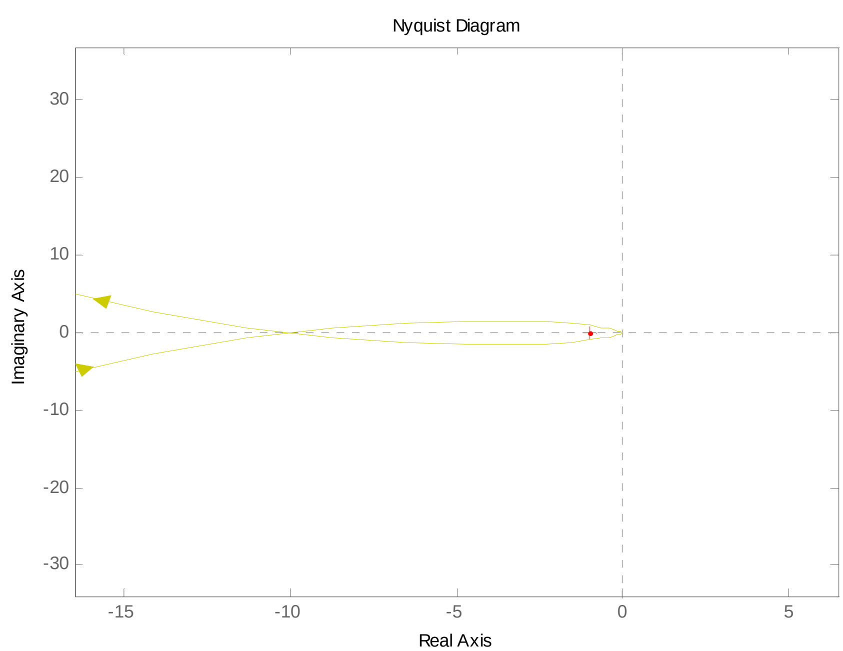
Part(a): small K resulted in stable system as shown below for K=0.219:





Part(b): Large K resulted in unstable system as shown below for K=20:


Root Locus Editor for Open Loop 1 (OL1)
Open-Loop Bode Editor for Open Loop 1 (OL1)
G.M.: -20 dB
Freq: 1 rad/sec
Unstable loop
Bode Editor for Closed Loop 1 (CL1)
P.M.: -47.8 deg
Freq: 2.59 rad/sec
Frequency (rad/sec)
Frequency (rad/sec)


Nyquist Diagram
Imaginary Axis
Real Axis

The system is stable for small value of K, since there is no encirclement of the s = -1

The system is unstable for large value of K, since the locus encirclement the $s = -1$ twice in CCW; which means two poles are in the right half s-plane.

8-32) (a) The transfer function (gain) for the sensor-amplifier combination is 10 V/0.1 in = 100 V/in. The velocity of flow of the solution is

$$v = \frac{10\ \text{in}^3/\text{sec}}{0.1\ \text{in}} = 100\ \text{in/sec}$$

The time delay between the valve and the sensor is $T_d = D/v$ sec. The loop transfer function is

$$G(s) = \frac{100Ke^{-T_d s}}{s^2 + 10s + 100}$$

(b) $K = 10$:

$$G(s) = G_1(s)e^{-T_d s} \qquad G(j\omega) = \frac{1000e^{-j\omega T_d}}{\left(100-\omega^2\right)+j10\omega}$$

Setting $\left|\dfrac{1000}{\left(100-\omega^2\right)+j10\omega}\right| = 1 \qquad \left(100-\omega^2\right)^2 + 100\omega^2 = 10^6$

Thus, $\omega^4 - 100\omega^2 - 990{,}000 = 0$ Real solutions: $\omega^2 = 1046.2$ $\omega = 32.35$ rad/sec

$$\angle G_1(j32.25) = -\tan^{-1}\left(\frac{10\omega}{100-\omega^2}\right)\Bigg|_{\omega=32.25} = -161^o$$

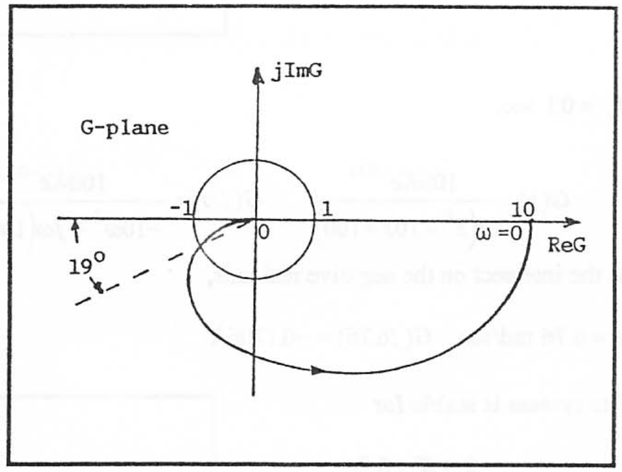
Thus,

$$32.35T_d = \frac{19^o \pi}{180^o} = 0.33 \text{ rad}$$

Thus,

$$T_d = 0.0103 \text{ sec}$$

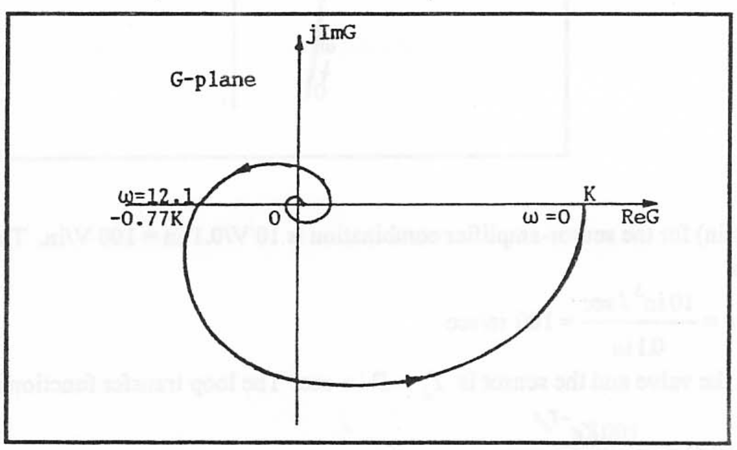
Maximum $D = vT_d = 100 \times 0.0103 = 1.03$ in



(c) $D = 10$ **in.**

$$T_d = \frac{D}{v} = 0.1 \text{ sec} \qquad G(s) = \frac{100Ke^{-0.1s}}{s^2+10s+100}$$

The Nyquist plot of $G(j\omega)$ intersects the negative real axis at $\omega = 12.1$ rad/sec.



8-33)

(a) The transfer function (gain) for the sensor-amplifier combination is 1 V/0.1 in = 10 V/in. The velocity of flow of the solutions is

$$v = \frac{10 \text{ in}^3/\text{sec}}{0.1 \text{ in}} = 100 \text{ in/sec}$$

The time delay between the valve and sensor is $T_d = D/v$ sec. The loop transfer function is

$$G(s) = \frac{10Ke^{-T_d s}}{s^2 + 10s + 100}$$

(b) ***K* = 10:**

$$G(s) = G_1(s)e^{-T_d s} \qquad G(j\omega) = \frac{100e^{-j\omega T_d}}{\left(100-\omega^2\right)+j10\omega}$$

Setting $\left|\frac{100}{\left(100-\omega^2\right)+j10\omega}\right| = 1 \qquad \left(100-\omega^2\right)^2 + 100\omega^2 = 10,000$

Thus, $\omega^4 - 100\omega^2 = 0 \qquad \text{Real solutions:} \quad \omega = 0, \quad \omega = \pm 10 \text{ rad/sec}$

$$\angle G_1(j10) = -\tan^{-1}\left(\frac{10\omega}{100-\omega^2}\right)\Bigg|_{\omega=10} = -90^o$$

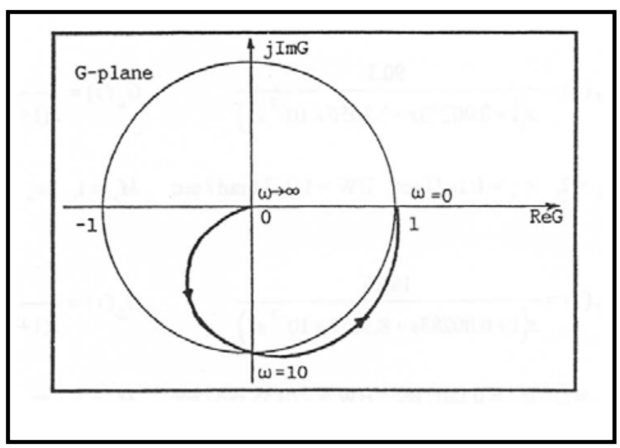
Thus,

$$10T_d = \frac{90^o \pi}{180^o} = \frac{\pi}{2} \text{ rad}$$

Thus,

$$T_d = \frac{\pi}{20} = 0.157 \text{ sec}$$

Maximum $D = vT_d = 100 \times 0.157 = 15.7$ in

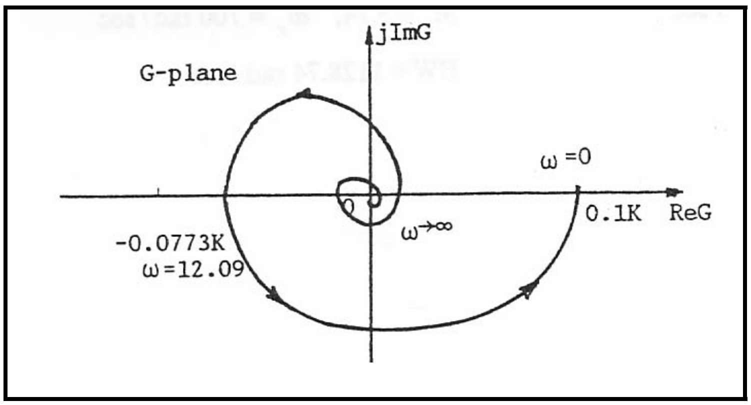


(c) ***D*** **= 10 in.**

$$T_d = \frac{D}{v} = \frac{10}{100} = 0.1 \text{ sec} \qquad G(s) = \frac{10Ke^{-0.1s}}{s^2 + 10s + 100}$$

The Nyquist plot of $G(j\omega)$ intersects the negative real axis at $\omega = 12.09$ rad/sec. $G(j) = -0.0773K$

For stability, the maximum value of *K* is 12.94 .



8-34)

The system (GH) has zero poles in the right of s plane: P=0.

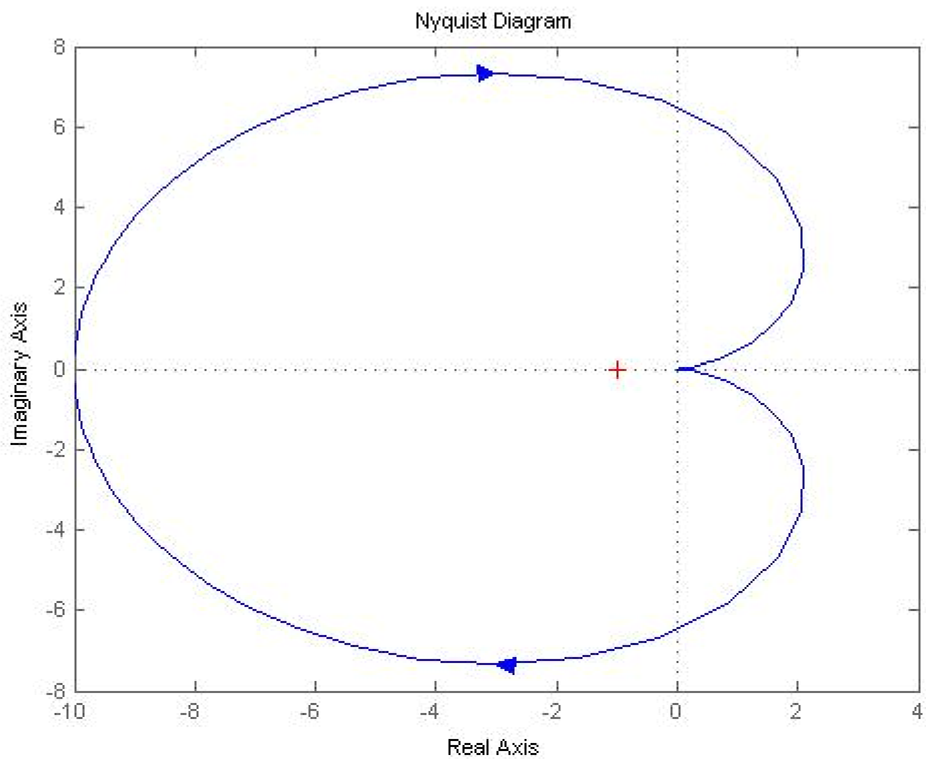
According to Nyquist criteria (Z=N+P), to ensure the stability which means the number of right poles of 1+ GH=0 should be zero (Z=0), we need N clockwise encirclements of Nyquist diagram about -1+0j point. That is N=-P or in other words, we need P counter-clockwise encirclement about -1+0j. In this case, we need 0 CCW encirclements.

8-34(a) According to Nyquist diagrams, this happens when K<-1. The three Nyquist diagrams are plotted with K=-10, K=-1, K=10 as examples:

MATLAB code:

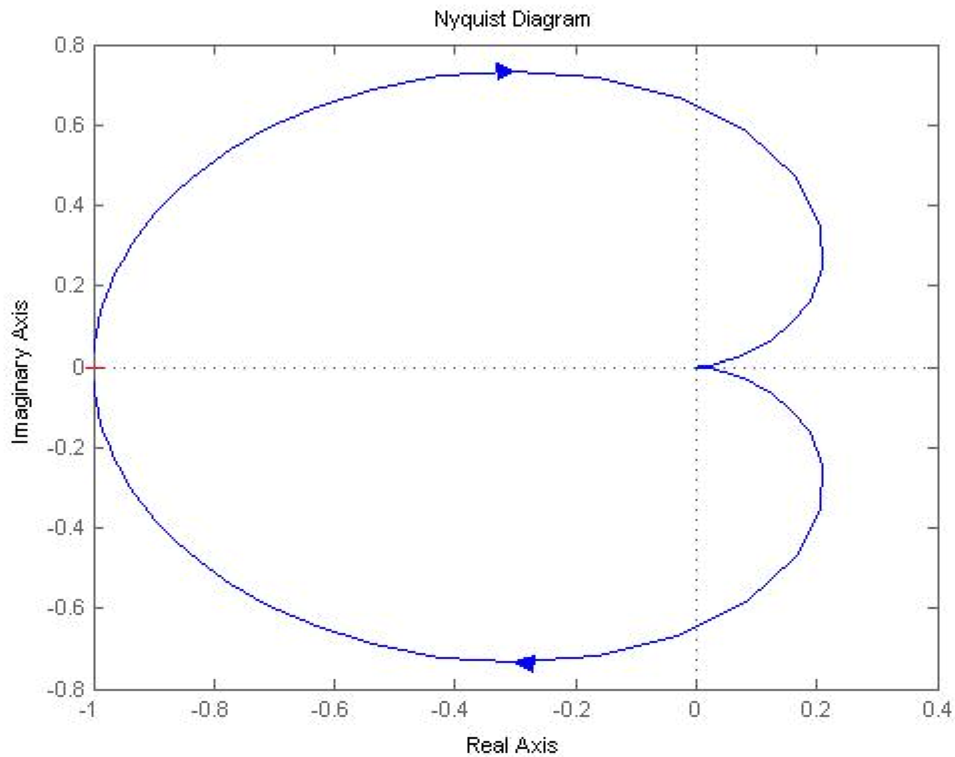
```
s = tf('s')
%a)
figure(1);
K=-10
num_G_a = K ;
```

```
den_G_a =(s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
OL_a = G_a*H_a
nyquist(OL_a)
```



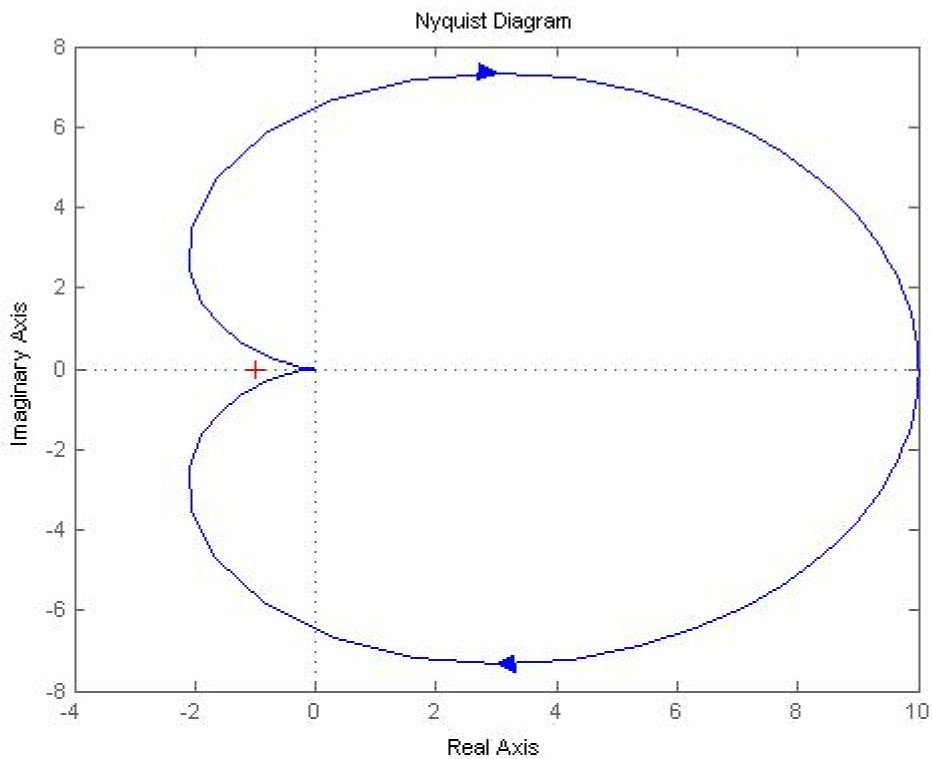
Case 1) Nyquist graph, K=-10: margin of stability K<-1 unstable

```
figure(2);
K=-1
num_G_a = K ;
den_G_a =(s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
%CL_a = G_a/(1 + G_a*H_a);
OL_a = G_a*H_a
nyquist(OL_a)
```



Case 2) Nyquist graph, K=-1: marginally unstable

```
figure(3);
K=10
num_G_a = K ;
den_G_a =(s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
%CL_a = G_a/(1 + G_a*H_a);
OL_a = G_a*H_a
nyquist(OL_a)
```



Case 3) Nyquist graph, K=10: stable case, -1<K no CCW encirclement about -1+0j point

8-34 (b)

For K<-1 (unstable), there will be 1 real pole in the right hand side of s-plane for the closed loop system, by running the following code.

```
K=-10
num_G_a = K ;
den_G_a =(s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
OL_a = G_a*H_a
nyquist(OL_a)
CL=1/(1+OL_a)
pole(CL)

K =

   -10


Transfer function:
     -10 s - 20
---------------------
s^3 + 3 s^2 + 4 s + 2


Transfer function:
s^3 + 3 s^2 + 4 s + 2
----------------------
s^3 + 3 s^2 - 6 s - 18


ans =

    2.4495
   -3.0000
   -2.4495
```

For K=-1 (marginally unstable), there will be 2 negative complex conjugate poles and a pole at zero for the closed loop system, by running the following code.

```
K=-1
num_G_a = K ;
den_G_a =(s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
```

```
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
OL_a = G_a*H_a
nyquist(OL_a)
CL=1/(1+OL_a)
pole(CL)
```

K =

    -1

Transfer function:

       -s - 2
----
s^3 + 3 s^2 + 4 s + 2

Transfer function:

s^3 + 3 s^2 + 4 s + 2
----
  s^3 + 3 s^2 + 3 s

ans =

        0
  -1.5000 + 0.8660i
  -1.5000 - 0.8660i

Note: you may also wish to use MATLAB `sisotool`.
See alternative solution to 8-38.

8-34(c) The Characteristic Equation is: $s^3 + 3s^2 + (4+K)s + 2+2K$

Using Routh criterions, the coefficient table is as follows:

s^3	1	$4+K$
s^2	3	$2K+2$
s^1	$K+10$	0
s^0	$2K+2$	0

The system is stable if the content of the 1st column is positive:

$$K+10>0 \rightarrow K>-10$$

$$2K+2>0 \rightarrow K>-1$$

which is consistent with the results of the Nyquist diagrams. For $K>-1$ system is **stable**.

8-35) (a) $M_r = 2.06, \quad \omega_r = 9.33 \text{ rad/sec}, \quad \text{BW} = 15.2 \text{ rad/sec}$

(b)

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 2.06 \qquad \zeta^4 - \zeta^2 + 0.0589 = 0$$ The solution for $\zeta < 0.707$ is $\zeta = 0.25$.

$$\omega_r\sqrt{1-2\zeta^2} = 9.33 \text{ rad/sec} \quad \text{Thus} \quad \omega_n = \frac{9.33}{0.9354} = 9.974 \text{ rad/sec}$$

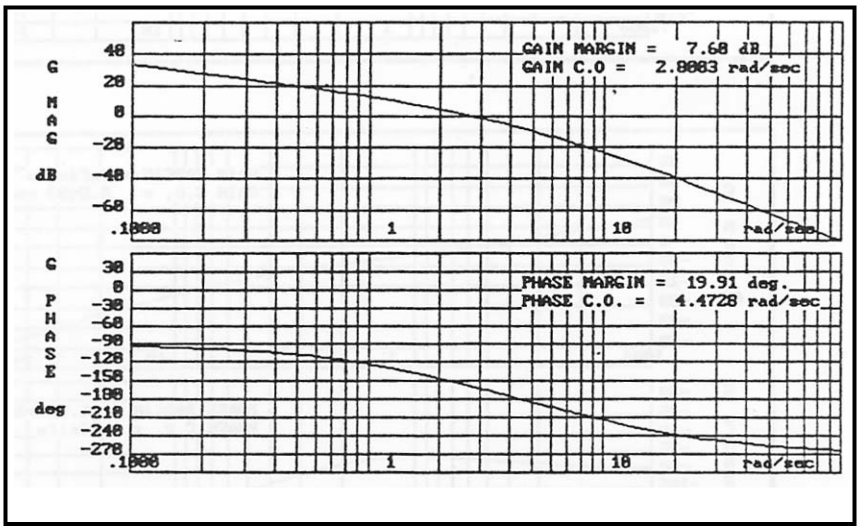
$$G_L(s) = \frac{\omega_n^2}{s\left(s+2\zeta\omega_n\right)} = \frac{99.48}{s(s+4.987)} = \frac{19.94}{s(1+0.2005s)}$$

BW = 15.21 rad/sec

8-36) Assuming a unity feedback loop (*H=1*), *G*(s)*H*(*s*)=*G*(*s*)

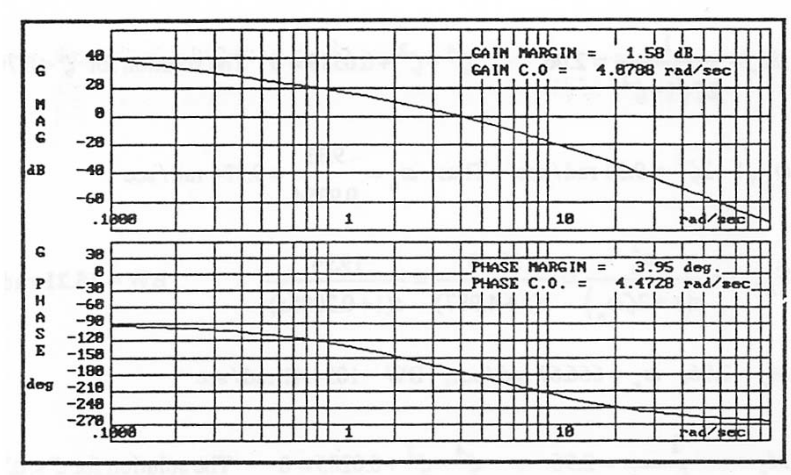
(a)

$$G(s) = \frac{5}{s(1+0.5s)(1+0.1s)}$$



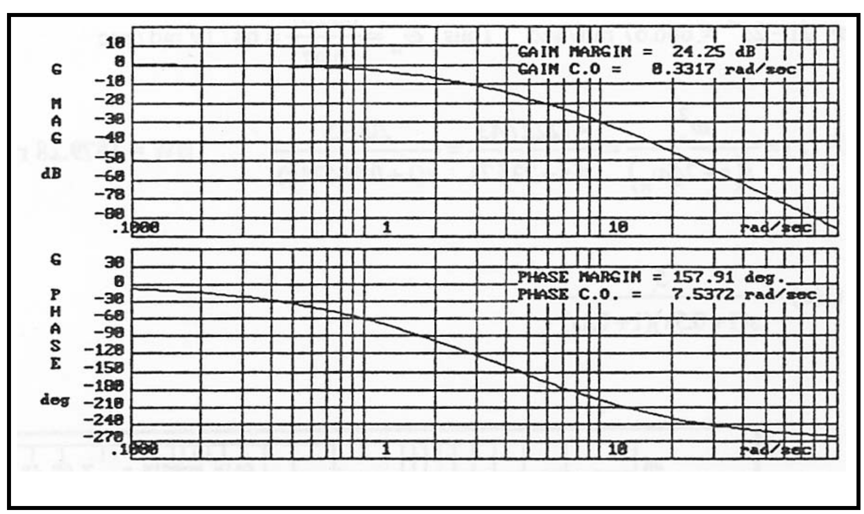
8-36 (b)

$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$



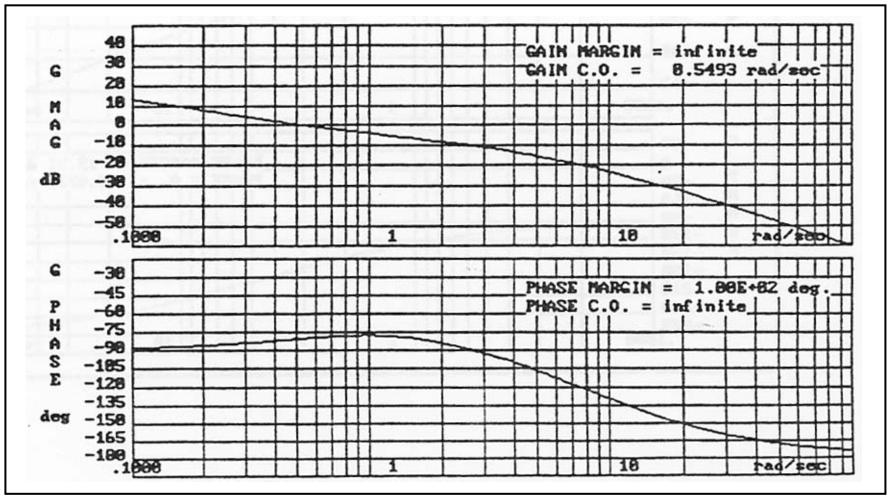
(c)

$$G(s) = \frac{500}{(s+1.2)(s+4)(s+10)}$$



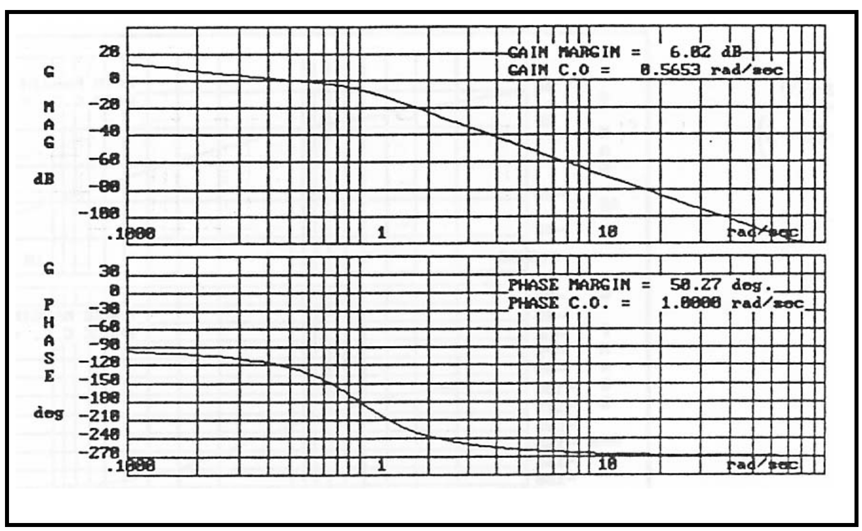
(d)

$$G(s) = \frac{10(s+1)}{s(s+2)(s+10)}$$



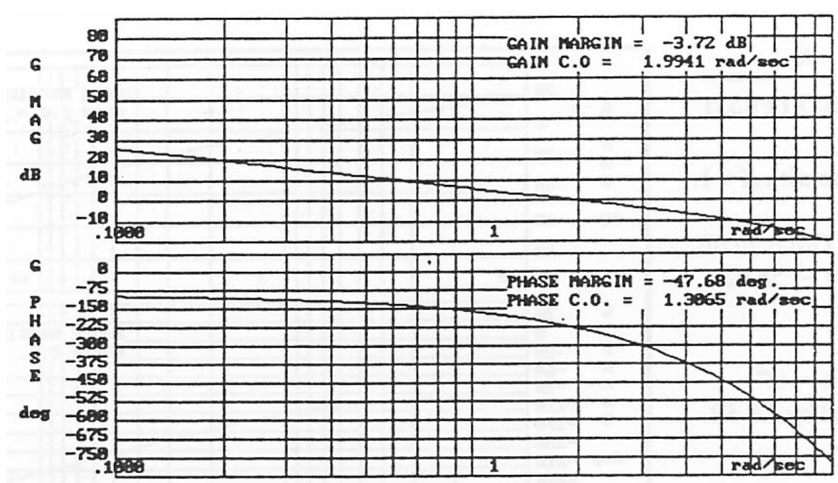
8-36 (e)

$$G(s) = \frac{0.5}{s\left(s^2+s+1\right)}$$



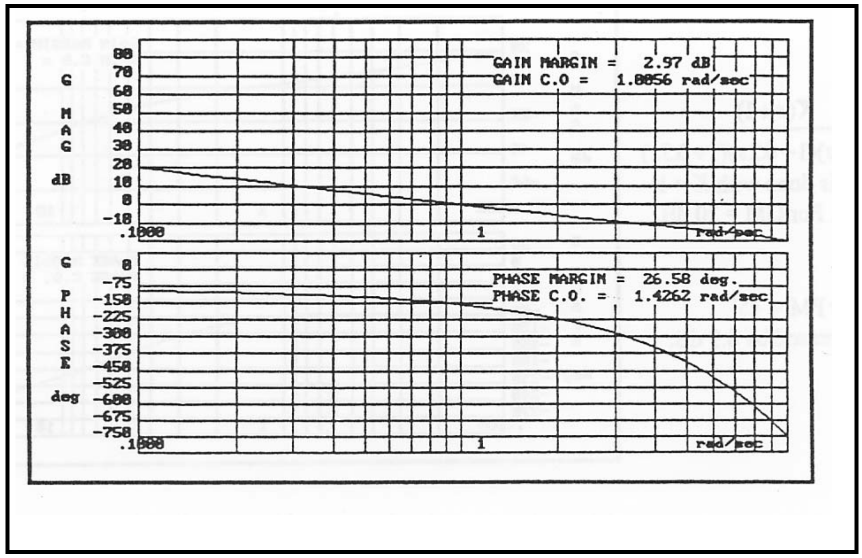
(f)

$$G(s) = \frac{100e^{-s}}{s\left(s^2 + 10s + 50\right)}$$



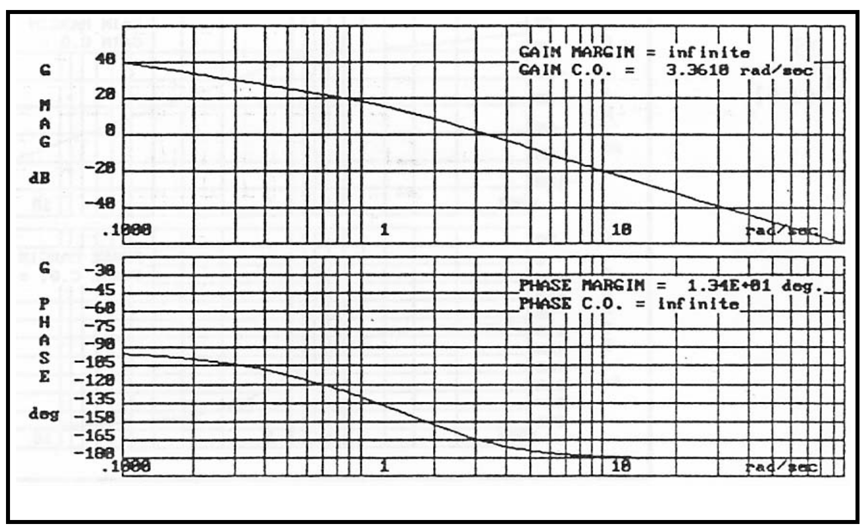
(g)

$$G(s) = \frac{100e^{-s}}{s\left(s^2 + 10s + 100\right)}$$



8-36 (h)

$$G(s) = \frac{10(s+5)}{s\left(s^2 + 5s + 5\right)}$$

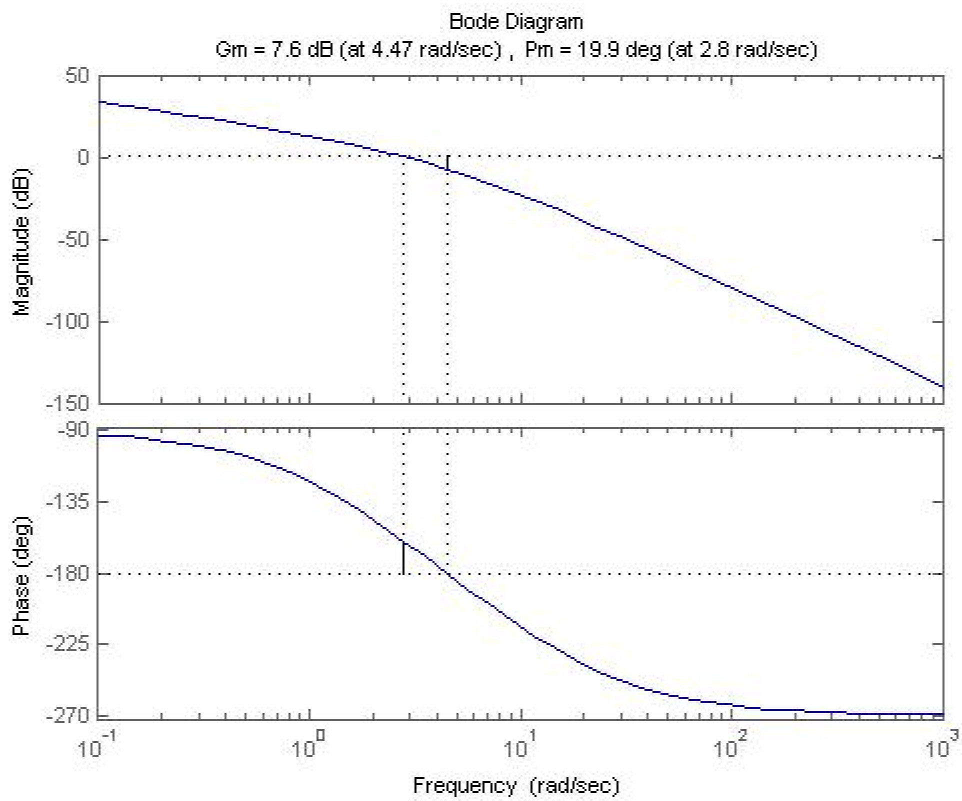


(a)

MATLAB code:

```
s = tf('s')
num_G_a= 5;
den_G_a=s*(0.5*s+1)*(0.1*s+1);
G_a=num_G_a/den_G_a
margin(G_a)
```

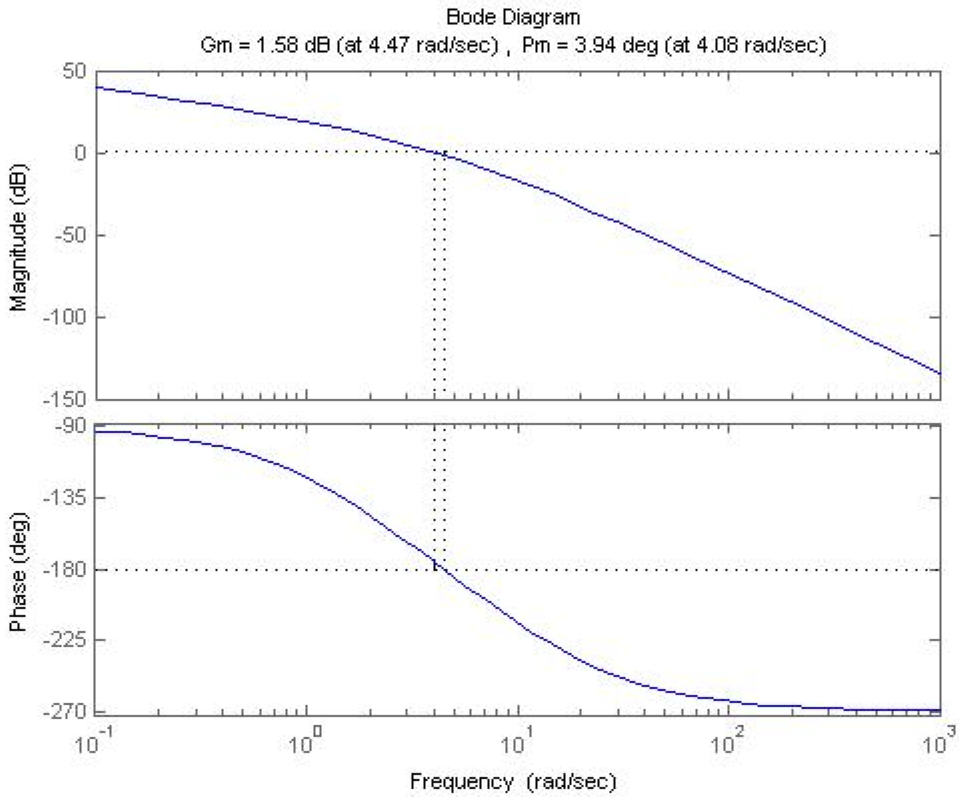
Bode diagram:



(b)

MATLAB code:

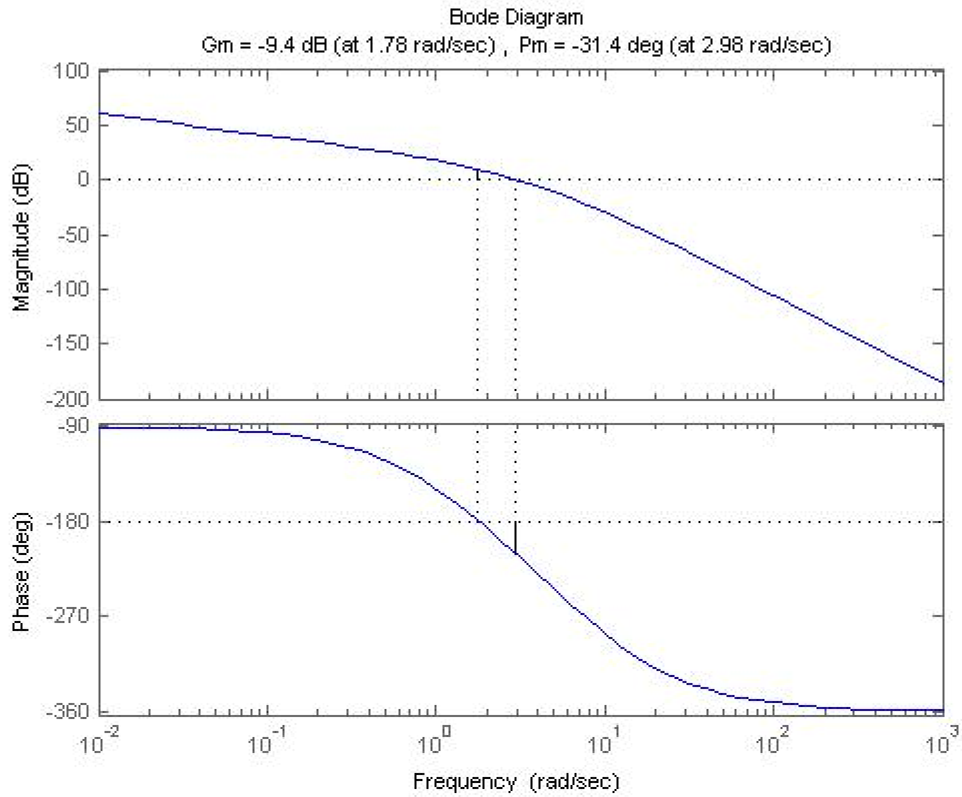
```
s = tf('s')
num_G_a= 10;
den_G_a=s*(1+0.5*s)*(1+0.1*s);
G_a=num_G_a/den_G_a
margin(G_a)
```



(c)

MATLAB code:

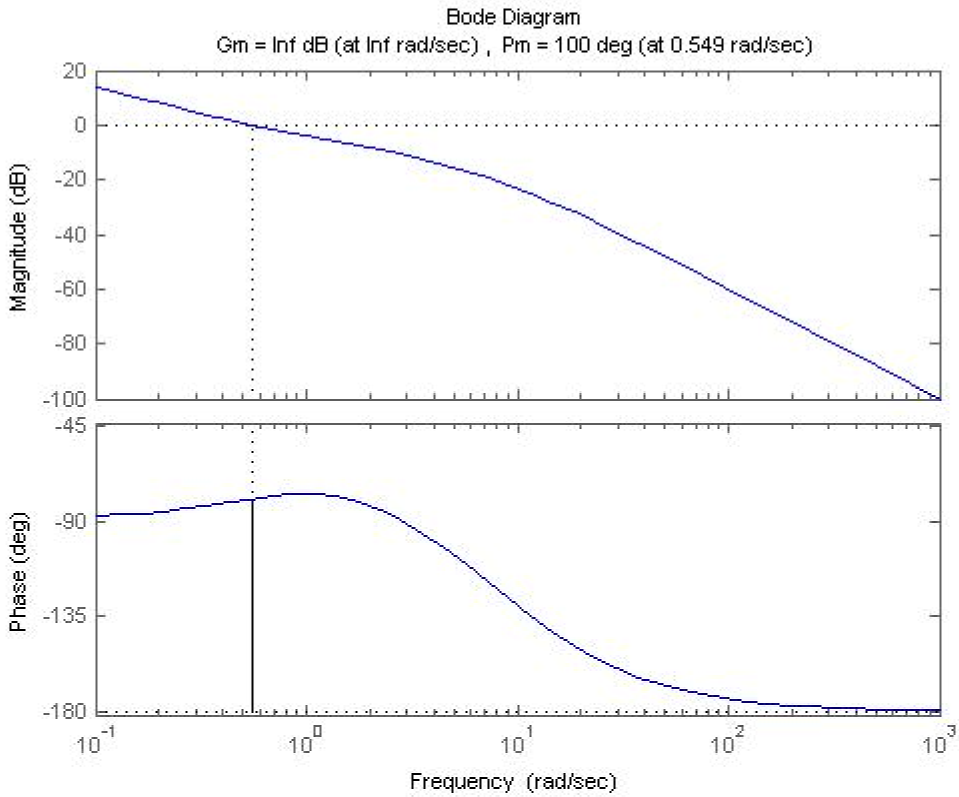
```
s = tf('s')
num_G_a= 500;
den_G_a=s*(s+1.2)*(s+4)*(s+10);
G_a=num_G_a/den_G_a
margin(G_a)
```



(d)

MATLAB code:

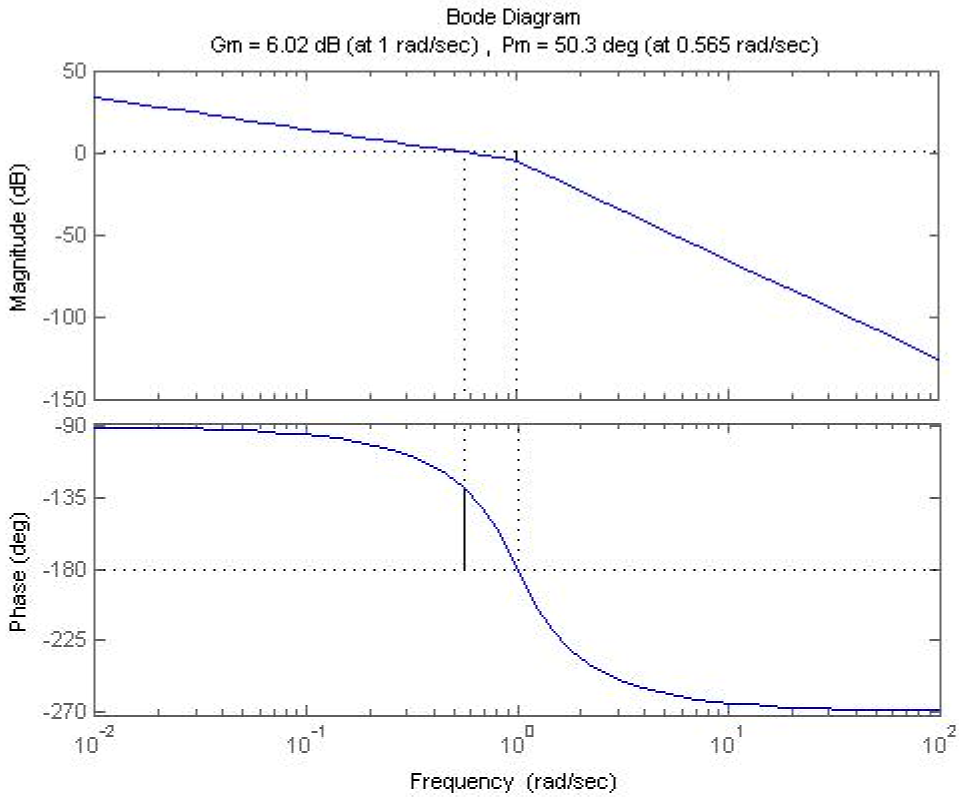
```
s = tf('s')
num_G_a= 10*(s+1);
den_G_a=s*(s+2)*(s+10);
G_a=num_G_a/den_G_a
margin(G_a)
```



(e)

MATLAB code:

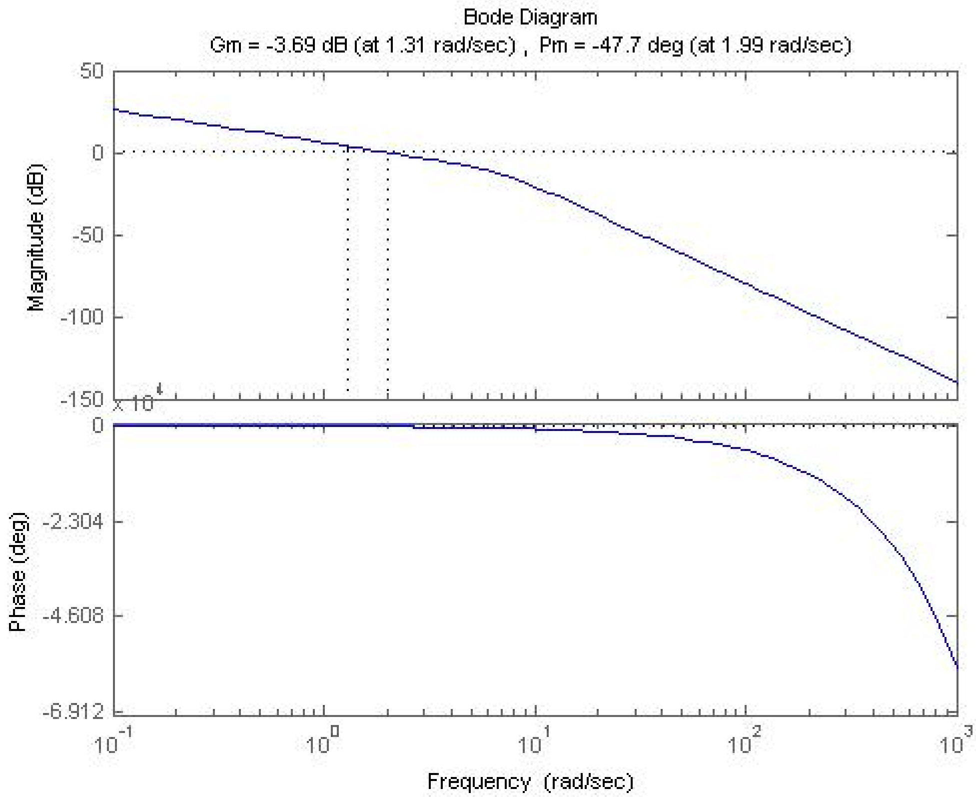
```
s = tf('s')
num_G_a= 0.5;
den_G_a=s*(s^2+s+1);
G_a=num_G_a/den_G_a
margin(G_a)
```



(f)

MATLAB code:

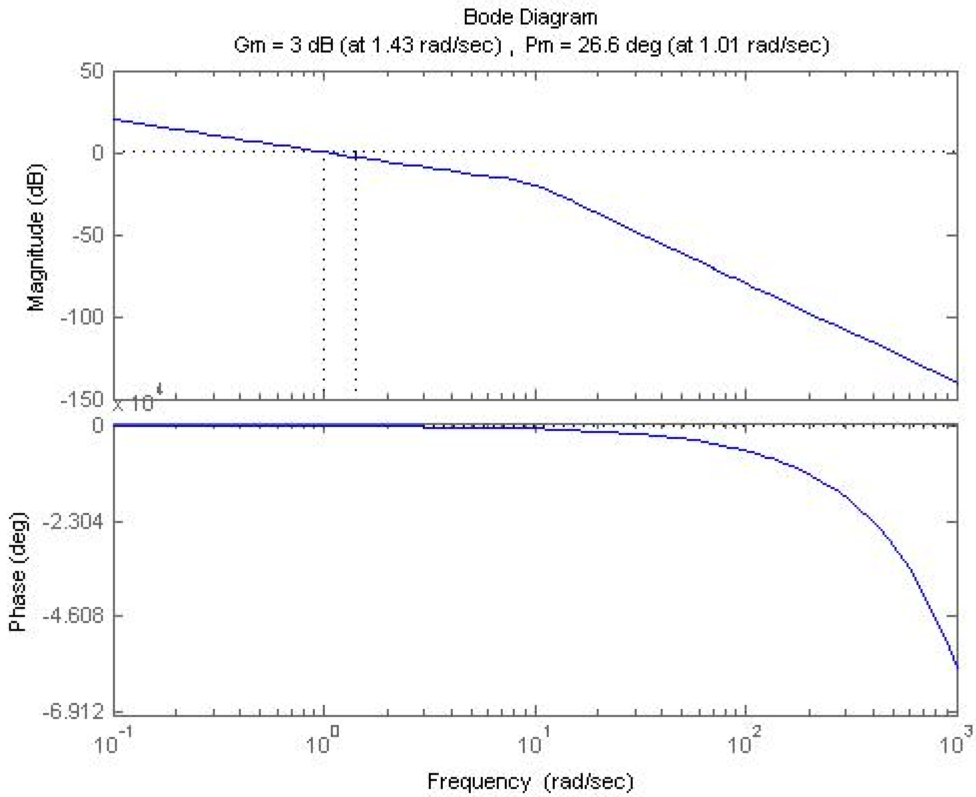
```
s = tf('s')
num_G_a= 100*exp(-s);
den_G_a=s*(s^2+10*s+50);
G_a=num_G_a/den_G_a
margin(G_a)
```



(g)

MATLAB code:

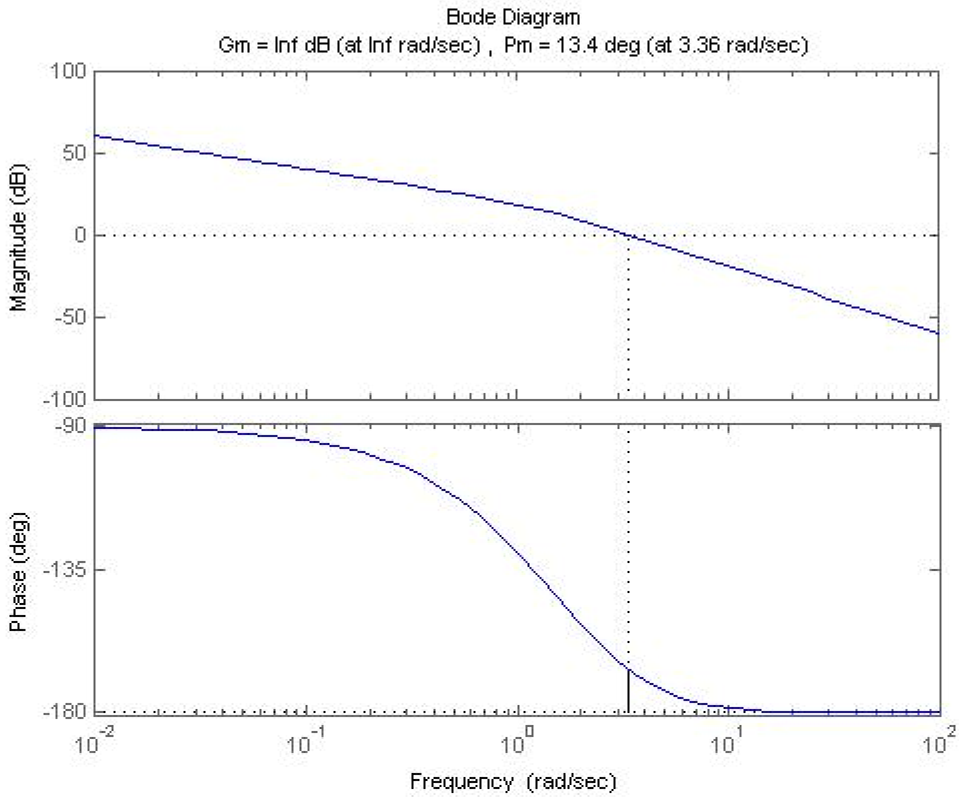
```
s = tf('s')
num_G_a= 100*exp(-s);
den_G_a=s*(s^2+10*s+100);
G_a=num_G_a/den_G_a
margin(G_a)
```



(h)

MATLAB code:

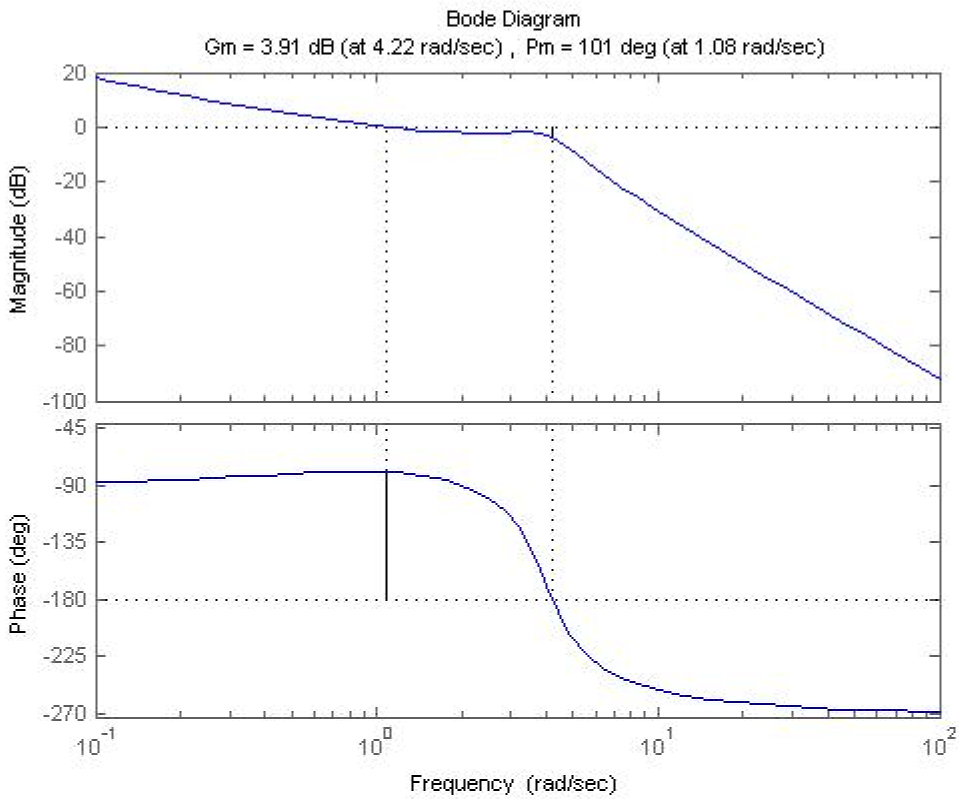
```
s = tf('s')
num_G_a= 10*(s+5);
den_G_a=s*(s^2+5*s+5);
G_a=num_G_a/den_G_a
margin(G_a)
```


Bode Diagram
Gm = Inf dB (at Inf rad/sec) , Pm = 13.4 deg (at 3.36 rad/sec)
100
50
0
-50
-100
Magnitude (dB)
-90
-135
-180
Phase (deg)
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

8-37)

MATLAB code:

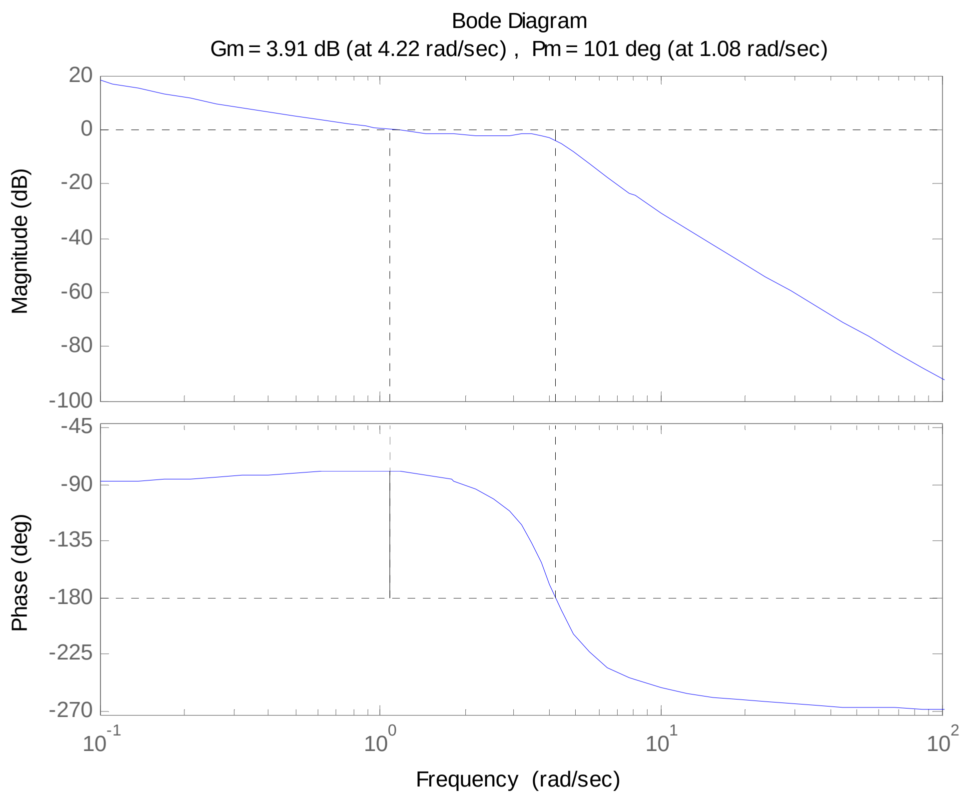
```
s = tf('s')
num_GH_a= 25*(s+1);
den_GH_a=s*(s+2)*(s^2+2*s+16);
GH_a=num_GH_a/den_GH_a
margin(GH_a)
```



8-38) MATLAB code:

```
s = tf('s')
num_G_a= 25*(s+1);
den_G_a=s*(s+2)*(s^2+2*s+16);
G_a=num_G_a/den_G_a
margin(G_a)
```

Bode diagram: PM=101 deg, GM=3.91 dB @ 4.22 rad/sec



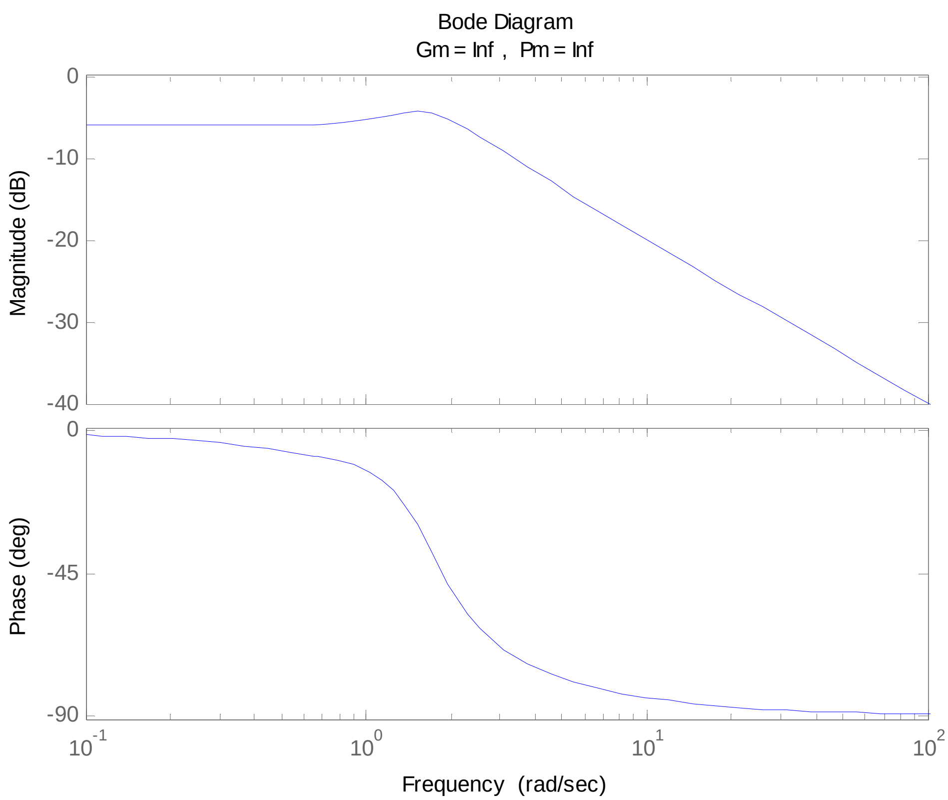
8-38 Alternative solution

MATLAB code:

```
s = tf('s')

%a)
figure(1);
num_G_a = 1 ;
den_G_a =(s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
```

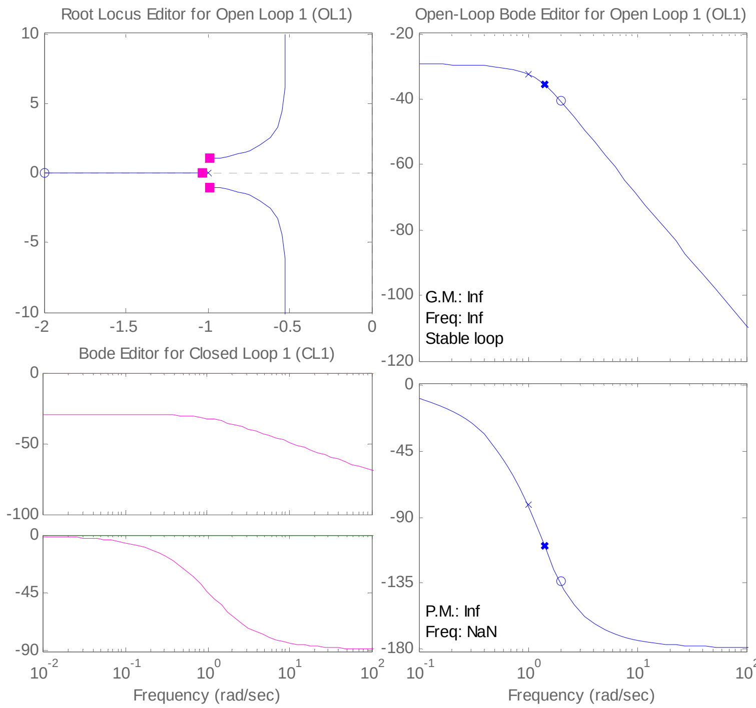
```
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
CL_a = G_a/(1 + G_a*H_a);
margin(CL_a)

sisotool
```

Bode diagram: for k=1

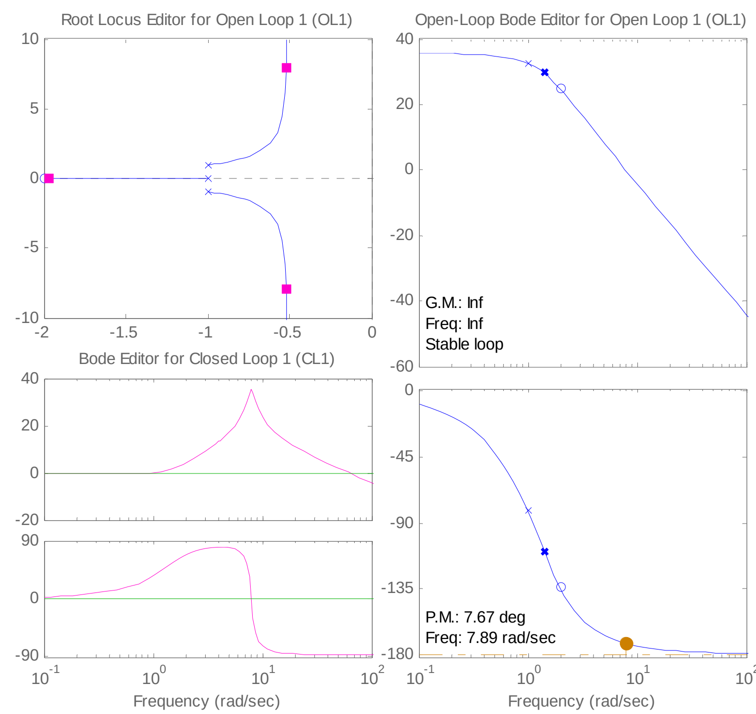


Using MATLAB sisotool, the transfer function gain can be iteratively changed in order to obtain different phase margins. By changing the gain K between very small and very big numbers, it was found that the closed loop system are stable (positive PM) ***for every positive K in this system***.

K=0.034


Root Locus Editor for Open Loop 1 (OL1)
Open-Loop Bode Editor for Open Loop 1 (OL1)
G.M.: Inf
Freq: Inf
Stable loop
Bode Editor for Closed Loop 1 (CL1)
P.M.: Inf
Freq: NaN
Frequency (rad/sec)
Frequency (rad/sec)

K=59.9

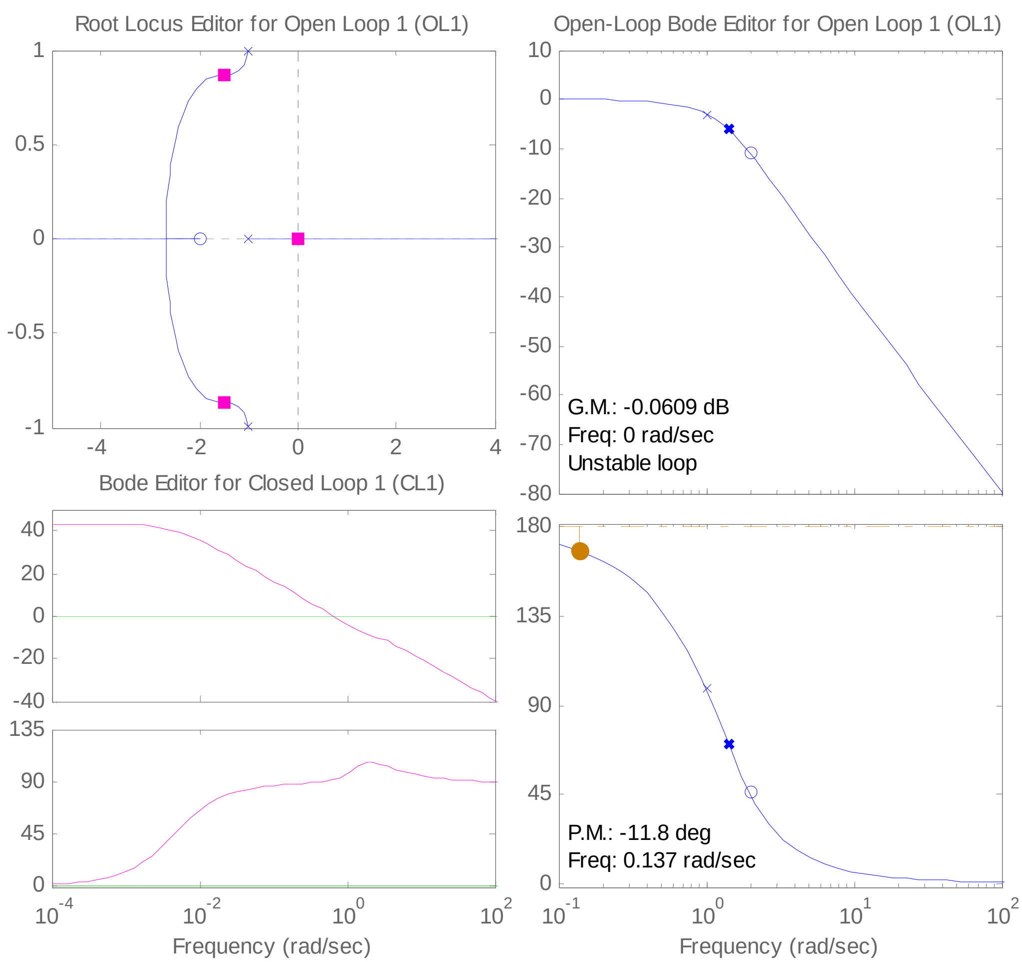

Root Locus Editor for Open Loop 1 (OL1)
Open-Loop Bode Editor for Open Loop 1 (OL1)
G.M.: Inf
Freq: Inf
Stable loop
Bode Editor for Closed Loop 1 (CL1)
P.M.: 7.67 deg
Freq: 7.89 rad/sec
Frequency (rad/sec)
Frequency (rad/sec)

In order to test the negative range of K, -1 was multiplied to the loop transfer function through the following code, and sisotool was used again.

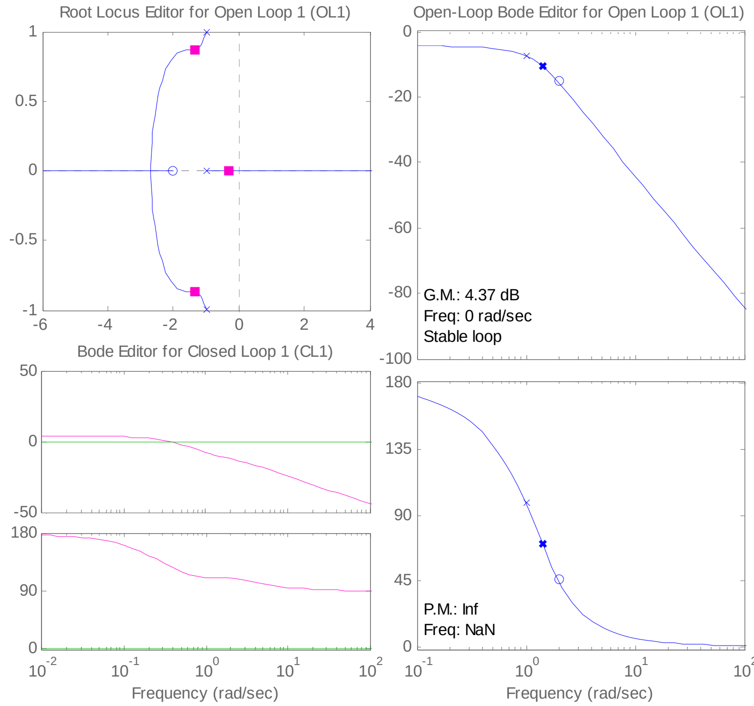
```
figure(1);
num_G_a = -1 ;
den_G_a =(s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
CL_a = G_a/(1 + G_a*H_a);
margin(CL_a)

sisotool
```

at K=-1, margin of stability is observed as PM~=0:

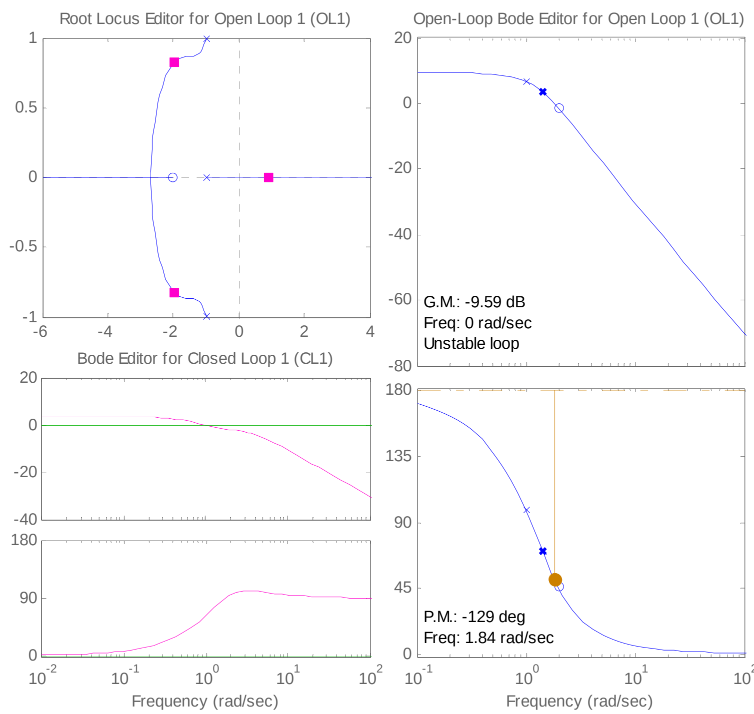
K= -1



The system is stable for K>-1 as follows: **K= -0.6**



And the system is unstable for K< -1: **K= -3**

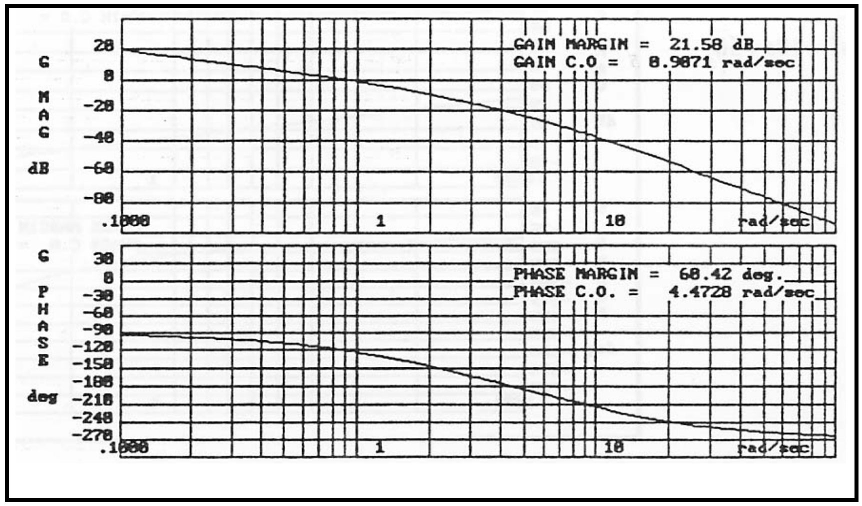


*Combining the individual ranges for K, the system will be stable in the range of K> -1

8-39 See sample MATLAB code in Part e. The MATLAB codes are identical to problem 8-36.

(a)

$$G(s) = \frac{K}{s(1+0.1s)(1+0.5s)}$$



The Bode plot is done with *K* = 1.

GM = 21.58 dB For GM = 20 dB,

K must be reduced by –1.58 dB.

Thus *K* = 0.8337

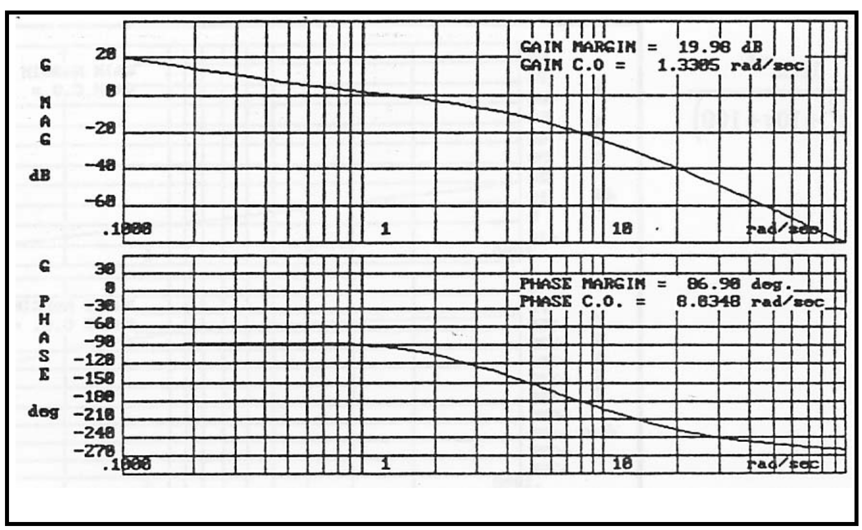
PM = 60.42^o. For PM = 45^o

K should be increased by 5.6 dB.

Or, *K* = 1.91

(b)

$$G(s) = \frac{K(s+1)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$ The Bode plot is done with *K* = 1.



GM = 19.98 dB. For GM = 20 dB,

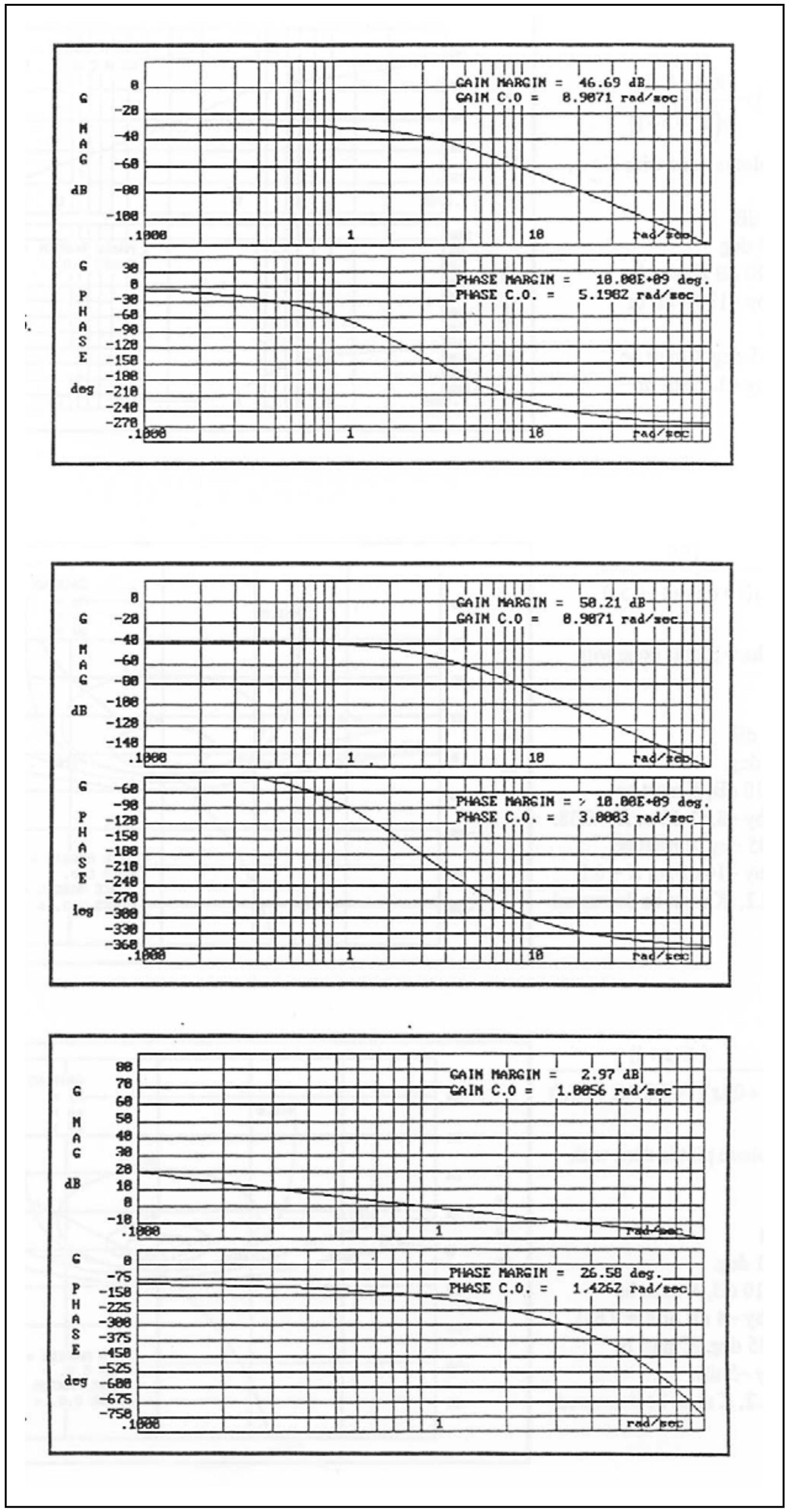
$K \cong 1$.

PM = 86.9^o. For PM = 45^o

K should be increased by 8.9 dB.

Or, *K* = 2.79.

8-39 (c) See the top plot



$$G(s) = \frac{K}{(s+3)^3}$$

The Bode plot is done with $K = 1$.

GM = 46.69 dB

PM = infinity.

For GM = 20 dB K can be increased by 26.69 dB or $K = 21.6$.

For PM = 45 deg. K can be increased by 28.71 dB, or $K = 27.26$.

(d) See the middle plot

$$G(s) = \frac{K}{(s+3)^4}$$

The Bode plot is done with $K = 1$.

GM = 50.21 dB

PM = infinity.

For GM = 20 dB K can be increased by 30.21 dB or $K = 32.4$

For PM = 45 deg. K can be increased by 38.24 dB, or $K = 81.66$

(e) See the bottom plot

The Bode plot is done with $K = 1$.

$$G(s) = \frac{Ke^{-s}}{s\left(1+0.1s+0.01s^2\right)}$$

GM=2.97 dB; PM = 26.58 deg

For GM = 20 dB *K* must be

decreased by –17.03 dB or

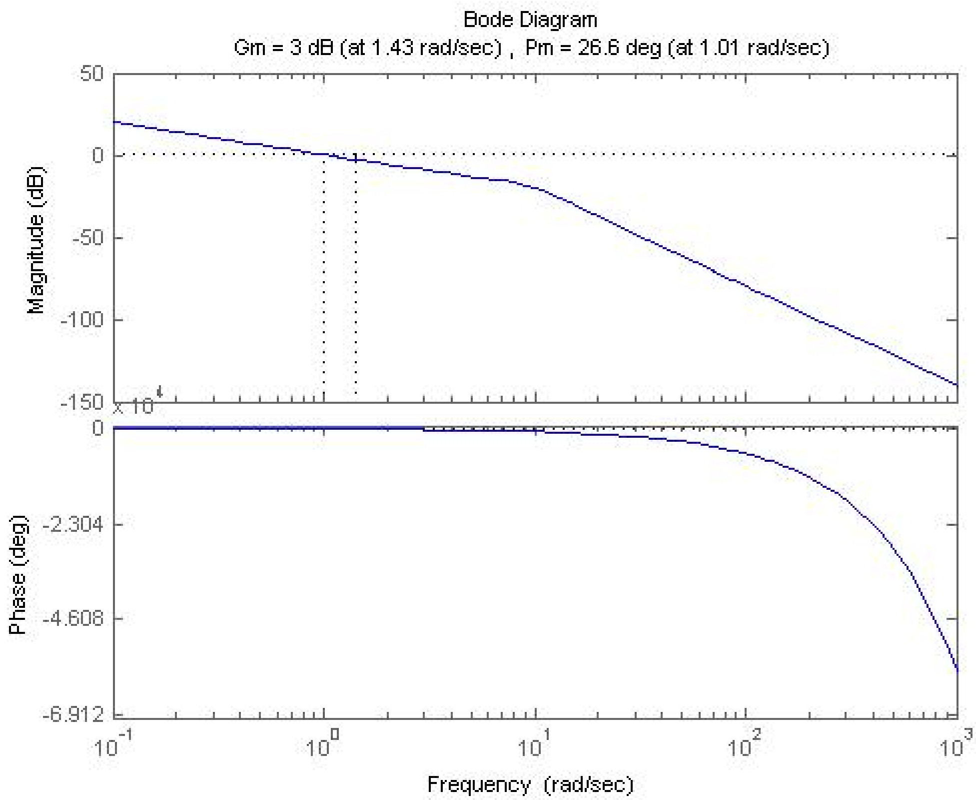
K = 0.141.

For PM = 45 deg. *K* must be

decreased by –2.92 dB or *K* = 0.71.

MATLAB code:

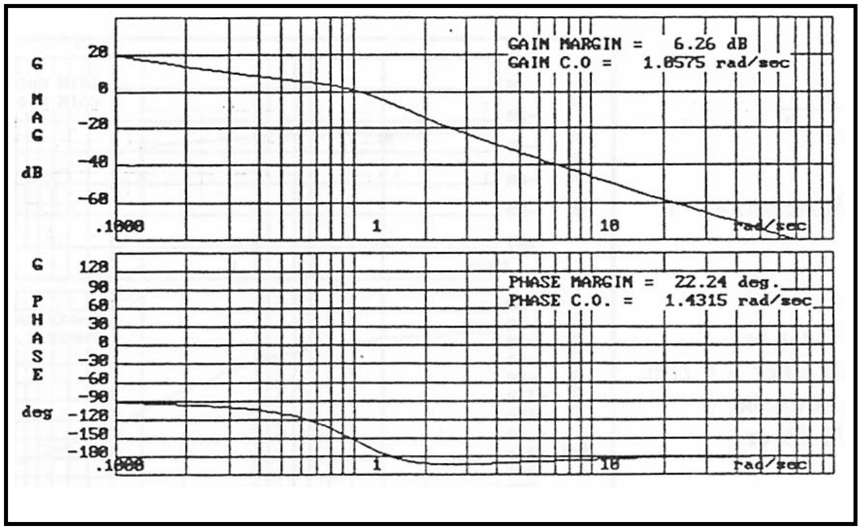
```
s = tf('s')
num_G_a= exp(-s);
den_G_a=s*(0.01*s^2+0.1*s+1);
G_a=num_G_a/den_G_a
margin(G_a)
```



8-39 (f)

$$G(s)=\frac{K(1+0.5s)}{s\left(s^{2}+s+1\right)}$$

The Bode plot is done with K = 1.



GM = 6.26 dB

PM = 22.24 deg

For GM = 20 dB K must be decreased by –13.74 dB or

K = 0.2055.

For PM = 45 deg K must be decreased by –3.55 dB or

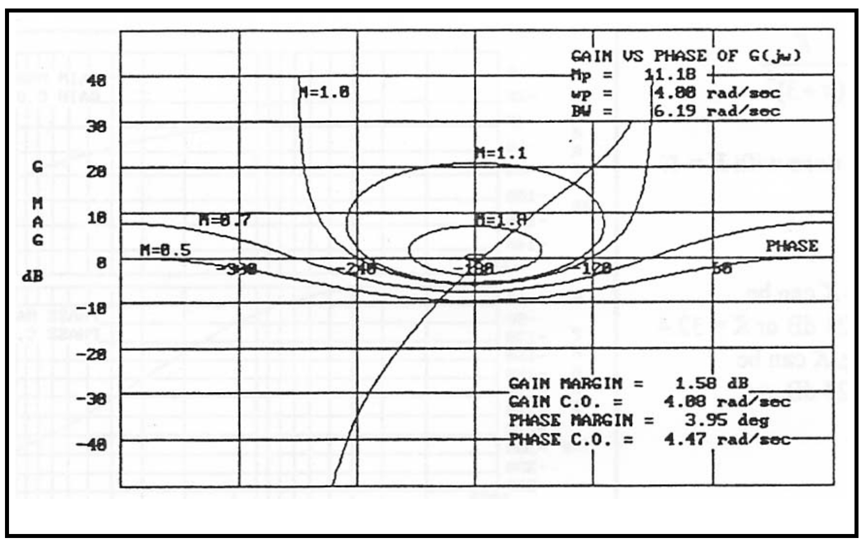
K = 0.665.

8-40 (a)

$$G(s)=\frac{10K}{s(1+0.1s)(1+0.5s)}$$

The gain-phase plot is done with

K = 1.



GM = 1.58 dB

PM = 3.95 deg.

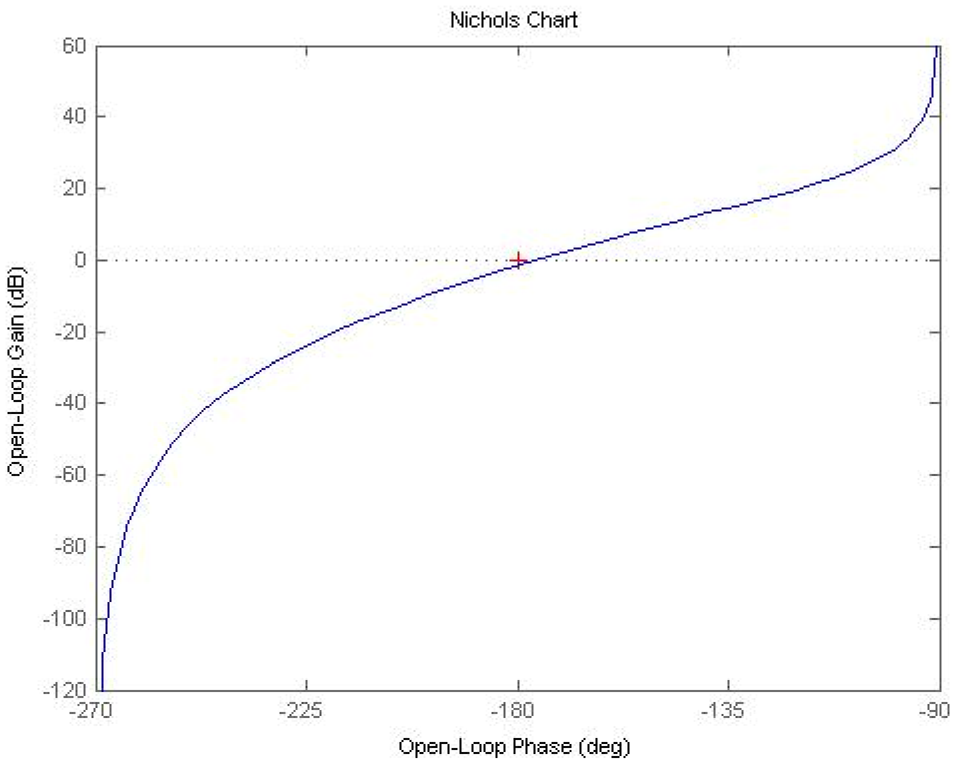
For GM = 10 dB, K must be decreased by –8.42 dB or K = 0.38.

For PM = 45 deg, K must be decreased by –14 dB, or K = 0.2.

For $M_r = 1.2$, K must be decreased to 0.16.

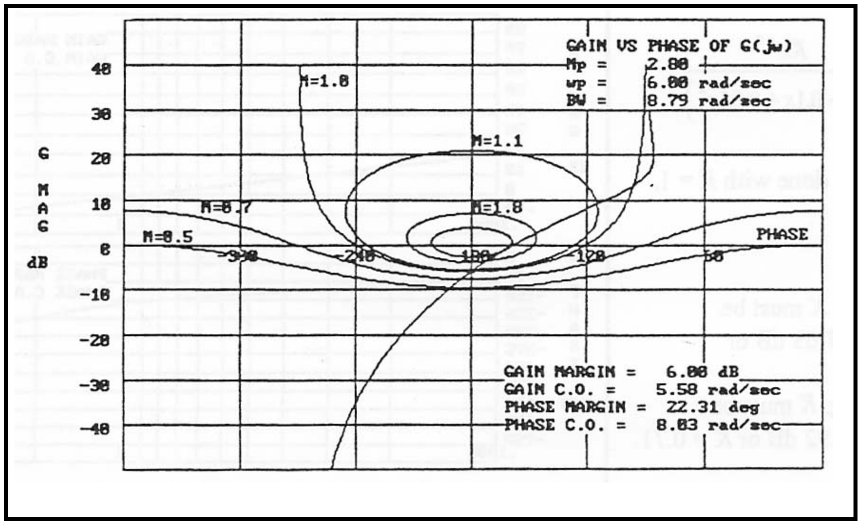
Sample MATLAB code:

```
s = tf('s')
num_G_a= 10;
den_G_a=s*(1+0.1*s)*(0.5*s+1);
G_a=num_G_a/den_G_a
nichols(G_a)
```



(b)

$$G(s) = \frac{5K(s+1)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$



The Gain-phase plot is done with

K = 1.

GM = 6 dB

PM = 22.31 deg.

For GM = 10 dB, K must be decreased by –4 dB or K = 0.631.

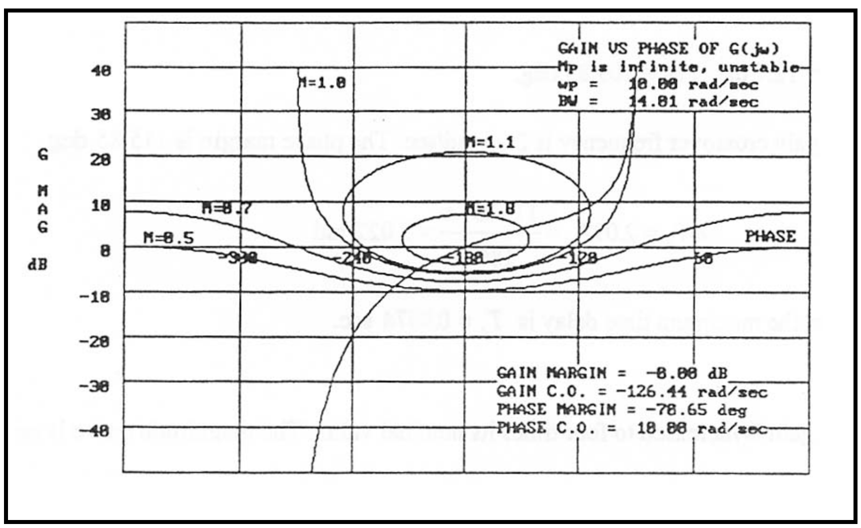
For PM = 45 deg, K must be

decrease by –5 dB.

For $M_r = 1.2$, K must be decreased to 0.48.

8-40 (c)

$$G(s) = \frac{10K}{s\left(1+0.1s+0.01s^2\right)}$$



The gain-phase plot is done for

K = 1.

GM = 0 dB $M_r = \infty$

PM = 0 deg

For GM = 10 dB, K must be decreased by –10 dB or K = 0.316.

For PM = 45 deg, K must be decreased by –5.3 dB, or

K = 0.543.

For $M_r = 1.2, K$ must be decreased to 0.2213.

(d)

$$G(s)=\frac{Ke^{-s}}{s\left(1+0.1s+0.01s^2\right)}$$

The gain-phase plot is done for

$K = 1$.

GM = 2.97 dB $M_r = 3.09$

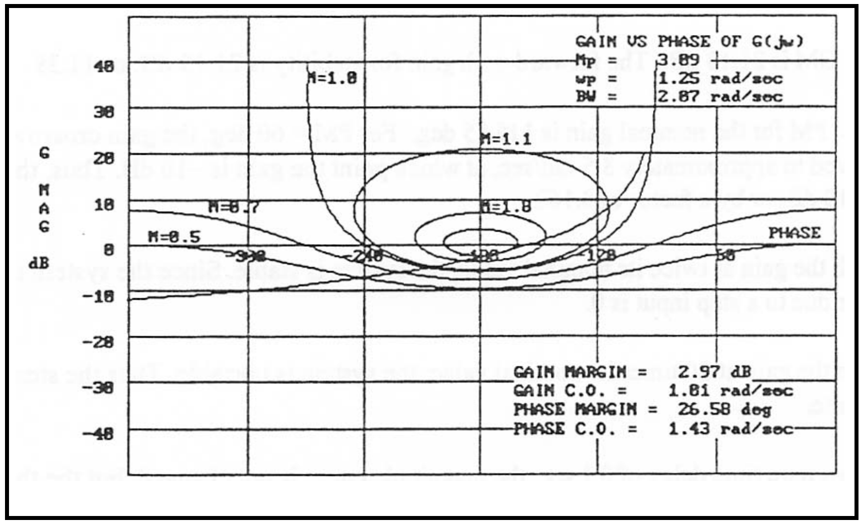
PM = 26.58 deg

For GM = 10 dB, K must be decreased by −7.03 dB, K = 0.445.

For PM = 45 deg, K must be decreased by −2.92 dB, or

K = 0.71.

For $M_r = 1.2,\ K = 0.61.$

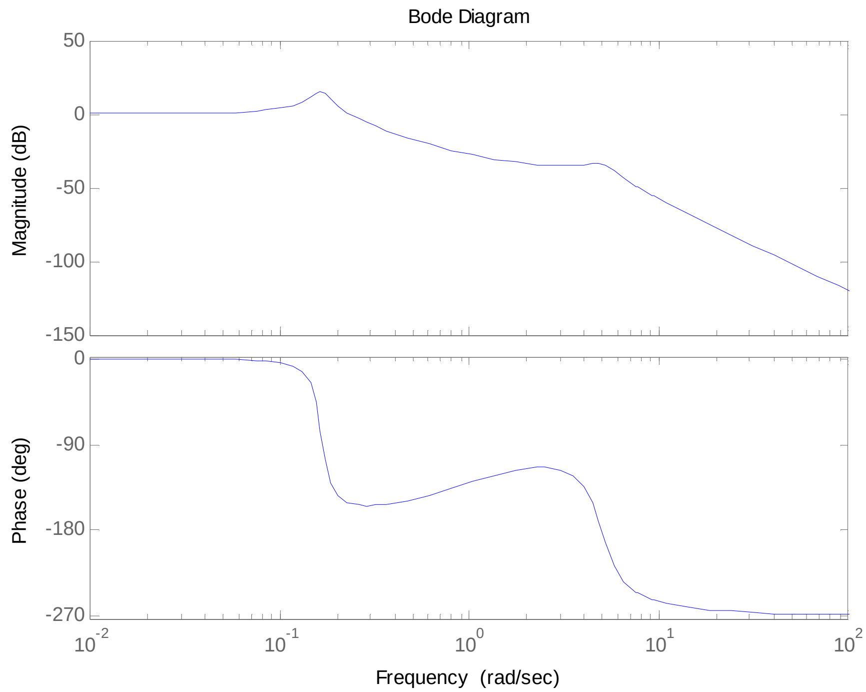


8-41

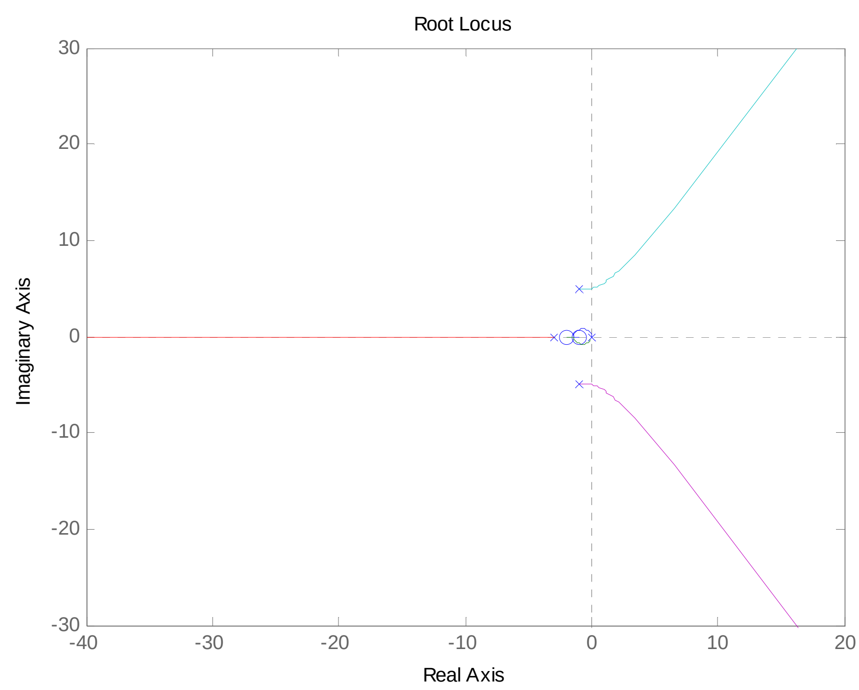
MATLAB code:

```
s = tf('s')
%a)
num_GH_a= 1*(s+1)*(s+2);
den_GH_a=s^2*(s+3)*(s^2+2*s+25);
GH_a=num_GH_a/den_GH_a;
CL_a = GH_a/(1+GH_a)
figure(1);
bode(CL_a)

%b)
figure(2);
rlocus(GH_a)

%c)
num_GH_c= 53*(s+1)*(s+2);
den_GH_c=s^2*(s+3)*(s^2+2*s+25);
GH_c=num_GH_c/den_GH_c;
figure(3);
nyquist(GH_c)
xlim([-2 1])
ylim([-1.5 1.5])

%d)
num_GH_d= (s+1)*(s+2);
den_GH_d=s^2*(s+3)*(s^2+2*s+25);
GH_d=num_GH_d/den_GH_d;
CL_d = GH_d/(1+GH_d)
figure(4);
margin(CL_d)

sisotool
```

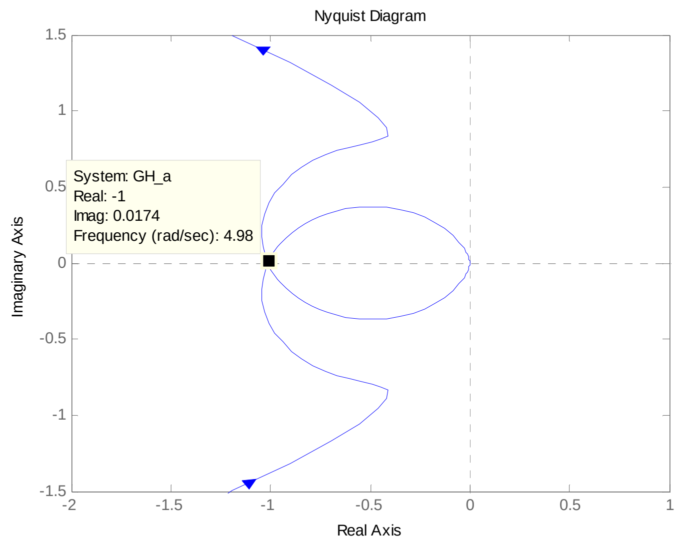
Part (a), Bode diagram:



Part (b), Root locus diagram:



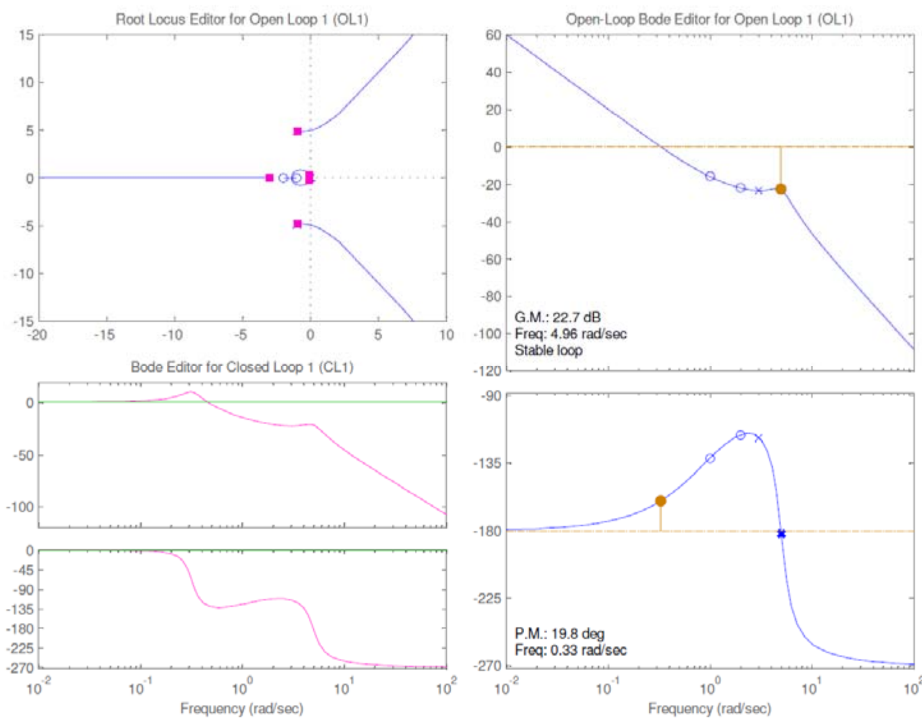
Part (c), Gain and frequency that instability occurs: Gain=53, Freq = 4.98 rad/sec, as seen in the data point in the figure:



Part (d), Gain and frequency that instability occurs:

Gain=0.127 at PM =20 deg:

By running sisotool command in MATLAB, the transfer functions are imported and the gain is iteratively changed until the phase margin of PM=20 deg is achieved. The corresponding Gain is K= 0.127.



Part (e): The corresponding gain margin is GM = 22.7 dB is seen in the figure

8-42 (a) Gain crossover frequency = 2.09 rad/sec PM = 115.85 deg

Phase crossover frequency = 20.31 rad/sec GM = 21.13 dB

(b) Gain crossover frequency = 6.63 rad/sec PM = 72.08 deg

Phase crossover frequency = 20.31 rad/sec GM = 15.11 dB

(c) Gain crossover frequency = 19.1 rad/sec PM = 4.07 deg

Phase crossover frequency = 20.31 rad/sec GM = 1.13 dB

(d) For GM = 40 dB, reduce gain by (40 – 21.13) dB = 18.7 dB, or gain = 0.116 $\times$ nominal value.

(e) For PM = 45 deg, the magntude curve reads –10 dB. This means that the loop gain can be increased by 10 dB from the nominal value. Or gain = 3.16 $\times$ nominal value.

(f) The system is type 1, since the slope of $|G(j\omega)|$ is –20 dB/decade as $\omega \to 0$.

(g) GM = 12.7 dB. PM = 109.85 deg.

(h) The gain crossover frequency is 2.09 rad/sec. The phase margin is 115.85 deg.

Set

$$\omega T_d = 2.09 T_d = \frac{115.85^o \pi}{180^o} = 2.022 \text{ rad}$$

Thus, the maximum time delay is $T_d = 0.9674$ sec.

8-43 **(a)** The gain is increased to four times its nominal value. The magnitude curve is raised by 12.04 dB.

Gain crossover frequency = 10 rad/sec PM = 46 deg

Phase crossover frequency = 20.31 rad/sec GM = 9.09 dB

(b) The GM that corresponds to the nominal gain is 21.13 dB. To change the GM to 20 dB we need to increase the gain by 1.13 dB, or 1.139 times the nominal gain.

(c) The GM is 21.13 dB. The forward-path gain for stability is 21.13 dB, or 11.39.

(d) The PM for the nominal gain is 115.85 deg. For PM = 60 deg, the gain crossover frequency must be moved to approximately 8.5 rad/sec, at which point the gain is –10 dB. Thus, the gain must be increased by 10 dB, or by a factor of 3.162.

(e) With the gain at twice its nominal value, the system is stable. Since the system is type 1, the steady-state error due to a step input is 0.

(f) With the gain at 20 times its nominal value, the system is unstable. Thus the steady-state error would be infinite.

(g) With a pure time delay of 0.1 sec, the magnitude curve is not changed, but the the phase curve is subject to a negative phase of -0.1ω rad. The PM is

$$\mathrm{PM} = 115.85 - 0.1 \times \text{gain crossover frequency} = 115.85 - 0.209 = 115.64 \text{ deg}$$

The new phase crossover frequency is approximately 9 rad/sec, where the original phase curve is reduced by –0.9 rad or –51.5 deg. The magnitude of the gain curve at this frequency is –10 dB. Thus, the gain margin is 10 dB.

(h) When the gain is set at 10 times its nominal value, the magnitude curve is raised by 20 dB. The new gain crossover frequency is approximately 17 rad/sec. The phase at this frequency is –30 deg. Thus, setting

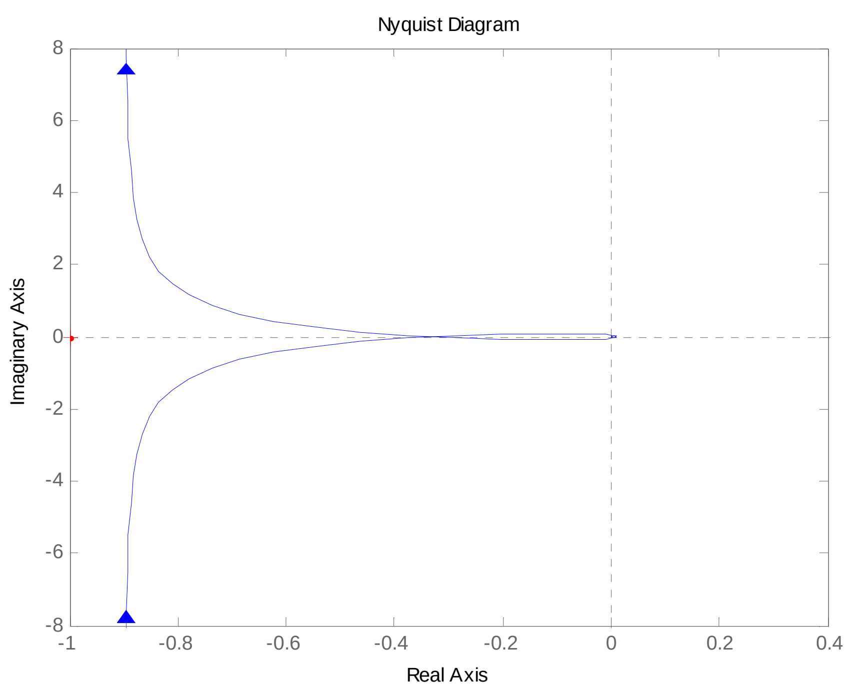
$$\omega T_d = 17T_d = \frac{30^o \pi}{180^o} = 0.5236 \qquad \text{Thus} \qquad T_d = 0.0308 \text{ sec.}$$

8-44

MATLAB code:

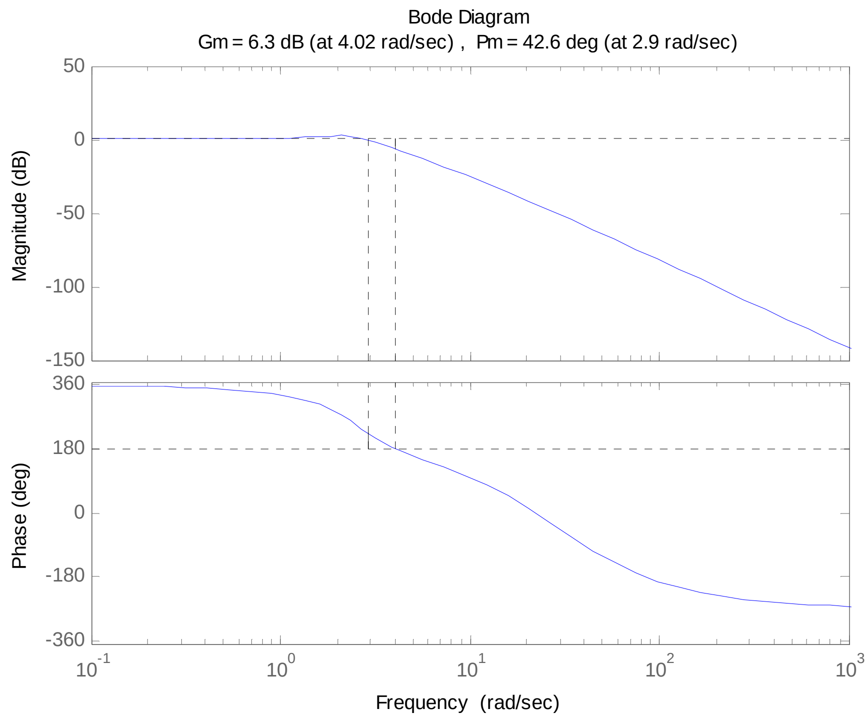
```
s = tf('s')
%using pade cammand for PADE approximation of exponential term
num_G_a= pade((80*exp(-0.1*s)),2);
den_G_a=s*(s+4)*(s+10);
G_a=num_G_a/den_G_a;
CL_a = G_a/(1+G_a)
OL_a = G_a*1;

%(a)
figure(1)
nyquist(OL_a)

%(b) and (c)
figure(2);
margin(CL_a)
```

Part (a), Nyquist diagram:



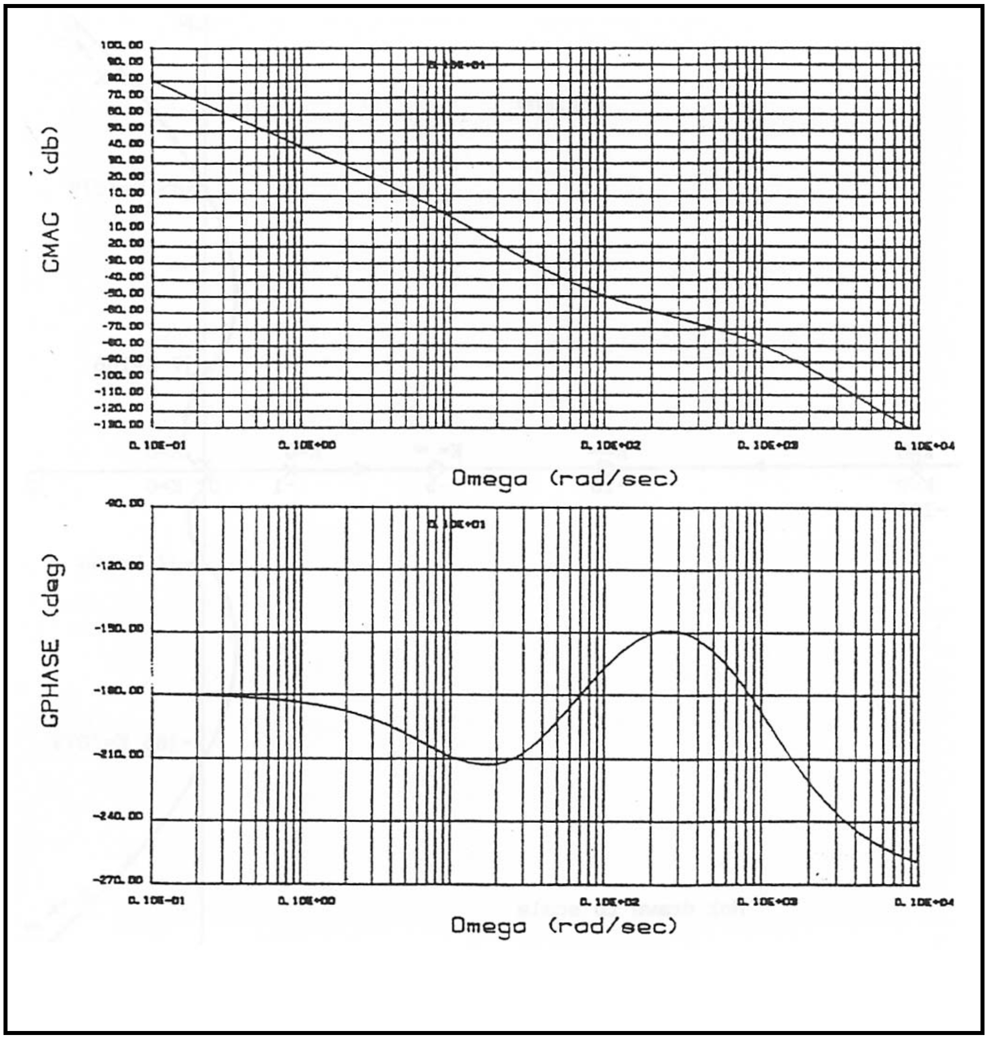
Part (b) and (c), Bode diagram:

Using Margin command, the gain and phase margins are obtained as GM = 6.3 dB, PM = 42.6 deg:

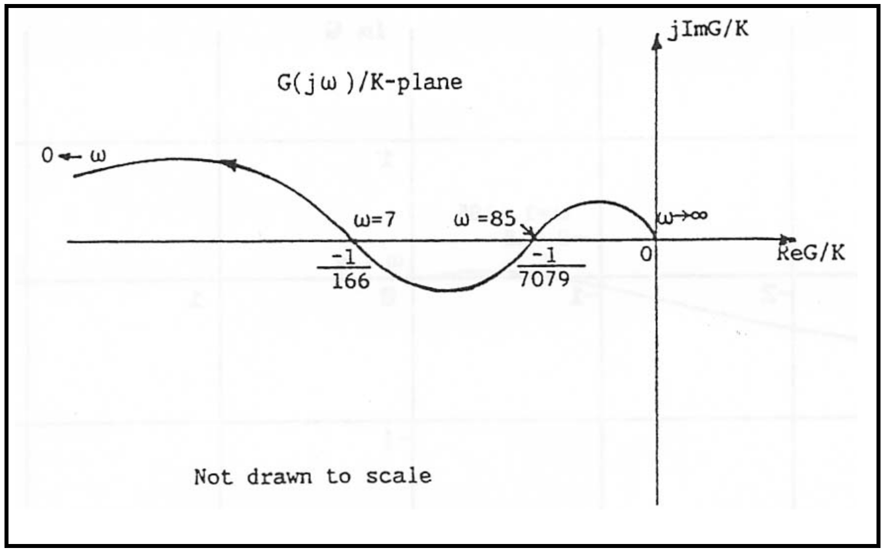

Bode Diagram
Gm = 6.3 dB (at 4.02 rad/sec) , Pm = 42.6 deg (at 2.9 rad/sec)
Magnitude (dB)
50
0
-50
-100
-150
Phase (deg)
360
180
0
-180
-360
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

8-45 (a) Bode Plot:

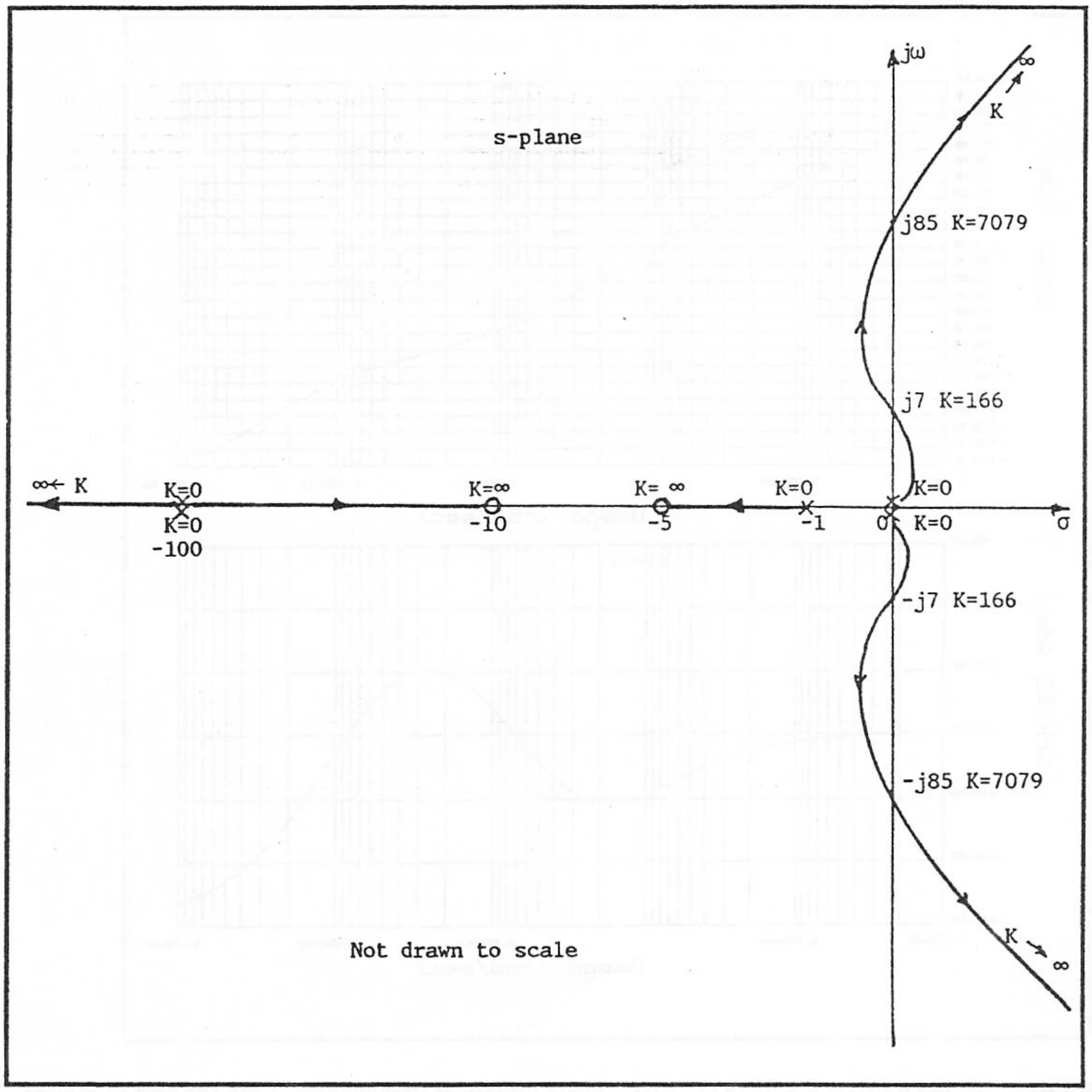
For stability: 166 (44.4 dB) < *K* < 7079 (77 dB) Phase crossover frequencies: 7 rad/sec and 85 rad/sec



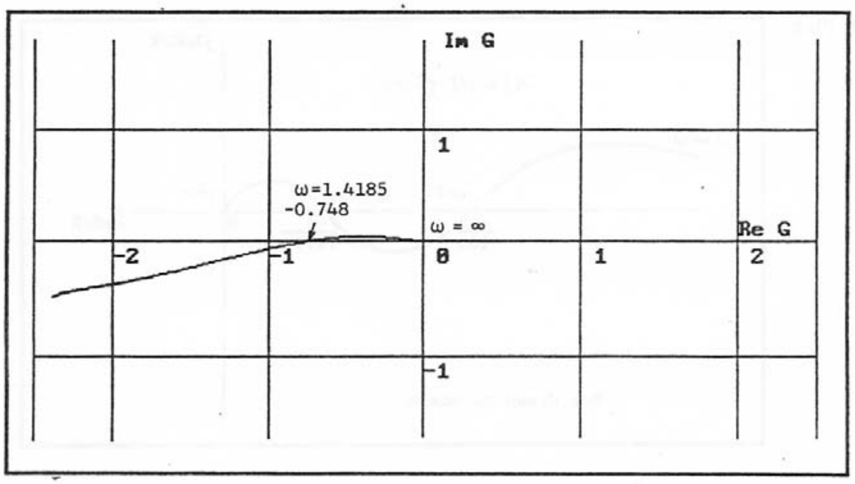
Nyquist Plot:



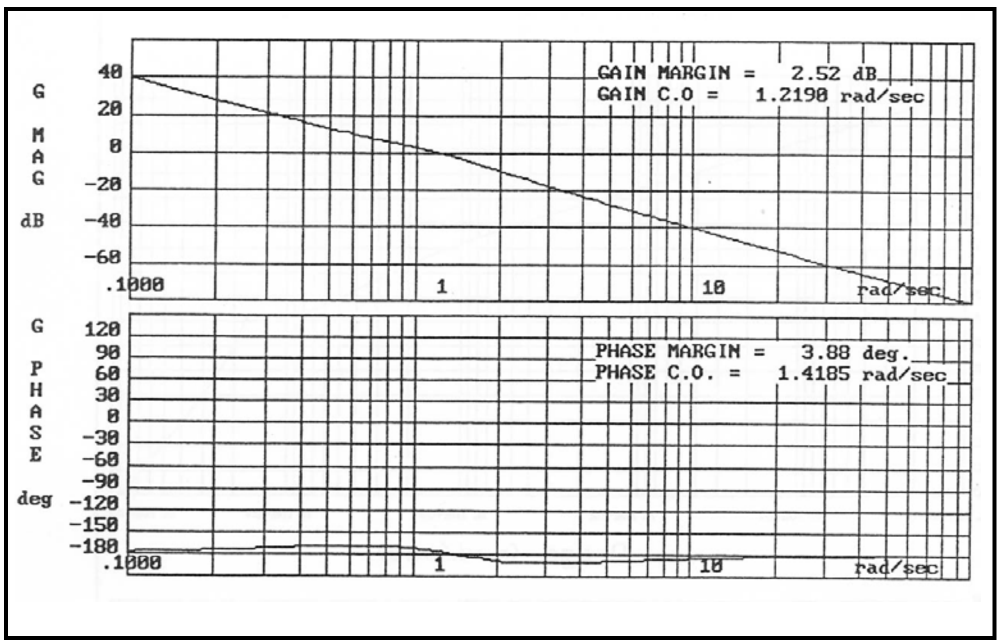
8-45 (b) **Root Loci.**



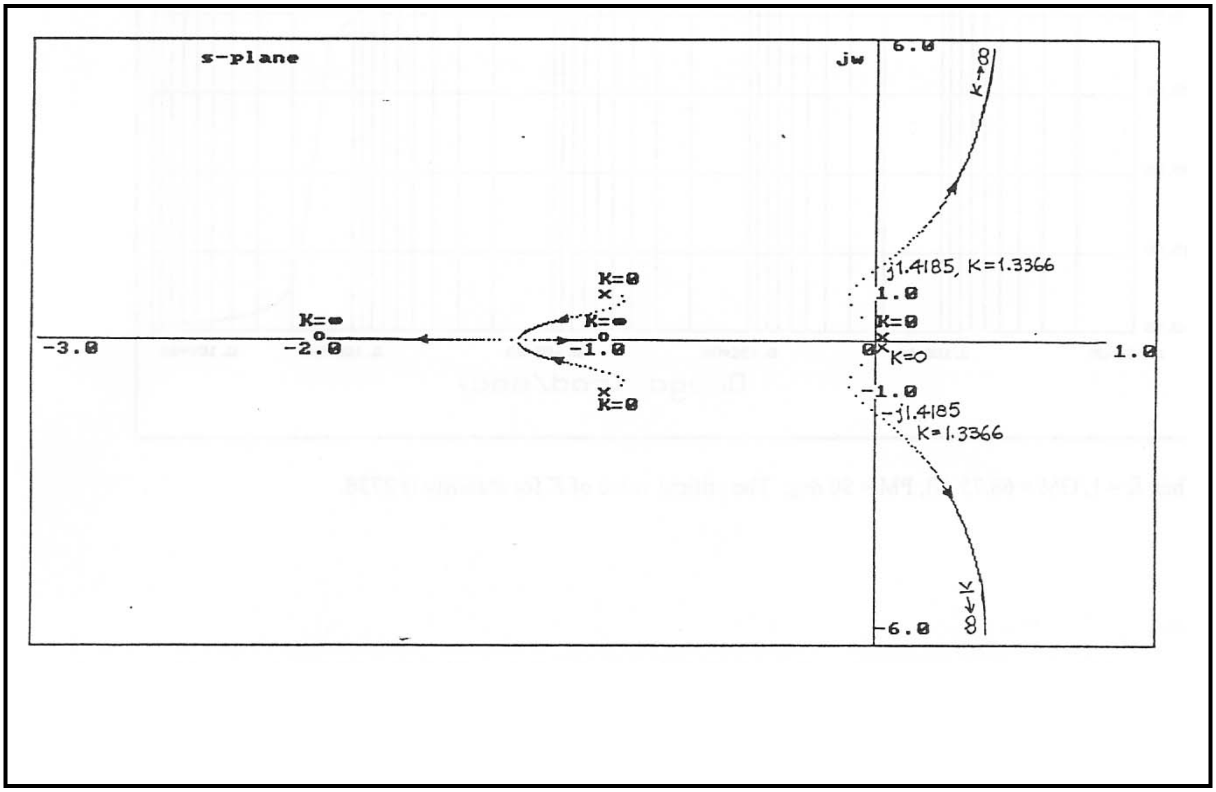
8-46 (a) Nyquist Plot



Bode Plot



Root Locus

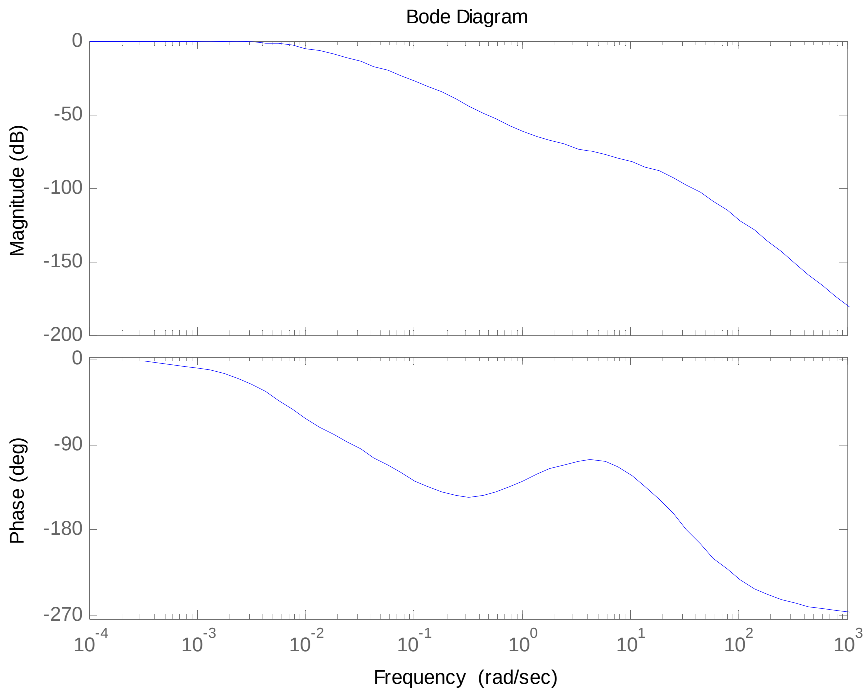

s-plane
jw
6.0
K→∞
j1.4185, K=1.3366
1.0
K=0
K=∞
K=0
K=∞
-3.0
-2.0
-1.0
0
K=0
1.0
-1.0
K=0
-j1.4185
K=1.3366
K→∞
-6.0

8-47

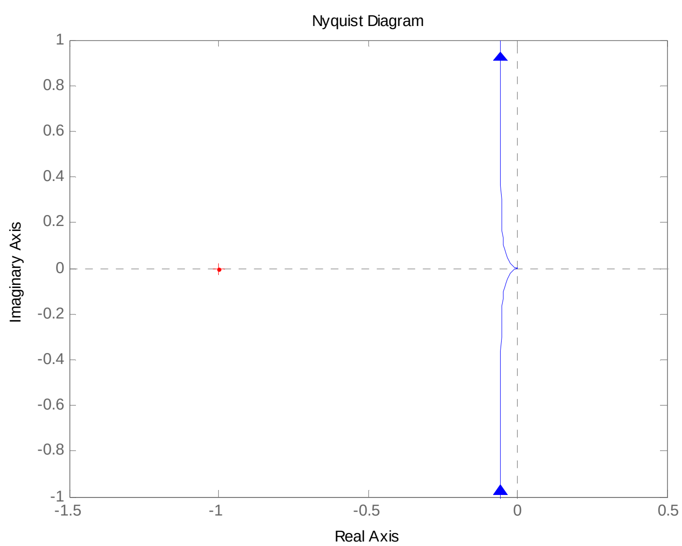
MATLAB code:

```
%a)
k=1
num_GH= k*(s+1)*(s+5);
den_GH=s*(s+0.1)*(s+8)*(s+20)*(s+50);
GH=num_GH/den_GH;
CL = GH/(1+GH)
figure(1);
bode(CL)
figure(2);
OL = GH;
nyquist(GH)
xlim([-1.5 0.5]);
ylim([-1 1]);

sisotool
```

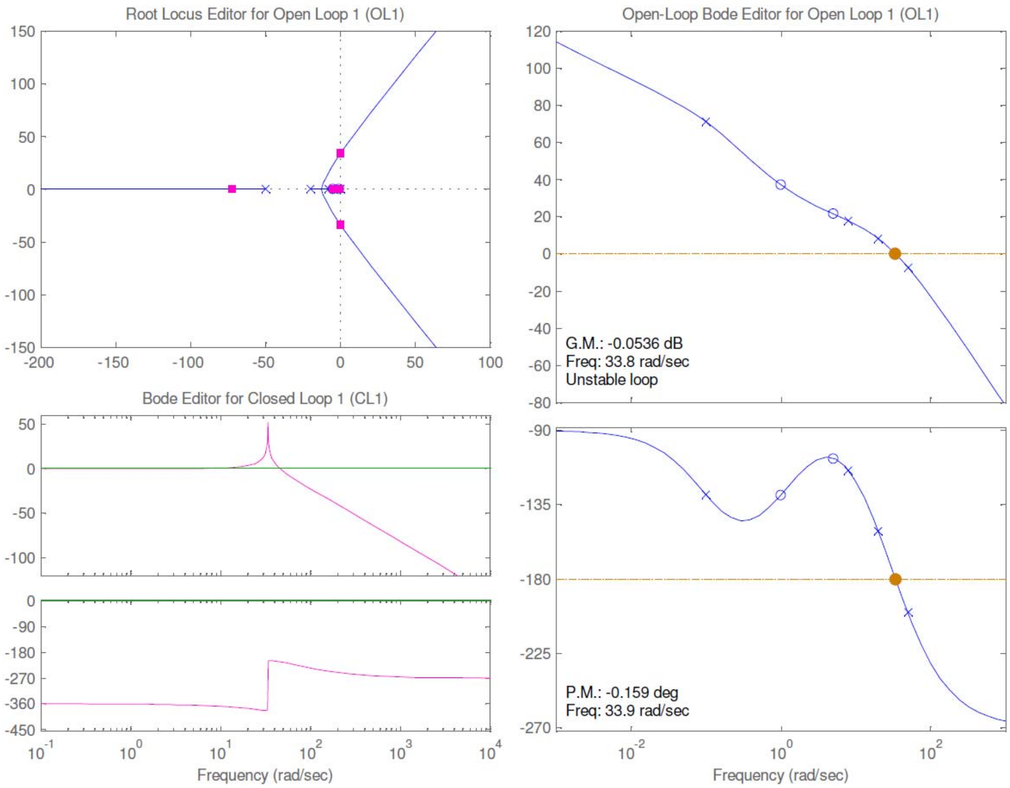
Part (a), Bode diagram:



Part (a), Nyquist diagram:

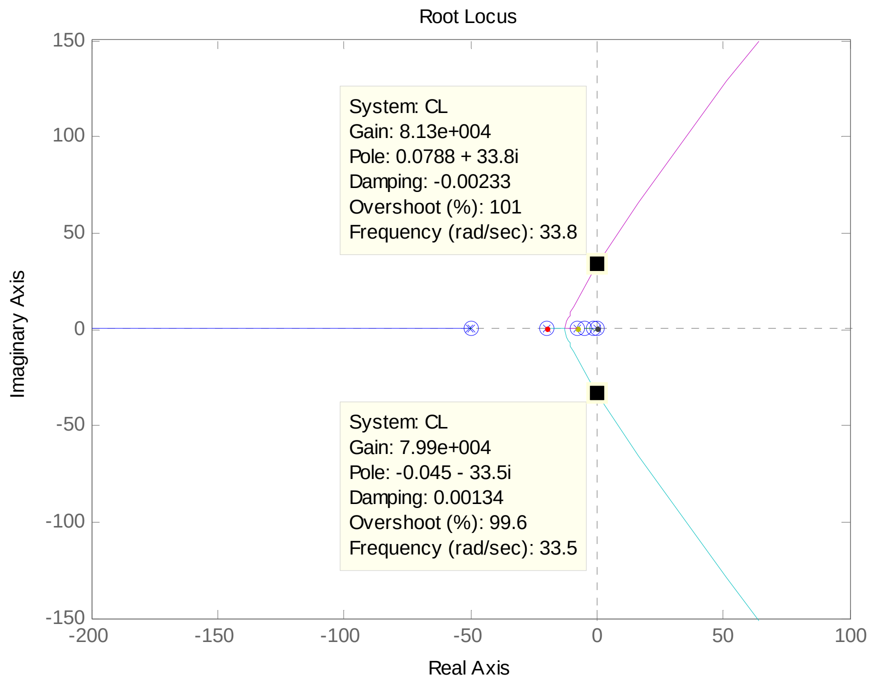


Part (a), range of K for stability: By running sisotool command in MATLAB, the transfer functions are imported and the gain is iteratively changed until the phase margin of PM=0 deg is achieved (where K =) which is the margin of stability. The stable rang for K is $K>8.16x10^4$:

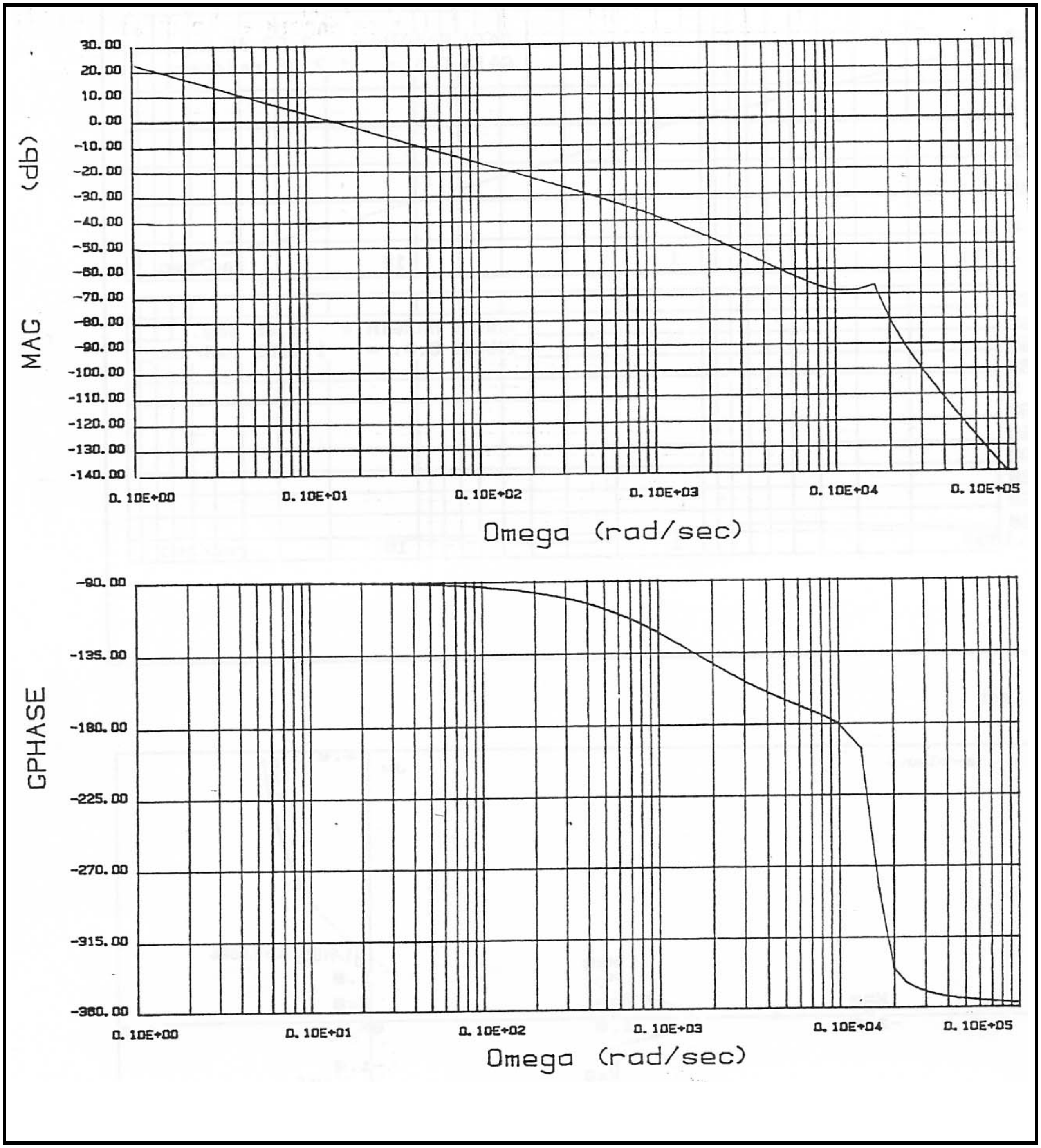


Part (b), Root-locus diagram, K and ω at the points where the root loci cross the $j\omega$-axis**:**

As can be seen in the figure at K=8.13x10^4 and ω=33.8 rad/sec, the poles cross the jω axis. Both of these values are consistent with the results of part(a) from sisotool.


Root Locus
System: CL
Gain: 8.13e+004
Pole: 0.0788 + 33.8i
Damping: -0.00233
Overshoot (%): 101
Frequency (rad/sec): 33.8
System: CL
Gain: 7.99e+004
Pole: -0.045 - 33.5i
Damping: 0.00134
Overshoot (%): 99.6
Frequency (rad/sec): 33.5
Imaginary Axis
Real Axis
150
100
50
0
-50
-100
-150
-200
-150
-100
-50
0
50
100

8-48 **Bode Diagram**



When $K = 1$, GM = 68.75 dB, PM = 90 deg. The critical value of K for stability is 2738.

8-49 (a) Forward-path transfer function:

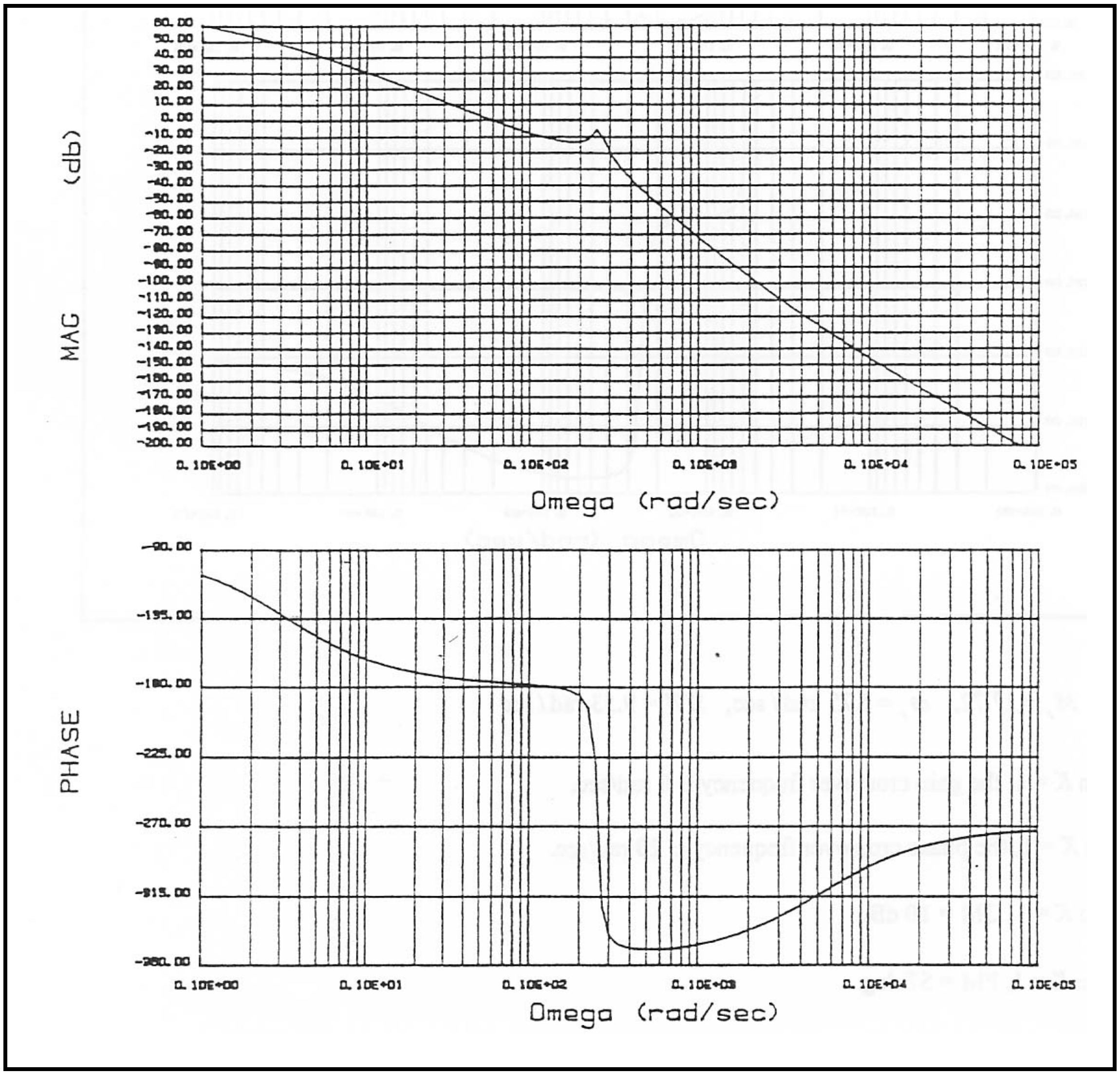
$$G(s) = \frac{\Theta_L(s)}{E(s)} = K_a G_p(s) = \frac{K_a K_i (Bs + K)}{\Delta_o}$$

where

$$\Delta_o = 0.12s\left(s + 0.0325\right)\left(s^2 + 2.5675s + 6667\right)$$
$$= s\left(0.12s^3 + 0.312s^2 + 80.05s + 26\right)$$

$$G(s) = \frac{43.33(s + 500)}{s\left(s^3 + 2.6s^2 + 667.12s + 216.67\right)}$$

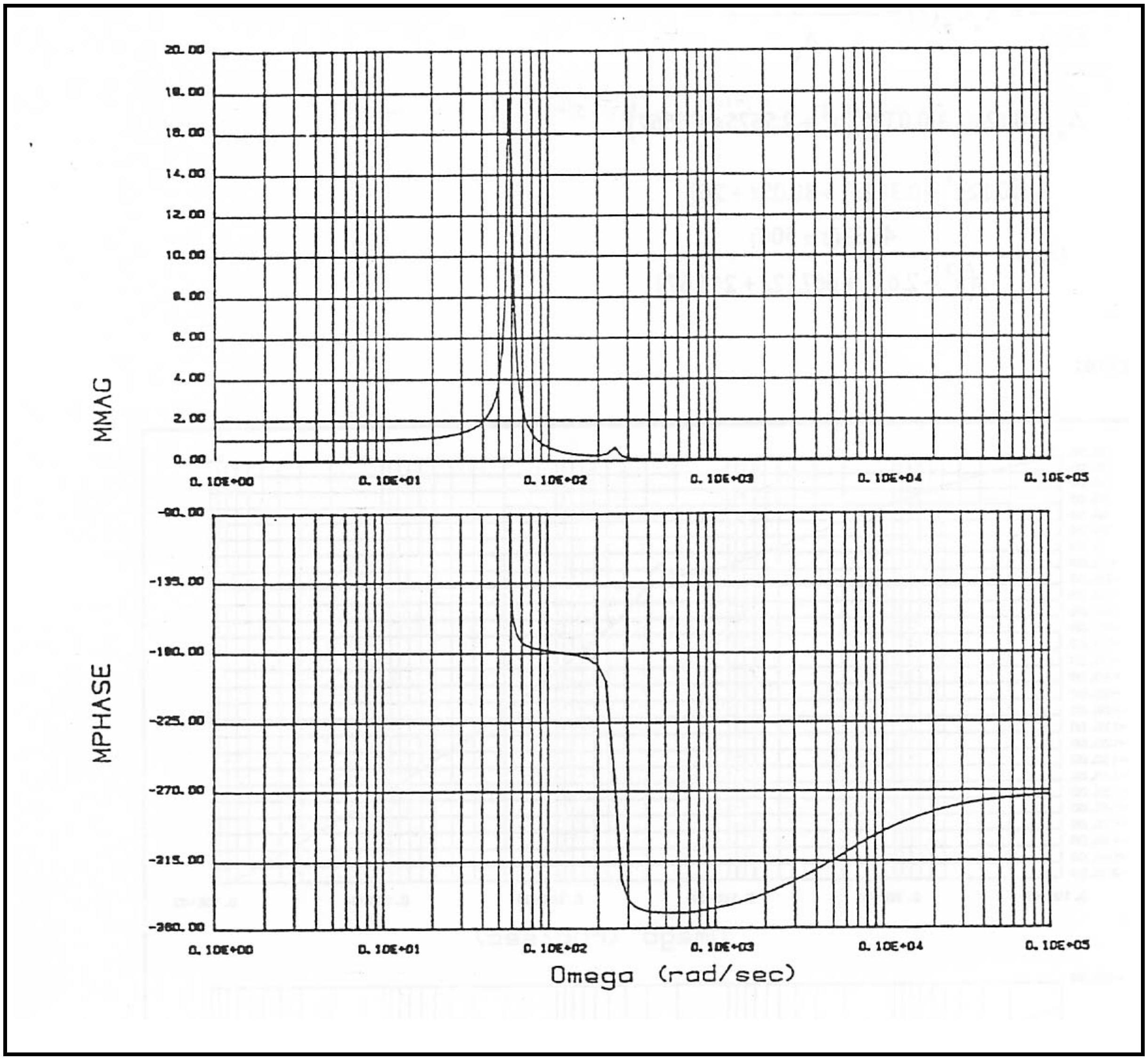
(b) Bode Diagram:



Gain crossover frequency = 5.85 rad/sec PM = 2.65 deg.

Phase crossover frequency = 11.81 rad/sec GM = 10.51 dB

8-49 (c) Closed-loop Frequency Response:



$M_r = 17.72, \quad \omega_r = 5.75 \text{ rad/sec}, \quad \text{BW} = 9.53 \text{ rad/sec}$

8-50

(a) $$G(s)H(s) = \frac{Kmgd}{L\left(\frac{J}{r^2}+m\right)s^2}$$

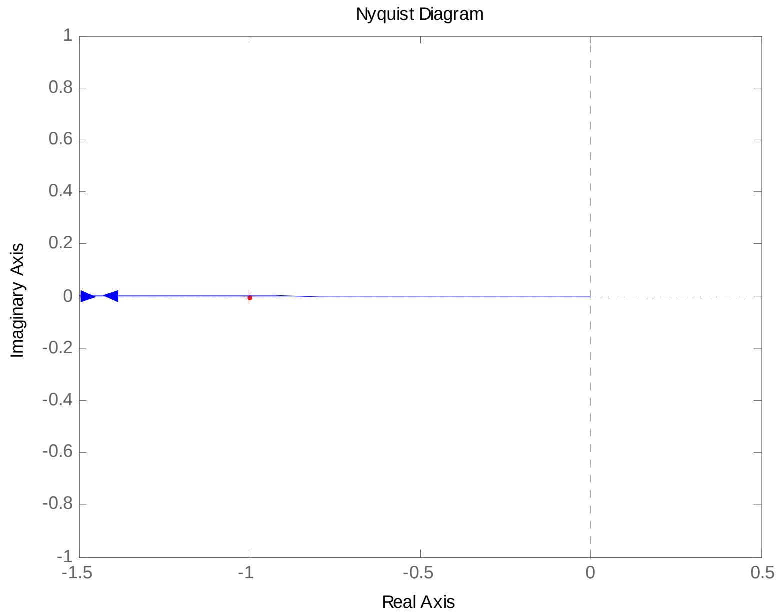
(b) $$\frac{G(s)H(s)}{1+G(s)H(s)} = \frac{Kmgd}{L\left(\frac{J}{r^2}+m\right)s^2 + Kmgd}$$

(c) to (e)

MATLAB code:

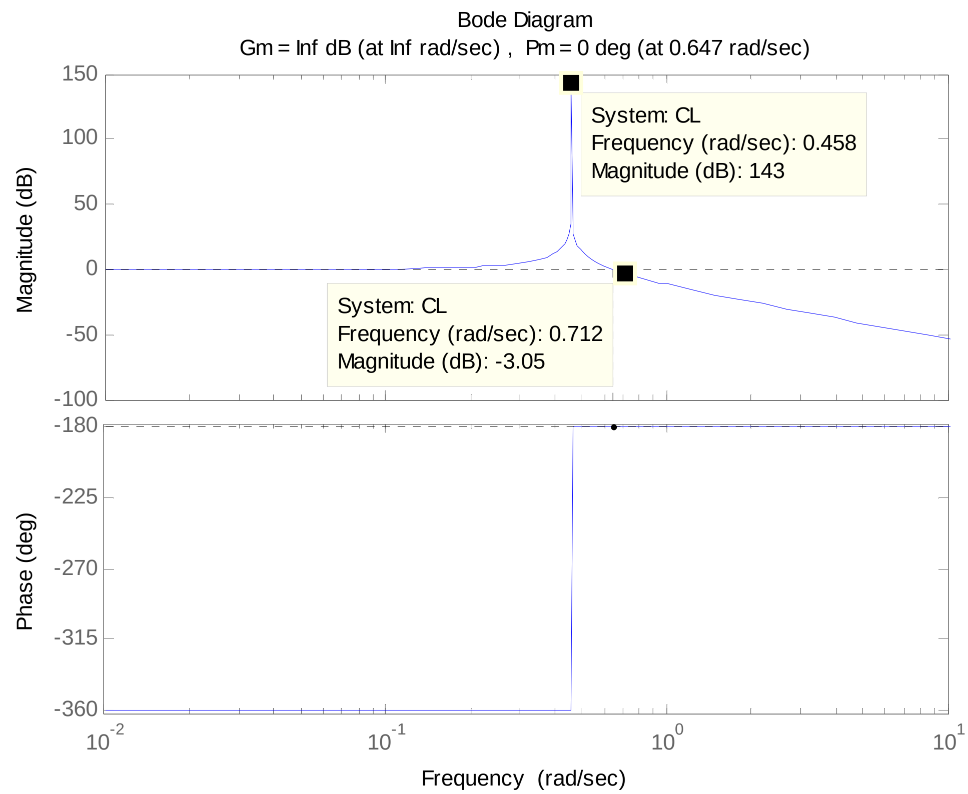
```
s = tf('s')

m = 0.11;
r = 0.015;
d = 0.03;
g = 9.8;
L = 1.0;
J = 9.99*10^-6

K=1
num_GH= K*m*g*d;
den_GH=L*(J/r^2+m)*s^2;
GH=num_GH/den_GH;
CL = GH/(1+GH)

%c)
figure(1);
nyquist(GH)
xlim([-1.5 0.5]);
ylim([-1 1]);

%d)
figure(2);
margin(CL)
```

Part (c): since the system is a double integrator ($1/s^2$), the phase is always -180 deg, and the system is always marginally stable for **any K**, leading to a complicated control problem.



Part (d), Bode diagram:

As explained in section (c), since the system is always marginally stable, GM=inf and PM = 0, as can be seen by MATLAB MARGIN command, resulting in the following figure:



Part (e), M_r,= 143 dB, ω_r = 0.458 rad/sec, and BW= 0.712 rad/sec as can be seen in the data points in the above figure.

8-51 (a) When K = 1, the gain crossover frequency is 8 rad/sec.

(b) When K = 1, the phase crossover frequency is 20 rad/sec.

(c) When K = 1, GM = 10 dB.

(d) When K = 1, PM = 57 deg.

(e) When K = 1, $M_r = 1.2$.

(f) When K = 1, $\omega_r = 3$ rad/sec.

(g) When K = 1, BW = 15 rad/sec.

(h) When K = –10 dB (0.316), GM = 20 dB

(i) When K = 10 dB (3.16), the system is marginally stable. The frequency of oscillation is 20 rad/sec.

(j) The system is type 1, since the gain-phase plot of $G(j\omega)$ approaches infinity at –90 deg. Thus, the steady-state error due to a unit-step input is zero.

8-52 When K = 5 dB, the gain-phase plot of $G(j\omega)$ is raised by 5 dB.

(a) The gain crossover frequency is ~10 rad/sec.

(b) The phase crossover frequency is ~20 rad/sec.

(c) GM = 5 dB.

(d) PM = ~34.5 deg.

(e) When K = 5, M_r= ~ 2 (smallest circle tangent to an ***M*** circle).

(f) $\omega_r = 15$ rad/sec

(g) BW = 30 rad/sec

(h) When K = –30 dB, the GM is 40 dB (shift the graph of K=1, 30 dbs down).

When ***K* = 10** dB, the gain-phase plot of $G(j\omega)$ is raised by 10 dB.

(a) The gain crossover frequency is 20 rad/sec.

(b) The phase crossover frequency is 20 rad/sec.

(c) GM = 0 dB.

(d) PM = 0 deg.

(e) When K = 10, M_r= ~ 1.1 (smallest circle tangent to an ***M*** circle).

(f) $\omega_r = 5$ rad/sec

(g) BW = ~40 rad/sec

(h) When K = –30 dB, the GM is 40 dB (shift the graph of K=1, 30 dbs down).

8-53

Since the function has exponential term, PADE command has been used to obtain the transfer function.

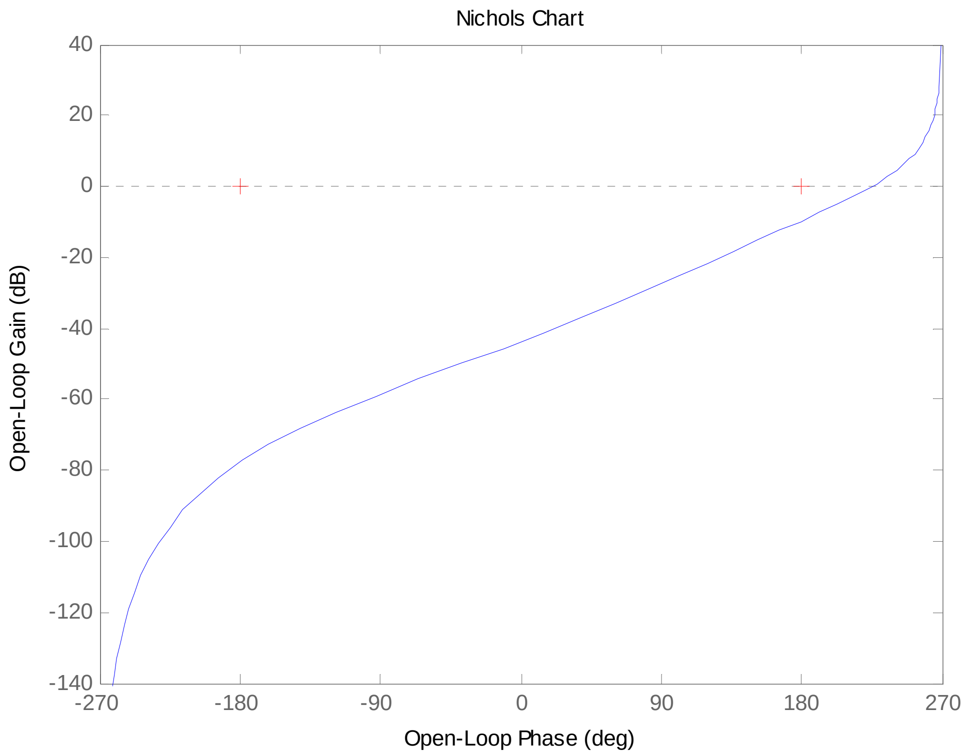
$$G(s)H(s) = \frac{80e^{-0.1s}}{s(s+4)(s+10)}$$

MATLAB code:

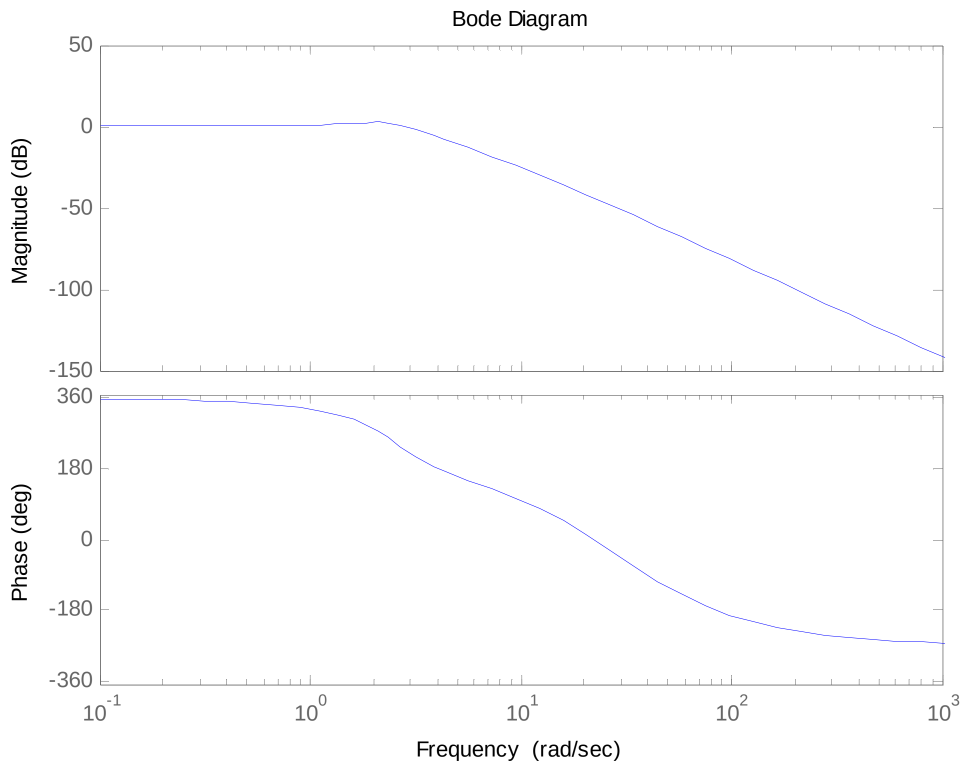
```
s = tf('s')
%a)
```

```
num_GH= pade(80*exp(-0.1*s),2);
den_GH=s*(s+4)*(s+10);
GH=num_GH/den_GH;
CL = GH/(1+GH)
BW = bandwidth(CL)
bode(CL)

%b)
figure(2);
nichols(GH)
```

Part(a), Nicholas diagram:



Part(b), Bode diagram:



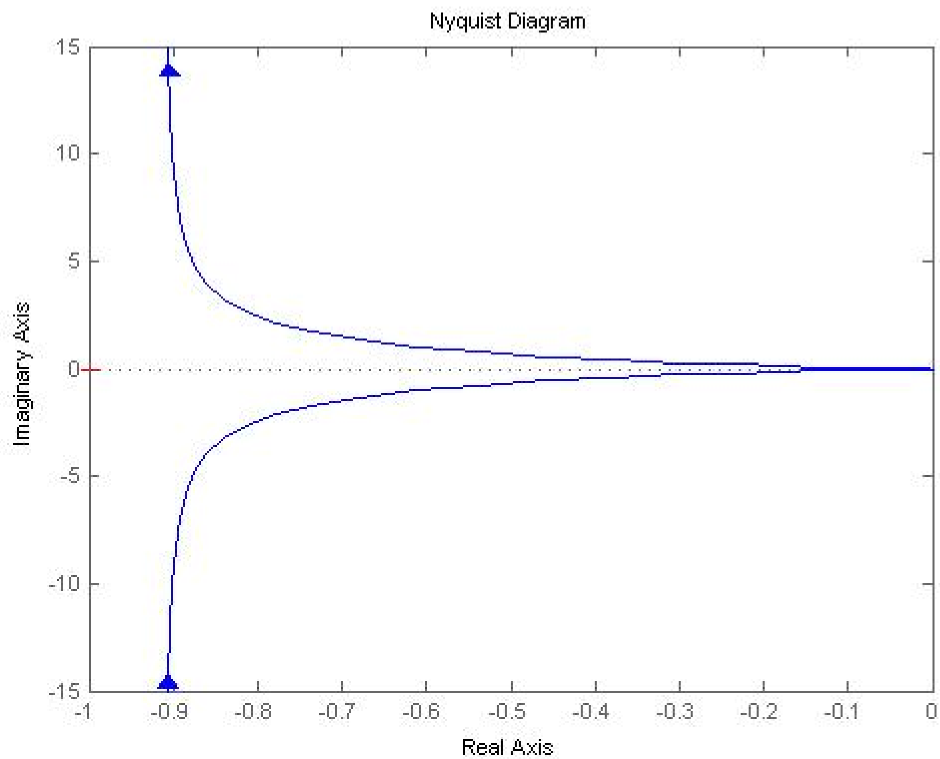
8- 54) Note: $G_{CL} = \dfrac{G}{1+G}$

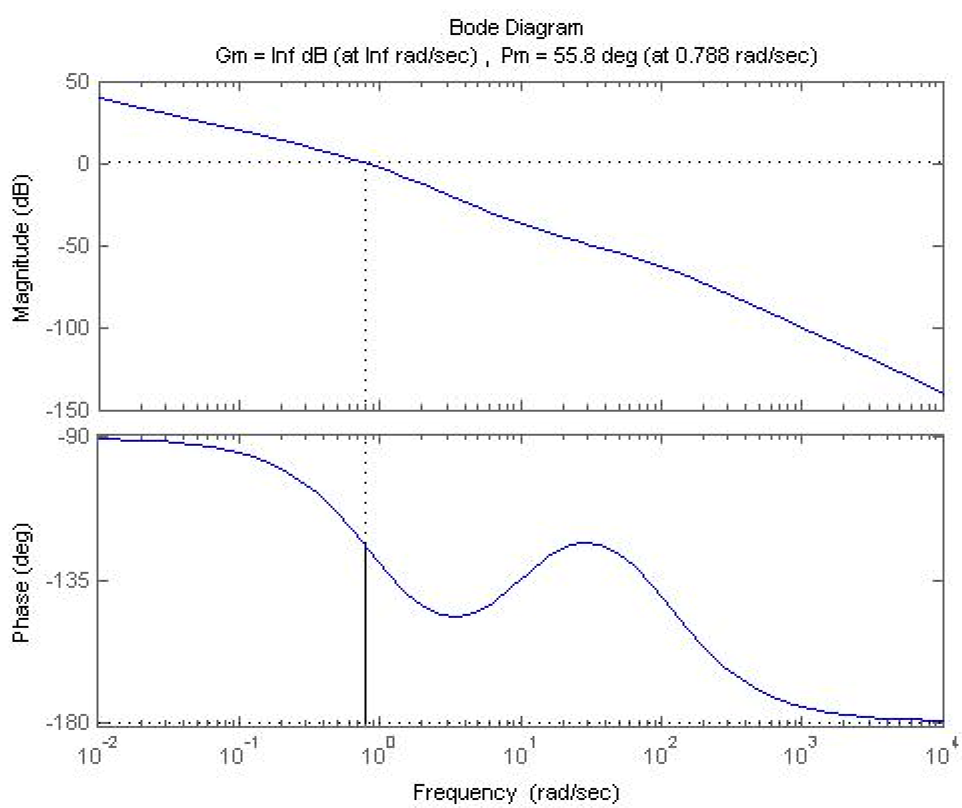
To draw the Bode and polar plots use the closed loop transfer function, G_{CL}, and find BW. Use G to obtain the gain-phase plots and Gm and Pm. Use the Bode plot to graphically obtain M_r.

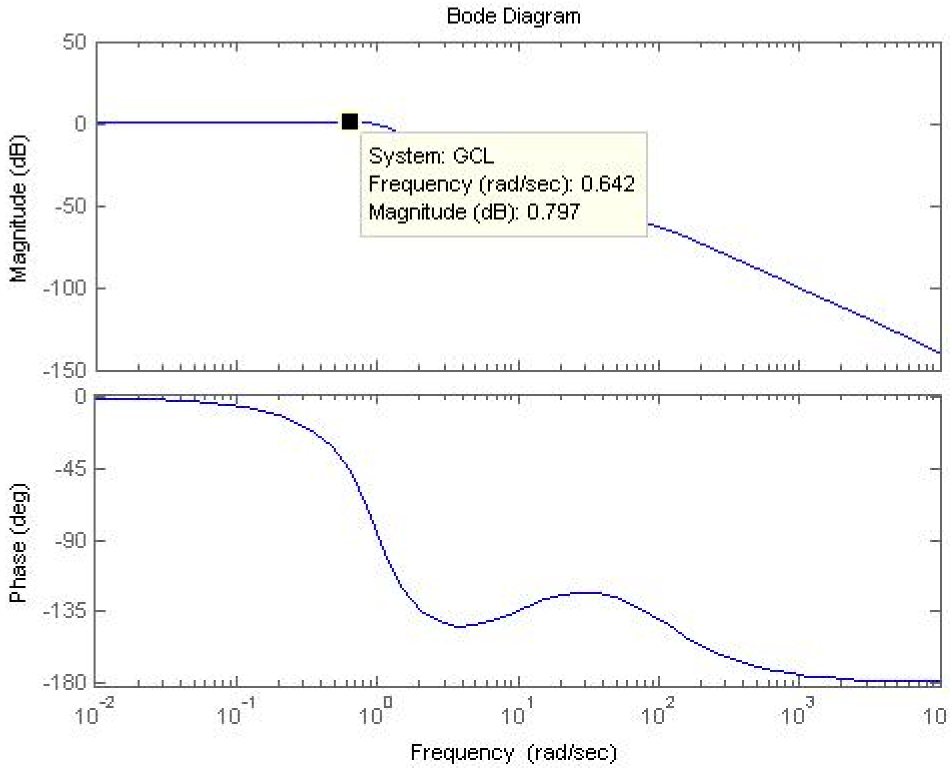
Sample MATLAB code:

```
s = tf('s')
%a)
num_G= 1+0.1*s;
den_G=s*(s+1)*(0.01*s+1);
G=num_G/den_G
figure(1)
nyquist(G)
figure(2)
margin(G)
GCL = G/(1+G)
BW = bandwidth(GCL)
figure(3)
bode(GCL)
```

```
Transfer function:
      0.1 s + 1
-----------------------
0.01 s^3 + 1.01 s^2 + s


Transfer function:
           0.001 s^4 + 0.111 s^3 + 1.11 s^2 + s
-------------------------------------------------------------
0.0001 s^6 + 0.0202 s^5 + 1.041 s^4 + 2.131 s^3 + 2.11 s^2 + s


BW =

    1.2235
```


Nyquist Diagram
Imaginary Axis
Real Axis
15
10
5
0
-5
-10
-15
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0


Bode Diagram
Gm = Inf dB (at Inf rad/sec) , Pm = 55.8 deg (at 0.788 rad/sec)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
50
0
-50
-100
-150
-90
-135
-180



8-55) (a) The phase margin with *K* = 1 and $T_d = 0$ sec is approximately 57 deg. For a PM of 40 deg, the time delay produces a phase lag of –17 deg. The gain crossover frequency is 8 rad/sec.

Thus,

$$\omega T_d = 17^o = \frac{17^o \pi}{180^o} = 0.2967 \text{ rad / sec} \qquad \text{Thus } \omega = 8 \text{ rad / sec}$$

$$T_d = \frac{0.2967}{8} = 0.0371 \text{ sec}$$

(b) With *K* = 1, for marginal stability, the time delay must produce a phase lag of –57 deg.

Thus, at $\omega = 8$ rad/sec,

$$\omega T_d = 57^o = \frac{57^o \pi}{180^o} = 0.9948 \text{ rad} \qquad T_d = \frac{0.9948}{8} = 0.1244 \text{ sec}$$

8-56 (a) The phase margin with K = 5 dB and $T_d = 0$ is approximately 34.5 deg. For a PM of 30 deg, the time delay must produce a phase lag of –4.5 deg. The gain crossover frequency is 10 rad/sec. Thus,

$$\omega T_d = 4.5^o = \frac{4.5^o \pi}{180^o} = 0.0785 \text{ rad} \qquad \text{Thus} \qquad T_d = \frac{0.0785}{10} = 0.00785 \text{ sec}$$

(b) With K = 5 dB, for marginal stability, the time delay must produce a phase lag of –34.5 deg.

Thus at $\omega = 10$ rad/sec,

$$\omega T_d = 34.5^o = \frac{34.5^o \pi}{180^o} = 0.602 \text{ rad} \qquad \text{Thus} \qquad T_d = \frac{0.602}{10} = 0.0602 \text{ sec}$$

8-57) For a GM of 5 dB, the time delay must produce a phase lag of –34.5 deg at $\omega = 10$ rad/sec. Thus,

$$\omega T_d = 34.5^o = \frac{34.5^o \pi}{180^o} = 0.602 \text{ rad} \qquad \text{Thus} \qquad T_d = \frac{0.602}{10} = 0.0602 \text{ sec}$$

8-58 (a) **Forward-path Transfer Function:**

$$G(s) = \frac{Y(s)}{E(s)} = \frac{e^{-2s}}{(1+10s)(1+25s)}$$

From the Bode diagram, phase crossover frequency = 0.21 rad/sec GM = 21.55 dB gain crossover frequency = 0 rad/sec PM = infinite

(b)

$$G(s) = \frac{1}{(1+10s)(1+25s)(1+2s+2s^2)}$$

From the Bode diagram, phase crossover frequency = 0.26 rad/sec GM = 25 dB

gain crossover frequency = 0 rad/sec PM = infinite

(c)

$$G(s) = \frac{1-s}{(1+s)(1+10s)(1+2s)}$$

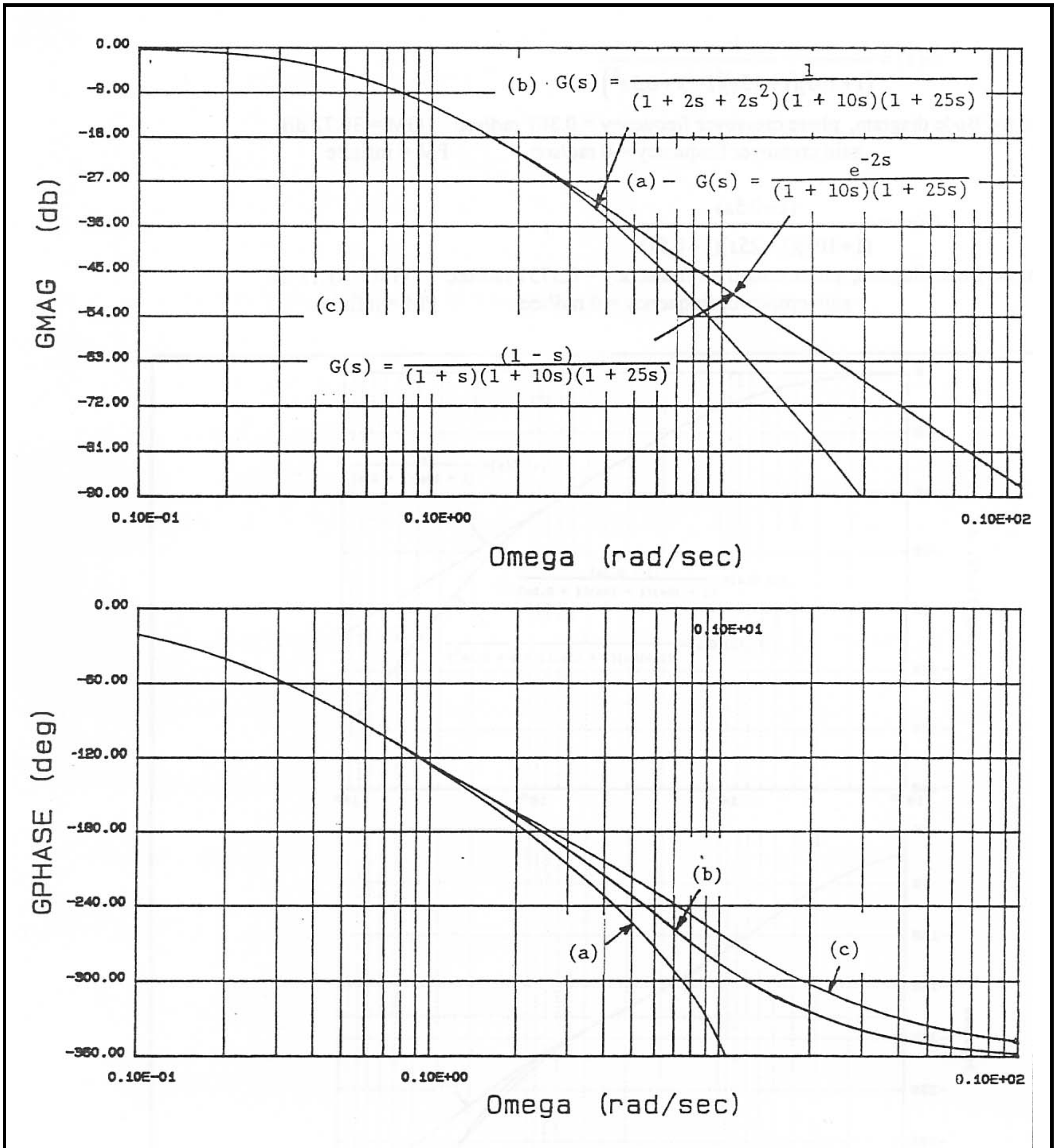
From the Bode diagram, phase crossover frequency = 0.26 rad/sec GM = 25.44 dB

gain crossover frequency = 0 rad/sec PM = infinite

Sample MATLAB code

```
s = tf('s')
%a)
num_G=exp(-2*s);
den_G=(10*s+1)*(25*s+1);
G=num_G/den_G
figure(1)
margin(G)
```

8-58 (continued) **Bode diagrams for all three parts.**



8-59 (a) Forward-path Transfer Function:

$$G(s) = \frac{e^{-s}}{(1+10s)(1+25s)}$$

From the Bode diagram, phase crossover frequency = 0.37 rad/sec GM = 31.08 dB

gain crossover frequency = 0 rad/sec PM = infinite

(b)

$$G(s) = \frac{1}{(1+10s)(1+25s)(1+s+0.5s^2)}$$

From the Bode diagram, phase crossover frequency = 0.367 rad/sec GM = 30.72 dB

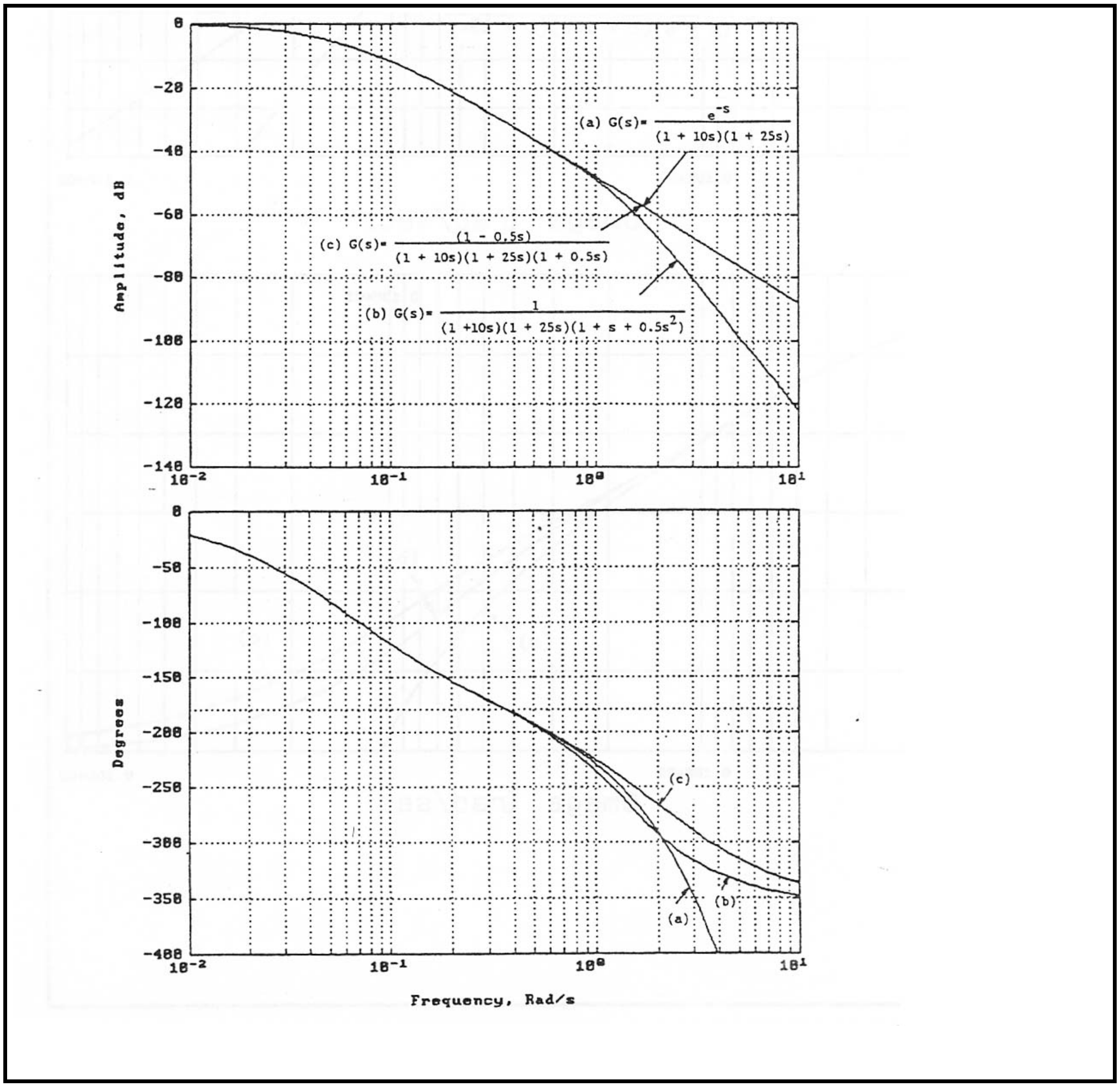
gain crossover frequency = 0 rad/sec PM = infinite

(c)

$$G(s) = \frac{(1-0.5s)}{(1+10s)(1+25s)(1+0.5s)}$$

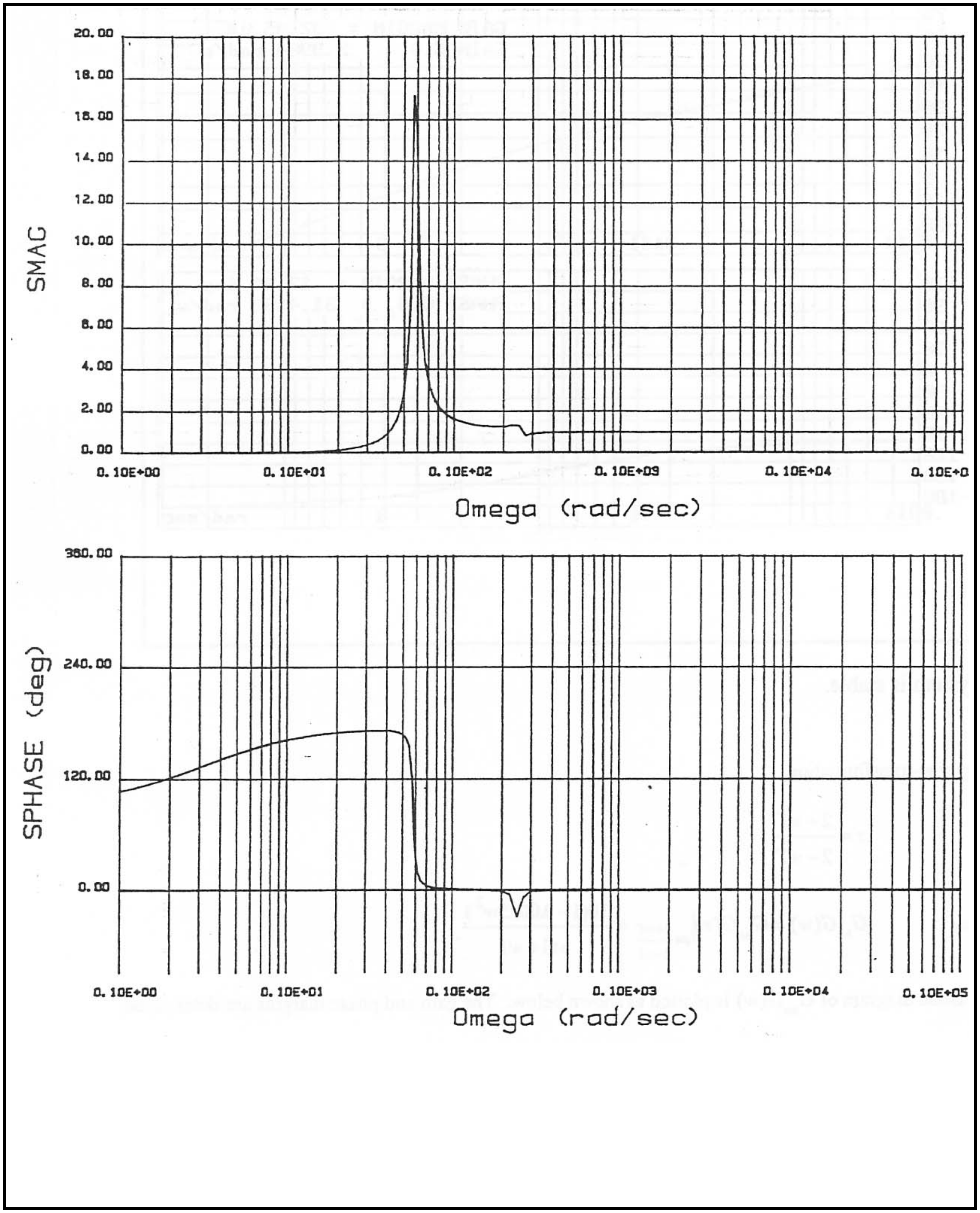
From the Bode diagram, phase crossover frequency = 0.3731 rad/sec GM = 31.18 dB

gain crossover frequency = 0 rad/sec PM = infinite


Amplitude, dB
(a) G(s)= e^-s / (1 + 10s)(1 + 25s)
(c) G(s)= (1 - 0.5s) / (1 + 10s)(1 + 25s)(1 + 0.5s)
(b) G(s)= 1 / (1 +10s)(1 + 25s)(1 + s + 0.5s^2)
Degrees
(a)
(b)
(c)
Frequency, Rad/s

Plots 8-59 (a-c)

8-60 **Sensitivity Plot:**



$$\left|S_G^M\right|_{\max} = 17.15 \qquad \omega_{\max} = 5.75 \text{ rad/sec}$$

8-61)

(a) $$G(s)H(s) = \frac{K(1.151s + 0.1774)}{s^3 + 0.739s^2 + 0.921s}$$

(b) $$\frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{K(1.151s + 0.1774)}{s^3 + 0.739s^2 + (0.921 + 1.151K)s + 0.1774K}$$

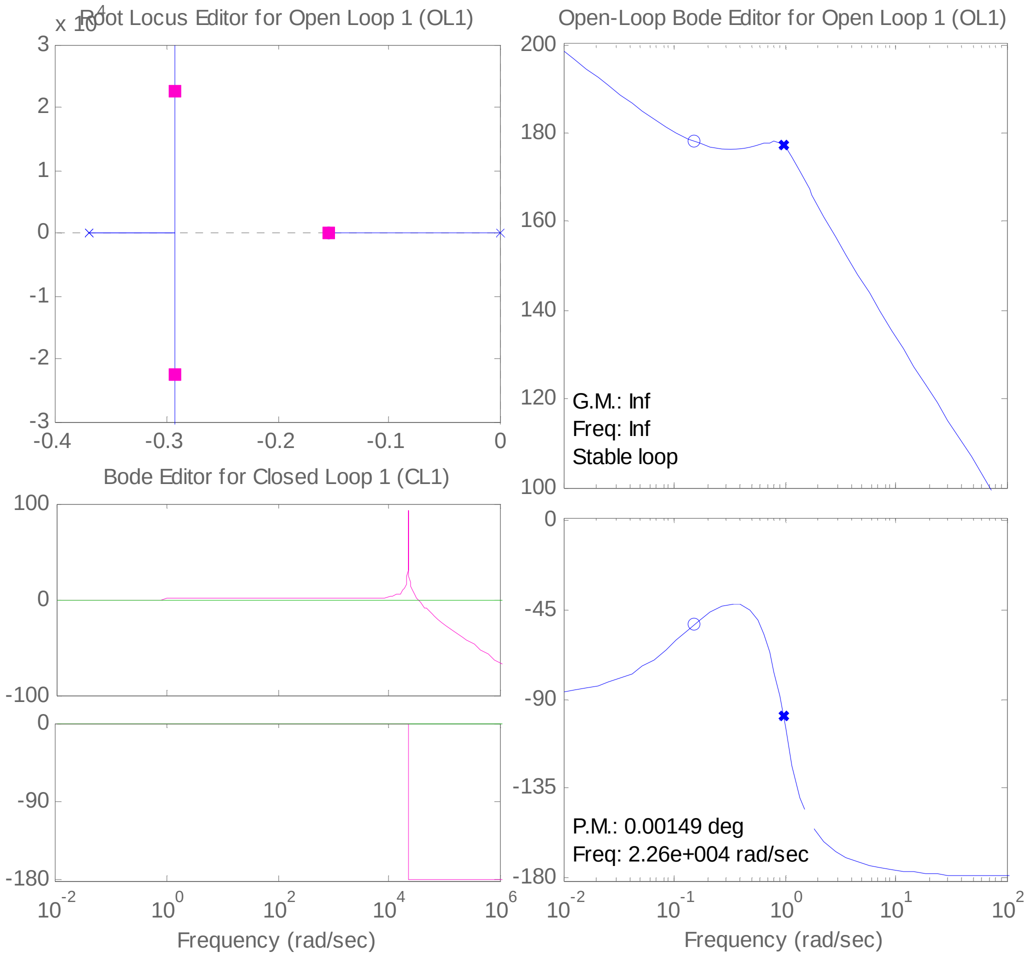
(c)&(d)

MATLAB code:

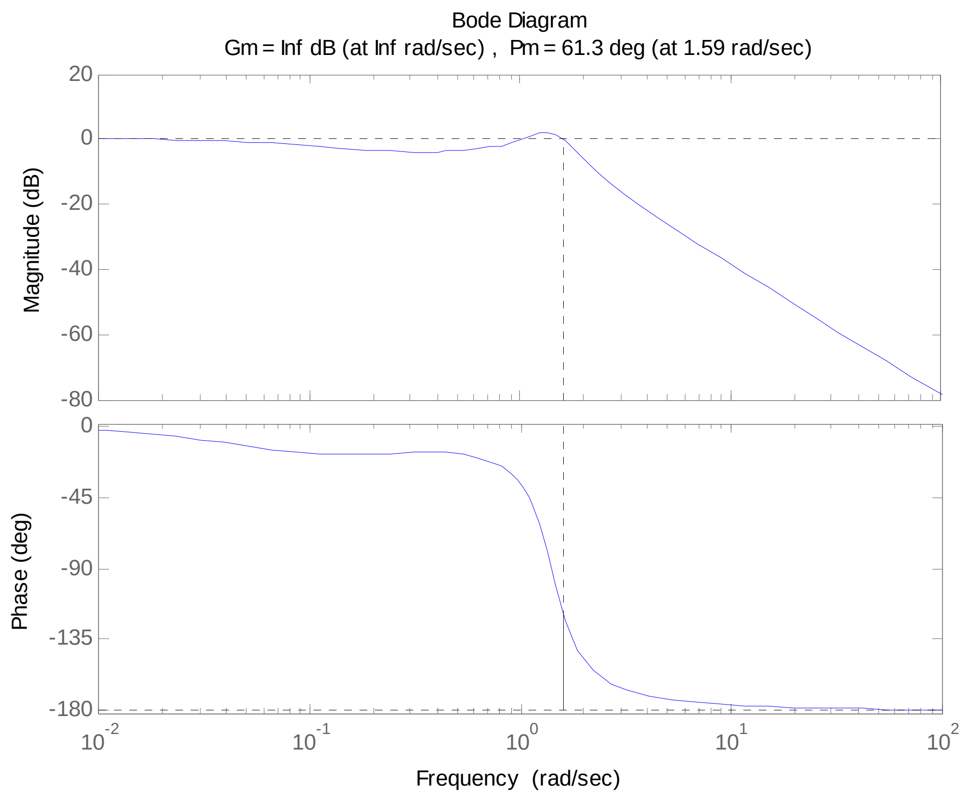
```
s = tf('s')
%c)
K = 1
num_GH= K*(1.151*s+0.1774);
den_GH=(s^3+0.739*s^2+0.921*s);
GH=num_GH/den_GH;
CL = GH/(1+GH)
sisotool
%(d)
figure(1)
margin(CL)
```

Part (c), range of K for stability:

Sisotool Result shows that by changing K between 0 and inf., all the roots of closed loop system remain in the left hand side plane and PM remains positive. Therefore, the system is stable for all positive K.



Part (d), Bode, GM & PM for K=1:


Bode Diagram
Gm = Inf dB (at Inf rad/sec) , Pm = 61.3 deg (at 1.59 rad/sec)
Magnitude (dB)
20
0
-20
-40
-60
-80
Phase (deg)
0
-45
-90
-135
-180
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

Chapter 9

9-1 Forward-path Transfer Function:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20+a)s^2 + (200+20a)s + 200a - K}$$

For type 1 system, $200a - K = 0$ Thus $K = 200a$

Ramp-error constant:

$$K_v = \lim_{s\to 0} sG(s) = \frac{K}{200+20a} = \frac{200a}{200+20a} = 5$$ Thus $a = 10$ $K = 2000$

The forward-path transfer function is

$$G(s) = \frac{2000}{s\left(s^2 + 30s + 400\right)}$$

The controller transfer function is

$$G_c(s) = \frac{G(s)}{G_p(s)} = \frac{20\left(s^2 + 10s + 100\right)}{\left(s^2 + 30s + 400\right)}$$

The maximum overshoot of the unit-step response is 0 percent.

9-2 Forward-path Transfer Function:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20+a)s^2 + (200+20a)s + 200a - K}$$

For type 1 system, $200a - K = 0$ Thus $K = 200a$

Ramp-error constant:

$$K_v = \lim_{s\to 0} sG(s) = \frac{K}{200+20a} = \frac{200a}{200+20a} = 9$$ Thus $a = 90$ $K = 18000$

The forward-path transfer function is

$$G(s) = \frac{18000}{s\left(s^2 + 110s + 2000\right)}$$

The controller transfer function is

$$G_c(s) = \frac{G(s)}{G_p(s)} = \frac{180\left(s^2 + 10s + 100\right)}{\left(s^2 + 110s + 2000\right)}$$

The maximum overshoot of the unit-step response is 4.3 percent.

From the expression for the ramp-error constant, we see that as *a* or *K* goes to infinity, K_v approaches 10. Thus the maximum value of K_v that can be realized is 10. The difficulties with very large values of *K* and *a* are that a high-gain amplifier is needed and unrealistic circuit parameters are needed for the controller.

9-3) The close loop transfer function is:

$$\frac{Y(s)}{X(s)} = \frac{K}{s^2+s+K} = \frac{K}{s^2+\frac{1}{\tau}s+\frac{K}{\tau}}$$

Comparing with second order system:

$$\omega_n = \sqrt{\frac{K}{\tau}} \text{ and } 2\,\xi\omega_n = \frac{1}{\tau}$$

$$M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.254 \quad \Rightarrow \quad \xi = 0.4$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 3 \quad \Rightarrow \quad \omega_n = 1.14$$

$$\tau = \frac{1}{2\,\xi\omega_n} \quad \Rightarrow \quad \tau = 1.09$$

$$K = \tau\omega_n^2 \quad \Rightarrow \quad K = 1.42$$

9-4) The forward path transfer function of the system is:

$$G(s)H(s) = \frac{24}{s(s+1)(s+6)}$$

1. The steady state error is less than to π/10 when the input is a ramp with a slope of 2π rad/sec

$$e_{ss} = \frac{\pi}{10} = \lim_{s\to 0} \frac{R}{sG_c(s)G(s)} = \lim_{s\to 0} \frac{2\pi}{s(K_p + K_d s)\frac{24}{s(s+1)(s+6)}} = \frac{2\pi}{4\mathrm{K_p}}$$

As a result $\mathrm{K_p} > 0.2$

2. The phase margin is between 40 to 50 degrees

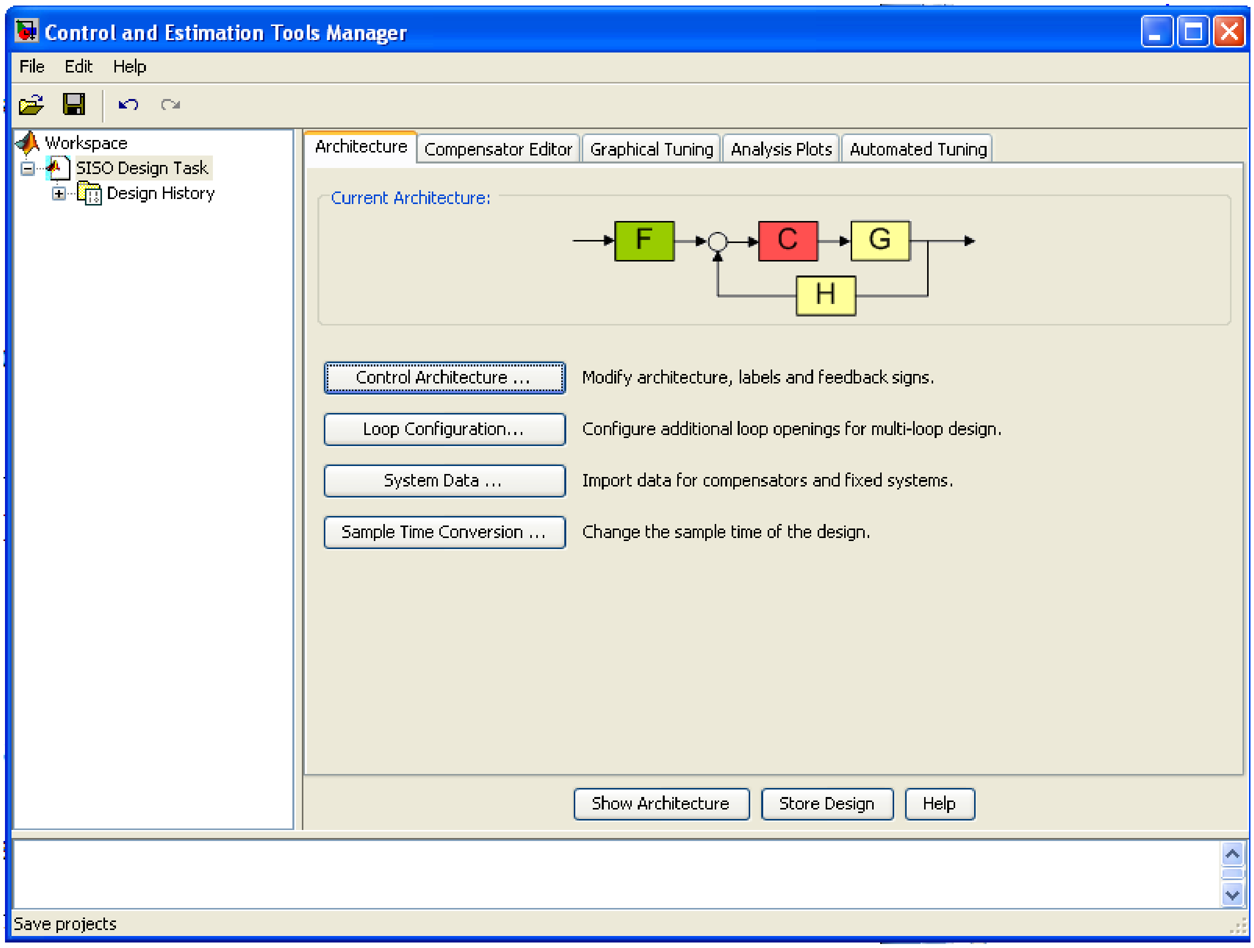
   In this part of the solution, MATLAB sisotool can be very helpful. More detailed instructions on using MATLAB sisotool is presented in the solutions for this particular problem. Similar guidelines could be used for similar questions of this chapter.

SISOTOOL quick instructions:

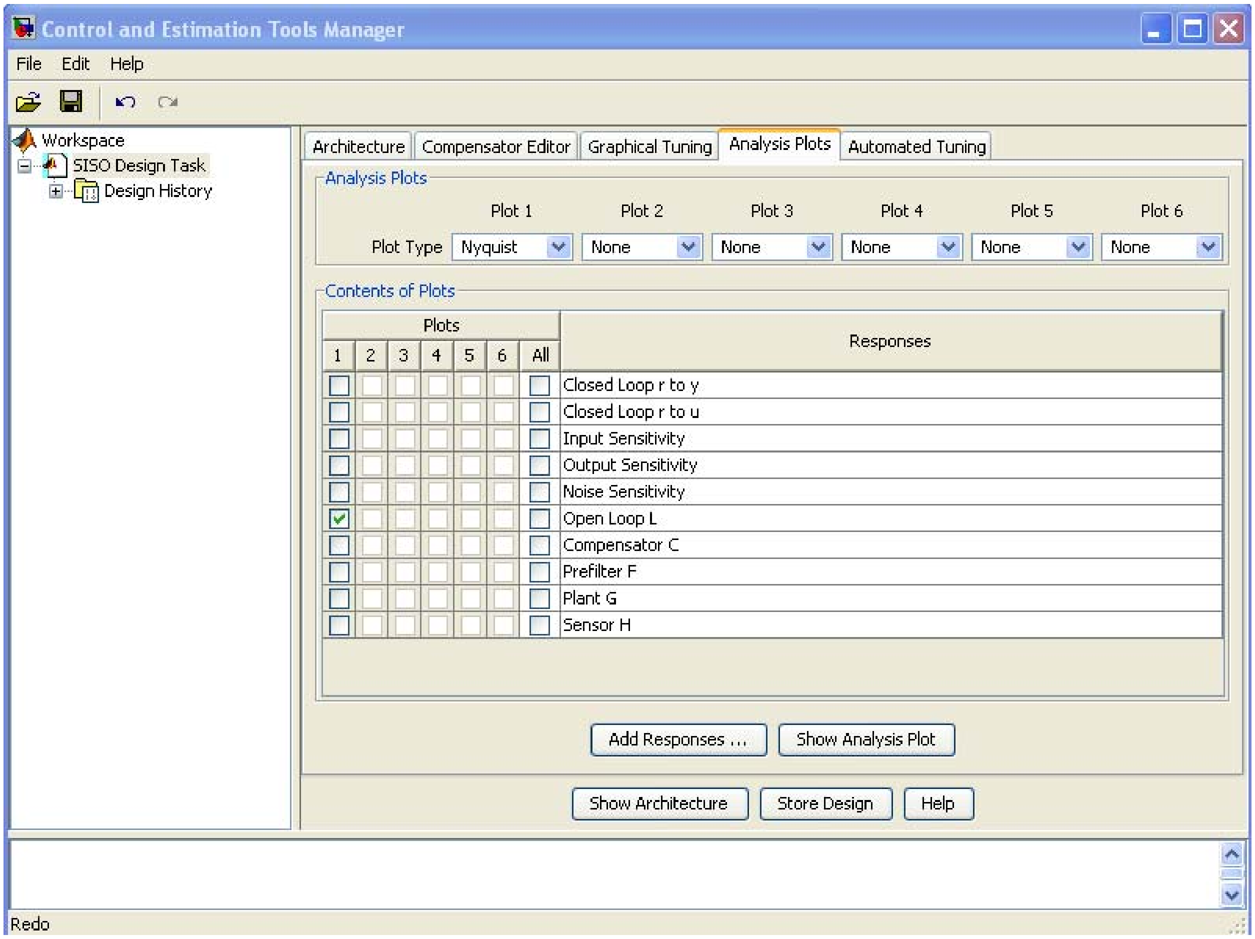
Once opening the sisotool by typing sisotool in MATLAB the command window (or in the “m” code), the following window pops up:

Where you can insert transfer functions for C, G, H, and F, or you can leave some of them as default value (1), by clicking on “System Data”. Once you substitute transfer functions, you will see a graph including the root-locus diagram, a closed loop Bode diagram, and an open loop Bode diagram indicating the Gain Margin and Phase Margin as well.

** You can drag the open-loop bode magnitude diagram up and down to see the effect of gain change on all of the graphs. Sisotool updates all these graph instantly. You can also drag the poles and zeros on the root locus diagram to observe the effect on the other diagrams.

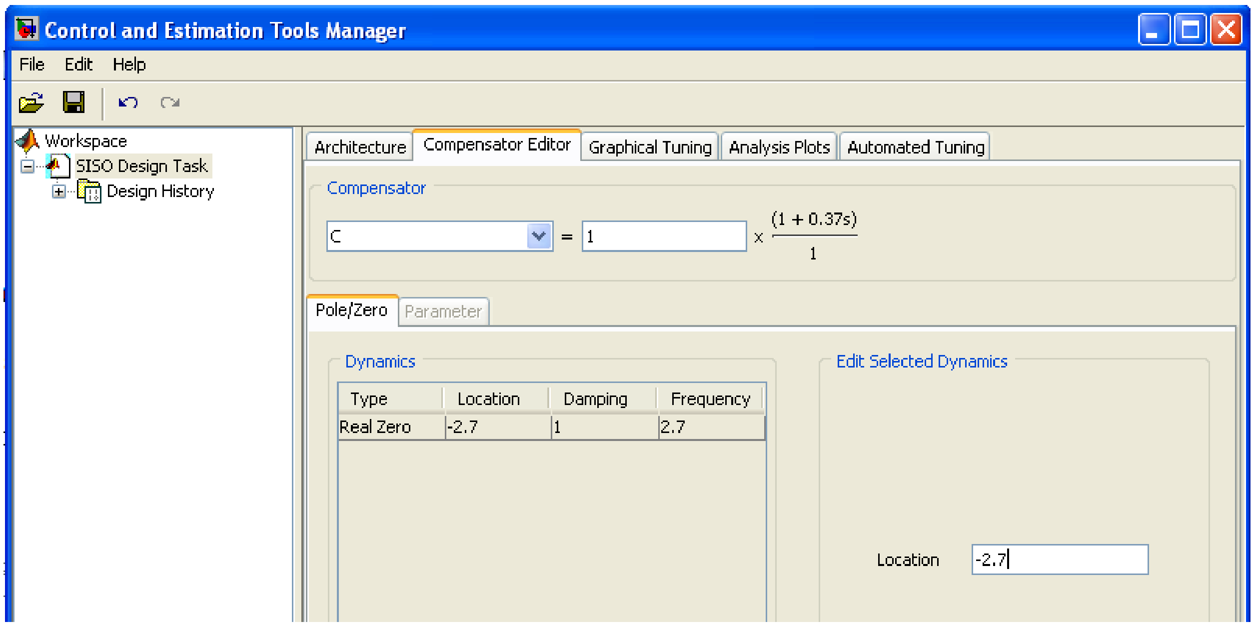


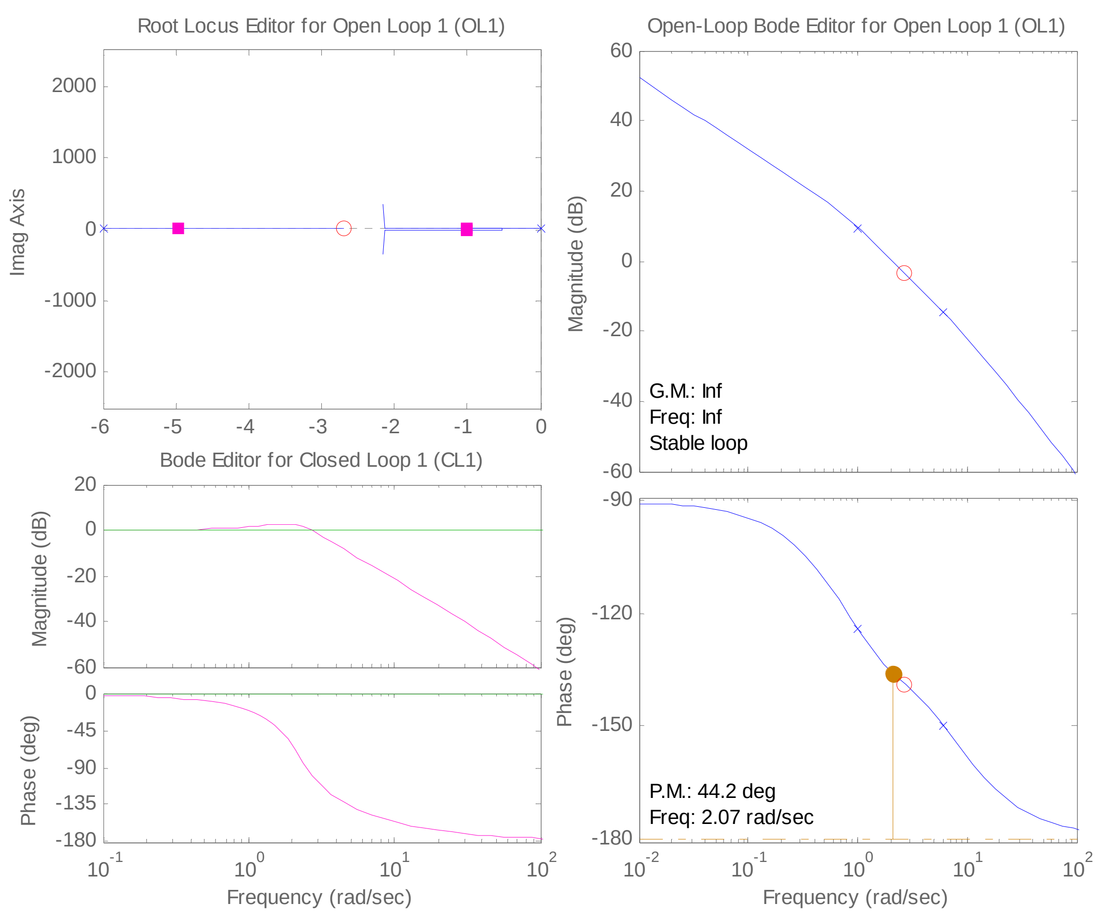
You can also open "Analysis Plots" tab to add other graphs such as Nyquist diagram as shown in the following figure:



In this particular question, you need to add a zero to include the effect of K_d. You can add a zero by using the "Compensator Editor" tab, as shown in the following graph. The last thing you need to do for this problem, is to drag the location of zero and gain in the following diagram (or edit these locations by assigning C gain and Zero location in the "Compensator Editor" tab), so it satisfies the PM of 40 to 50 deg; while gain of C is kept above 0.2 ($K_p \geq 0.2$, from part (a)).

In the following snapshot of "Compensator Editor", C gain or K_p is set to 1, and the zero location is set to -2.7, resulting in 44.2 [deg] phase margin, presented in the following figure.





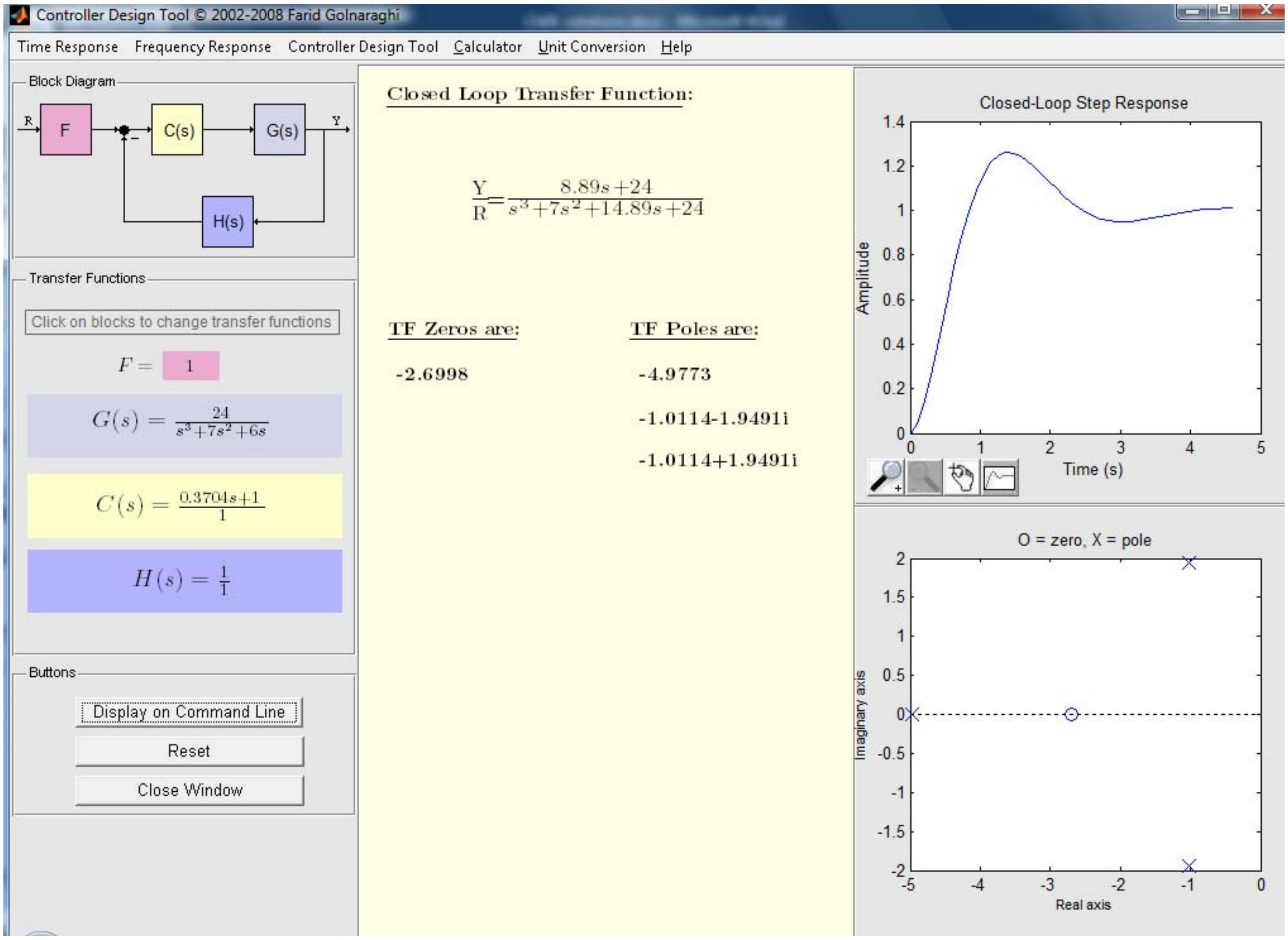
Final answers: $K_p = 1,\ K_d = 0.3704$

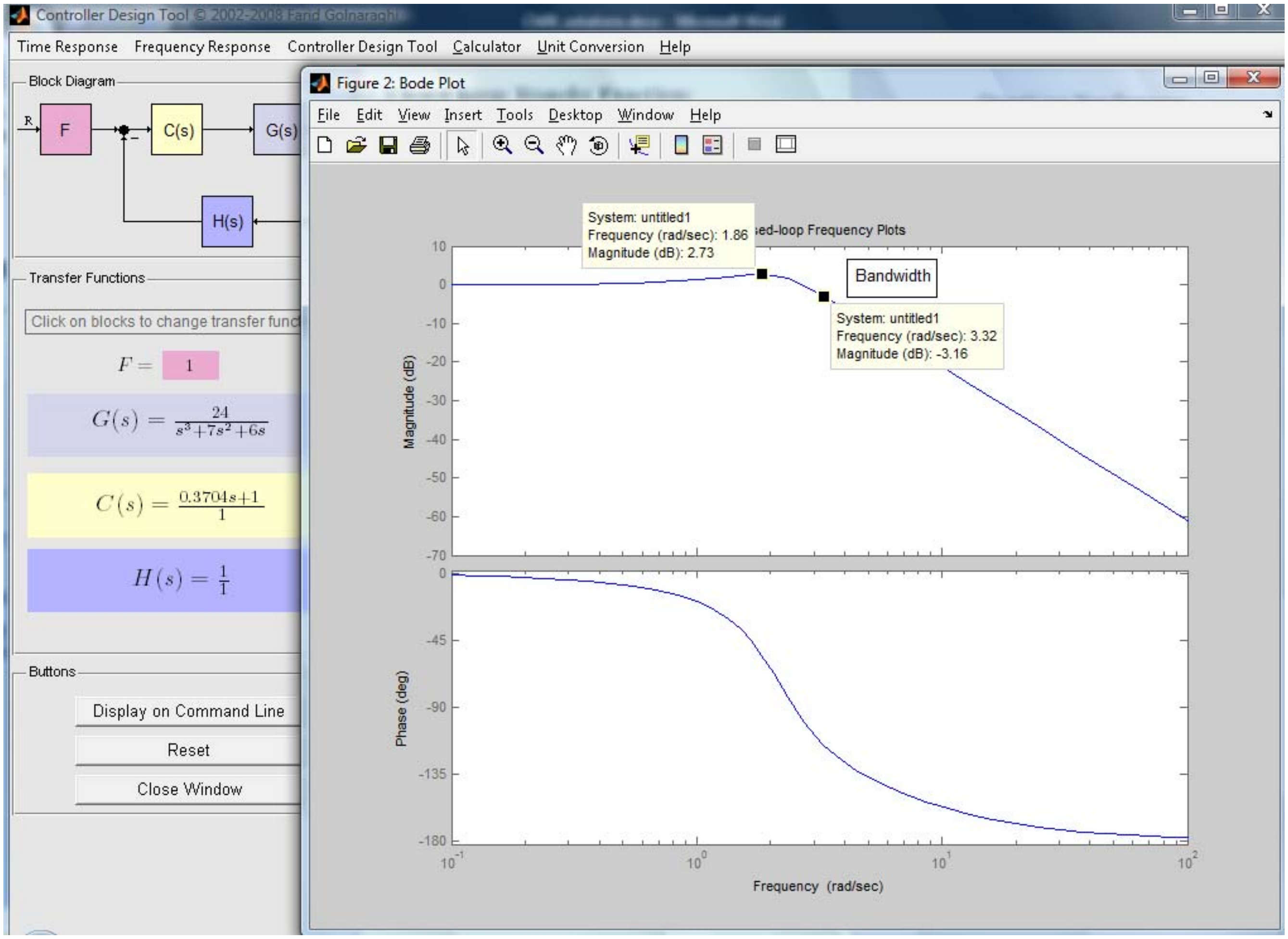
Preliminary MATLAB code for 9-4:

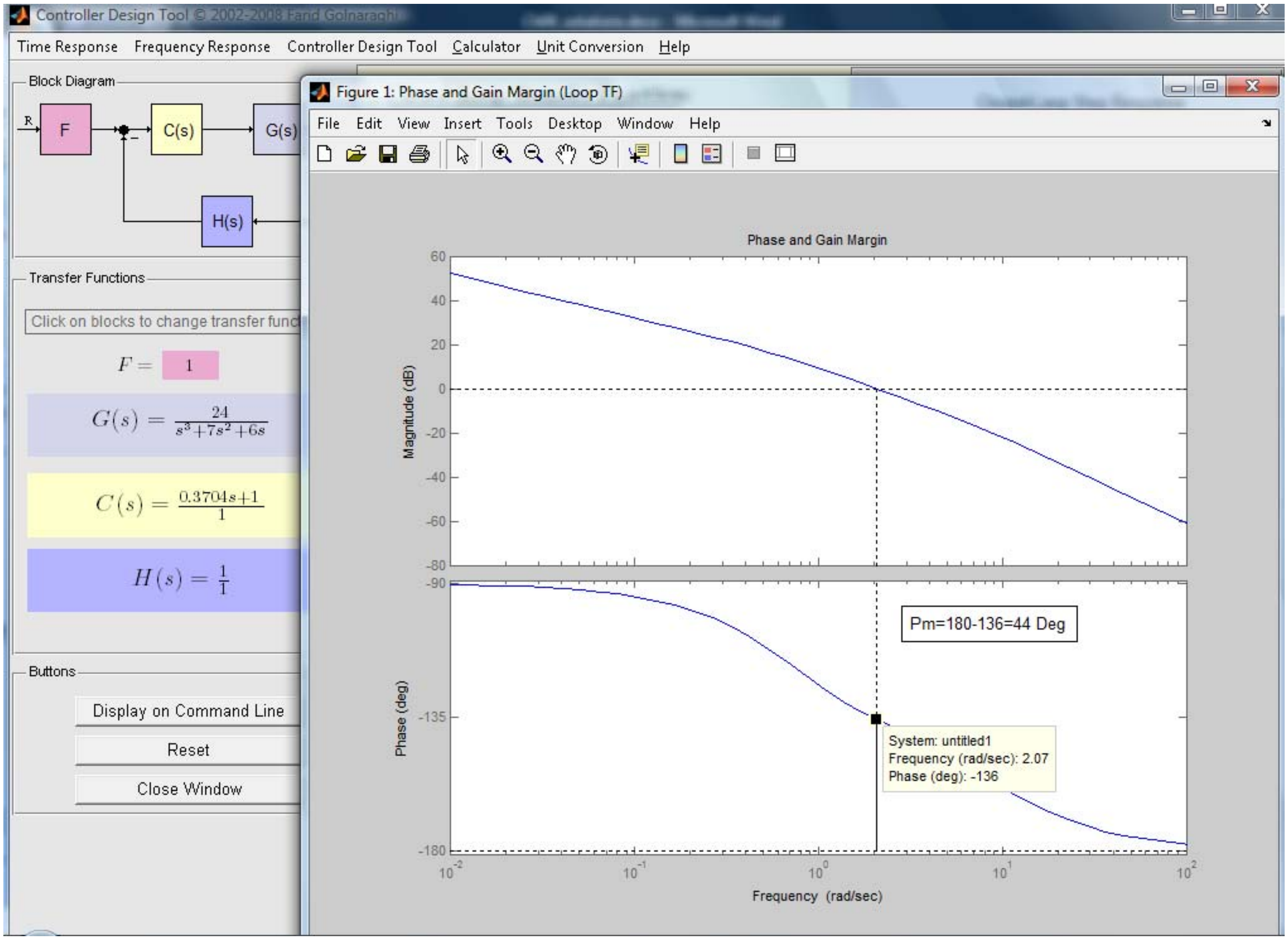
```
%solving for k:

syms kc

omega=1.5

sol=eval(solve('0.25*kc^2=0.7079^2*((-0.25*omega^3+omega)^2+(-
0.375*omega^2+0.5*kc)^2)',kc))


%ploting bode with K=1.0370

s = tf('s')

K=1.0370;

num_G_a= 0.5*K;

den_G_a=s*(0.25*s^2+0.375*s+1);

G_a=num_G_a/den_G_a;

CL_a = G_a/(1+G_a)

BW = bandwidth(CL_a)

bode(CL_a);


sisotool;
```

Alternatively we can use ACSYS.

$K_p = 1,\ K_d = 0.3704$


Controller Design Tool © 2002-2008 Farid Golnaraghi
Time Response
Frequency Response
Controller Design Tool
Calculator
Unit Conversion
Help
Block Diagram
R
F
C(s)
G(s)
Y
H(s)
Transfer Functions
Click on blocks to change transfer functions
F = 1
G(s) = 24/(s^3+7s^2+6s)
C(s) = (0.3704s+1)/1
H(s) = 1/1
Buttons
Display on Command Line
Reset
Close Window
Closed Loop Transfer Function:
Y/R = (8.89s+24)/(s^3+7s^2+14.89s+24)
TF Zeros are:
-2.6998
TF Poles are:
-4.9773
-1.0114-1.9491i
-1.0114+1.9491i
Closed-Loop Step Response
Amplitude
Time (s)
O = zero, X = pole
Imaginary axis
Real axis


Controller Design Tool
Time Response
Frequency Response
Controller Design Tool
Calculator
Unit Conversion
Help
Block Diagram
F
C(s)
G(s)
H(s)
Transfer Functions
Click on blocks to change transfer func
$F = 1$
$G(s) = \frac{24}{s^3+7s^2+6s}$
$C(s) = \frac{0.3704s+1}{1}$
$H(s) = \frac{1}{1}$
Buttons
Display on Command Line
Reset
Close Window
Figure 2: Bode Plot
System: untitled1
Frequency (rad/sec): 1.86
Magnitude (dB): 2.73
Bandwidth
System: untitled1
Frequency (rad/sec): 3.32
Magnitude (dB): -3.16
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)



9-5) (a) **Ramp-error Constant:**

$$K_v = \lim_{s \to 0} s \frac{1000(K_P + K_D s)}{s(s+10)} = \frac{1000 K_P}{10} = 100 K_P = 1000 \qquad \text{Thus} \qquad K_P = 10$$

Characteristic Equation: $s^2 + (10 + 1000K_D)s + 1000K_P = 0$

$$\omega_n = \sqrt{1000K_P} = \sqrt{10000} = 100 \text{ rad/sec} \qquad 2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.5 \times 100 = 100$$

Thus $K_D = \dfrac{90}{1000} = 0.09$

9-5 (b) For $K_v = 1000$ and $\zeta = 0.707$, and from part (a), $\omega_n = 100$ rad/sec,

$$2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.707 \times 100 = 141.4 \qquad \text{Thus} \qquad K_D = \frac{131.4}{1000} = 0.1314$$

(c) For $K_v = 1000$ and $\zeta = 1.0$, and from part (a), $\omega_n = 100$ rad/sec,

$$2\zeta\omega_n = 10 + 1000K_D = 2 \times 1 \times 100 = 200 \qquad \text{Thus} \qquad K_D = \frac{190}{1000} = 0.19$$

9-6) The ramp-error constant:

$$K_v = \lim_{s \to 0} s \frac{1000\left(K_P + K_D s\right)}{s(s+10)} = 100K_P = 10,000 \qquad \text{Thus} \quad K_P = 100$$

The forward-path transfer function is: $G(s) = \dfrac{1000\left(100 + K_D s\right)}{s(s+10)}$

K_D	PM (deg)	GM	M_r	BW (rad/sec)	Max overshoot (%)
0	1.814	∝	13.5	493	46.6
0.2	36.58	∝	1.817	525	41.1
0.4	62.52	∝	1.291	615	22
0.6	75.9	∝	1.226	753	13.3
0.8	81.92	∝	1.092	916	8.8
1.0	84.88	∝	1.06	1090	6.2

The phase margin increases and the maximum overshoot decreases monotonically as K_D increases.

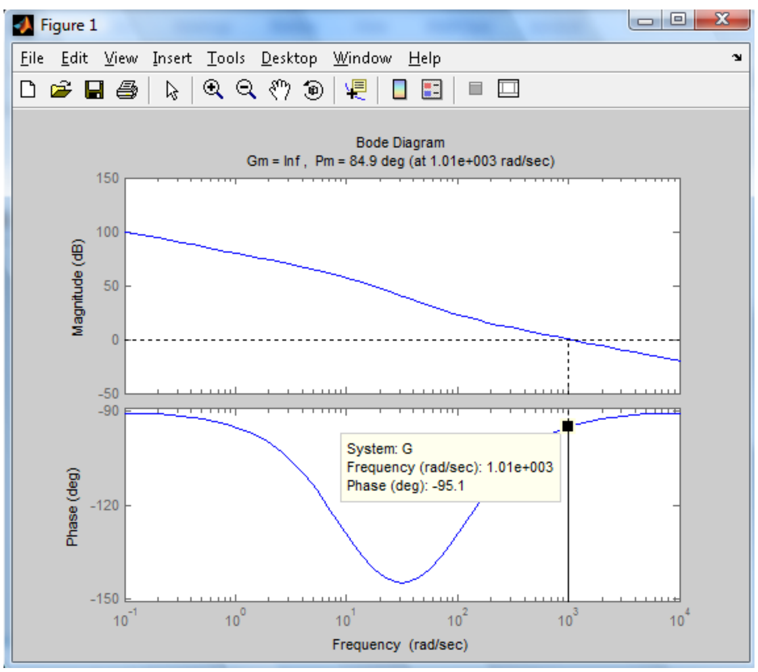
Sample MATLAB CODE for time frequency responses

```
clear all
KD=1.0;
num = [-100/KD];
den = [0 -10];
G=zpk(num,den,1000);
figure(1)
margin(G)
M=feedback(G,1)
figure(2)
step(M);
figure(3)
bode(M)
```

Zero/pole/gain:

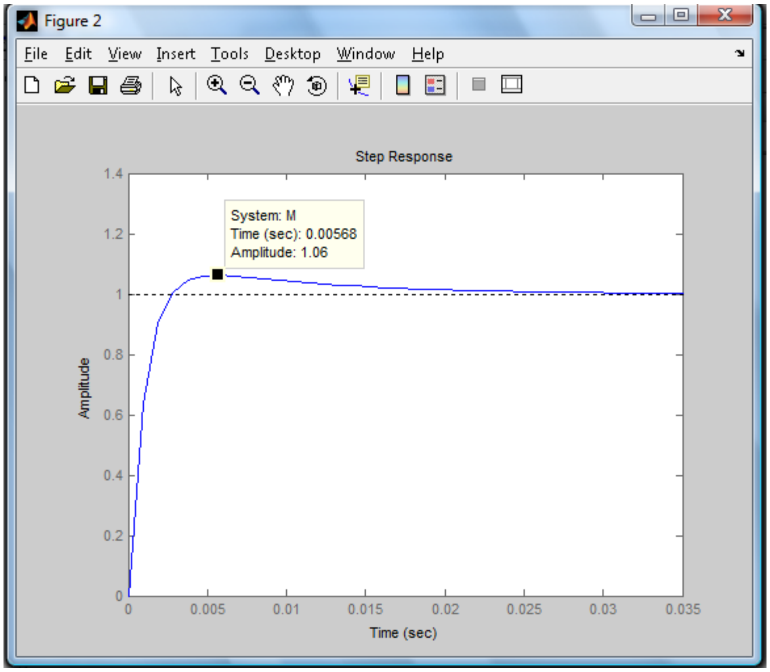
1000 (s+100)

(s+111.3) (s+898.7)



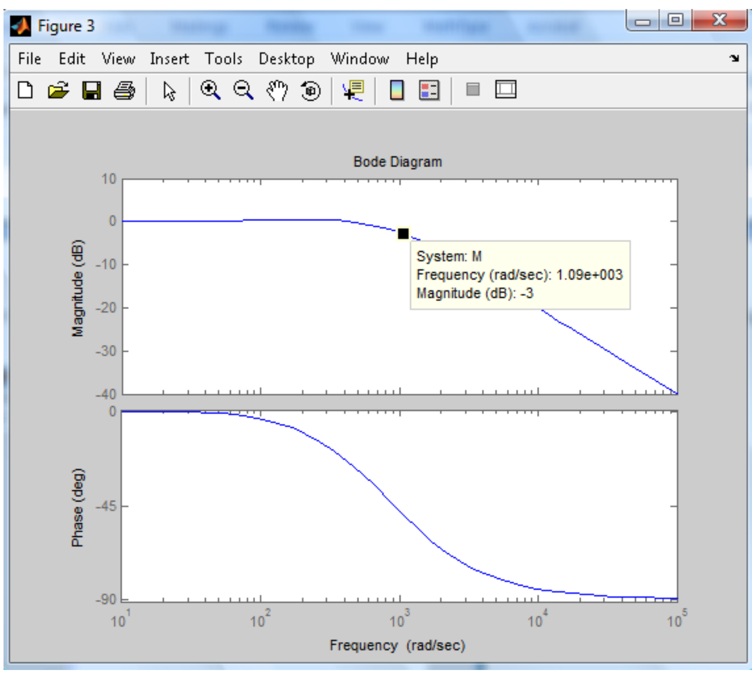
Phase margin is Pm=180-95=85 Deg

Gm=∞



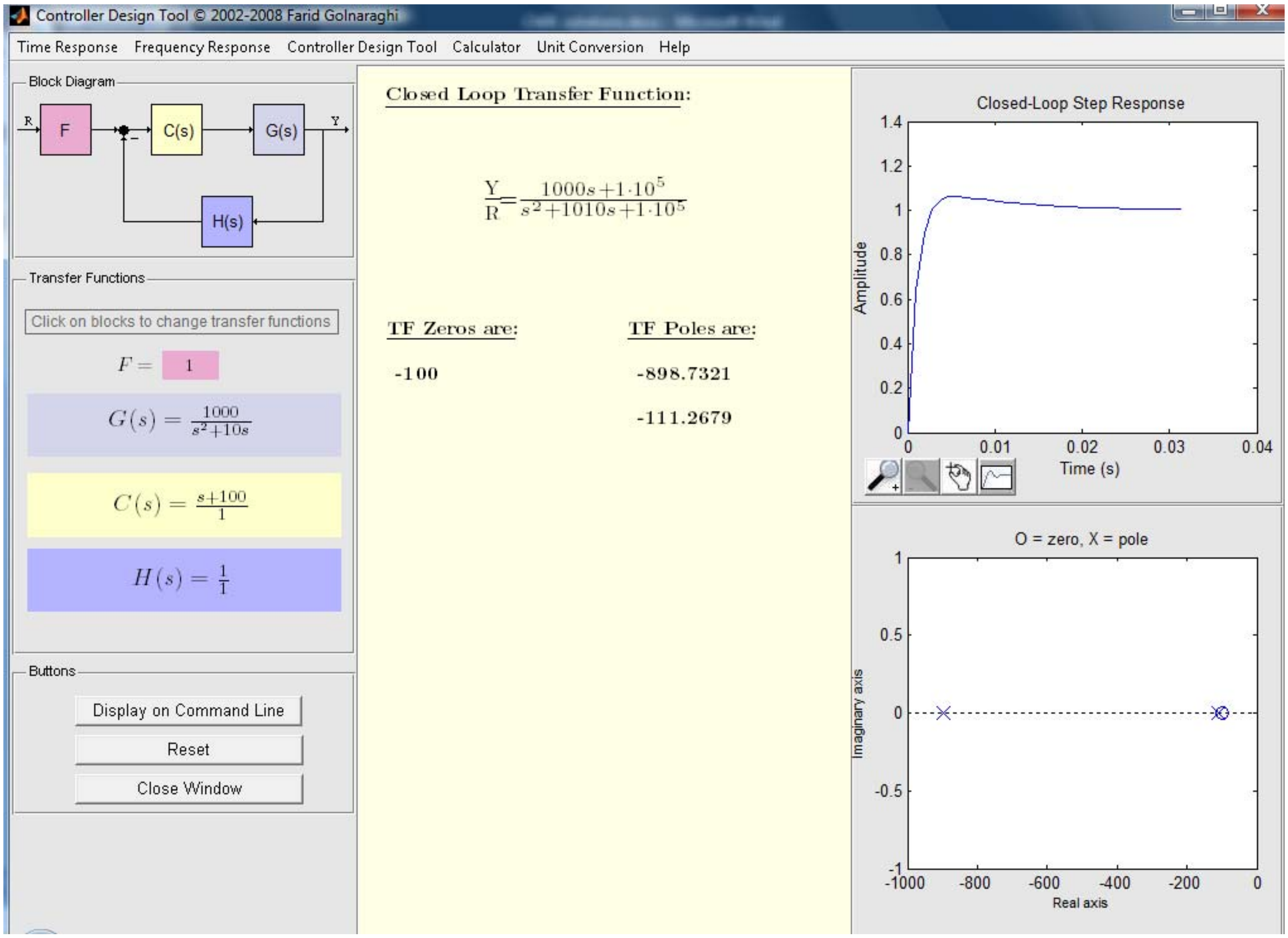
Use the cursor to obtain the PO and tr values.

KD increase results in the minimum overshoot.



Bandwidth is 1090 rads/s.

Use ACSYS to find the same results:



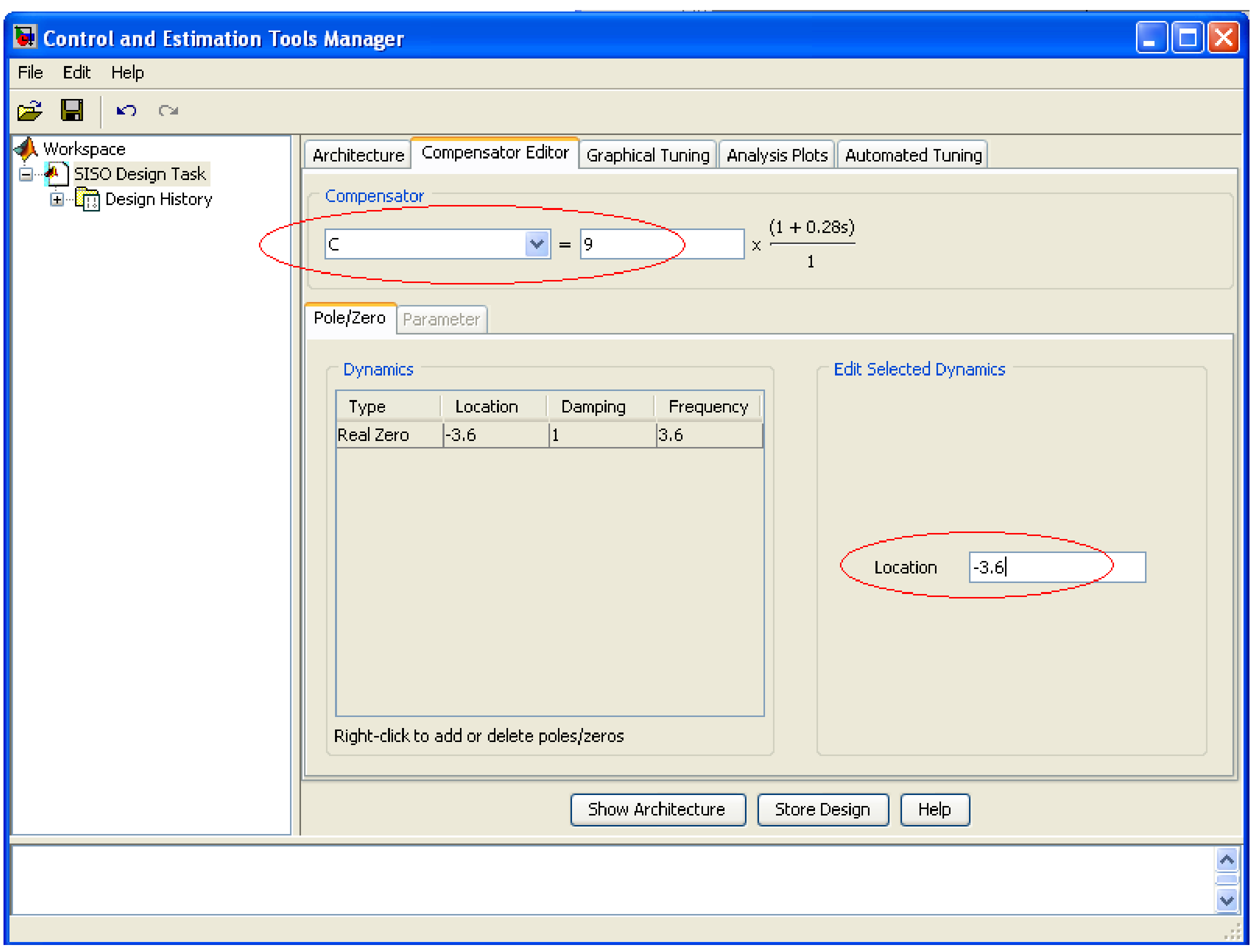
9-7)

PD controller design

The open-loop transfer function of a system is:

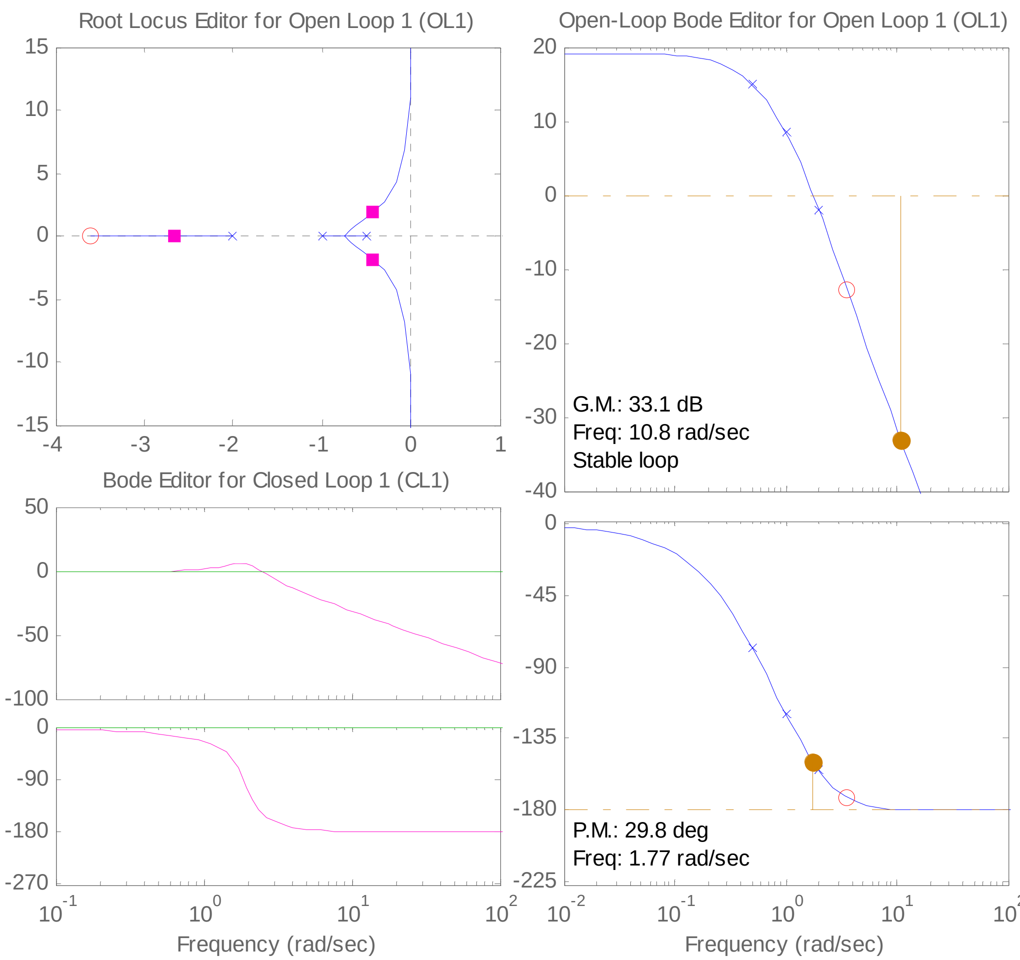
$$G(s)H(s) = \frac{1}{(2s+1)(s+1)(0.5s+1)}$$

The solution is very similar to 9-4. The transfer functions are inserted into sisotool, where another real zero is added to represent the effect of K_d. That is $C(s) = K_p + K_d s = K_p(1 + K_d s / K_p)$, which is called the compensator transfer function in sisotool. The place of real zero is $Z = -K_p / K_d$, and the gain of the compensator is equal to K_p, as noted in the following sisotool window:



By fixing the gain to 9, and starting to change the zero location, PM can be adjusted to above 25 [deg] as required by the question. The current setting has a zero at -3.5, which resulted in 30 [deg] phase margin and 33.1 dB gain margin as seen in the following diagrams.

The design requires $K_p = 9$ **and** $K_d = -K_p / Z = -9 / -3.6 = 2.5$



Preliminary MATLAB code for 9-7:

```
s = tf('s')
Kp = 1
num_GH= Kp*1;
den_GH=(2*s+1)*(s+1)*(0.5*s+1);
```

```
GH=num_GH/den_GH;

CL = GH/(1+GH)


Sisotool
```

9-8: PD controller design: The open-loop transfer function of a system is:

$$G(s)H(s) = \frac{60}{s(0.4s+1)(s+1)(s+6)}$$

(a) Design a PD controller to satisfy the following specifications:

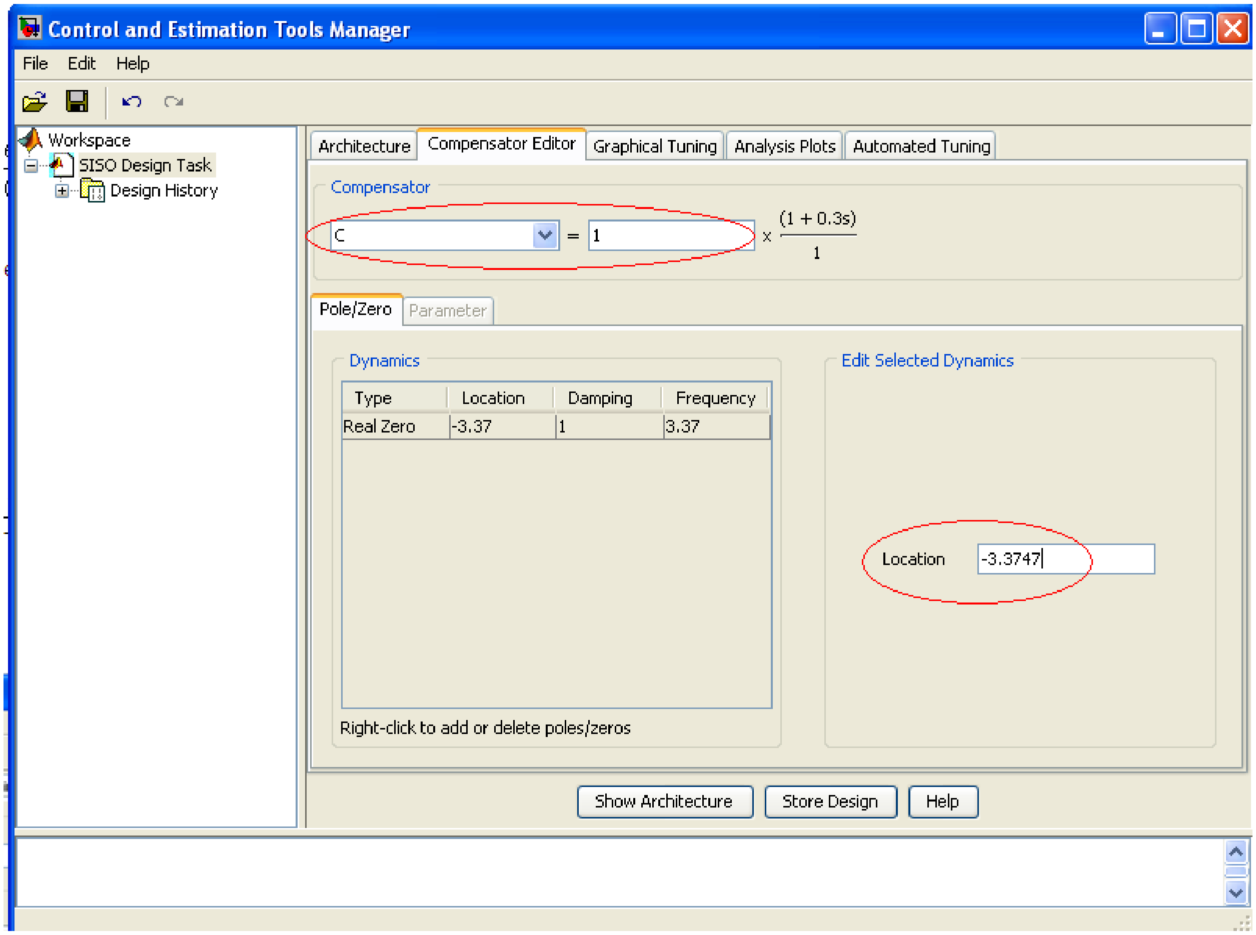
(i) $K_v = 10$

(ii) the phase margin is 45 degrees.

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{60(K_D s + K_p)}{(0.4s+1)(s+1)(s+6)} = 10K_p = 10$$

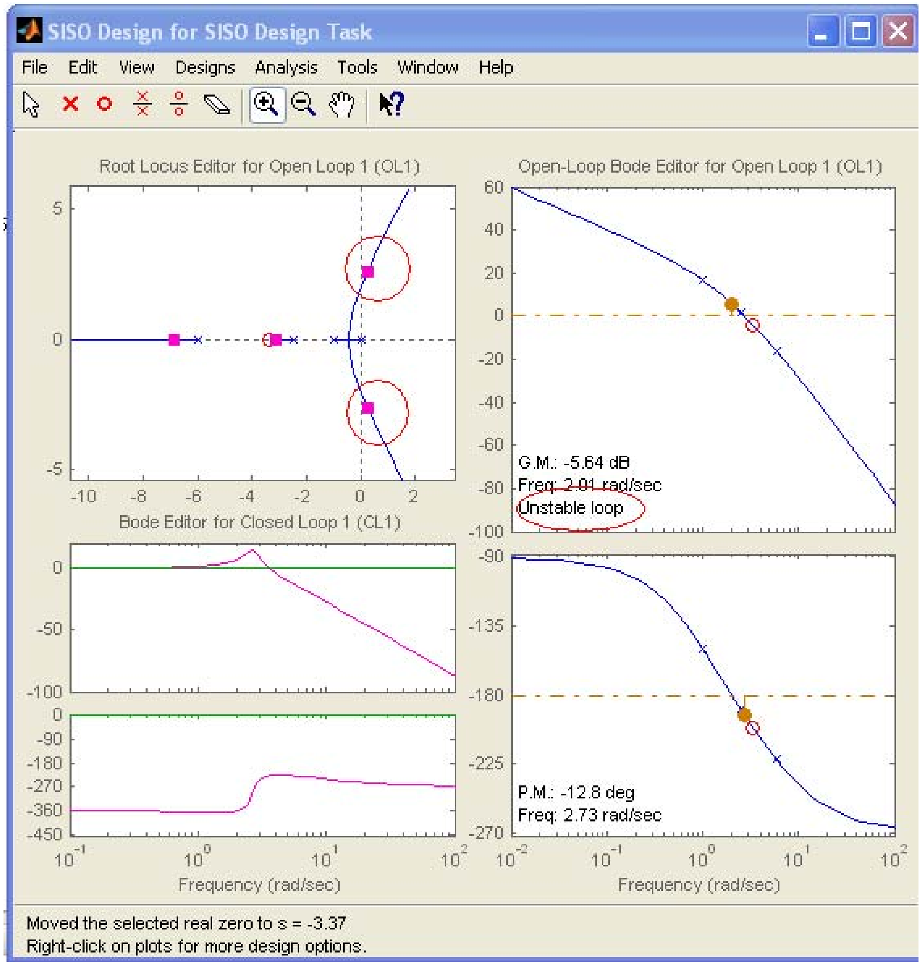
As a result: $K_p = 1$

The rest of the procedure is similar to 9-7:



The transfer functions are inserted into sisotool, where another real zero is added to represent the effect of K_d. That is $C(s) = K_p + K_d s = K_p(1 + K_d s / K_p)$, which is called the compensator transfer function in sisotool. The place of real zero is $Z = -K_p / K_d$, and the gain of the compensator is equal to K_p, as noted in the sisotool window:

$K_p = 1$ fixed to 1, and zero location was changed in the entire real axis. However, 2 of the closed loop poles remained in the right hand side of S plane in the root locus diagram, indicating instability for all Kd values. **Solution for $K_v = 10$ with PD controller and PM=45 [deg] does not exist.** Unstable close loop poles are indicated in the root locus diagram of the following figure:



9-8)

Preliminary MATLAB code for 9-8:

```
s = tf('s')

Kp = 1

num_GH= Kp*60;

den_GH=s*(0.4*s+1)*(s+1)*(s+6);

GH=num_GH/den_GH;

CL = GH/(1+GH)


sisotool
```

9-9) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{4500K\left(K_D + K_P s\right)}{s(s+361.2)}$$

Ramp Error Constant: $K_v = \lim_{s\to 0} sG(s) = \frac{4500KK_P}{361.2} = 12.458KK_P$

$e_{ss} = \frac{1}{K_v} = \frac{0.0802}{KK_P} \leq 0.001$ Thus $KK_P \geq 80.2$ Let $K_P = 1$ and $K = 80.2$

Attributes of Unit-step Response:

K_D	t_r (sec)	t_s (sec)	Max Overshoot (%)
0	0.00221	0.0166	37.1
0.0005	0.00242	0.00812	21.5
0.0010	0.00245	0.00775	12.2
0.0015	0.0024	0.0065	6.4
0.0016	0.00239	0.00597	5.6
0.0017	0.00238	0.00287	4.8
0.0018	0.00236	0.0029	4.0
0.0020	0.00233	0.00283	2.8

Select $K_D \geq 0.0017$

(b) BW must be less than 850 rad/sec.

K_D	GM	PM (deg)	M_r	BW (rad/sec)
0.0005	$\propto$	48.45	1.276	827
0.0010	$\propto$	62.04	1.105	812
0.0015	$\propto$	73.5	1.033	827
0.0016	$\propto$	75.46	1.025	834
0.0017	$\propto$	77.33	1.018	842
0.00175	$\propto$	78.22	1.015	847
0.0018	$\propto$	79.07	1.012	852

Select $K_D \cong 0.00175$. A larger K_D would yield a BW larger than 850 rad/sec.

9-10)

The forward-path Transfer Function: *N* = 20

$$G(s) = \frac{200\left(K_P + K_D s\right)}{s(s+1)(s+10)}$$

To stabilize the system, we can reduce the forward-path gain. Since the system is type 1, reducing the gain does not affect the steady-state liquid level to a step input. Let $K_P = 0.05$

$$G(s) = \frac{200\left(0.05 + K_D s\right)}{s(s+1)(s+10)}$$

Unit-step Response Attributes:

K_D	t_s (sec)	Max Overshoot (%)
0.01	5.159	12.7
0.02	4.57	7.1
0.03	2.35	3.2
0.04	2.526	0.8
0.05	2.721	0
0.06	3.039	0
0.10	4.317	0

When $K_D = 0.05$ the rise time is 2.721 sec, and the step response has no oversho

9-11)

(a) For $e_{ss} = 1$,

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{200\left(K_P + K_D s\right)}{s(s+1)(s+10)} = 20K_P = 1 \qquad \text{Thus} \quad K_P = 0.05$$

Forward-path Transfer Function:

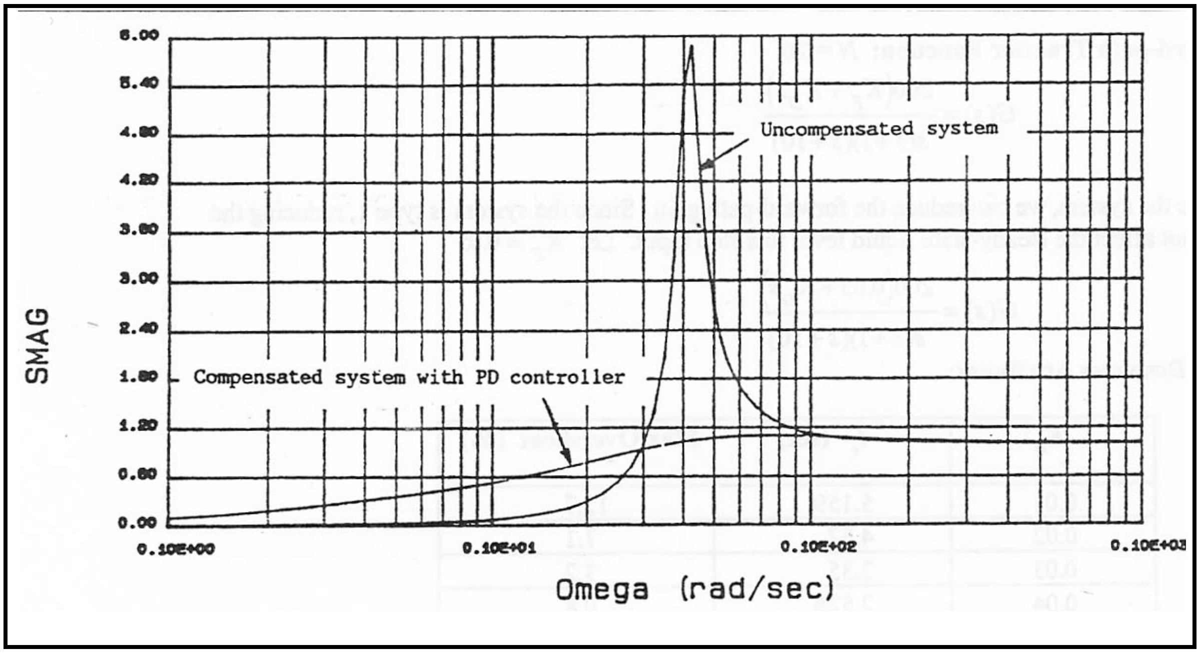
$$G(s) = \frac{200\left(0.05 + K_D s\right)}{s(s+1)(s+10)}$$

Attributes of Frequency Response:

K_D	PM (deg)	GM (deg)	M_r	BW (rad/sec)
0	47.4	20.83	1.24	1.32
0.01	56.11	∝	1.09	1.24
0.02	64.25	∝	1.02	1.18
0.05	84.32	∝	1.00	1.12
0.09	93.80	∝	1.00	1.42
0.10	93.49	∝	1.00	1.59
0.11	92.71	∝	1.00	1.80
0.20	81.49	∝	1.00	4.66
0.30	71.42	∝	1.00	7.79
0.50	58.55	∝	1.03	12.36

For maximum phase margin, the value of K_D is 0.09. PM = 93.80 deg. GM = ∝, $M_r = 1$, and BW = 1.42 rad/sec.

(b) Sensitivity Plots:



The PD control reduces the peak value of the sensitivity function $\left|S_G^M(j\omega)\right|$

9-12)

PD controller design: The open loop transfer function of a system is:

$$G(s)H(s) = \frac{100}{s(0.1s+1)(0.02s+1)}$$

Design the PD controller so that the phase margin is greater than 50 degrees and the BW is greater than 20 rad/sec.

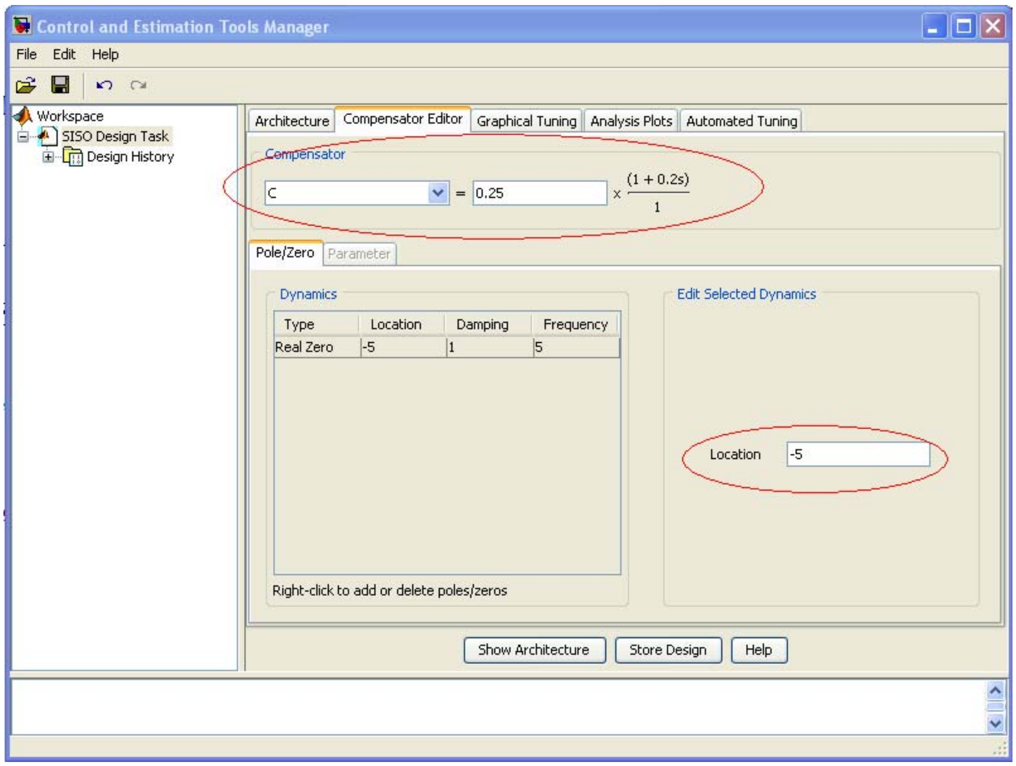
The transfer functions are generated and imported in sisotool as in 9-4:

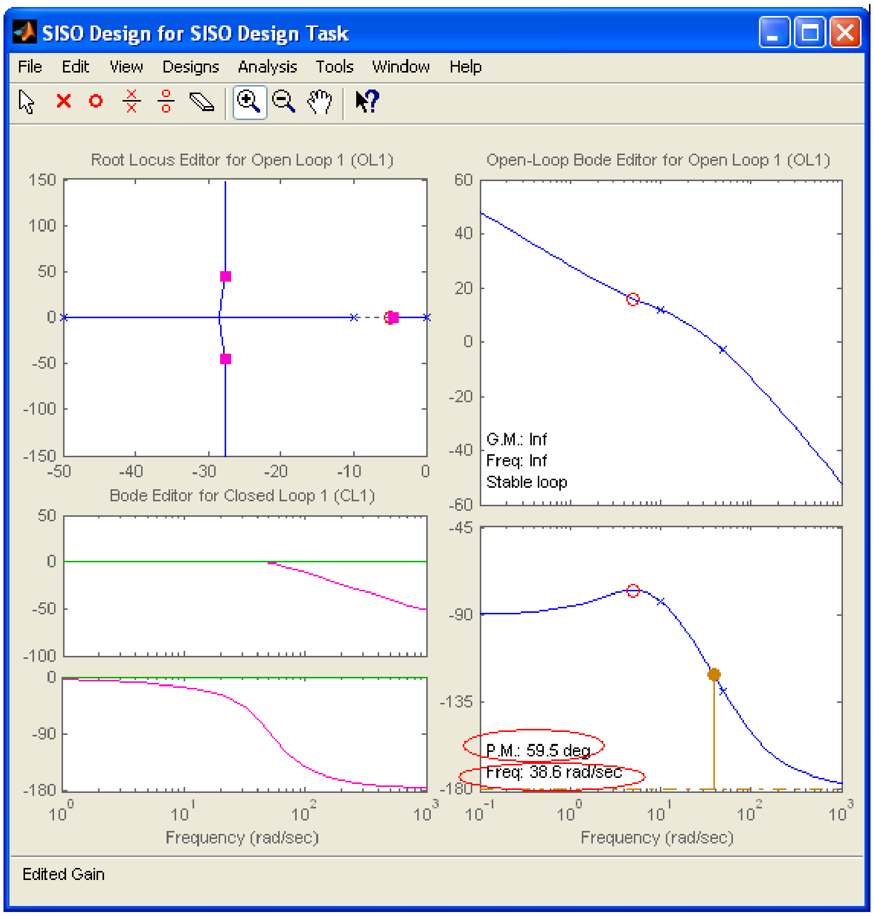
MATLAB code:

```
s = tf('s')

Kp = 1
num_GH= Kp*100;
den_GH=s*(0.1*s+1)*(0.02*s+1);
GH=num_GH/den_GH;
CL = GH/(1+GH)

sisotool
```

Following similar steps in 9-4, the loop transfer functions are inserted into sisotool. Another real zero is added to represent the effect of K_d. That is $C(s) = K_p + K_d s = K_p(1 + K_d s / K_p)$, which is called the compensator transfer function in sisotool. The place of real zero is $Z = -K_p / K_d$, and the gain of the compensator is equal to K_p. The zero location and K_p gain were changed interactively in sisotool until the desired PM (59.5 [deg]) and BW is achieved. Following figures shows this PM at cross over frequency of 38.6 rad/sec, which insures BW of higher than 38.6 rad/sec (as the bandwidth is @ -3dB rather than 0 DB, i.e. bandwidth occurs at higher frequency compared to cross over frequency).

Final possible answer: $K_p = 0.25$ **and** $K_d = -K_p / Z = -0.25 / -5 = 0.05$




SISO Design for SISO Design Task
File Edit View Designs Analysis Tools Window Help
Root Locus Editor for Open Loop 1 (OL1)
Open-Loop Bode Editor for Open Loop 1 (OL1)
G.M.: Inf
Freq: Inf
Stable loop
Bode Editor for Closed Loop 1 (CL1)
P.M.: 59.5 deg
Freq: 38.6 rad/sec
Frequency (rad/sec)
Frequency (rad/sec)
Edited Gain

9-13)

Lead compensator design:

$$G(s)H(s) = \frac{1000K}{s(0.2s+1)(0.005s+1)}$$

Design a compensator such that the steady state error to the unit step input is less than 0.01 and the closed loop damping ratio ζ>0.4.

The transfer functions are generated and imported in sisotool as in 9-4:

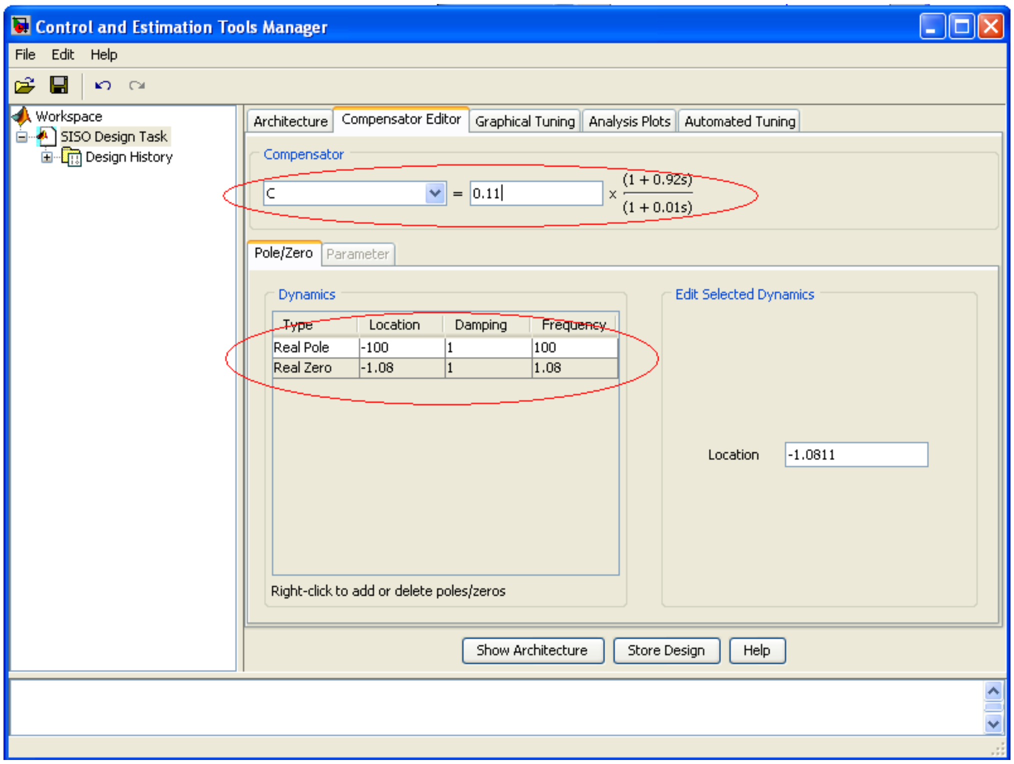
$e_{ss} = \frac{1}{1+1000K_p} < 0.01 \Rightarrow 1000K_p > 101 \Rightarrow$ Therefore, K_p is selected as 150: (K_p =0.11)

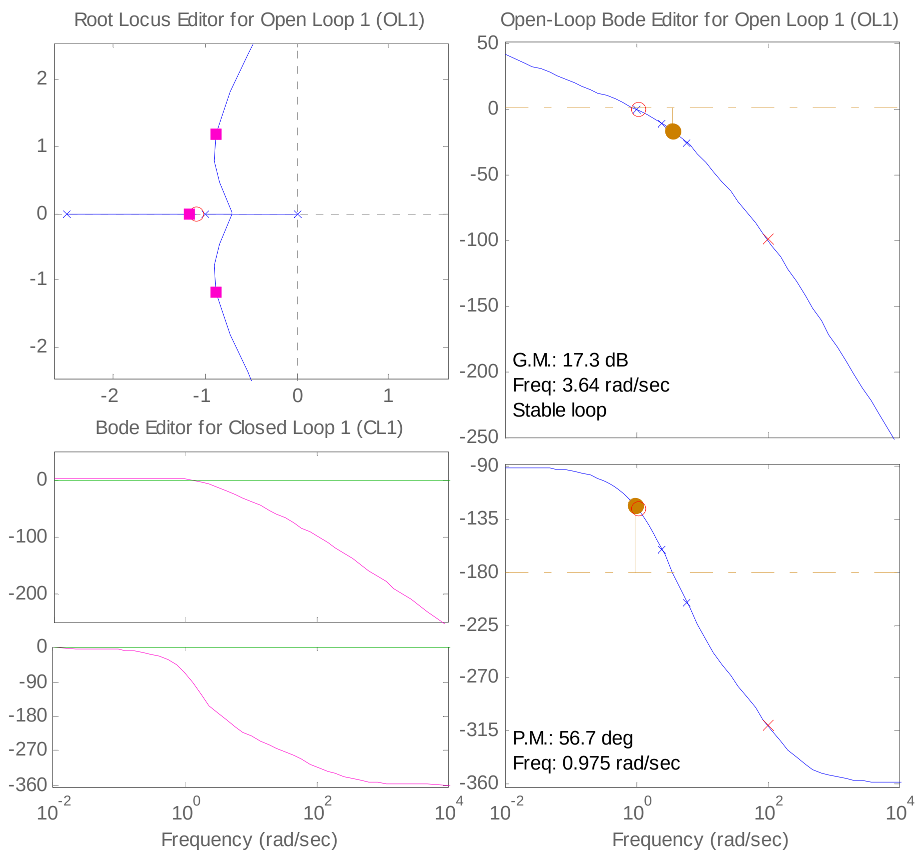
To achieve the required damping ratio, the poles of the closed loop system are placed with an angle of less than ArcCos(ζ=0.4), in the root locus diagram of sisotool. This is done by iteratively change the location of poles and zeros of a lead compensator and setting K_p =0.11. The pole and zero (which perform as a lead compensator when the pole is further away from zero to the left) are inserted in sisotool as explained in 9-4. The lead compensator will introduce some phase lead at lower frequencies about the zero location which improves the closed loop response in terms of damping and phase margin. Following is the chosen location for lead compensator pole and zero:

Pole @ -100 rad/sec

Zero @ -1.08 rad/sec

Which resulted in smallest angles of dominant pole locations (the ones closer to imaginary axis) with the real axis. This small angle means higher damping of the poles as ζ=ArcCos(pole's angle with real axis).


Control and Estimation Tools Manager
File Edit Help
Workspace
SISO Design Task
Design History
Architecture
Compensator Editor
Graphical Tuning
Analysis Plots
Automated Tuning
Compensator
C = 0.11 × (1 + 0.92s) / (1 + 0.01s)
Pole/Zero
Parameter
Dynamics
Type Location Damping Frequency
Real Pole -100 1 100
Real Zero -1.08 1 1.08
Edit Selected Dynamics
Location -1.0811
Right-click to add or delete poles/zeros
Show Architecture
Store Design
Help


Root Locus Editor for Open Loop 1 (OL1)
Open-Loop Bode Editor for Open Loop 1 (OL1)
G.M.: 17.3 dB
Freq: 3.64 rad/sec
Stable loop
Bode Editor for Closed Loop 1 (CL1)
P.M.: 56.7 deg
Freq: 0.975 rad/sec
Frequency (rad/sec)
Frequency (rad/sec)

In this particular case, closed loop complex poles can be observed in the shown root locus diagram at about -0.8±1.2j. This corresponds to damping of about:

cos(atan(1.2/0.8))=0.554 → $\zeta \approx 0.55$

Preliminary MATLAB code for 9-13:

```
s = tf('s')
Kp = 1
num_GH= Kp*60;
den_GH=s*(0.4*s+1)*(s+1)*(s+6);
GH=num_GH/den_GH;
CL = GH/(1+GH)

figure(1)
margin(CL)

sisotool
```

9-14)

PD controller designed for a maximum overshoot and a maximum steady state error

$$e_{ss} = \frac{1}{\lim_{s\to 0} sG_c(s)G(s)} \leq -0.005$$

Therefore:

$$\frac{1}{250K_p} \leq 0.005 \Rightarrow K_p > \frac{1}{0.005 * 250} \Rightarrow K_p > \frac{4}{5}$$

Let $K_p = 1$, then:

$$M_p = \exp(-\frac{\pi\xi}{\sqrt{1-\xi^2}}) < 0.20 \Rightarrow \xi > 0.45$$

Let $\zeta = 0.6$; then:

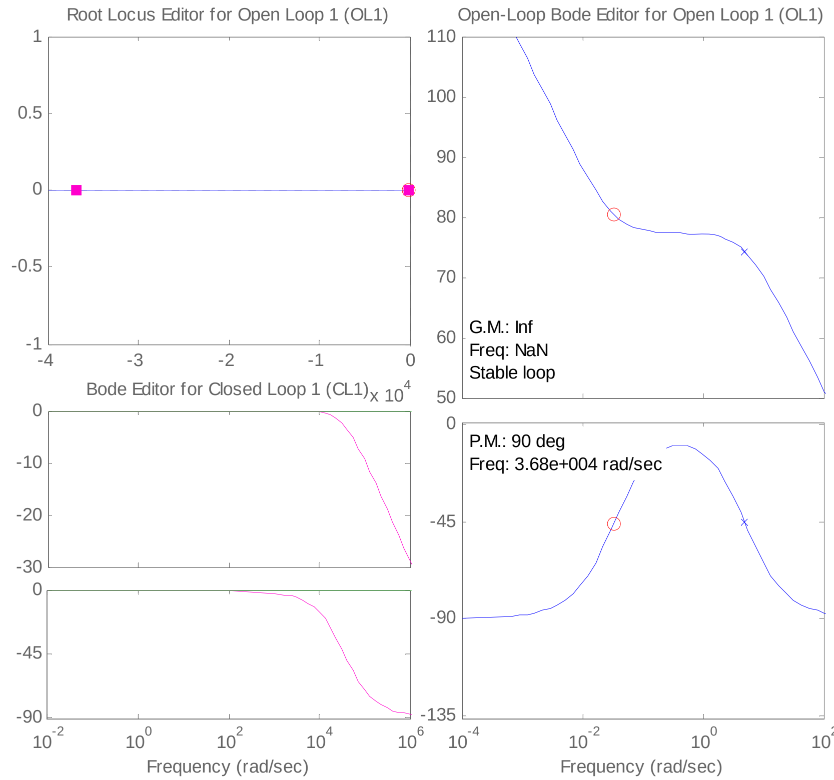
$$\frac{Y(s)}{X(s)} = \frac{250(K_D s + 1)}{0.2s^2 + (250K_D + 1) + 250} = \frac{1250(K_D s + 1)}{s^2 + (1250K_D + 5)s + 1250}$$

Accordingly, $\omega_n^2 = 1250$, or $\omega_n = 35.35$. therefore,

$$(1250K_D + 5) = 2\,\xi\omega_n \Rightarrow 1250K_D = (2)(0.6)(0.35)$$

which gives: $K_D \approx 0.034$

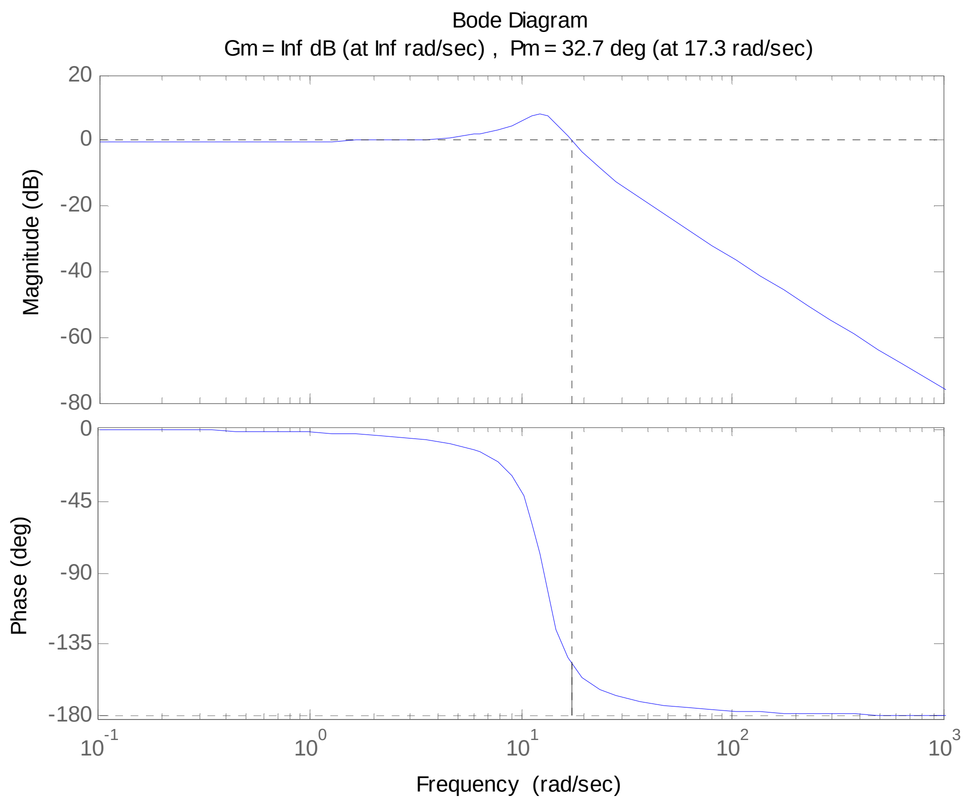
The design characteristics can be observed in the diagram below:



9-15)

Lead compensator controller design

The bode diagram of the system without lead compensator is shown below:



Indicating a PM of 32.7 [deg]. To reach 45 [deg] phase margin, additional 12.3 [deg] phase lead is needed. At ω = 17.3 rad/s crossover frequency, if Φ in = 12.3 then

$$r = \frac{1 - \sin \Phi_{in}}{1 + \sin \Phi_{in}} = 0.6488$$

As $\omega = \frac{1}{\tau\sqrt{r}} = 17.3$, then τ = 0.0718

Since the gain is lowered by $\left|\frac{r(1+\tau j\omega)}{1+r\tau j\omega}\right|_{\omega=17.3} = 1.2838$

A gain compensator with gain of 0.7789 is required, where,

$$G_c(s) = \frac{Kr(\tau s + 1)}{r\tau s + 1}$$

9-16) If r = 0.1 ⇨ $\Phi_m = \sin^{-1}\frac{1-r}{1+r} \approx 55$

$\Phi_m(\omega) = \tan^{-1}\omega\tau - \tan^{-1} r\omega\tau$ ⇨ $\Phi_m(17.3) = \tan^{-1} 17.3\tau - \tan^{-1} 1.73\tau = 12.3$

then from trial and error we found τ = 0.014088 and required gain would be 9.7185

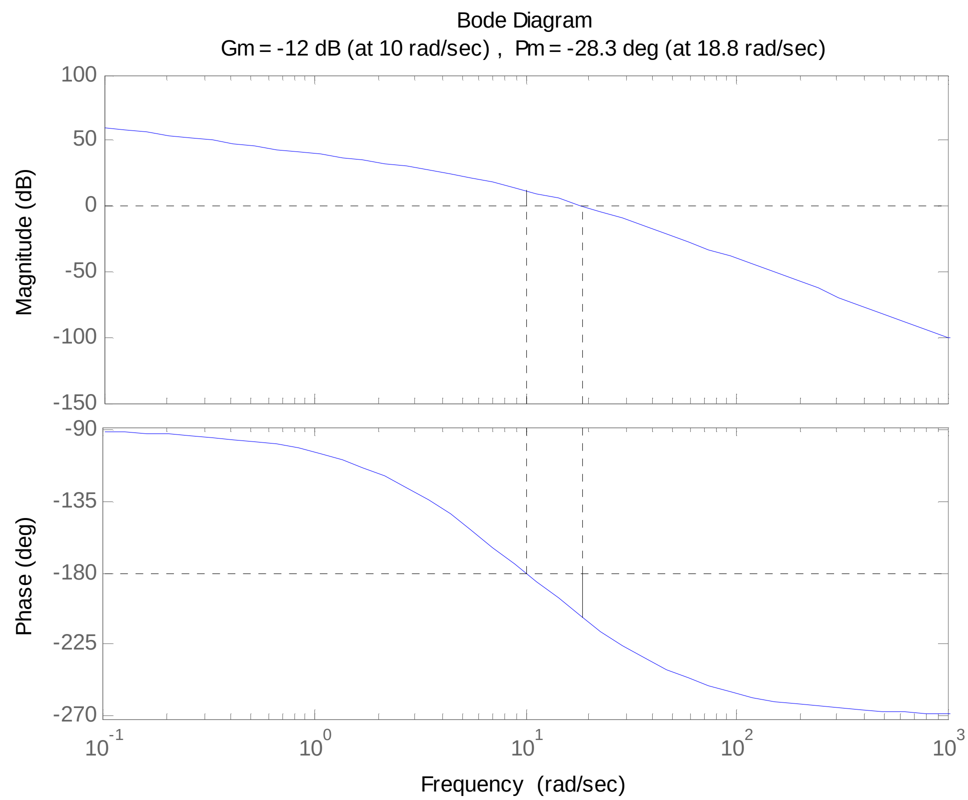
9-17) $K_v = \lim_{s\to 0} sG_i(s)G(s) = 100K \geq 100$ ⇨ $K \geq 1$

First plot the bode diagram of uncompensated system when K = 1

Bode diagram for Loop T.F. is included for K=1

MATLAB CODE:

```
s = tf('s')

Kp = 1
num_GH= Kp*100;
den_GH=s*(0.2*s+1)*(0.05*s+1);
GH=num_GH/den_GH
lag_tf=(s/2+1)/(s/0.2+1)
lead_tf=(s/4+1)/(s/50+1)
LL=lag_tf*lead_tf
OL=GH*LL
CL =OL/(1+OL);

figure(1)
margin(GH)
figure(2)
margin(OL)
figure(3)
bode(CL)
grid on;
```



The bode diagram with K=1 shows -28 deg PM at 18.8 rad/sec.

According to the requirements the gain must be greater than $\frac{1}{0.004}$ or 250 for $\omega_1 \leq 0.2 rad/s$ and must be less than $\frac{1}{100}$ or 0.01 for $\omega_2 \geq 200\ rad/s$

In order to achieve above requirements, a lead-lag compensator will be appropriate.

Using a lag compensator will allow lower gain at frequencies less then ω_1 and using a lead compensator will allow to increase phase margin

For the lag compensator, $\alpha = 1/10$ is chosen to boost the low frequency amplitude

$$Lag = \frac{1+\alpha Ts}{1+Ts} = \frac{s/2+1}{s/0.2+1}$$

In order to introduce some phase lead to obtain the require PM, a lead compensator is also designed as:

$$T = \frac{1}{\sqrt{10}\omega} = \frac{1}{28.3\sqrt{10}} = 0.0112$$

Where ω is overlaid with the crossover frequency (28.3 rad/sec) for applying the maximum phase lead at this frequency. The Lead compensator T.F. will be as follows:

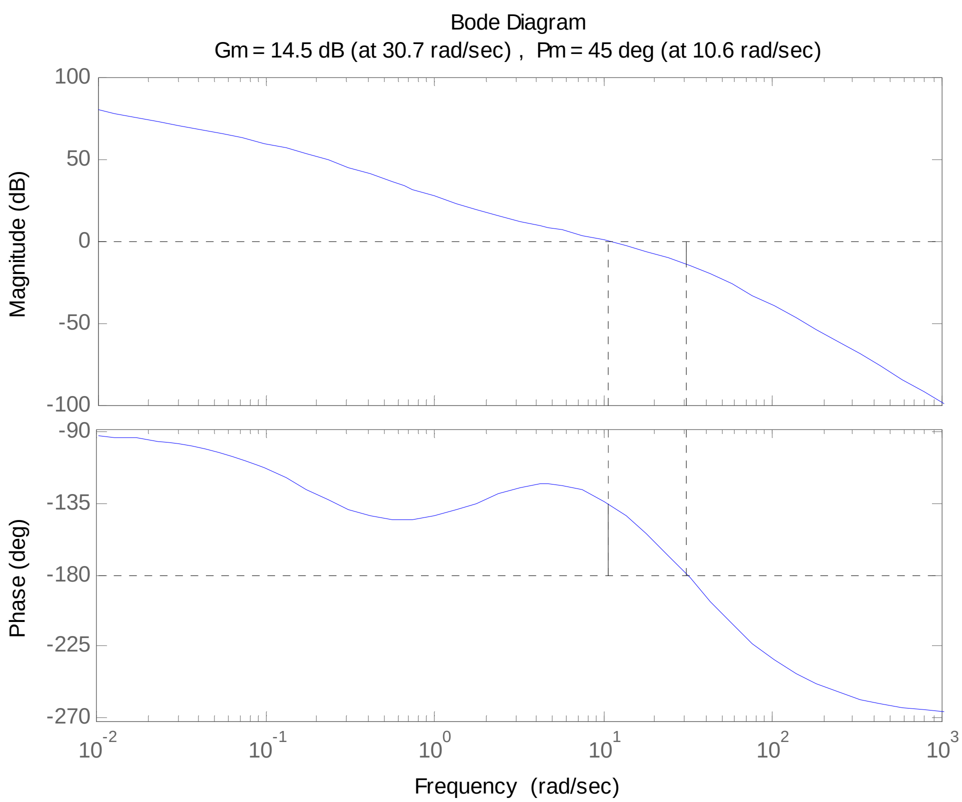
$$Lead = \frac{1+\alpha Ts}{1+Ts} = \frac{s/5+1}{s/50+1}$$

Resulting in the following Bode diagram for the compensated system, showing 44 deg PM:

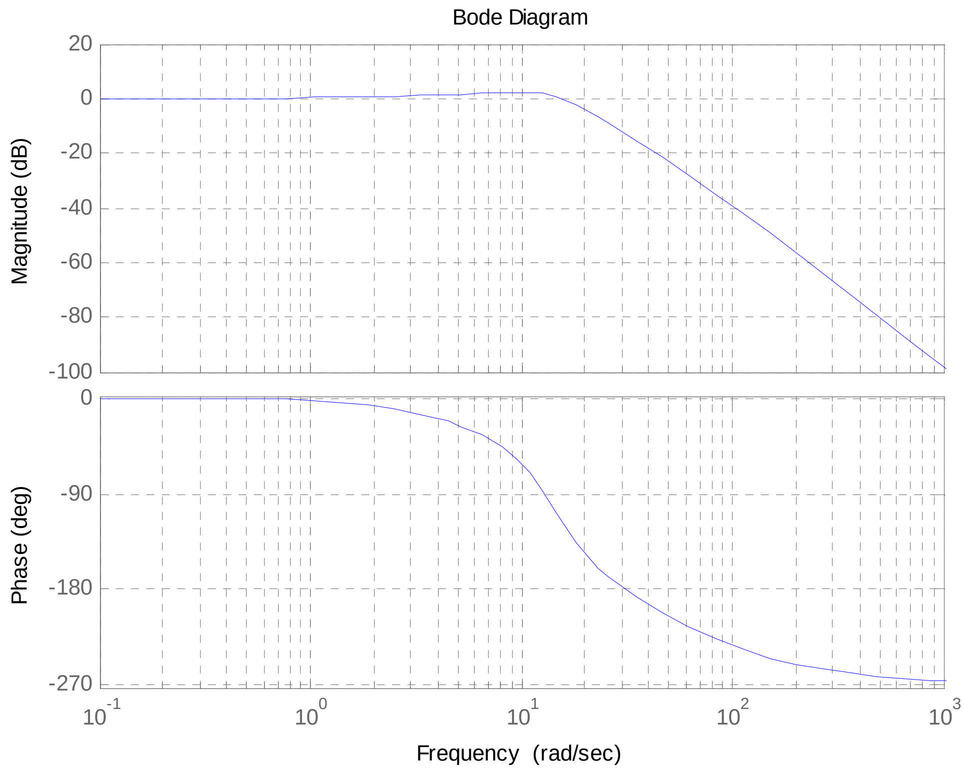
To obtain a slightly higher PM, lead compensator zero was re-tuned, where the zero is pulled closer to imaginary axis from -5 to -4:

$$Lead = \frac{1+\alpha Ts}{1+Ts} = \frac{s/4+1}{s/50+1}$$

This resulted in a higher PM as shown in the following bode diagram of loop transfer function:



Correspondingly, the Bode diagram of closed loop system can be shown as:



9-18) See Chapter 5 solutions for MATLAB codes for this problem.

(a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P + \frac{K_I}{s}\right)}{s^2 + 10s + 100}$$

For $K_v = 10$, $$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 10$$

Thus $K_I = 10.$

(b) Let the complex roots of the characteristic equation be written as $s = -\sigma + j15$ and $s = -\sigma - j15.$

The quadratic portion of the characteristic equation is $s^2 + 2\sigma s + \left(\sigma^2 + 225\right) = 0$

The characteristic equation of the system is $s^3+10s^2+\left(100+100K_P\right)s+1000=0$

The quadratic equation must satisfy the characteristic equation. Using long division and solve for zero remainder condition.

$$
\begin{array}{r|l}
 & s+(10-2\sigma) \\
s^2+2\sigma s+\sigma^2+225 & s^3+10s^2+\left(100+100K_P\right)s+1000 \\
 & s^3+2\sigma s^2+\left(\sigma^2+225\right)s \\
\hline
 & (10-2\sigma)s^2+\left(100K_P-\sigma^2-125\right)s+1000 \\
 & (10-2\sigma)s^2+\left(20\sigma-4\sigma^2\right)s+(10-2\sigma)\left(s^2+225\right) \\
\hline
 & \left(100K_P+3\sigma^2-20\sigma-125\right)s+2\sigma^3-10\sigma^2+450\sigma-1250
\end{array}
$$

For zero remainder, $2\sigma^3-10\sigma^2+450\sigma-1250=0$ (1)

and $100K_P+3\sigma^2-20\sigma-125=0$ (2)

The real solution of Eq. (1) is $\sigma=2.8555$. From Eq. (2),

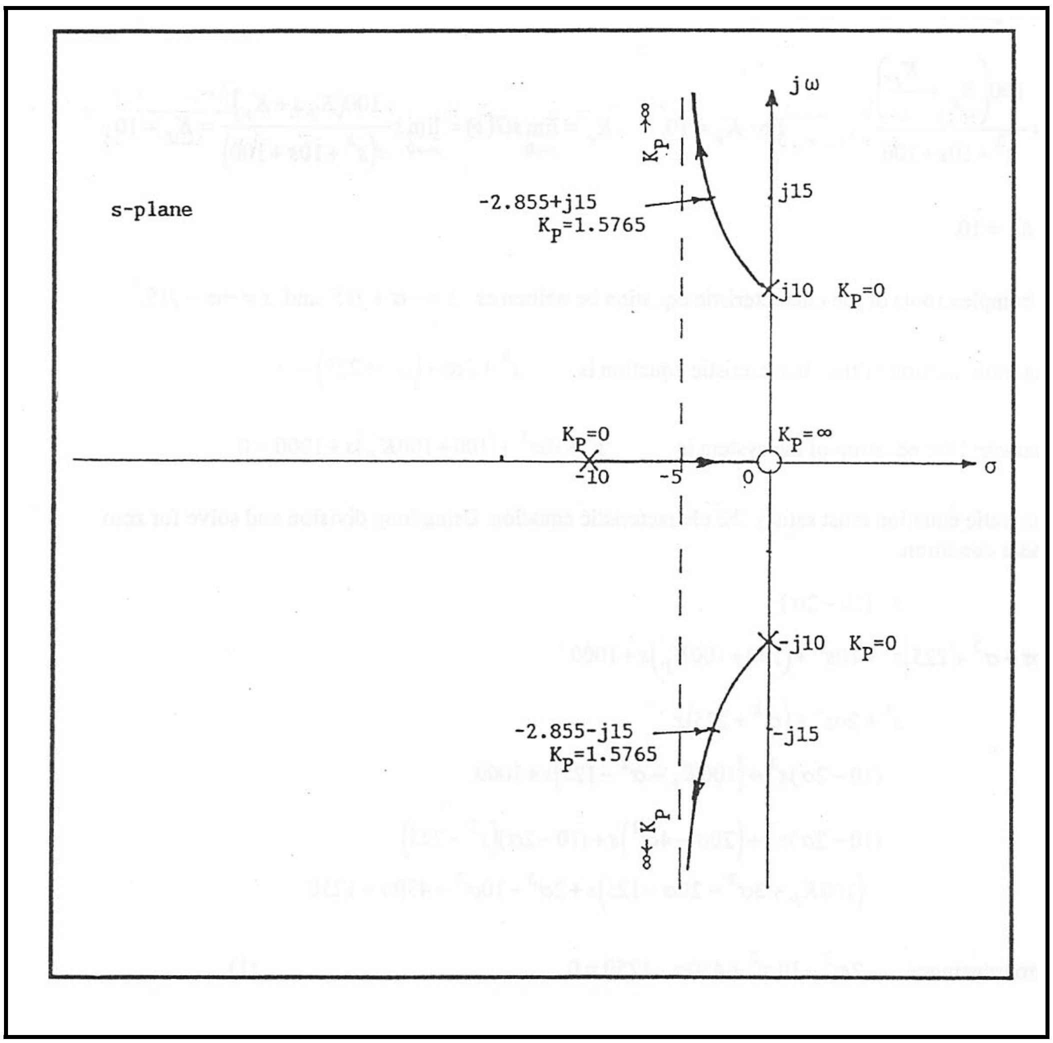
$$K_P=\frac{125+20\sigma-3\sigma^2}{100}=1.5765$$

The characteristic equation roots are: $s=-2.8555+j15,\quad -2.8555-j15,$ and $s=-10+2\sigma=-4.289$

(c) Root Contours:

$$G_{eq}(s) = \frac{100K_p s}{s^3 + 10s^2 + 100s + 1000} = \frac{100K_p s}{(s+10)(s^2+100)}$$

Root Contours:



9-19)

(a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P + \frac{K_I}{s}\right)}{s^2 + 10s + 100} \qquad \text{For } K_v = 10, \qquad K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 10$$

Thus the forward-path transfer function becomes

$$G(s) = \frac{10\left(1 + 0.1K_P s\right)}{s\left(1 + 0.1s + 0.01s^2\right)}$$

Attributes of the Frequency Response:

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.1	5.51	1.21	10.05	14.19
0.5	22.59	6.38	2.24	15.81
0.6	25.44	8.25	1.94	16.11
0.7	27.70	10.77	1.74	16.38
0.8	29.40	14.15	1.88	16.62

0.9	30.56	20.10	1.97	17.33
1.0	31.25	$\propto$	2.00	18.01
1.5	31.19	$\propto$	1.81	20.43
1.1	31.51	$\propto$	2.00	18.59
1.2	31.40	$\propto$	1.97	19.08

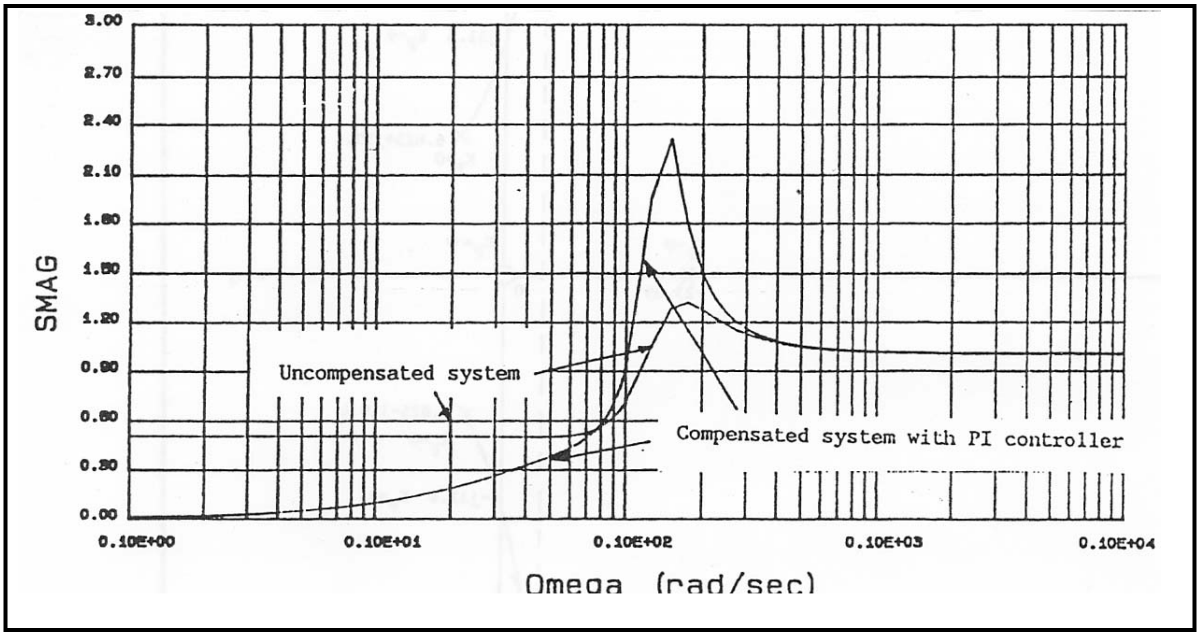
When $K_P = 1.1$ and $K_I = 10$, $K_v = 10$, the phase margin is 31.51 deg., and is maximum.

The corresponding roots of the characteristic equation roots are:

$$-5.4, \quad -2.3 + j13.41, \text{ and } \; -2.3 - j13.41$$

Referring these roots to the root contours in Problem 10-8(c), the complex roots corresponds to a relative damping ratio that is near optimal.

(b) Sensitivity Function:



In the present case, the system with the PI controller has a higher maximum value for the sensitivity function.

9-20)

(a) **Forward-path Transfer Function:**

$$G(s) = \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)}$$

For $K_v = 100,$

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} = K_I = 100 \qquad \text{Thus} \quad K_I = 100.$$

(b) The characteristic equation is $s^3 + 10s^2 + \left(100 + 100K_P\right)s + 100K_I = 0$

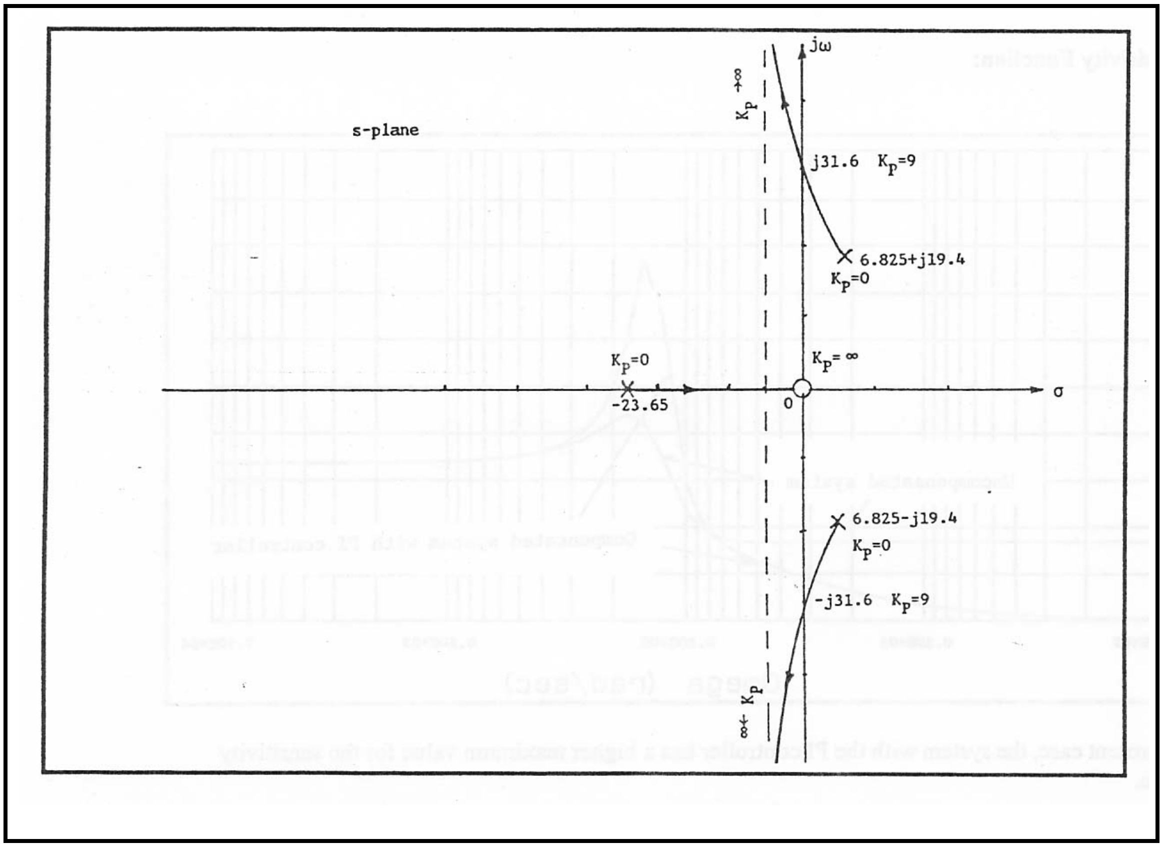
Routh Tabulation:

s^3	1	$100 + 100K_P$
s^2	10	10,000
s^1	$100K_P - 900$	0
s^0	10,000	

For stability, $100K_P - 900 > 0 \qquad \text{Thus} \quad K_P > 9$

Root Contours:

$$G_{eq}(s) = \frac{100K_P s}{s^3 + 10s^2 + 100s + 10{,}000} = \frac{100K_P s}{(s + 23.65)(s - 6.825 + j19.4)(s - 6.825 - j19.4)}$$



(c) $K_I = 100$

$$G(s) = \frac{100\left(K_P s + 100\right)}{s\left(s^2 + 10s + 100\right)}$$

The following maximum overshoots of the system are computed for various values of K_P.

K_P	15	20	22	24	25	26	30	40	100	1000
y_{max}	1.794	1.779	1.7788	1.7785	1.7756	1.779	1.782	1.795	1.844	1.859

When K_P = 25, minimum y_{max} = 1.7756

9-21)

(a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)} \qquad \text{For } K_v = \frac{100K_I}{100} = 10, \qquad K_I = 10$$

(b) Characteristic Equation: $s^3 + 10s^2 + 100\left(K_P + 1\right)s + 1000 = 0$

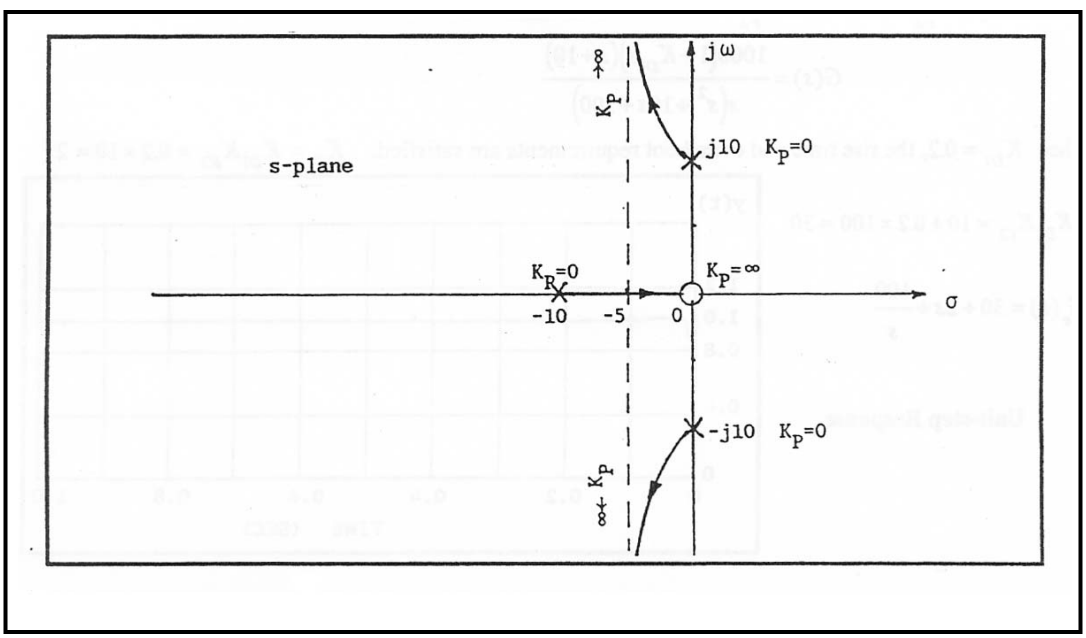
Routh Tabulation:

s^3	1	$100 + 100K_P$
s^2	10	1000
s^1	$100K_P$	0
s^0	1000	

For stability, $K_P > 0$

Root Contours:

$$G_{eq}(s) = \frac{100K_P s}{s^3 + 10s^2 + 100s + 1000}$$



(c) The maximum overshoots of the system for different values of K_P ranging from 0.5 to 20 are computed and tabulated below.

K_P	0.5	1.0	1.6	1.7	1.8	1.9	2.0	3.0	5.0	10	20
y_{max}	1.393	1.275	1.2317	1.2416	1.2424	1.2441	1.246	1.28	1.372	1.514	1.642

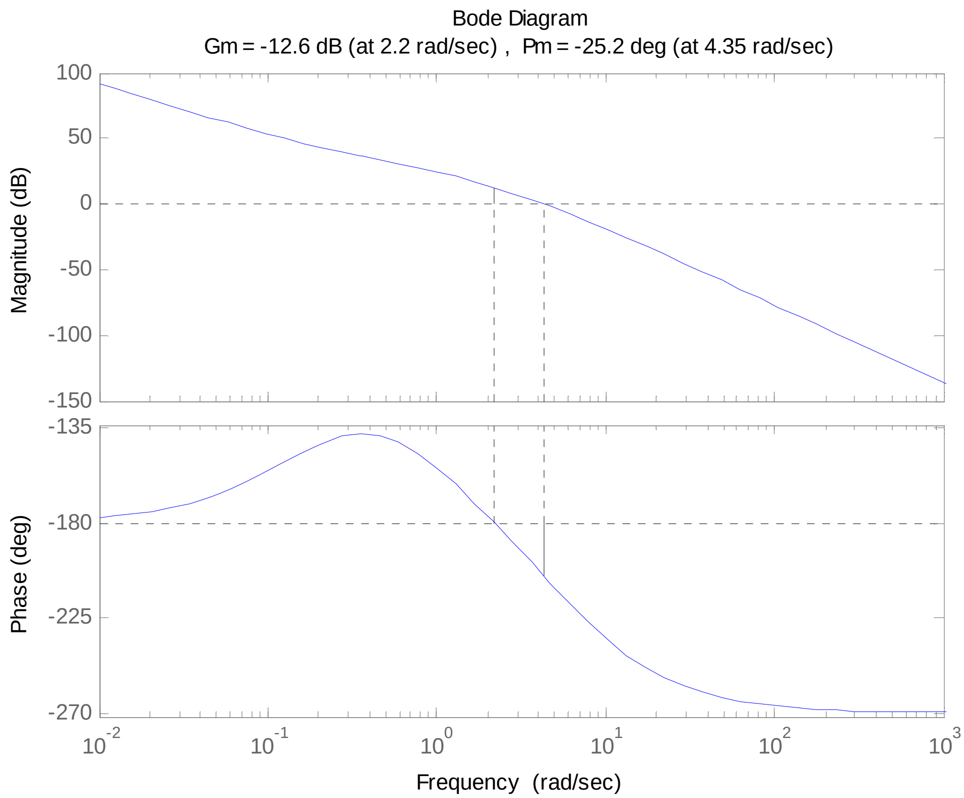
When K_P = 1.7, maximum y_{max} = 1.2416

9-22) $K_v = \lim_{s\to 0} sG_c(s)G(s) > 20 \Rightarrow \frac{24K}{6} > 20 \Rightarrow K > 5$

let K = 6 and targeted PM = 45° . To include some integral action, Ki is set to 1.

First , let's take a look at uncompensated system:

The open loop bode shows as PM of -25.2 @ 4.35 rad/sec:



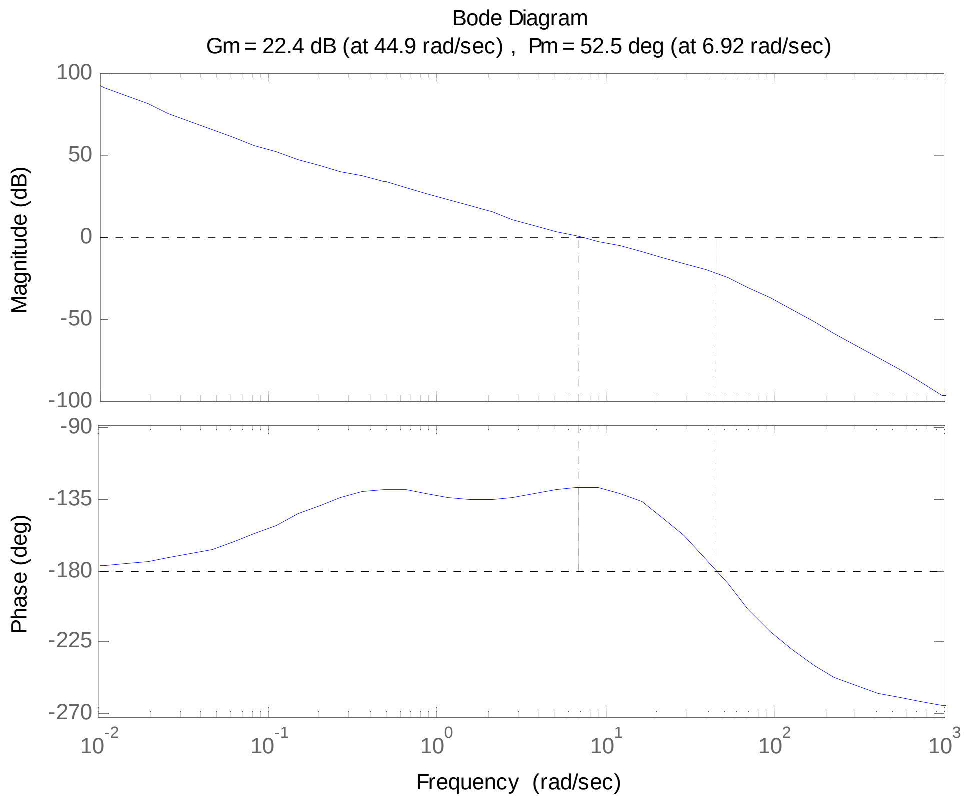
To achieve the PM of 45 deg, we need to add a phase lead of (45 – (-25.2)=70.2). By try and error, 2 compensators (a double lead compensator) each with phase lead of 55 deg was found suitable. Considering the change in cross over frequency after applying the lead filters, overall, a PM of 52 deg was obtained as seen in the bode diagram of compensated loop:

Double Lead filter design:

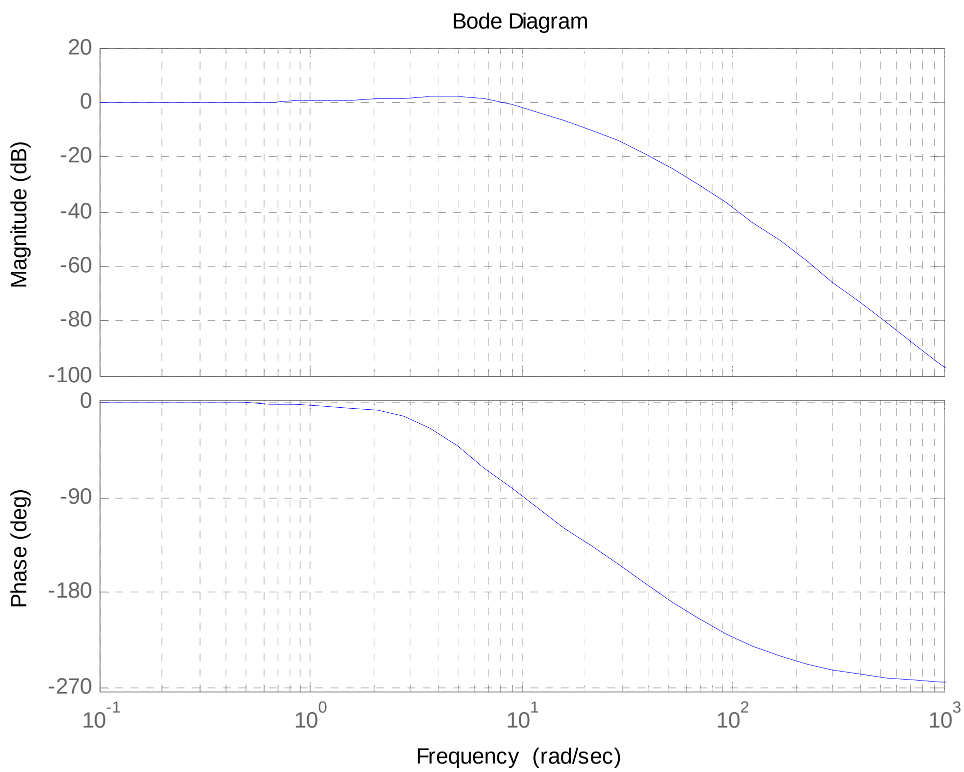
$$\alpha = \frac{1+\sin\Phi_m}{1-\sin\Phi_m} = \frac{1+\sin 55}{1-\sin 55} = 10.0590, \qquad T = \frac{1}{\sqrt{10.059}\omega} = \frac{1}{15\sqrt{10.0590}} = 0.0210$$

The maximum phase lead of the compensators are placed at 15 rad/sec, which resulted in a larger PM (=52.5 deg) compared to applying this phase lead at original cross over frequency of 4.35 rad/sec. This was due to the shape of phase diagram affected by integral action (i.e. phase starts at -180 @ $\omega = 0$ rad/sec.

The gain crossover frequency is $\omega = 6.92$ rad/sec. Bode diagram of compensated loop transfer function can be observed in the following figure, showing a PM pf 52.5 deg:



Correspondingly, the Bode diagram of closed loop system can be shown as:



MATLAB code:

```
s = tf('s')
Kp = 6;
Ki = 1;
num_GH= 24*(Kp+Ki/s);
den_GH=s*(s+1)*(s+6);
GH=num_GH/den_GH;
%lead design
PL=55
CRover=15
alpha=(1+sin(PL/180*pi))/(1-sin(PL/180*pi))
T=1/alpha^0.5/CRover
lead=(1+T*alpha*s)/(1+T*s)
LT=GH*lead*lead  %double lead compensation
CL = LT/(1+LT);
figure(1)
Margin(GH)
figure(2)
Margin(LT)
figure(3)
Bode(CL)
grid on;
```

9-23)

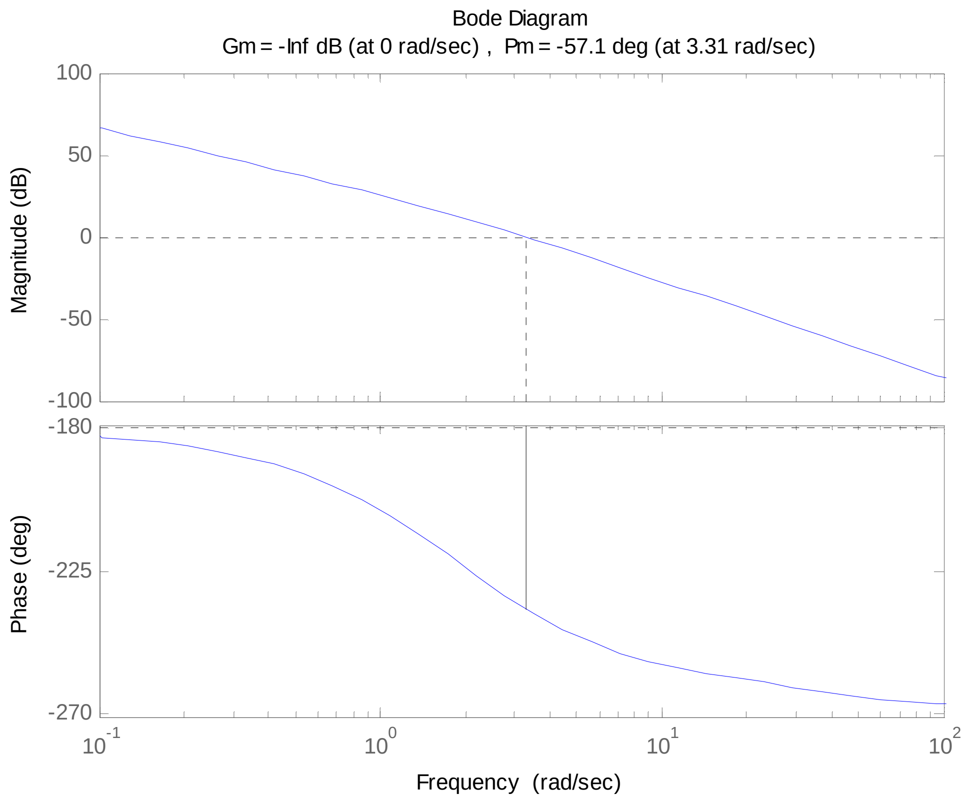
$$e_{ss} = \frac{1}{K_a} \leq 0.05$$

$$K_a = \lim_{s\to 0} s^2 G_c(s)G(s) = \lim_{s\to 0} \frac{40\left(K_p s + K_I\right)}{(s+2)(s+20)} = K_I > 20$$

Let's consider $K_I = 21$

As gain crossover frequency is $\omega = 1$ ⇨ $|G_c G(j\omega)|_{\omega=1} = 1$ ⇨ $K_p = 1.25$

Let's see if the PM is in the required range. The bode of the loop transfer function shows a PM of -57 deg at 3.31.



By try and error, a double lead compensator, each with phase lead of 53 deg was found suitable. Considering the change in cross over frequency after applying the lead filters, overall, a PM of 35.4 deg was obtained as seen in the bode diagram of compensated loop:

Double Lead filter design:

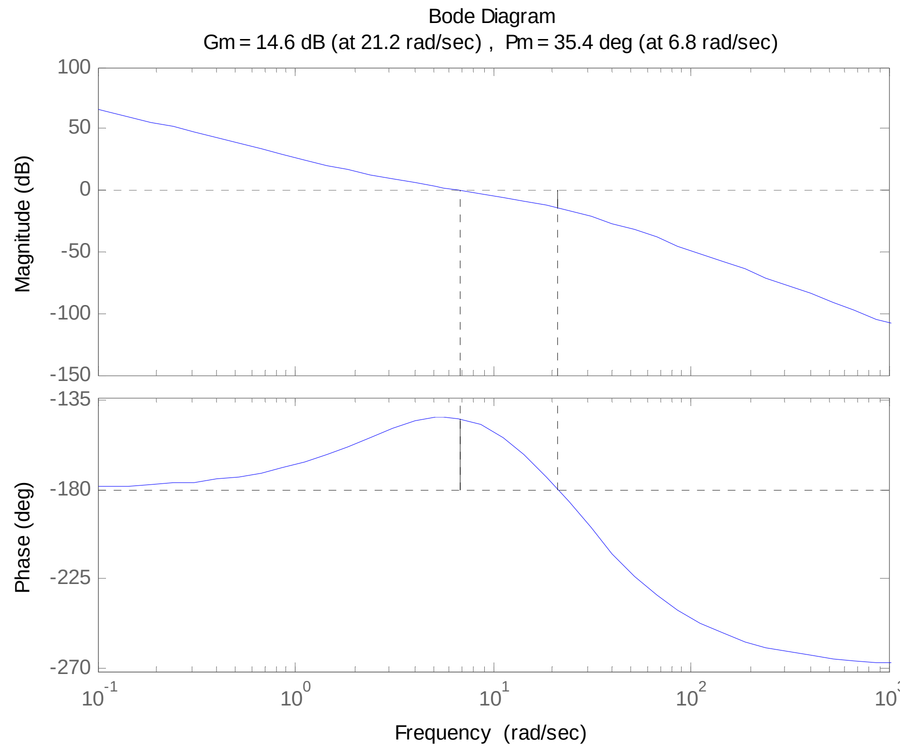
$$\alpha = \frac{1+\sin\Phi_m}{1-\sin\Phi_m} = \frac{1+\sin 53}{1-\sin 53} = 8.9322, \qquad T = \frac{1}{\sqrt{8.9322}\omega} = \frac{1}{7.5\sqrt{8.9322}} = 0.0446$$

The maximum phase lead of the compensators are placed at 7.5 rad/sec, resulting in a larger PM (= 52.5 deg) compared to applying this phase lead at original cross over frequency of 3.31 rad/sec. This was due to the shape of phase diagram affected by integral action (i.e. phase starts at -180 @ ω = 0 rad/sec.

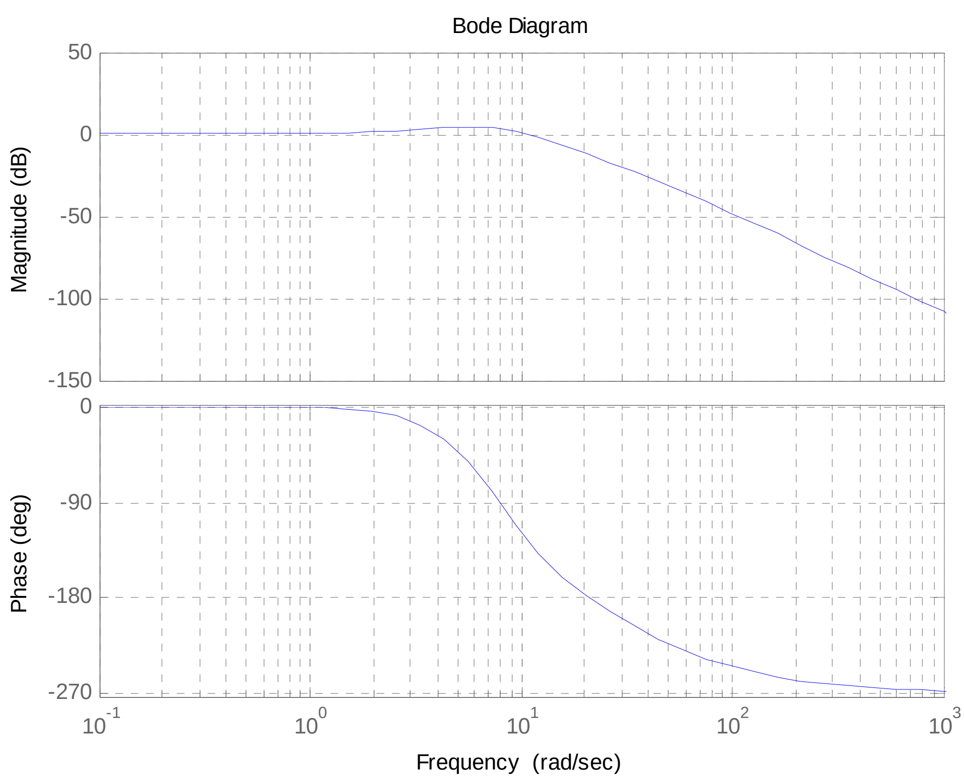
Then the gain crossover frequency is $\omega = 6.8$ rad/sec. Bode diagram of compensated loop can be observed in the following figure, showing a PM pf 35.4 deg:

MATLAB code:

```
s = tf('s')
Kp = 1.25;
Ki = 21;
num_GH= 40*(Kp+Ki/s);
den_GH=s*(s+2)*(s+20);
GH=num_GH/den_GH;
CL = GH/(1+GH);
%lead design
PL=53
CRover=7.5
alpha=(1+sin(PL/180*pi))/(1
-sin(PL/180*pi))
T=1/alpha^0.5/CRover
lead=(1+T*alpha*s)/(1+T*s)
LT=GH*lead*lead
CL = LT/(1+LT);

figure(1)
Margin(GH)
figure(2)
Margin(LT)
figure(3)
Bode(CL)
grid on;
```



Correspondingly, the Bode diagram of closed loop system can be shown as:

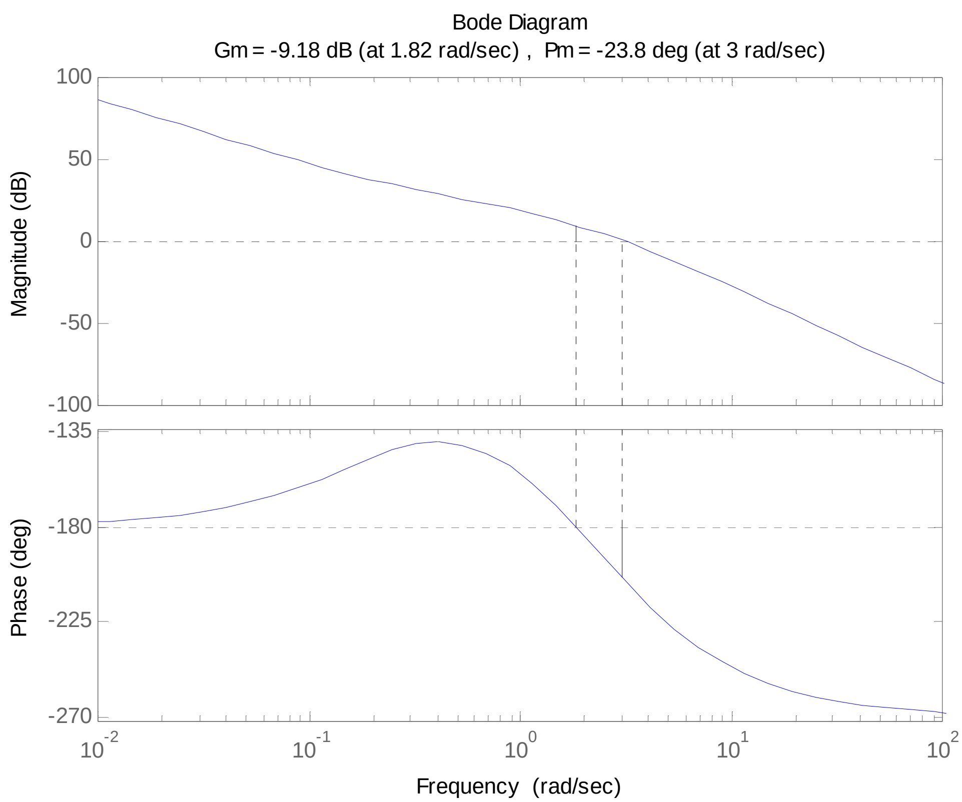


9-24)

To satisfy the unity DC gain (G(0)/(1+Kp*G(0)) = 1), Kp should be equal to 1: Kp=1

In order to add some integral action, $K_I = 0.2$ was chosen as the integral gain.

First, the bode plot of the Loop transfer function is obtained demonstrating a PM of -23.8 deg at 3 rad/sec cross over frequency:



First, the bode plot of the Loop transfer function is obtained demonstrating a PM of -23.8 deg at 3 rad/sec cross over frequency:

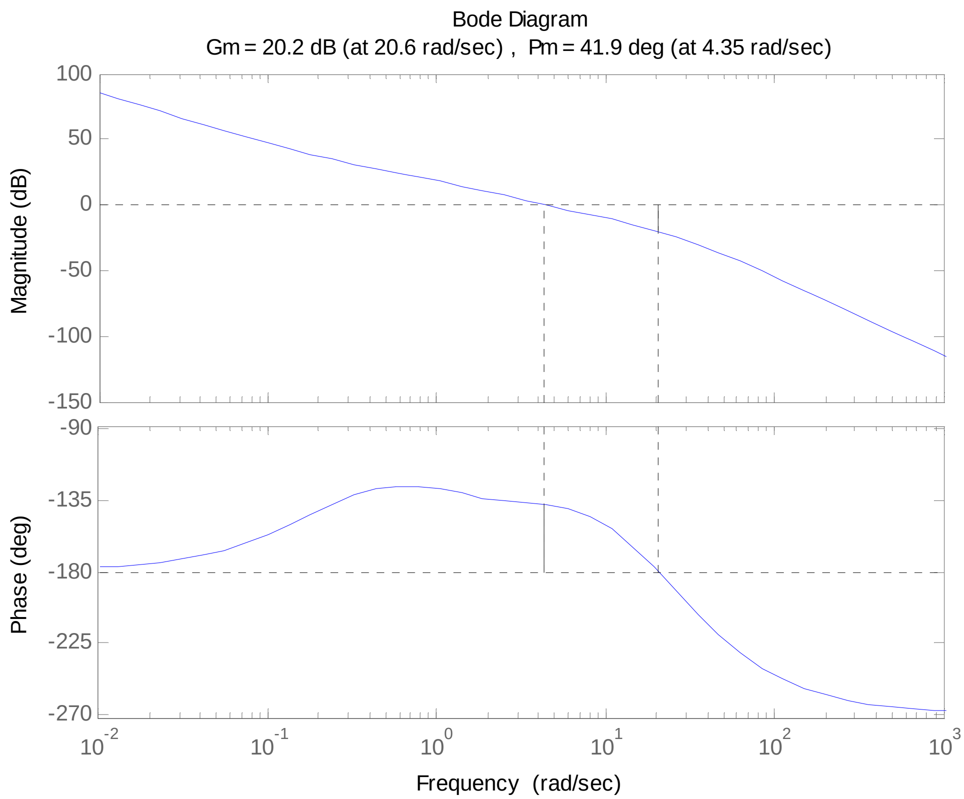
By try and error, a double lead compensator, each with phase lead of 48 deg was found suitable. Considering the change in cross over frequency after applying the lead filters, overall, a PM of 41.9 deg was obtained as seen in the bode diagram of compensated loop:

Double Lead filter design:

$$\alpha = \frac{1+\sin\Phi_m}{1-\sin\Phi_m} = \frac{1+\sin 48}{1-\sin 48} = 6.7865, \qquad T = \frac{1}{\sqrt{6.7865}\omega} = \frac{1}{9\sqrt{8.9322}} = 0.0427$$

The maximum phase lead of the compensators are placed at 9 rad/sec, resulting in a larger PM (= 41.9 deg) compared to applying this phase lead at original cross over frequency of 3 rad/sec. This was due to the shape of phase diagram affected by integral action (i.e. phase starts at -180 @ ω = 0 rad/sec.

Then the gain crossover frequency is $\omega = 4.35$ rad/sec. Bode diagram of compensated loop can be observed in the following figure, showing a PM pf 35.4 deg:

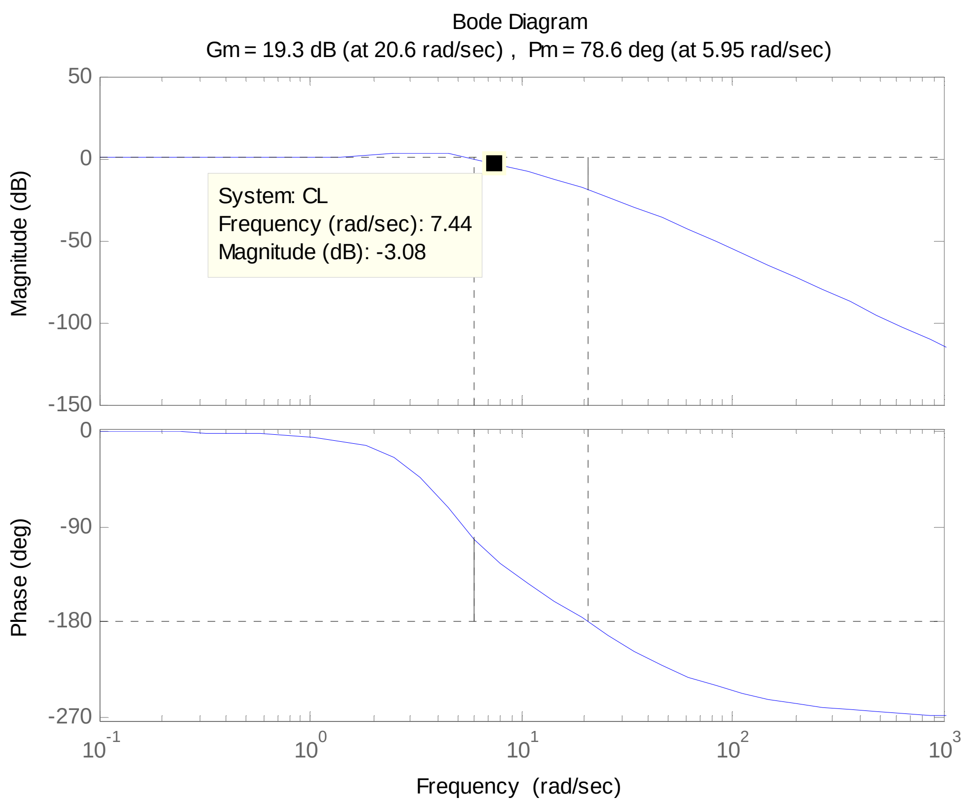


MATLAB code:

```
s = tf('s')
Kp=1;
Ki=0.2;
num_GH= 210*(Kp+Ki/s);
den_GH=s*(5*s+7)*(s+3);
GH=num_GH/den_GH;

%lead design
PL=48
CRover=9
alpha=(1+sin(PL/180*pi))/(1-sin(PL/180*pi))
T=1/alpha^0.5/CRover
lead=(1+T*alpha*s)/(1+T*s)

LT=GH*lead*lead
CL = LT/(1+LT);
```

```
figure(1)
Margin(GH)
figure(2)
Margin(LT)
figure(3)
Margin(CL)
```

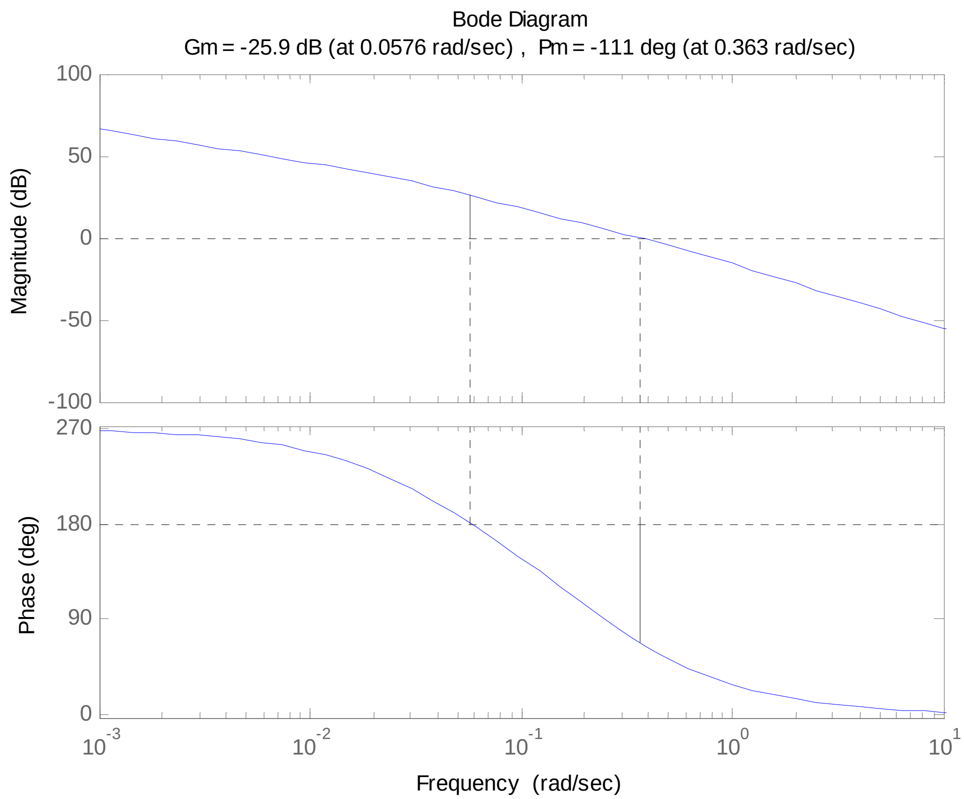
Correspondingly, the Bode diagram of closed loop system can be shown as:



The Bandwidth can be obtained from -3dB in magnitude diagram of the Bode plot. He above data point in the figure shows BW = 7.44 rad/sec

9-25) $K_v = lim_{s\to 0}\, sG_c(s)G(s) = \frac{2353K(71)}{(71)(13)(181)} = 2$, therefore, $K > 2$

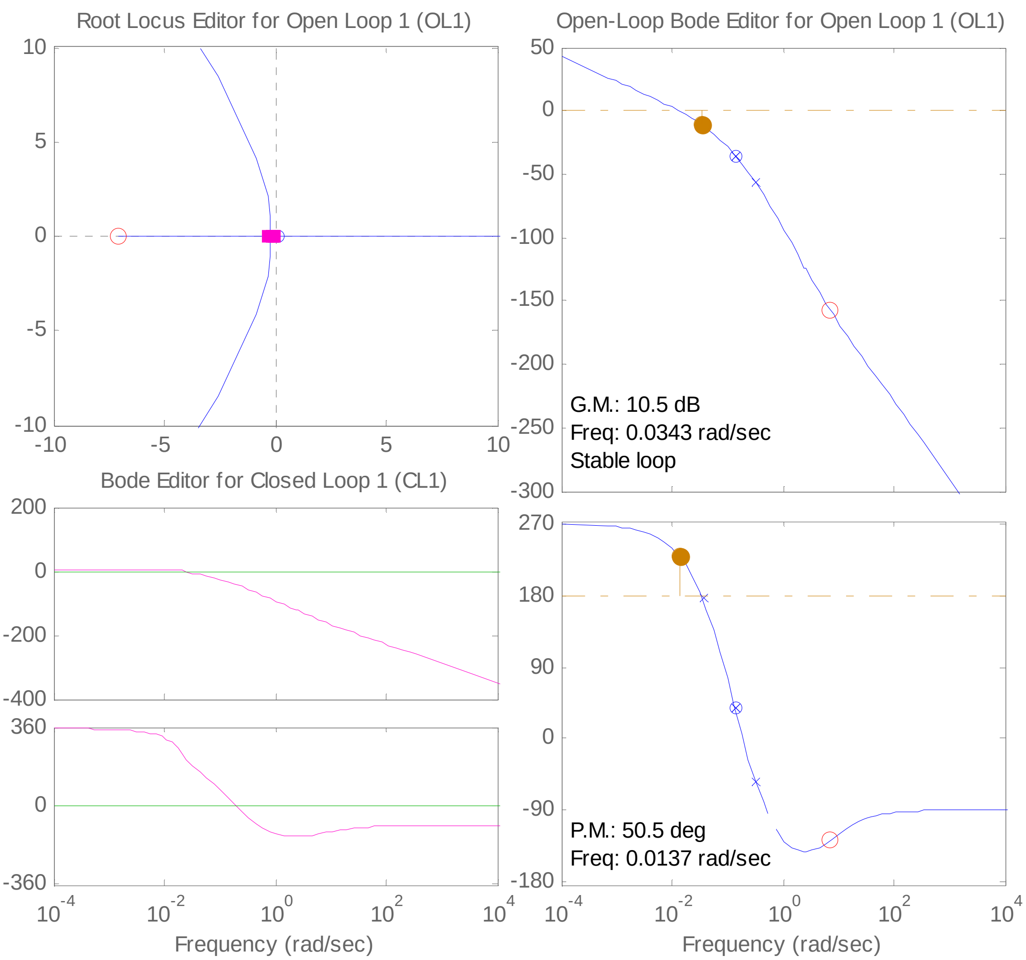
From bode plot of uncompensated loop, we have PM = -111 at ω = 0.363 rad/s:



A PI controller can be expressed as $\mathrm{PI} = (\mathrm{K_p} + \mathrm{sK_i})$. The effect is similar to adding a Zero at $\frac{-\mathrm{K_p}}{\mathrm{K_i}}$. Let's place this zero at 71/500 to cancel the Phase lag originating from the unstable zero of G at +71/500: $(G(s) = \dfrac{2353K(71-500s)}{71s(40s+13)(5000s+181)})$.

The compensator can be expressed as:

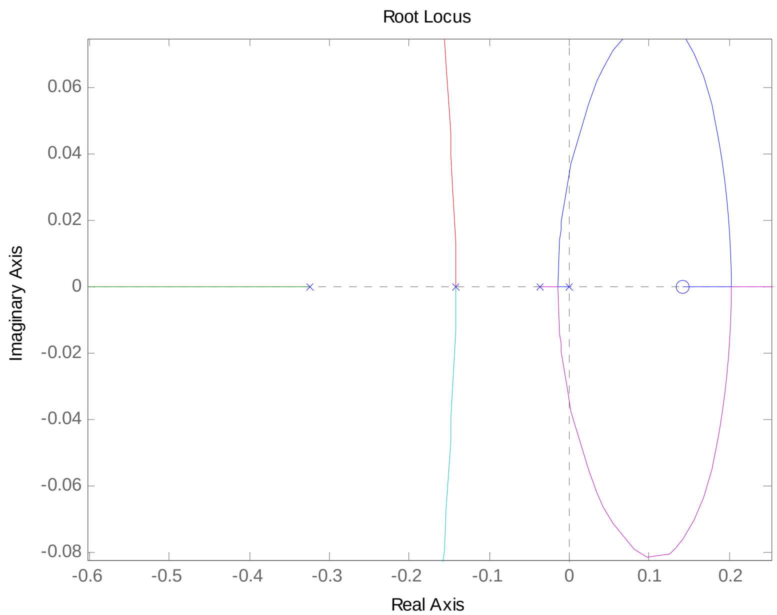
$\mathrm{PI} = \mathrm{K_p}\left(\mathrm{s} + \frac{\mathrm{K_p}}{\mathrm{K_i}}\right) = \mathrm{K_p}(\mathrm{s} + 71/500)$, where $\mathrm{K_p}$ can be adjusted in sisotool as the overall gain of the loop, until the required PM is achieved. At $\mathrm{K_p} = 37$, PM=50 deg as seen in the following sisotool results:



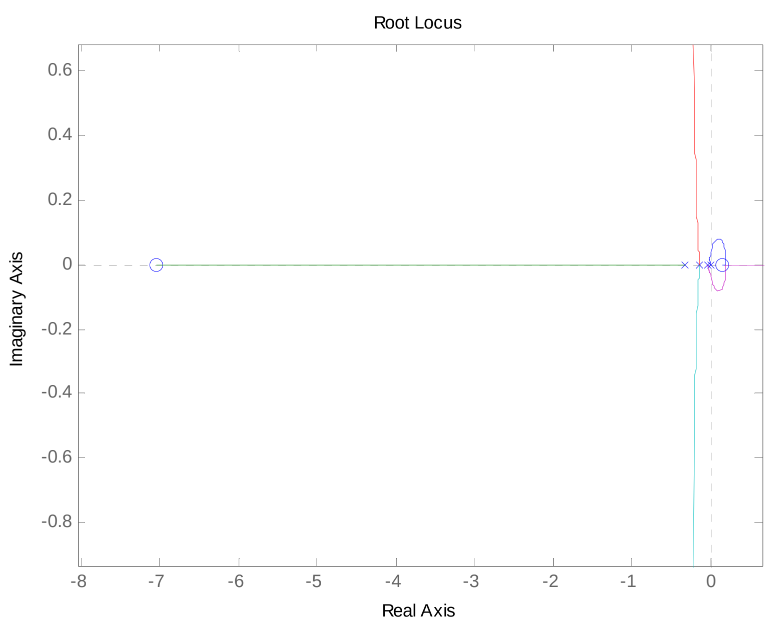
Considering the slow dynamic of the ship, and the RHS zero, the crossover frequency is relatively low.

The root locus diagram can be seen as:

Zoom in



Zoom out



MATAB Code:

```
s = tf('s')
Kp=1;
Ki=1;

num_G= 2353*2*(71-500*s)
den_G=71*s*(40*s+13)*(5000*s+181)*(71+500*s)^2;
G=num_G/den_G;

%PI design
```

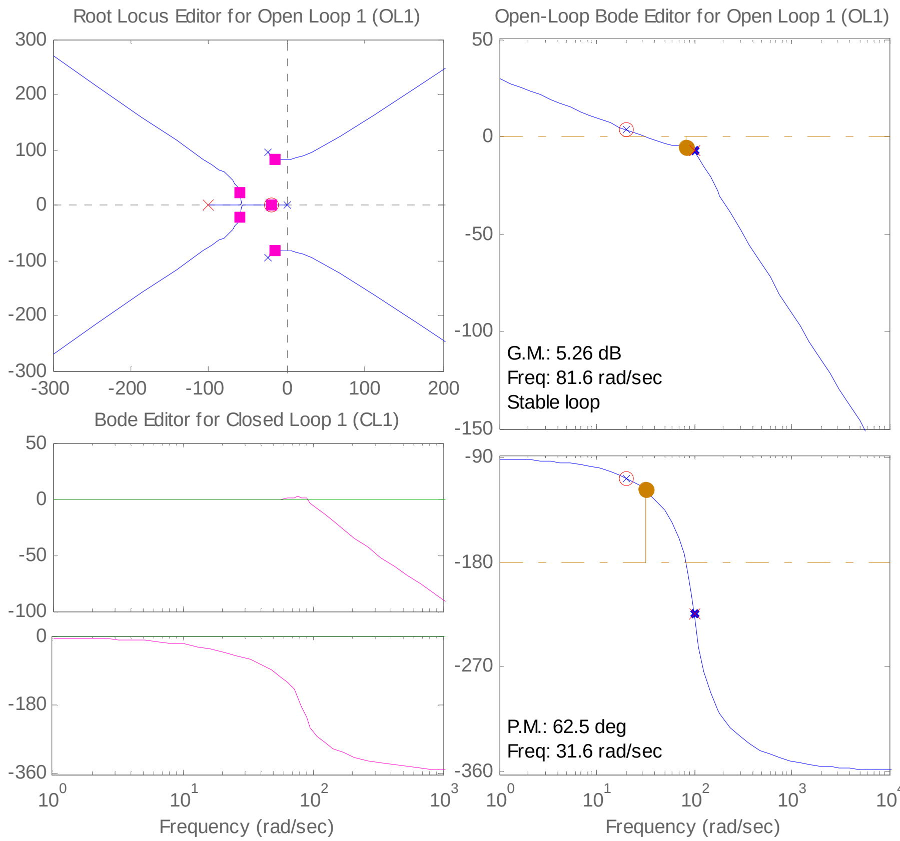
```
Kp=1
Ki=71/500
PI=Kp+Ki*s

figure(100)
Margin(G)

figure(101)
rlocus(G*PI)

sisotool
```

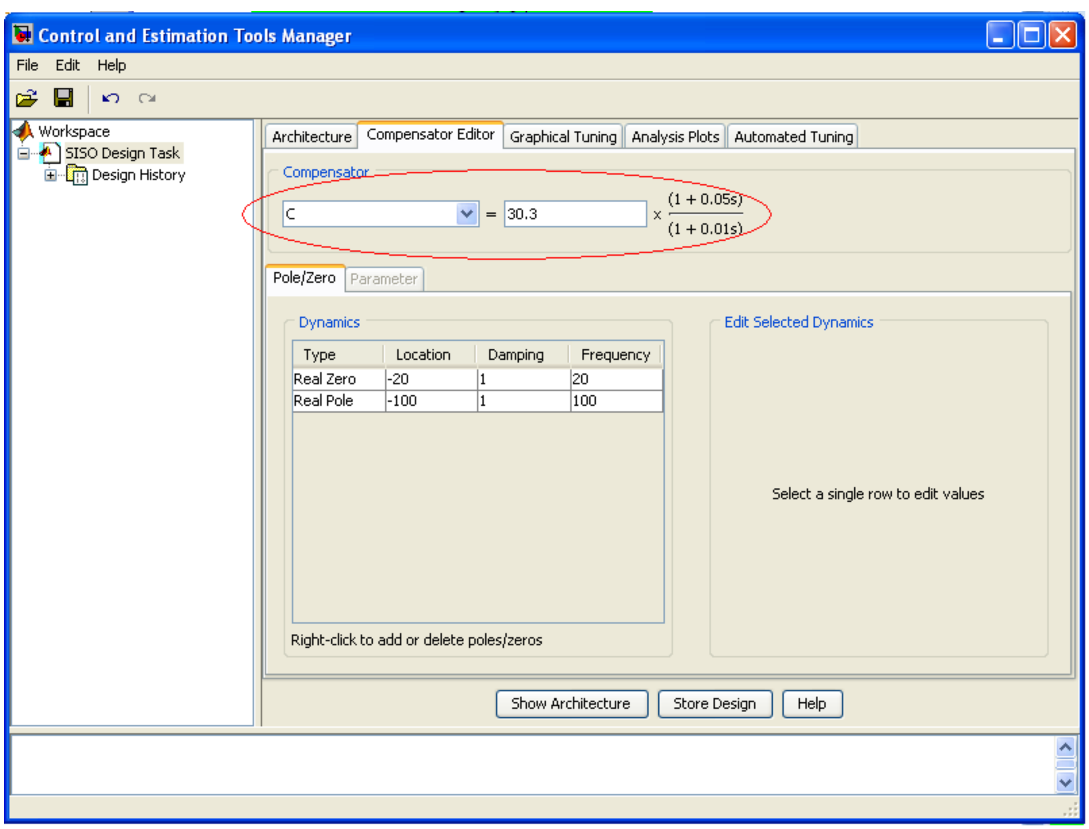
9-26) a) Transfer functions G and H are generated in MATLAB and imported into sisotool:

MATLAB Code:

```
s = tf('s')

num_G= 2*10^5;
den_G=s*(s+20)*(s^2+50*s+10000);
G=num_G/den_G;

num_c1= 0.05*s+1;
den_c1=0.01*s+1;
c1=num_c1/den_c1;

num_c2= s/0.316+1;
den_c2=s/3.16+1;
c2=num_c2/den_c2;

sisotool
```

(a) The gain was changed until the cross over frequency matches 31.6 rad/sec as a requirement. At K=30.3, the desired cross over frequency of 31.6 rad/sec happens as can be seen in the following sisotool results:


Root Locus Editor for Open Loop 1 (OL1)
Open-Loop Bode Editor for Open Loop 1 (OL1)
G.M.: 5.26 dB
Freq: 81.6 rad/sec
Stable loop
Bode Editor for Closed Loop 1 (CL1)
P.M.: 62.5 deg
Freq: 31.6 rad/sec
Frequency (rad/sec)
Frequency (rad/sec)


Control and Estimation Tools Manager
File Edit Help
Workspace
SISO Design Task
Design History
Architecture
Compensator Editor
Graphical Tuning
Analysis Plots
Automated Tuning
Compensator
C = 30.3 x (1 + 0.05s) / (1 + 0.01s)
Pole/Zero
Parameter
Dynamics
Type Location Damping Frequency
Real Zero -20 1 20
Real Pole -100 1 100
Right-click to add or delete poles/zeros
Edit Selected Dynamics
Select a single row to edit values
Show Architecture
Store Design
Help

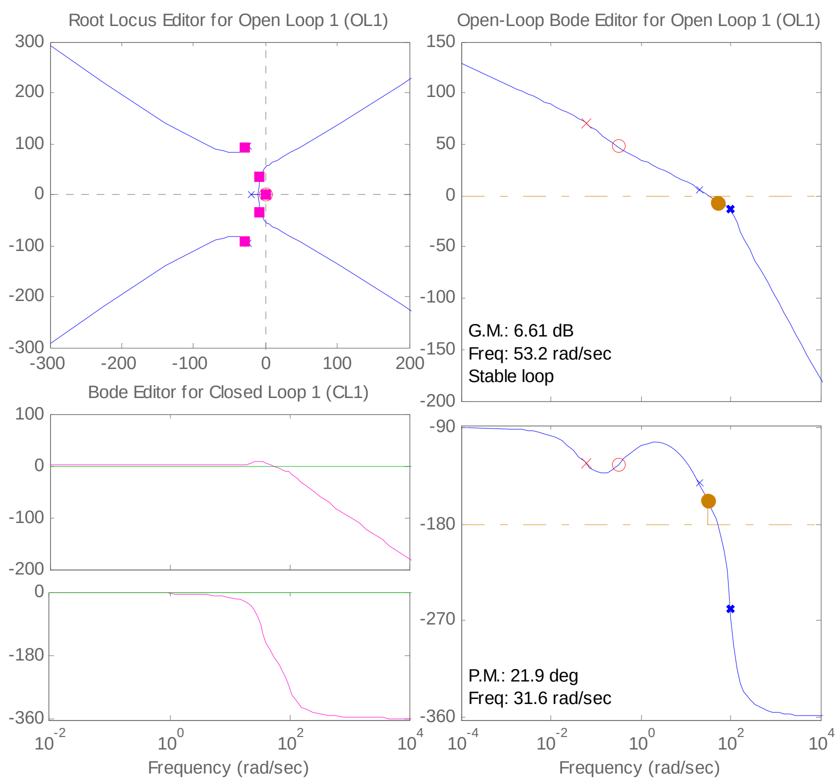
b) $K_v = \lim_{s\to0} sG_c(s)G(s)$ $\rightarrow K_v = \text{Lim}_{s\to0}\, s\left(\frac{0.05s+1}{0.01s+1}\right)\left(\frac{2\times10^5\times K}{s(s+20)(s^2+50s+10000)}\right) = K = 30.3$, then $K_v = 30.3$

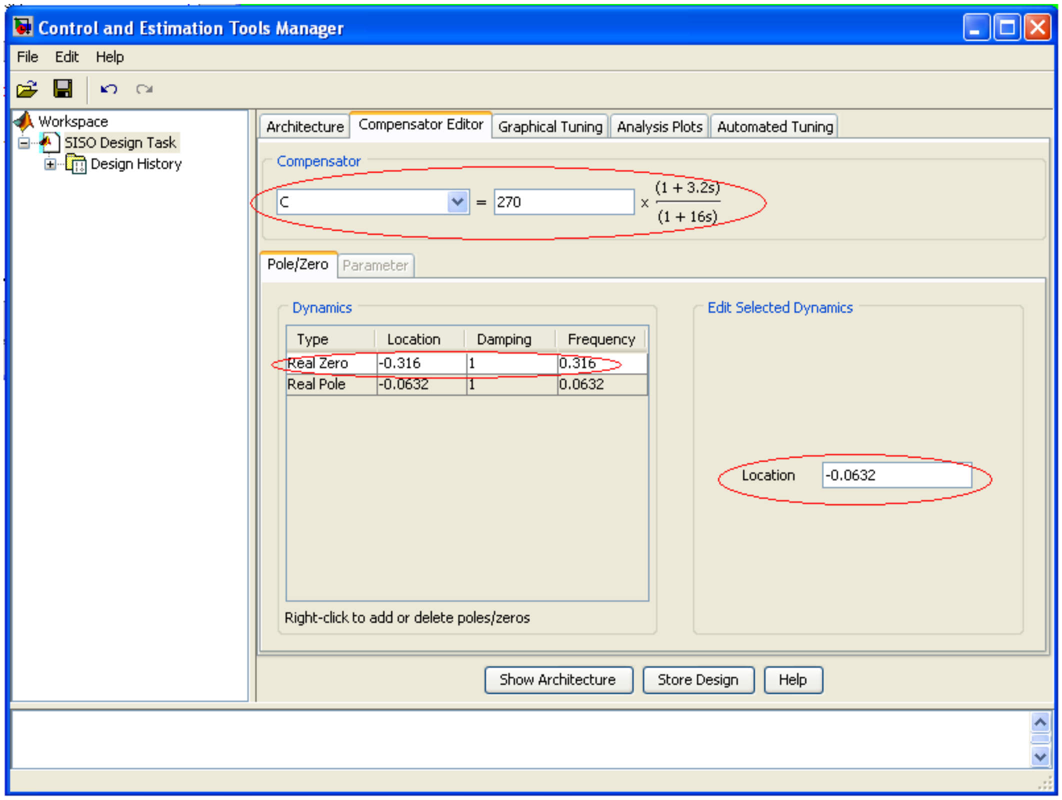
c) Again, $K_v = \lim_{s\to0} sG_c(s)G(s)$ $\rightarrow K_v = \text{Lim}_{s\to0}\, s\left(\frac{r(\tau s+1)}{r\tau s+1}\right)\left(\frac{2\times10^5\times r}{s(s+20)(s^2+50s+10000)}\right) = r$. To have $K_v = 100$, the overall gain of the PI controller should be equal to 100 (r = 100).

d & e) In this part, the PI pole is asked to be placed at -3.16 rad/sec and the crossover frequency needs to be at 31.6 rad/sec. The zero and the gain of the PI controller needs to be designed.

Considering the structure of the PI controller given in the question, $H(s) = \dfrac{r(\tau s+1)}{(r\tau s+1)}$, the corresponding pole is set to -3.16 in sisotool. The place of the zero and the overall gain is iteratively changed in the MATLAB sisotool to achieve the crossover frequency of 31.6 rad/sec.

With a zero at -0.06321 rad/sec and overall gain of K=270, required crossover frequency (31.6 rad/sec) and PM of 21.9 deg is obtained as shown in the following sisotool results:





e) the presented sisotool figure shows the compensated bode diagram and 21.9 deg of PM.

9-27)

$$G_c(s) = K_P + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_P s + K_I}{s} = \left(1 + K_{D1}s\right)\left(K_{P2} + \frac{K_{I2}}{s}\right)$$

where

$$K_P = K_{P2} + K_{D1}K_{I2} \qquad K_D = K_{D1}K_{P2} \qquad K_I = K_{I2}$$

Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{100\left(1 + K_{D1}s\right)\left(K_{P2}s + K_{I2}\right)}{s\left(s^2 + 10s + 100\right)} \qquad K_v = \lim_{s \to 0} sG(s) = K_{I2} = 100$$

Thus

$$K_I = K_{I2} = 100$$

Consider only the PI controller, (with $K_{D1} = 0$)

Forward-path Transfer Function: **Characteristic Equation:**

$$G(s) = \frac{100\left(K_{P2}s + 100\right)}{s\left(s^2 + 10s + 100\right)} \qquad s^3 + 10s^2 + \left(100 + 100K_{P2}\right)s + 10,000 = 0$$

For stability, $K_{P2} > 9$. Select $K_{P2} = 10$ for fast rise time.

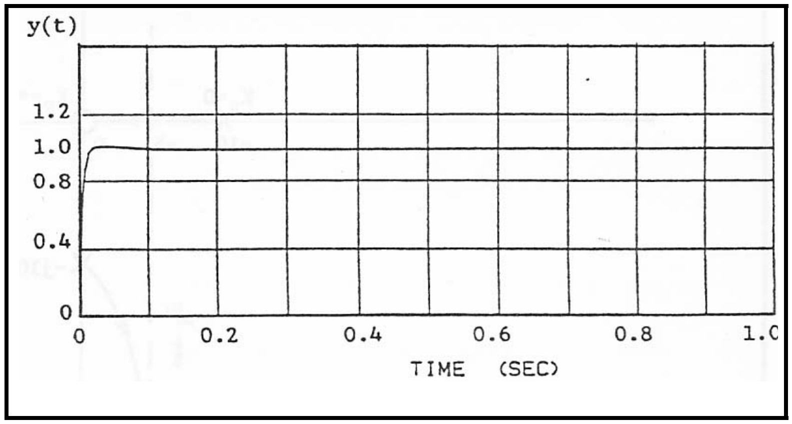
$$G(s) = \frac{1000\left(1 + K_{D1}s\right)\left(s + 10\right)}{s\left(s^2 + 10s + 100\right)}$$

When $K_{D1} = 0.2$, the rise time and overshoot requirements are satisfied.

$K_D = K_{D1}K_{P2} = 0.2 \times 10 = 2$

$$K_P = K_{P2} + K_{D1}K_{I2} = 10 + 0.2 \times 100 = 30$$

$$G_c(s) = 30 + 2s + \frac{100}{s}$$



Unit-step Response

9-28)

Process Transfer Function:

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{e^{-0.2s}}{1+0.25s} \cong \frac{1}{(1+0.25s)(1+0.2s+0.02s^2)}$$

(a) PI Controller:

$$G(s) = G_c(s)G_p(s) \cong \frac{K_P + \frac{K_I}{s}}{(1+0.25s)(1+0.2s+0.02s^2)} = \frac{200(K_P s + K_I)}{s(s+4)(s^2+10s+50)}$$

For $K_v = 2,$ $K_v = \lim_{s\to 0} sG(s) = \frac{200K_I}{4\times 50} = K_I = 2$ Thus $K_I = 2$

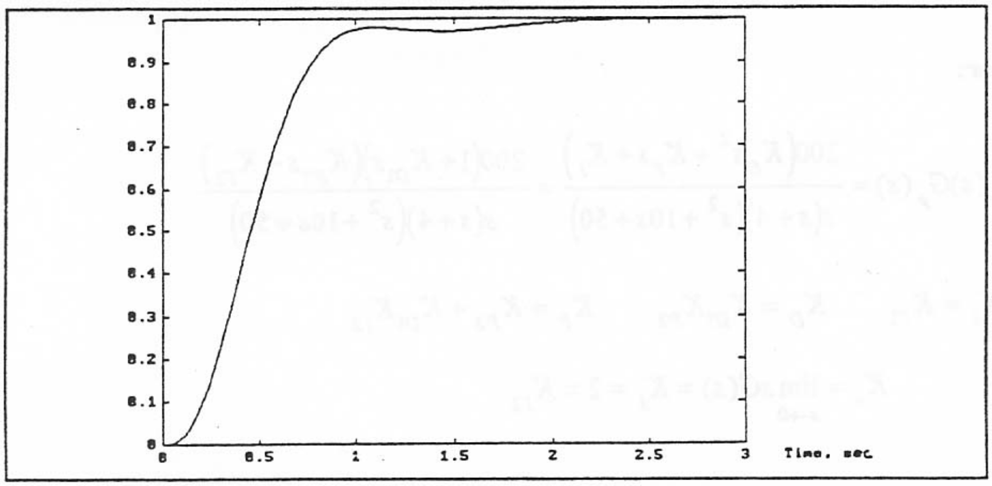
Thus $G(s) = \frac{200(2+K_P s)}{s(s+4)(s^2+10s+50)}$

The following values of the attributes of the unit-step response are computed for the system with various values for K_P.

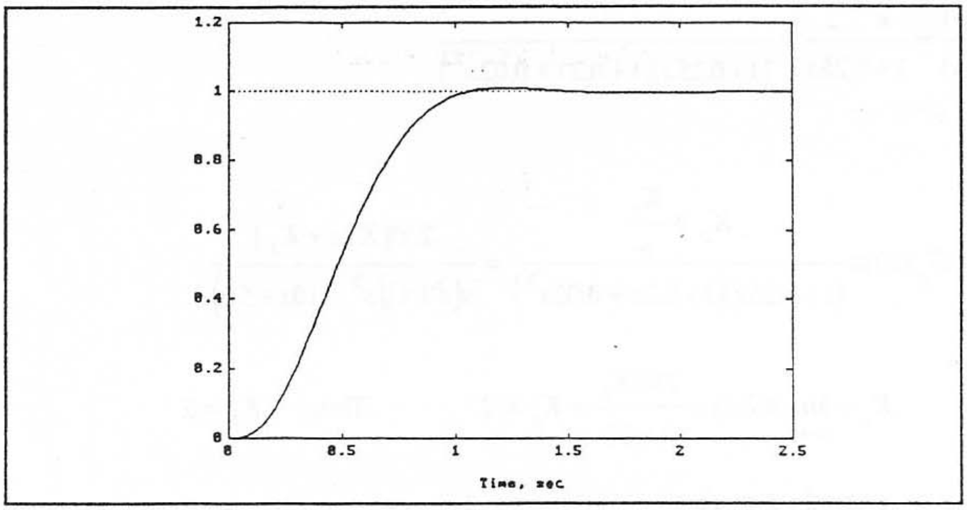
K_P	Max overshoot (%)	t_s (sec)	t_s (sec)
0.1	19.5	0.61	2.08
0.2	13.8	0.617	1.966
0.3	8.8	0.615	1.733
0.4	4.6	0.606	0.898
0.5	1.0	0.5905	0.878
0.6	0	0.568	0.851
0.7	0	0.541	1.464
0.8	0	0.5078	1.603
1.0	0	0.44	1.618

The settling time t_s is minimum (0.851 sec) when $K_P = 0.6$. Statistically, $K_P = 0.6$ is the best choice. The unit-step response is shown below. However, a better response is obtained when $K_P = 0.5$.

Unit-step Response: ($K_P = 0.6$, $K_I = 2$)



Unit-step Response: ($K_P = 0.5,\ K_I = 2$)



For stability check we perform the Routh tabulation. The characteristic equation with $K_I = 2$ is

$$s^4 + 14s^3 + 90s^2 + (200 + 200K_P)s + 400 = 0$$

Routh Tabulation:

s^4	1	90	400
s^3	14	$200 + 200K_P$	
s^2	$75.714 - 14.284K_P$	400	
s^1	$\dfrac{9542.8 + 12285.66K_P - 2857.14K_P^2}{75.714 - 14.284K_P}$		
s^0	400		

For the coefficients in the first row to be positive, from the s^2 row, $K_P < 5.3$. From the s^1 row,

$$9542.8 + 12285.66K_P - 2857.14K_P^2 > 0 \quad \text{or} \quad (K_P - 4.9718)(K_P + 0.6718) < 0$$

Thus $K_P < 4.9718$ which is the condition for stability.

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{200(K_D s^2 + K_P s + K_I)}{s(s+4)(s^2+10s+50)} = \frac{200(1 + K_{D1}s)(K_{P2}s + K_{I2})}{s(s+4)(s^2+10s+50)}$$

where

$$K_I = K_{I2} \qquad K_D = K_{D1}K_{P2} \qquad K_P = K_{P2} + K_{D1}K_{I2}$$

$$K_v = \lim_{s \to 0} sG(s) = K_I = 2 = K_{I2}$$

$$G(s) = \frac{200(1+K_{D1}s)(K_{P2}s+2)}{s(s+4)(s^2+10s+50)} = \frac{200(K_D s^2 + K_P s + K_I)}{s(s+4)(s^2+10s+50)}$$

From the results in part (a), we set $K_P = 0.6$. The following attributes of the unit-step response show that adding derivative control does not provide any further improvement to the system response.

K_D	Max Overshoot (%)	t_r (sec)	t_s (sec)
0.1	1.1	0.9568	1.247
0.05	0.1	0.792	1.14
0.01	0	0.608	0.9075
0.005	0	0.588	0.8828
0.001	0	0.572	0.8753
0.0005	0	0.570	0.8778

9-29)

Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{(K_P s + K_I)e^{-0.2s}}{s(1+0.25s)} \qquad K_v = \lim_{s \to 0} sG(s) = K_I = 2 \qquad \text{Thus} \quad K_I = 2$$

The attributes of the frequency response for various values of K_P are computed and tabulated below.

K_P	PM (deg)	GM (deg)	M_r	BW (rad/sec)
0.1	49.72	10.91	1.196	3.55
0.2	54.5	12.58	1.092	3.54
0.3	59.0	13.15	1.027	3.56
0.5	67.07	11.88	1.000	3.81
0.6	70.50	10.92	1.000	4.20
0.7	73.41	9.98	1.000	5.09
0.8	75.65	9.10	1.000	6.62
0.9	76.93	8.27	1.000	7.99
1.0	77.04	7.50	1.000	9.05
1.1	75.81	6.78	1.033	9.90
2.0	31.08	2.03	4.029	13.64
2.4	8.51	0.52	12.55	14.52
2.5545	0	0	∞	

Maximum phase margin of 77.04 deg is obtained when $K_P = 1.0$. In Problem 10-13(a), $K_P = 0.6$ is chosen for 0 maximum overshoot and minimum settling time.

The critical value of K_P for stability is 2.5545. In Problem 10-13(a), the critical value of K_P is 4.9718.

9-30) If there is no disturbance then

$$G(s) = \frac{0.9}{s^2}\left(\frac{2}{s+2}\right)$$

Let's consider PID controller as:

$$G_c(s) = \left(1 + K_D(s)\right)\left(K_p + K_I(s)\right) = \frac{K}{s}[(\tau_D s + 1)(s + 1/\tau_I)]$$

If τ_D is sufficiently smaller than τ_I then the τ_I has minor effect in PID controller. Let's examine PID controller when τ_D is varied.

$$\text{If}\begin{cases} \tau_D \leq 0.5\ then\ the\ PM < 180^o \\ \tau_D \geq 100\ then\ the\ PM < -90^o \\ \tau_D = 10\ then\ PM = 65^o\ and\ \omega = 0.5\frac{rad}{sec} \\ \tau_D \leq 20\ then\ max\{\omega_c\} = 1\frac{rad}{sec} \end{cases}$$

so let's consider $\tau_D = 10$, then $\frac{1}{\tau_D} = 0.1$

As τ_I to be large enough with respect to τ_D, let $\tau_I = 20\ \tau_D = 200 \Rightarrow \frac{1}{\tau_I} = 0.005$

Now we have to determine the value of K so that the gain at the crossover frequency remains at 1. If K=1 then $|G_c(s)G(s)|_{\omega=0.5} = 20$. Therefore, $\frac{1}{K} = 20$, or K=0.05

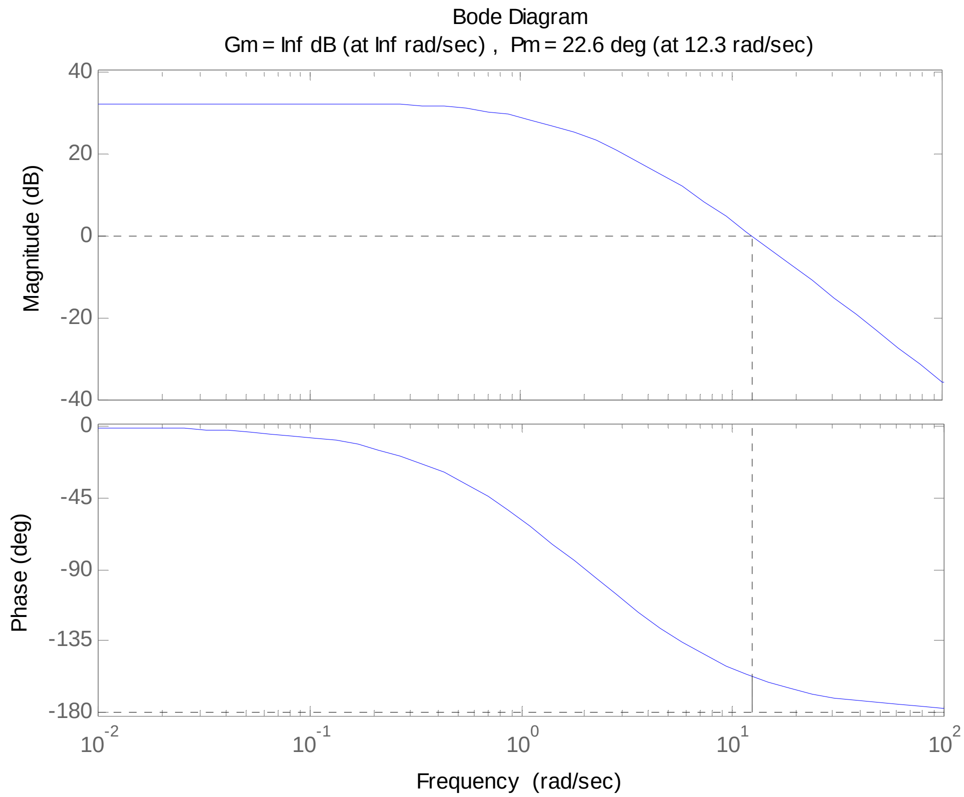
9-31) Let

$$G_c(s) = \frac{1+\tau_2 s}{1+\frac{\tau_2}{\alpha_2}s}\frac{K\alpha_1(1+\tau_1 s)}{1+\alpha_1\tau_1 s}$$

where $e_{ss} \leq 0.01$. therefore, $lim_{s\to 0}\, G_c(s)G(s) = 10K\ \alpha_1 + 1 > 100$

which gives$K\alpha_1 = 10$. Now $K\alpha_1 = 10$,then PM = -40^o. As a result, 91° phase lead is required to achieve PM = 45^o

The crossover frequency is 12.3 rad/sec as can be seen in the uncompensated bode diagram. The lag compensator must position$\omega_c = 5\text{rad/sec}$, where its gain is 17.5 dB. Therefore the ratio of lag compensator can be chosen for this purpose as $3 < \frac{1}{\alpha_2} < 10$



Now the gain, which is obtained from combination of lead and lag compensator, is

$$Gain = K\alpha_1 \left|\frac{1+j5\tau_1}{1+j5\alpha_1\tau_1}\right| \cdot \left|\frac{1+j5\,\tau_2}{1+j5\,\frac{\tau_2}{\alpha_2}}\right|$$

or

$$gain_{dB} = 20\,log\left[K\alpha_1\left|\frac{1+j5\,\tau_1}{1+j5\,\alpha_1\tau_1}\right|\right] + 20log\left|\frac{1+j5\,\tau_2}{1+j5\,\frac{\tau_2}{\alpha_2}}\right|$$

where $\left|\frac{1+j5\,\tau_1}{1+j5\,\alpha_1\tau_1}\right| > 1$ for $\alpha_1 < 1$. Since it is required that the final gain is increased by 17.5 dB, let's choose $\alpha_2 = \frac{1}{15}$.

On the other hand, the corner frequency is $\frac{1}{\tau_2} = 0.5\frac{rad}{sec}$ ➔ $\tau_2 = 2$. Therefore,

$$20log\left|\frac{1+j5\tau_1}{1+j5\alpha_1\tau_1}\right|_{\substack{\alpha_2=\frac{1}{15}\\ \tau_2=2}} = 23.5$$

$$20log\left|\frac{1+j5\,\tau_2}{1+j5\dfrac{\tau_2}{\alpha_2}}\right| = 23.5 - 17.5 = 6dB$$

As a result, the actual phase reduction is $\Phi = tan^{-1}\left(\frac{1}{\alpha_2}\tau_2\omega\right) - tan^{-1}(\tau_2\omega) = 5.33$

The required phase lead is $\Phi = 45 + 5.33 + 3 = 53.33$, where $\Phi_m(\omega) = tan^{-1}(\omega\tau_1) - tan^{-1}(\omega\alpha_1\tau_1) = 53.33$

By trial and error, we can find $\alpha_1 = 0.068$ and $\tau_1 = 0.35$. Therefore $K = 147$ where $K = \frac{10}{\alpha_1}$

9-32)

(a)

$$G_p(s) = \frac{Z(s)}{F(s)} = \frac{1}{Ms^2 + K_s} = \frac{1}{150s^2 + 1} = \frac{0.00667}{s^2 + 0.00667}$$

The transfer function $G_p(s)$ has poles on the $j\omega$ axis. The natural undamped frequency is $\omega_n = 0.0816$ rad/sec.

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{0.00667\left(K_D s^2 + K_P s + K_I\right)}{s\left(s^2 + 0.00667\right)}$$

$K_v = \lim_{s \to 0} sG(s) = K_I = 100$ Thus $K_I = 100$

Characteristic Equation: $s^3 + 0.00667K_D s^2 + 0.00667\left(1 + K_P\right)s + 0.00667K_I = 0$

For $\zeta = 0.707$ and $\omega_n = 1$ rad / sec, the second-order term of the characteristic equation is $s^2 + 1.414s + 1 = 0$. Divide the characteristic equation by the second-order term.

$$s + \left(0.00667K_D - 1.414\right)$$

$$s^2 + 1.414s + 1 \left| s^3 + 0.00667K_D s^2 + \left(0.00667 + 0.00667K_P\right)s + 0.00667K_I \right.$$

$$s^3 + 1.414s^2 + 1$$

$$\left(0.00667K_D - 1.414\right)s^2 + \left(0.00667K_P - 0.99333\right)s + 0.00667K_I$$

$$\left(0.00667K_D - 1.414\right)s^2 + \left(0.00943K_D - 2\right)s + 0.00667K_d - 1.414$$

$$\left(0.00667K_P - 0.00943K_D + 1.00667\right)s + 0.00667K_I - 0.00667K_D + 1.41$$

For zero remainder, $0.00667K_I - 0.00667K_D + 1.414 = 0$ (1)

and $0.00667K_P - 0.00943K_D + 1.00667 = 0$ (2)

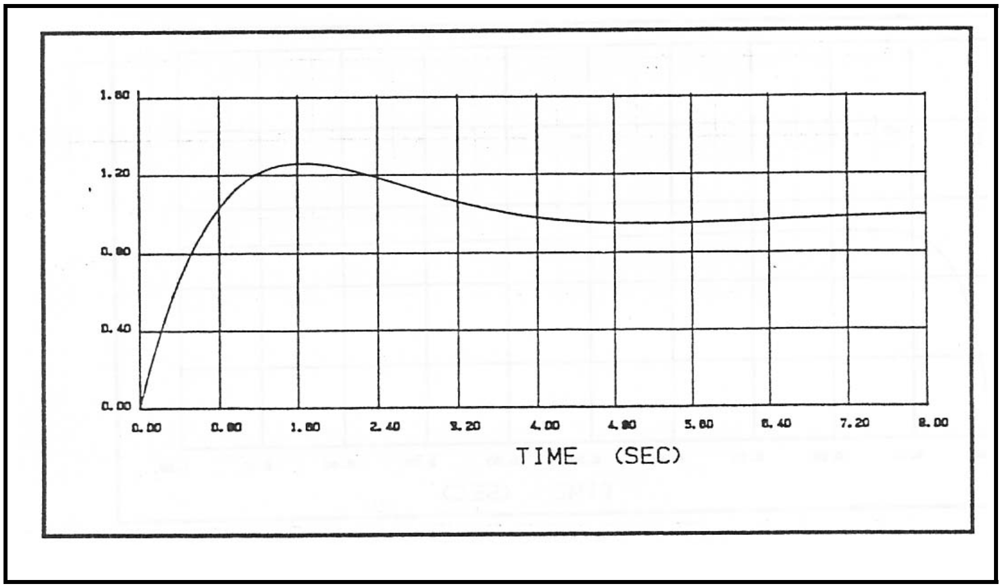
From Eq. (1), $K_D = \frac{2.081}{0.00667} = 312$

From Eq. (2), $K_P = \frac{0.00943K_D - 1.00667}{0.00667} - 290.18$

The forward-path transfer function becomes,

$$G(s) = \frac{2.081s^2 + 1.9355s + 0.667}{s\left(s^2 + 0.00667\right)}$$

Unit-step Response.



The unit-step response shows a maximum overshoot of 26%. Although the relative damping ratio of the complex roots is 0.707, the real pole of the third-order system transfer function is at –0.667 which adds to the overshoot.

(c)

$$G(s) = G_c(s)G_p(s) = \frac{0.00667\left(1+K_{D1}s\right)\left(K_{P2}s+K_{I2}\right)}{s\left(s^2+0.00667\right)}$$

For $K_v = 100, K_{I2} = K_I = 100.$ Let us select $K_{P2} = 50.$ Then

$$G(s) = \frac{0.00667\left(1+K_{D1}s\right)\left(50s+100\right)}{s\left(s^2+0.00667\right)}$$

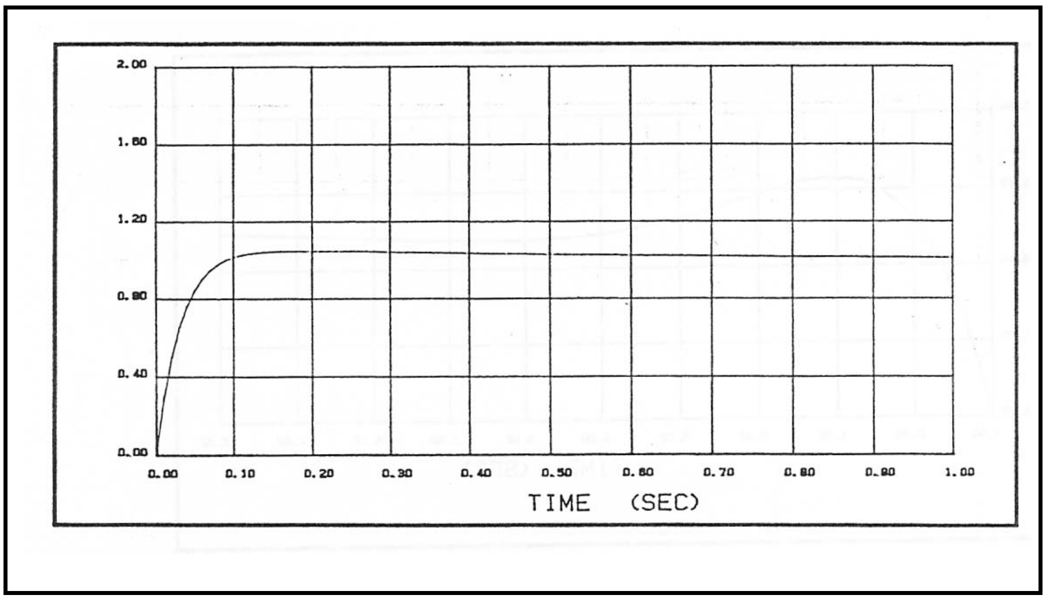
For a small overshoot, K_{D1} must be relatively large. When $K_{D1} = 100,$ the maximum overshoot is approximately 4.5%. Thus,

$$K_P = K_{P2} + K_{D1}K_{I2} = 50 + 100 \times 100 = 10050$$
$$K_D = K_{D1}K_{P2} = 100 \times 50 = 5000$$
$$K_I = 100$$

System Characteristic Equation: $s^3 + 33.35s^2 + 67.04s + 0.667 = 0$

Roots: **–0.01, –2.138, –31.2**

Unit-step Response.



9-33)(a)

$$G_p(s) = \frac{Z(s)}{F(s)} = \frac{1}{Ms^2 + K_s} = \frac{1}{150s^2 + 1} = \frac{0.00667}{s^2 + 0.00667}$$

The transfer function $G_p(s)$ has poles on the $j\omega$ axis. The natural undamped frequency is

$\omega_n = 0.0816$ rad/sec.

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{0.00667\left(K_D s^2 + K_P s + K_I\right)}{s\left(s^2 + 0.00667\right)}$$

$$K_v = \lim_{s \to 0} sG(s) = K_I = 100 \quad \text{Thus} \quad K_I = 100$$

Characteristic Equation: $s^3 + 0.00667K_D s^2 + 0.00667\left(1 + K_P\right)s + 0.00667K_I = 0$

For $\zeta = 1$ and $\omega_n = 1$ rad/sec, the second-order term of the characteristic equation is $s^2 + 2s + 1.$

Dividing the characteristic equation by the seond-order term.

$$\begin{array}{r|l}
 & s + \left(0.00667K_D - 2\right) \\
\hline
s^2 + 2s + 1 & s^3 + 0.00667K_D s^2 + \left(0.00667 + 0.00667K_P\right)s + 0.00667K_I \\
 & s^3 + 2s^2 + s \\
\hline
 & \left(0.00667K_D - 2\right)s^2 + \left(0.00667K_P - 0.99333\right)s + 0.00667K_I \\
 & \left(0.00667K_D - 2\right)s^2 + \left(0.01334K_D - 4\right)s + 0.00667K_D - 2 \\
\hline
 & \left(0.00667K_P - 0.01334K_D + 3.00667\right)s + 0.00667K_I - 0.00667K_D + 2
\end{array}$$

For zero remainder,

$$0.00667K_P - 0.01334K_D + 3.00667 = 0 \qquad (1)$$

$$-0.00667K_D + 0.00667K_I + 2 = 0 \qquad (2)$$

From Eq. (2),

$$0.00667K_D = 0.00667K_I + 2 = 2.667 \quad \text{Thus} \quad K_D = 399.85$$

From Eq. (1),

$$0.00667K_P = 0.01334K_D - 3.00667 = 2.3273 \qquad \text{Thus} \quad K_P = 348.93$$

Forward-path Transfer Function:

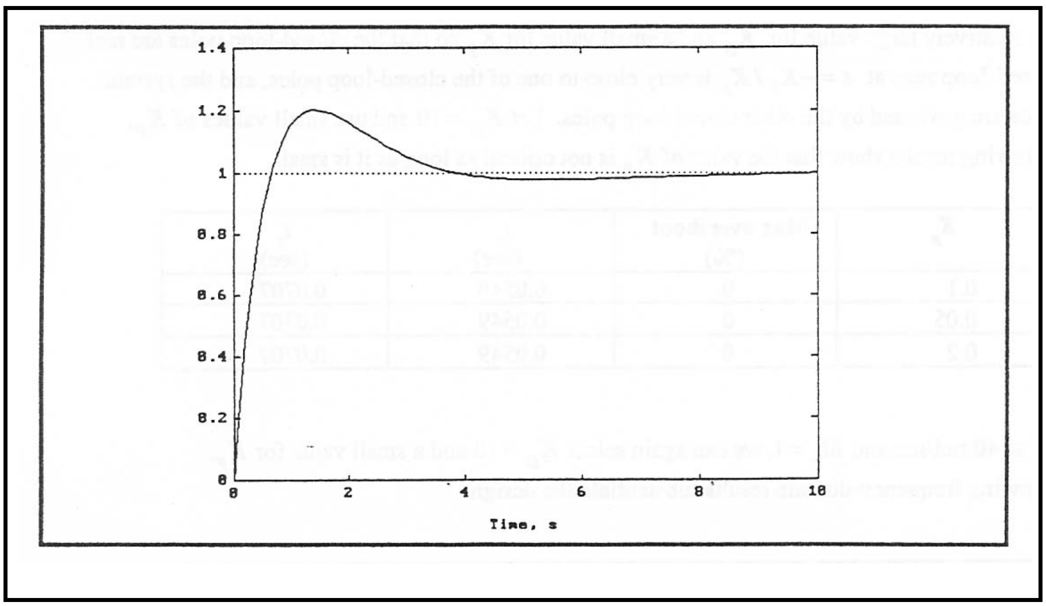
$$G(s) = \frac{0.00667\left(399.85s^2 + 348.93s + 100\right)}{s\left(s^2 + 0.00667\right)}$$

Characteristic Equation:

$$s^3 + 2.667s^2 + 2.334s + 0.667 = \left(s+1\right)^2\left(s+0.667\right) = 0$$

Roots: −1, −1, −0.667

Unit-step Response.



The maximum overshoot is 20%.

9-34) a) As $M\dot{v} + \mu v = u(t)$, therefore, $(Ms + \mu)V(s) = U(s)$ or $G(s) = \frac{V(s)}{U(s)} = \frac{1}{Ms+\mu}$

b) $\frac{V(s)}{U(s)} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)}$

According to the second order system:

$$\begin{cases} \omega_n \geq \dfrac{1.8}{t_r} \rightarrow \omega_n \geq \dfrac{1.8}{5} \rightarrow \omega_n \geq 0.36 \\ \xi \geq \sqrt{\dfrac{\left(\dfrac{lnM_p}{\pi}\right)^2}{1+\left(\dfrac{lnM_p}{K}\right)^2}} \rightarrow \xi \geq 0.6 \end{cases}$$

Let's first add a PD controller with $G_c(s) = 1 + K_D s$, and find K_D which satisfy the maximum overshoot requirement.

After writing the closed-loop transfer function including the PD controller, the characteristic equation (denominator of the closed loop T.F.) is: $s^2 + \frac{(\mu+K_d)}{M}s + \frac{K_p}{M}$

Therefore:

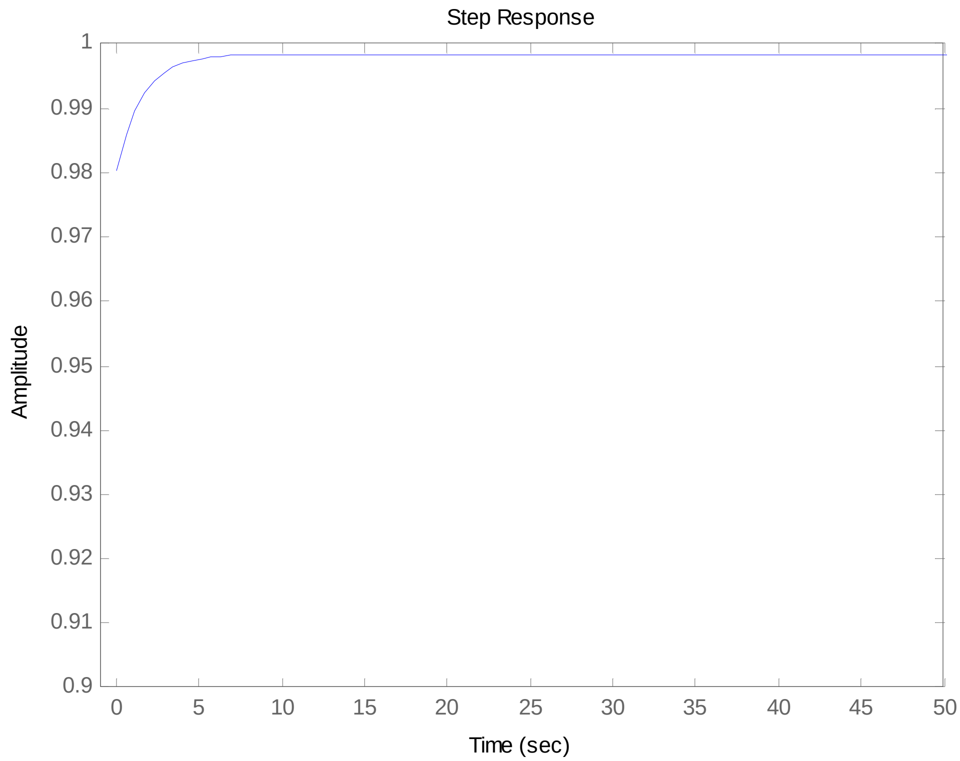
$$K_p = M\omega_n^2 = 1000(0.36^2) = 129.6$$

Also, $\frac{(\mu+K_d)}{M} = 2\xi\omega_n \rightarrow K_d = 2M\xi\omega_n - \mu = 2(1000)(0.6)(0.36) - 50 = 382$ Nsec/m

Now let's add a PI controller with $G_c(s) = K_p + \frac{K_I}{s}$, and find K_p and K_I by using following table.

Now, for the PI part, K_I should be selected so that the additional pole at $-\frac{K_I}{K_p}$ does not interfere with the system dynamics. This pole is usually placed at least 1 decade lower (frequency wise) than the slowest existing poles of the system. In this case, since $\frac{\mu}{M} = \frac{50}{1000} = 0.05$, let's have $\frac{K_I}{K_p} = 0.005$, resulting in $K_I = K_p(0.005) = 129.6(0.005) = 0.648$

The step response is obtained through the following MATLAB code, showing the rise time of less than 5 sec, and almost no overshoot



MATLAB code:

```
s = tf('s')

Kp = 129.6
Kd = 382
Ki = 0.648

num_GH= (Kp*Kd*s)*(1+Ki/s);
den_GH=(1000*s+50);
GH=num_GH/den_GH;
CL = GH/(1+GH)

figure(1)
step(CL)
xlim([-1 50])
ylim([0.9 1])
```

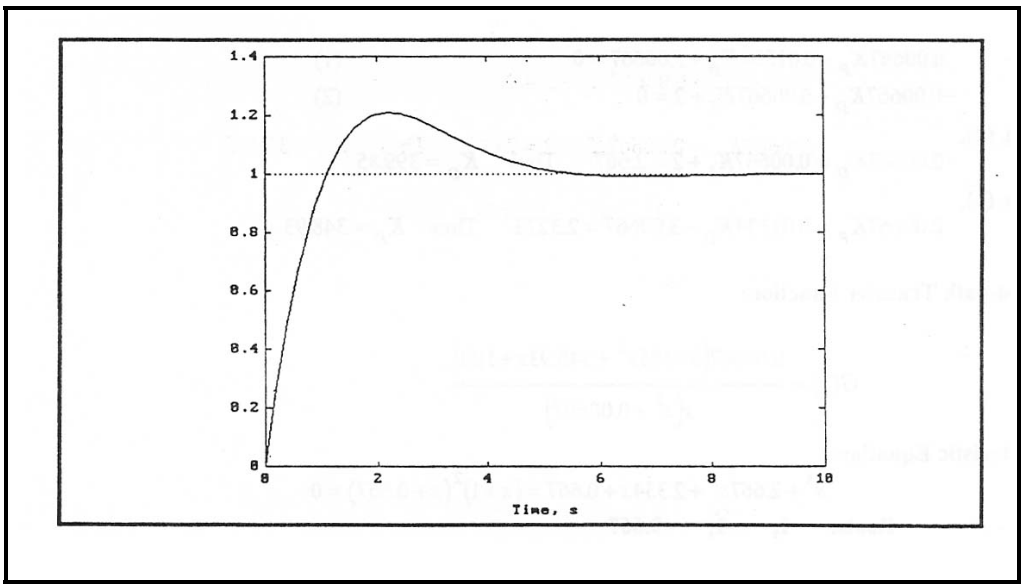
9-35)(a) Process Transfer Function: **Forward-path Transfer Function**

$$G_p(s) = \frac{4}{s^2} \qquad G(s) = G_c(s)G_p(s) = \frac{4\left(K_P + K_D s\right)}{s^2}$$

Characteristic Equation: $s^2 + 4K_D s + 4K_P = s^2 + 1.414s + 1 = 0$ for $\zeta = 0.707,\ \omega_n = 1$ rad/sec

$K_P = 0.25$ and $K_D = 0.3535$

Unit-step Response.



Maximum overshoot = 20.8%

(b) Select a relatively large value for K_D and a small value for K_P so that the closed-loop poles are real.

The closed-loop zero at $s = -K_P / K_D$ is very close to one of the closed-loop poles, and the system dynamics are governed by the other closed-loop poles. Let $K_D = 10$ and use small values of K_P.

The following results show that the value of K_P is not critical as long as it is small.

K_p	**Max overshoot (%)**	t_r **(sec)**	t_s **(sec)**
0.1	0	0.0549	0.0707
0.05	0	0.0549	0.0707
0.2	0	0.0549	0.0707

(c) For BW $\leq$ 40 rad/sec and $M_r = 1$, we can again select $K_D = 10$ and a small value for K_P.

The following frequency-domain results substantiate the design.

K_p	**PM (deg)**	M_r	**BW (rad/sec)**
0.1	89.99	1	40
0.05	89.99	1	40
0.2	89.99	1	40

9-36) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{10,000\left(K_P + K_D s\right)}{s^2\left(s+10\right)}$$

Characteristic Equation:

$$s^3 + 10s^2 + 10{,}000K_D s + 10{,}000K_P = 0$$

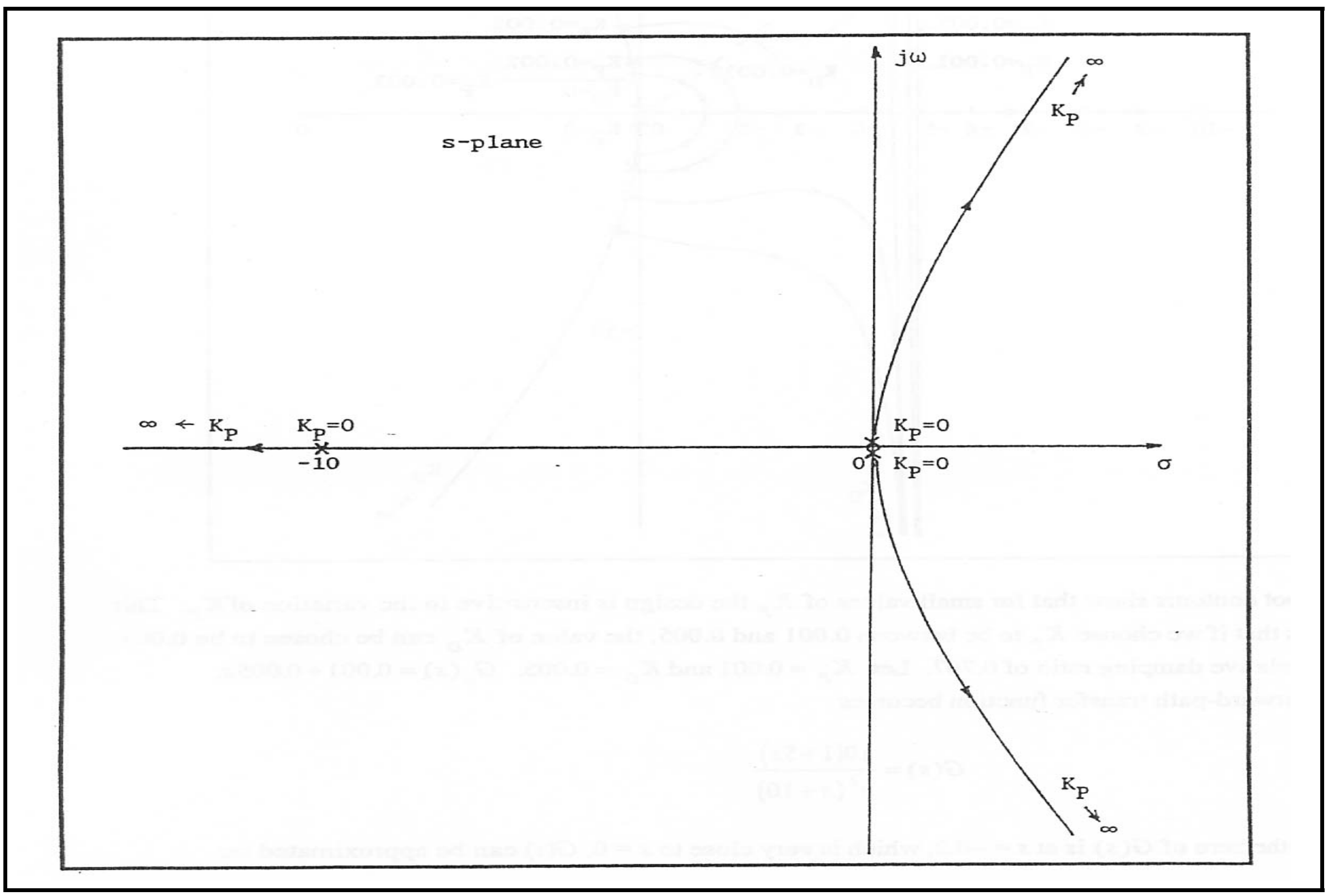
Routh Tabulation:

s^3	1	$10{,}000K_D$
s^2	10	$10{,}000K_P$
s^1	$10{,}000K_D - 1000K_P$	0
s^0	$10{,}000K_P$	

The system is stable for $K_P > 0$ and $K_D > 0.1K_P$

(b) Root Locus Diagram:

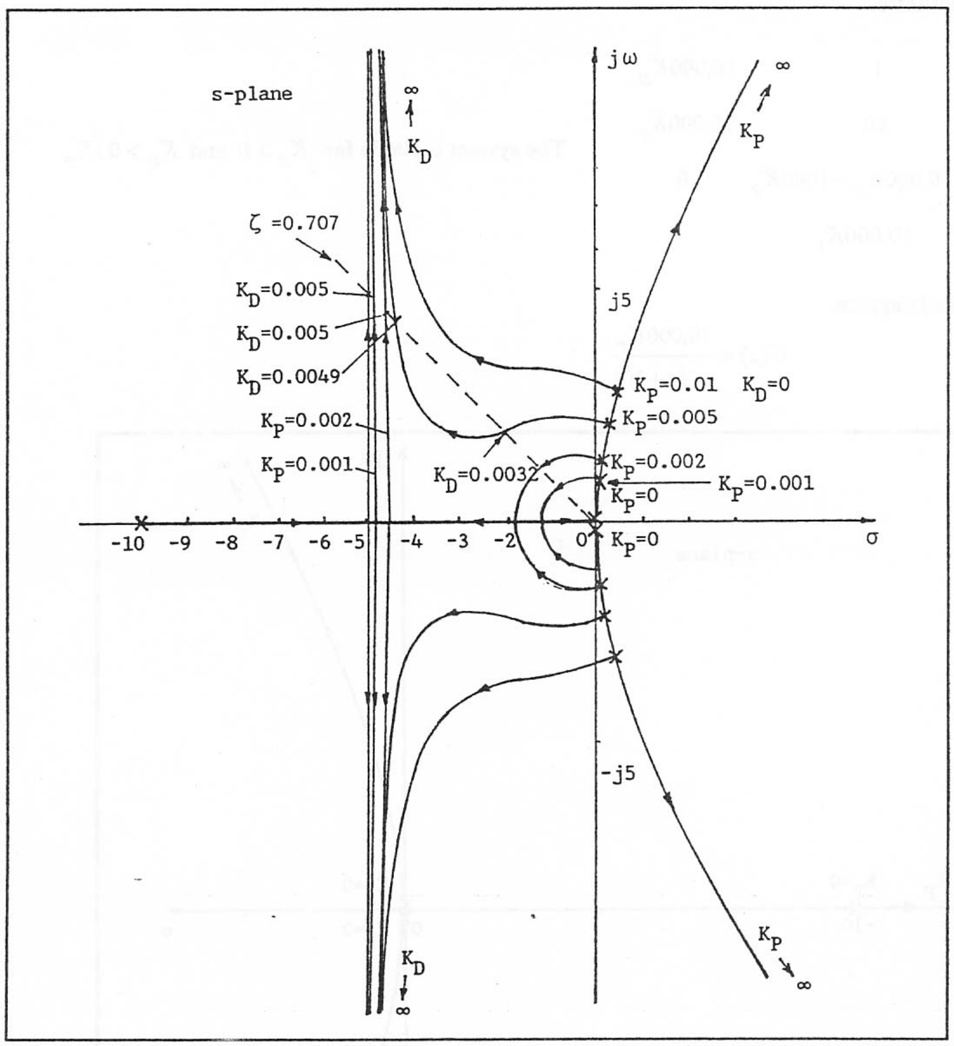
$$G(s) = \frac{10,000K_P}{s^2\left(s+10\right)}$$



Root Contours:

$0 \le K_D < \infty$, $K_P = 0.001, 0.002, 0.005, 0.01.$

$$G_{eq}(s) = \frac{10{,}000 K_D s}{s^3 + 10s^2 + 10{,}000 K_P}$$



(c) The root contours show that for small values of K_P the design is insensitive to the variation of K_P. This means that if we choose K_P to be between 0.001 and 0.005, the value of K_D can be chosen to be 0.005 for a relative damping ratio of 0.707. Let $K_P = 0.001$ and $K_D = 0.005$. $G_c(s) = 0.001 + 0.005s$. The forward-path transfer function becomes

$$G(s) = \frac{10(1+5s)}{s^2(s+10)}$$

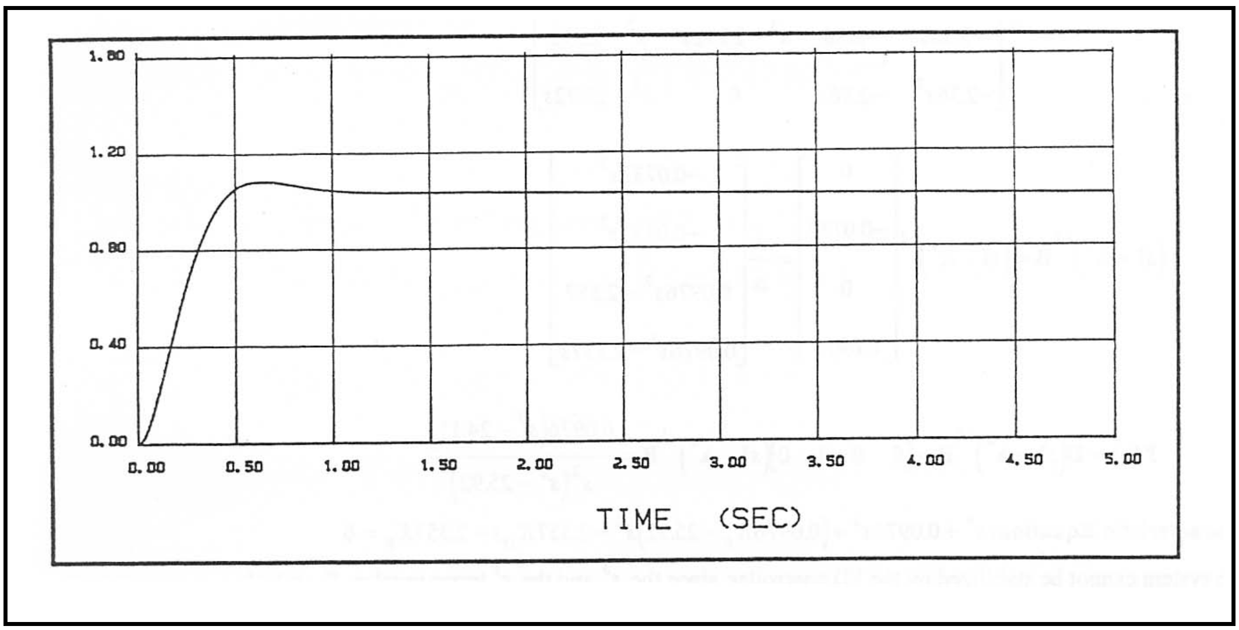
Since the zero of $G(s)$ is at $s = -0.2$, which is very close to $s = 0$, $G(s)$ can be approximated as:

$$G(s) \cong \frac{50}{s(s+10)}$$

For the second-order system, $\zeta = 0.707$. Using Eq. (7-104), the rise time is obtained as

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.306 \text{ sec}$$

Unit-step Response:

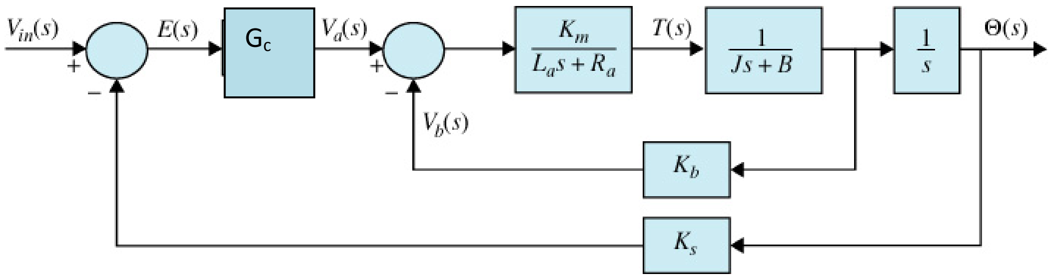


(d) Frequency-domain Characteristics:

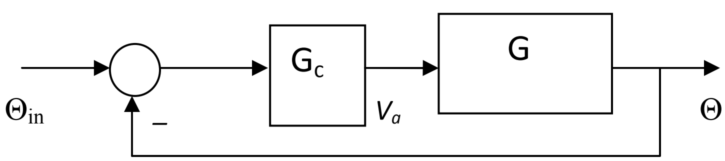
$$G(s) = \frac{10(1+5s)}{s^2(s+10)}$$

PM (deg)	GM (dB)	M_r	BW (rad/sec)
63	$\propto$	1.041	7.156

9-37) This problem is extensively discussed in Chapters 5 and 6. Use the transfer function (5-123) for the open-loop system, and a series PID compensator in a unity feedback system.



Reduce to:



Where:

$$G(s) = \frac{\Theta(s)}{V_a(s)} = \frac{K_t}{s\left(L_a J s^2 + (L_a B + R_a J)s + R_a B + K_t K_b\right)}$$

$$G_c(s) = K_p + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_p s + K_I}{s}$$

With
the rotor inertia (J) = 0.01 kg.m2/s2
damping ratio of the mechanical system (B) = 0.1 Nms
back-emf constant (Kb) = 0.01 Nm/Amp
torque constant (Kt) = 0.01 Nm/Amp
armature resistance (Ra) = 1 Ω
armature inductance (La) = 0.5 H

Starting systematically, set K_I=K_D=0. Assume a small electric time constant (or small inductance) and simplify to Equation (5-126):

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{\dfrac{KK_mK_s}{R_a}}{(\tau_e s+1)\left\{J_m s^2 + \left(B + \dfrac{K_bK_m}{R_a}\right)s + \dfrac{KK_mK_s}{R_a}\right\}} \qquad (5\text{-}125)$$

Where K_s is the sensor gain, and, as before, $\tau_e = (L_a/R_a)$ may be neglected for small L_a.

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{\frac{KK_tK_s}{R_aJ}}{s^2 + \left(\frac{R_aB + K_tK_b}{R_aJ}\right)s + \frac{KK_tK_s}{R_aJ}}$$

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{\frac{KK_tK_s}{R_aJ}}{s^2 + \left(\frac{R_aB + K_tK_b}{R_aJ}\right)s + \frac{KK_tK_s}{R_aJ}} \qquad (5\text{-}126)$$

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{\frac{K_p 0.1}{0.01}}{s^2 + \left(\frac{0.1 + (0.01)(0.01)}{0.01}\right)s + \frac{K_p 0.1}{0.01}}$$

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{10K_p}{s^2 + 12\,s + 10K_p}$$

Where K_s=0.

Using $t_s \cong \frac{3.2}{\zeta\omega_n}$; for a less than 2 sec settling time $\zeta\omega_n \le 1.6$

For a PO of 4.3, ζ=0.707, resulting in ω_n=2.26.

Then a standard 2nd order prototype system that will have the desired response, with zero steady state error, takes the following form

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{5.12}{s^2 + 3.2s + 5.12}$$

For obvious reasons

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{10K_p}{s^2 + 12\,s + 10K_p} \ne \frac{5.12}{s^2 + 3.2s + 5.12}$$

Let's add a PD controller

$$G_c(s) = K_p + K_D s$$

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{10\left(K_D s + K_p\right)}{s^2 + \left(12 + 10K_D\right)\, s + 10K_p} \neq \frac{5.12}{s^2 + 3.2s + 5.12}$$

$$\left(12 + 10K_D\right) = 5.12$$

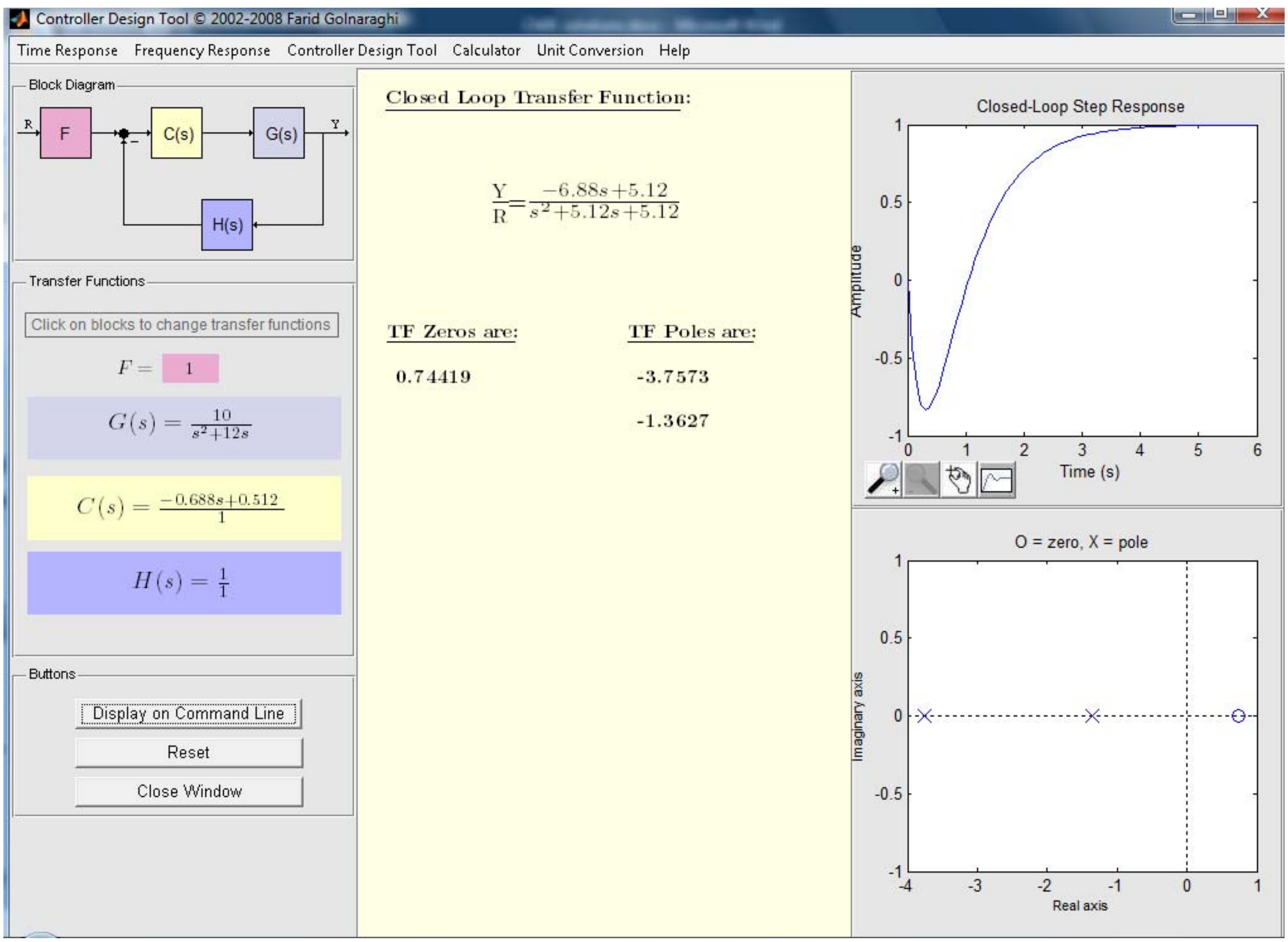
$$10K_p = 5.12$$

$$K_D = -0.688$$

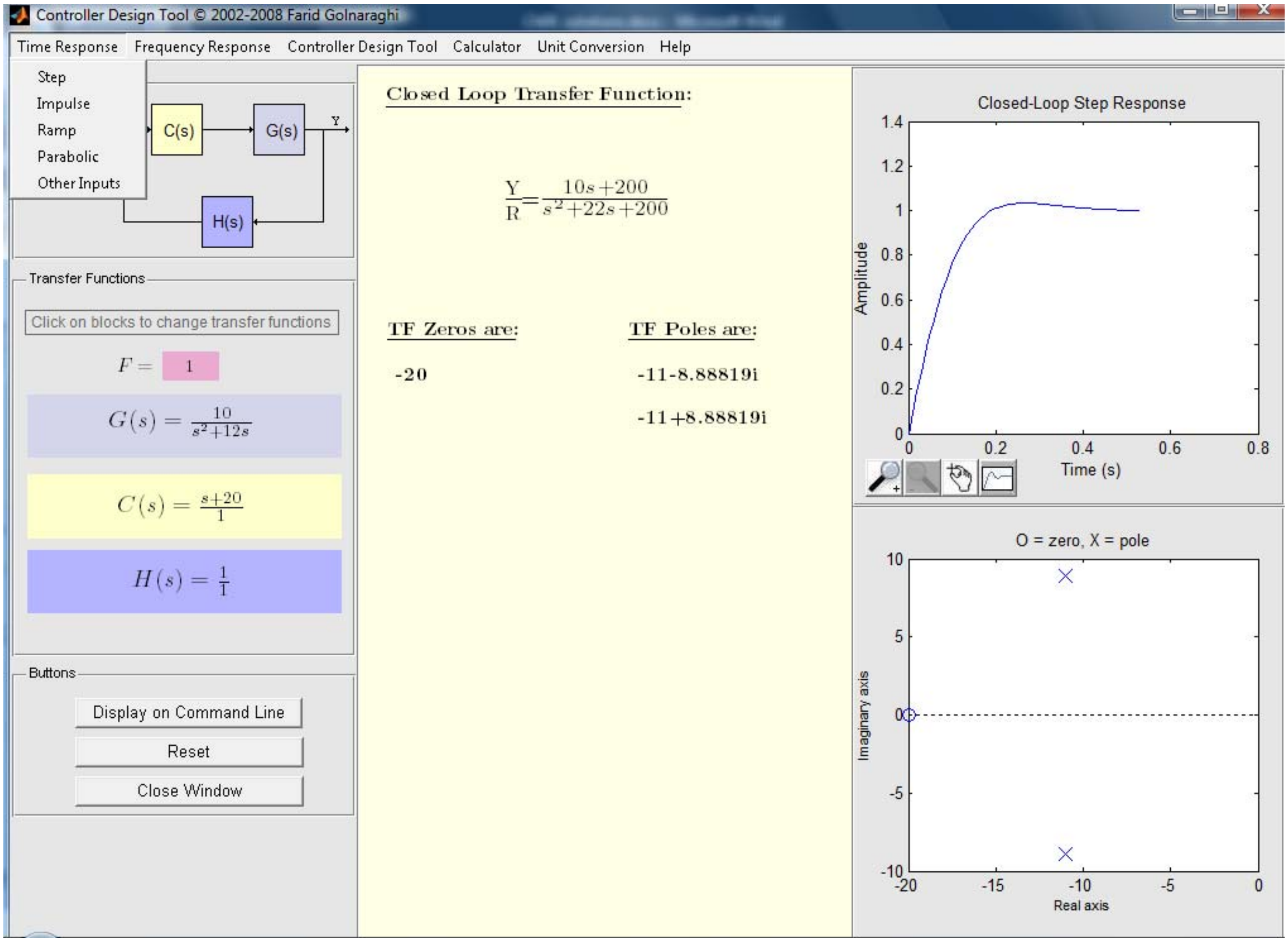
$$K_p = 0.512$$

Although the two systems are not the same because one has a zero, we chose the controller gain values by matching the two characteristic equations – as an initial approximation. The resulting zero in the right hand plane is troubling.

Lets find the response of the system through ACSYS:



Looking at the TF poles, it seems prudent to design the controller by placing its zero farther to LHS of the s-plane. Set z=-Kp/KD=-20 and vary KD to find the root locus or the response.



Done!

9-38) The same as 9-37

9-39)

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \qquad s\mathbf{I} - \mathbf{A}^* = \begin{bmatrix} s & -1 & 0 & 0 \\ -25.92 & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 2.36 & 0 & 0 & s \end{bmatrix}$$

$$\Delta = \left| s\mathbf{I} - \mathbf{A}^* \right| = s\begin{vmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{vmatrix} + \begin{vmatrix} -25.92 & 0 & 0 \\ 0 & s & -1 \\ 2.36 & 0 & s \end{vmatrix} = s^2\left(s^2 - 25.92\right)$$

$$\left(s\mathbf{I} - \mathbf{A}^*\right)^{-1} = \frac{1}{\Delta}\begin{bmatrix} s^3 & s^2 & 0 & 0 \\ 25.92s^2 & s^3 & 0 & 0 \\ -2.36s & -2.36 & s^3 - 25.92s & s^2 - 25.92 \\ -2.36s^2 & -2.36s & 0 & s^3 - 25.92s \end{bmatrix}$$

$$\left(s\mathbf{I} - \mathbf{A}^*\right)^{-1}\mathbf{B} = \left(s\mathbf{I} - \mathbf{A}^*\right)^{-1}\begin{bmatrix} 0 \\ -0.0732 \\ 0 \\ 0.0976 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -0.0732s^2 \\ -0.0732s^3 \\ 0.0976s^2 - 2.357 \\ 0.0976s^3 - 2.357s \end{bmatrix}$$

$$Y(s) = \mathbf{D}\left(s\mathbf{I} - \mathbf{A}^*\right)^{-1}\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\left(s\mathbf{I} - \mathbf{A}^*\right)^{-1}\mathbf{B} = \frac{0.0976\left(s^2 - 24.15\right)}{s^2\left(s^2 - 25.92\right)}$$

Characteristic Equation: $s^4 + 0.0976s^3 + \left(0.0976K_P - 25.92\right)s^2 - 2.357K_D s - 2.357K_P = 0$

The system cannot be stabilized by the PD controller, since the s^3 and the s^1 terms involve K_D which require opposite signs for K_D.

9-40)

Let us first attempt to compensate the system with a PI controller.

$$G_c(s) = K_P + \frac{K_I}{s} \qquad \text{Then} \qquad G(s) = G_c(s)G_p(s) = \frac{100\left(K_P s + K_I\right)}{s\left(s^2 + 10s + 100\right)}$$

Since the system with the PI controller is now a type 1 system, the steady-state error of the system due to a step input will be zero as long as the values of K_P and K_I are chosen so that the system is stable.

Let us choose the ramp-error constant $K_v = 100$. Then, $K_I = 100$. The following frequency-domain performance characteristics are obtained with $K_I = 100$ and various value of K_P ranging from 10 to 100.

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)
10	1.60	$\propto$	29.70	50.13
20	6.76	$\propto$	7.62	69.90
30	7.15	$\propto$	7.41	85.40
40	6.90	$\propto$	8.28	98.50
50	6.56	$\propto$	8.45	106.56
100	5.18	$\propto$	11.04	160.00

The maximum phase margin that can be achieved with the PI controller is only 7.15 deg when $K_P = 30$.

Thus, the overshoot requirement cannot be satisfied with the PI controller alone.

Next, we try a PID controller.

$$G_c(s) = K_P + K_D s + \frac{K_I}{s} = \frac{(1+K_{D1}s)(K_{P2}s+K_{I2})}{s} = \frac{(1+K_{D1}s)(K_{P2}s+100)}{s}$$

Based on the PI-controller design, let us select $K_{P2} = 30.$ Then the forward-path transfer function becomes

$$G(s) = \frac{100(30s+100)(1+K_{D1}s)}{s(s^2+10s+100)}$$

The following attributes of the frequency-domain performance of the system with the PID controller are obtained for various values of K_{D1} ranging from 0.05 to 0.4.

K_{D1}	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.05	85.0	∝	1.04	164.3
0.10	89.4	∝	1.00	303.8
0.20	90.2	∝	1.00	598.6
0.30	90.2	∝	1.00	897.0
0.40	90.2	∝	1.00	1201.0

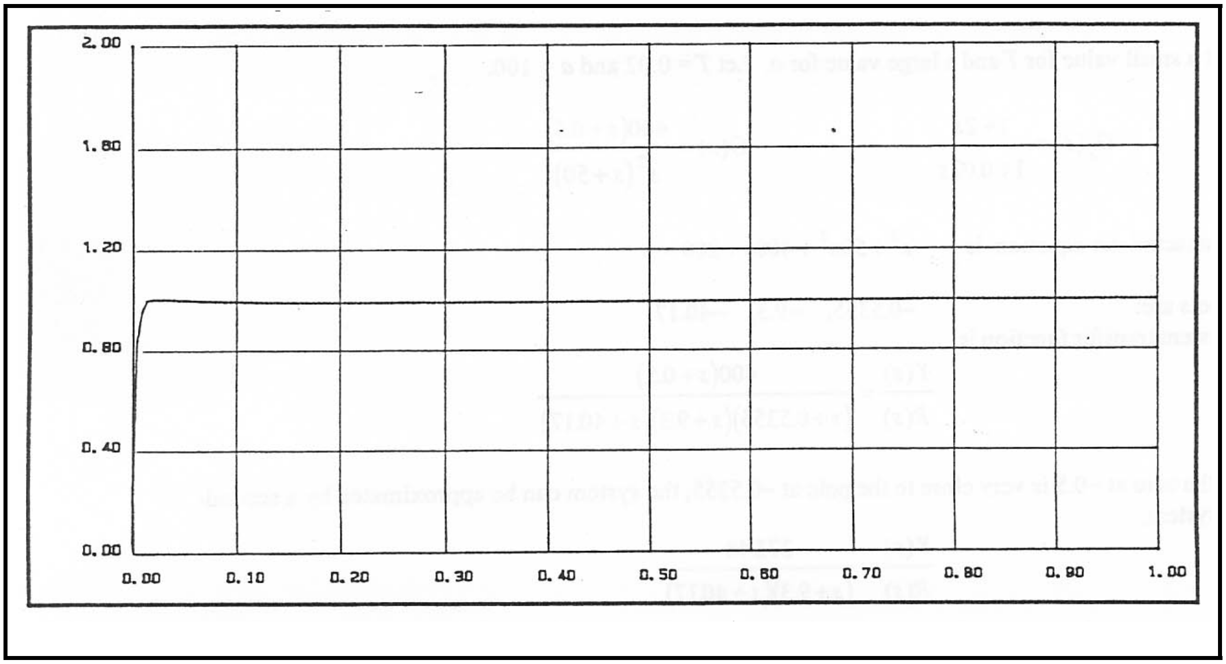
We see that for values of K_{D1} greater than 0.2, the phase margin no longer increases, but the bandwidth increases with the increase in K_{D1}. Thus we choose

$$K_{D1} = 0.2, \quad K_I = K_{I2} = 100, \quad K_D = K_{D1}K_{P2} = 0.2 \times 30 = 6,$$

$$K_P = K_{P2} + K_{D1}K_{I2} = 30 + 0.2 \times 100 = 50$$

The transfer function of the PID controller is $G_c(s) = 50 + 6s + \frac{100}{s}$

The unit-step response is show below. The maximum overshoot is zero, and the rise time is 0.0172 sec.


2.00
1.60
1.20
0.80
0.40
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00

9-41)

$$\frac{Y(s)}{D(s)} = \frac{s}{s(s^2 + 3.6s + 9) + K(\tau_1 s + 1)(\tau_2 s + 1)}$$
$$= \frac{s}{s^3 + (3.6 + K\tau_1\tau_2)s^2 + (9 + K\tau_1 + K\tau_2)s + K}$$

Let's consider the characteristic equation like:

$$s^3 + (3.6 + K\tau_1\tau_2)s^2 + (9 + K\tau_1 + K\tau_2)s + K = (s + p)(s^2 + 2\,\xi\omega_n + \ \omega_n^2)$$

$t_s = \frac{4}{\xi\omega_n}$ for 2% settling time.

Therefore we can choose $\xi = 0.5$ and $\omega_n = 4\frac{rad}{sec}$ where $2 < t_s < 3$. Now, we can choose pole p far enough from pole dominant of second order. Let p = 10, then the characteristic equation would be:

$$s^3 + (36 + K\tau_1\tau_2)s^2 + (9 + K\tau_1\tau_2)s + K = s^3 + 14s^2 + 56s + 160$$

where $K = 160$, $\tau_1 + \ \tau_2 = 0.29$, and $\tau_1\tau_2 = 0.065$

$$G_c(s) = \frac{160(0.065s^2 + 0.29s + 1)}{s}$$

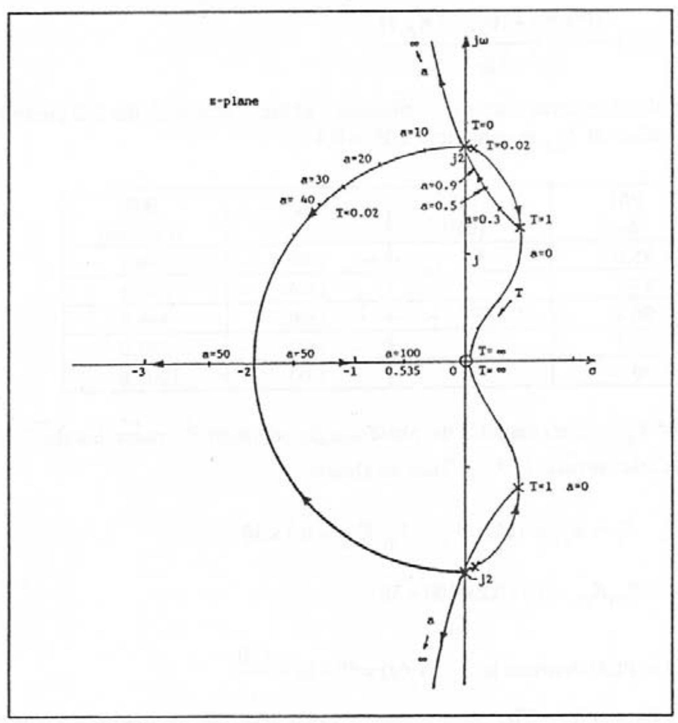
Verify using MATLAB

9-42)

(a)

$$G_p(s) = \frac{4}{s^2} \qquad G(s) = G_c(s)G_p(s) = \frac{4(1+aTs)}{s^2(1+Ts)} \qquad G_{eq}(s) = \frac{4aTs}{Ts^3 + s^2 + 4}$$

Root Contours: (*T* is fixed and *a* varies)



Select a small value for *T* and a large value for *a*. Let *T* = 0.02 and *a* = 100.

$$G_c(s) = \frac{1+2s}{1+0.02s} \qquad G(s) = \frac{400(s+0.5)}{s^2(s+50)}$$

The characteristic equation is $s^3 + 50s^2 + 400s + 200 = 0$

The roots are: –0.5355, –9.3, –40.17

The system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{400(s+0.5)}{(s+0.5355)(s+9.3)(s+40.17)}$$

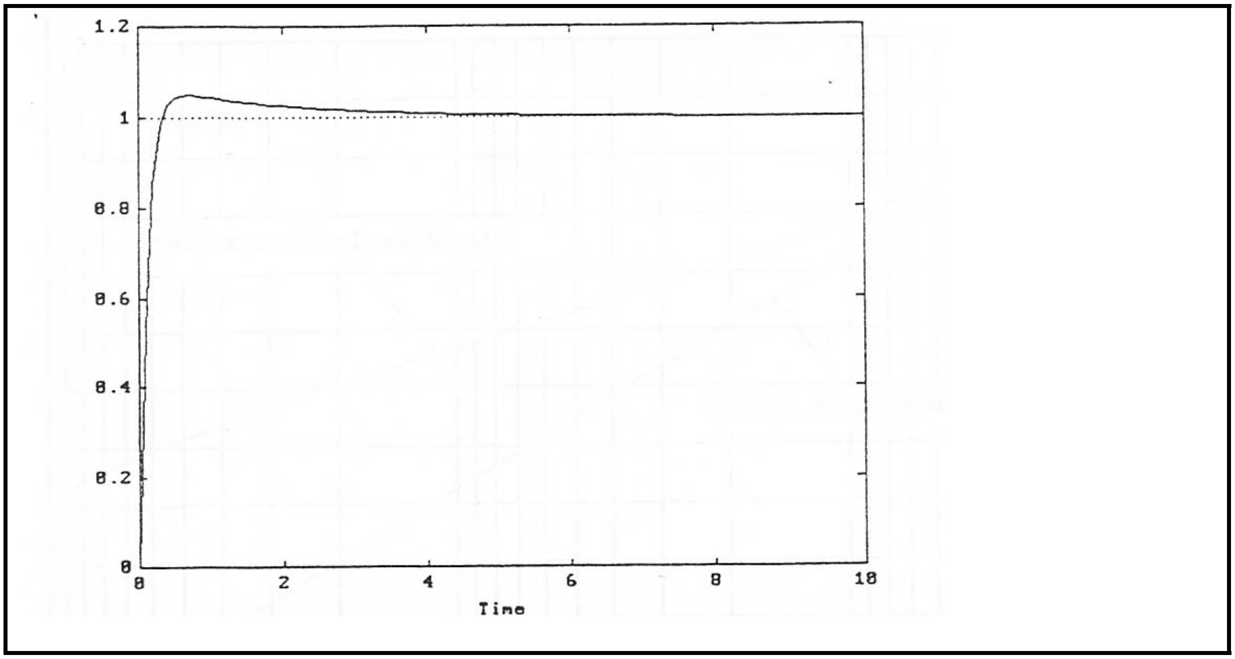
Since the zero at –0.5 is very close to the pole at –0.5355, the system can be approximated by a second-order system,

$$\frac{Y(s)}{R(s)} = \frac{373.48}{(s+9.3)(s+40.17)}$$

The unit-step response is shown below. The attributes of the response are:

Maximum overshoot = 5% $t_s = 0.6225$ sec $t_r = 0.2173$ sec

Unit-step Response.



The following attributes of the frequency-domain performance are obtained for the system with the phase-lead controller.

PM = 77.4 deg   GM = infinite M_r = 1.05   BW = 9.976 rad/sec

(b) The Bode plot of the uncompensated forward-path transfer function is shown below. The diagram shows that the uncompensated system is marginally stable. The phase of $G(j\omega)$ is –180 deg at all frequencies. For the phase-lead controller we need to place ω_m at the new gain crossover frequency to realize the desired phase margin which has a theoretical maximum of 90 deg.

For a desired phase margin of 80 deg,

$$a = \frac{1+\sin 80^o}{1-\sin 80^o} = 130$$

The gain of the controller is $20\log_{10} a = 42$ dB. The new gain crossover frequency is at

$$|G(j\omega)| = -\frac{42}{2} = -21 \text{ dB}$$

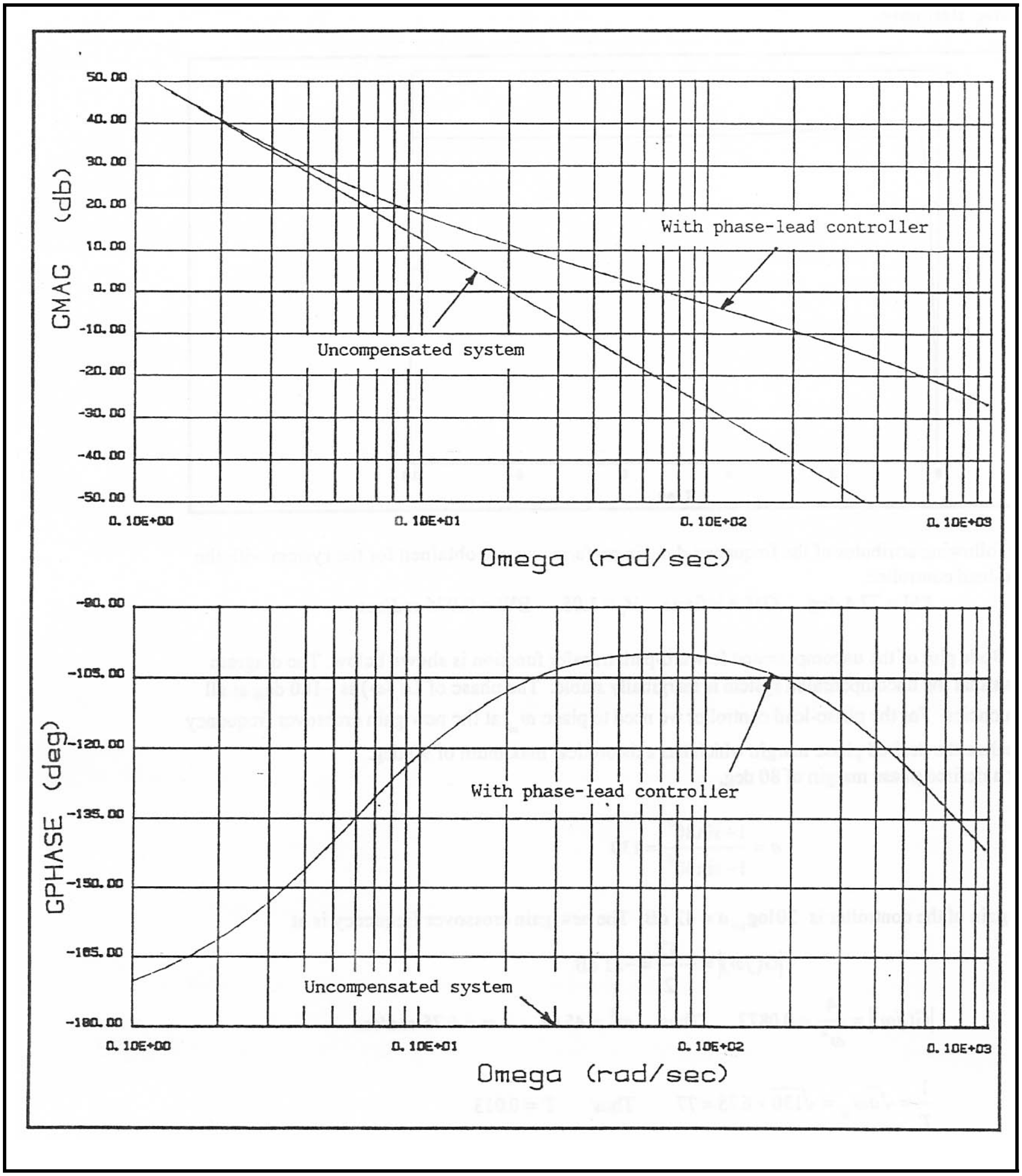
Or $\quad |G(j\omega)| = \frac{4}{\omega^2} = 0.0877 \quad$ Thus $\quad \omega^2 = 45.61 \quad \omega = 6.75$ rad/sec

$$\frac{1}{T} = \sqrt{a}\omega_m = \sqrt{130} \times 6.75 = 77 \qquad \text{Thus} \qquad T = 0.013$$

$$\frac{1}{aT} = 0.592 \qquad \text{Thus} \qquad aT = 1.69$$

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+1.702s}{1+0.0131s} \qquad G(s) = \frac{4(1+1.702s)}{s^2(1+0.0131s)}$$

Bode Plot.



50.00
40.00
30.00
20.00
10.00
0.00
-10.00
-20.00
-30.00
-40.00
-50.00
GMAG (db)
With phase-lead controller
Uncompensated system
0.10E+00
0.10E+01
0.10E+02
0.10E+03
Omega (rad/sec)
-90.00
-105.00
-120.00
-135.00
-150.00
-165.00
-180.00
GPHASE (deg)
With phase-lead controller
Uncompensated system
0.10E+00
0.10E+01
0.10E+02
0.10E+03
Omega (rad/sec)

9-43) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{1000(1+aTs)}{s(s+10)(1+Ts)} = \frac{1000a\left(s+\frac{1}{aT}\right)}{s(s+10)\left(s+\frac{1}{T}\right)}$$

Set $1/aT = 10$ so that the pole of $G(s)$ at $s = -10$ is cancelled. The characteristic equation of the system becomes

$$s^2 + \frac{1}{T}s + 1000a = 0$$

$\omega_n = \sqrt{1000a}$ $\qquad 2\zeta\omega_n = \frac{1}{T} = 2\sqrt{1000a}$ $\qquad$ Thus $a = 40$ and $T = 0.0025$

Controller Transfer Function: **Forward-path Transfer Function:**

$$G_c(s) = \frac{1+0.01s}{1+0.0025s} \qquad G(s) = \frac{40,000}{s(s+400)}$$

The attributes of the unit-step response of the compensated system are:

Maximum overshoot = 0 $\quad t_r = 0.0168 \text{ sec} \quad t_s = 0.02367 \text{ sec}$

(b) Frequency-domain Design

The Bode plot of the uncompensated forward-path transfer function is made below.

$$G(s) = \frac{1000}{s(s+10)}$$

The attributes of the system are PM = 17.96 deg, GM = infinite.

M_r = 3.117, and BW = 48.53 rad/sec.

To realize a phase margin of 75 deg, we need more than 57 deg of additional phase. Let us add an additional 10 deg for safety. Thus, the value of ϕ_m for the phase-lead controller is chosen to be 67 deg. The value of *a* is calculated from

$$a = \frac{1+\sin 67^o}{1-\sin 67^o} = 24.16$$

The gain of the controller is $20\log_{10} a = 20\log_{10} 24.16 = 27.66$ dB. The new gain crossover frequency is at

$$\left|G(j\omega_m^{'})\right| = -\frac{27.66}{2} = -13.83 \text{ dB}$$

From the Bode plot $\omega_m^{'}$ is found to be 70 rad/sec. Thus,

$$\frac{1}{T} = \sqrt{a}T = \sqrt{24.16} \times 70 = 344 \quad \text{or} \quad T = 0.0029 \quad aT = 0.0702$$

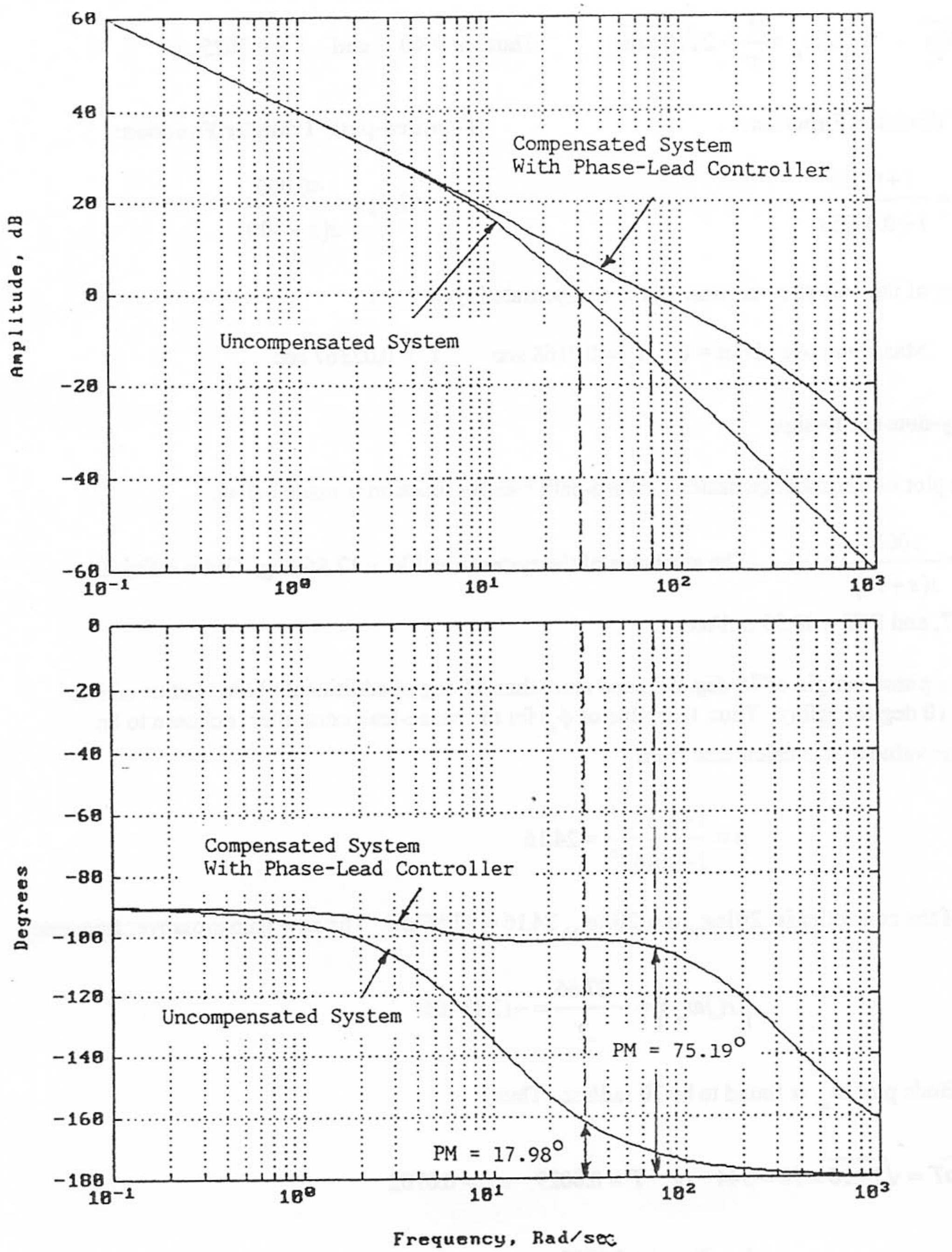
Thus $$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.0702s}{1+0.0029s}$$

The compensated system has the following frequency-domain attributes:

PM = 75.19 deg GM = infinite M_r = 1.024 BW = 91.85 rad/sec

The attributes of the unit-step response are:

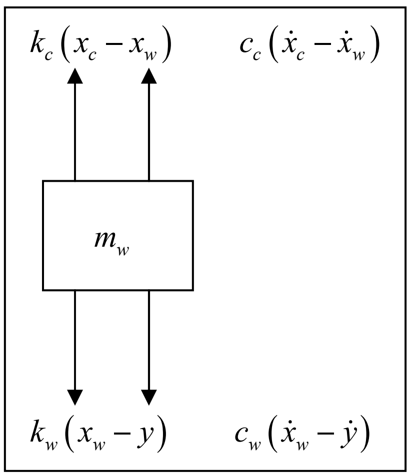
Rise time t_r = 0.02278 sec Settling time t_s = 0.02828 sec Maximum overshoot = 3.3%

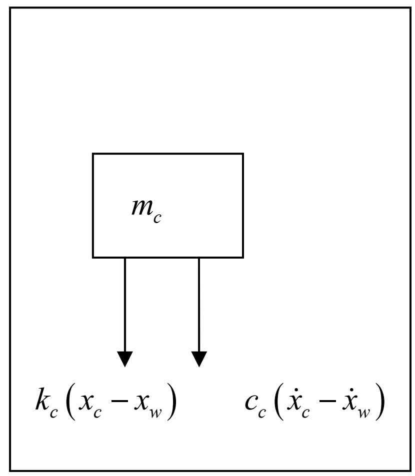

60
40
20
0
-20
-40
-60
Amplitude, dB
Compensated System
With Phase-Lead Controller
Uncompensated System
10-1
100
101
102
103
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
Degrees
Compensated System
With Phase-Lead Controller
Uncompensated System
PM = 75.19°
PM = 17.98°
Frequency, Rad/sec

9-44) Also see Chapter 6 for solution to this problem.

Mathematical Model:

Draw free body diagrams (Assume both x_c and x_w are positive and are measured from equilibrium). Refer to Chapter 4 problems for derivation details.





$$\begin{bmatrix} m_w & 0 \\ 0 & m_c \end{bmatrix}\begin{bmatrix} \ddot{x}_w \\ \ddot{x}_c \end{bmatrix} + \begin{bmatrix} c_w + c_c & -c_c \\ c_c & c_c \end{bmatrix}\begin{bmatrix} \dot{x}_w \\ \dot{x}_c \end{bmatrix} + \begin{bmatrix} k_w + k_c & -k_c \\ k_c & k_c \end{bmatrix}\begin{bmatrix} x_w \\ x_c \end{bmatrix} = \begin{bmatrix} c_w & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} k_w & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix}$$

To Solve we need to simplify, since the problem is very difficult.

Assume the wheel is very stiff; hence $k_w = \infty$, which implies x_w=y. Then

$$m_c\ddot{x}_c + c_c\dot{x}_c + k_c x_c = c_c\dot{y} + k_c y$$
$$or$$
$$\ddot{x}_c + 2\zeta\omega_n\dot{x}_c + \omega_n^2 x_c = 2\zeta\omega_n\dot{y} + \omega_n^2 y$$

Placing an actuator between the two masses (ignore actuator dynamics for simplicity), and use a PD control: the control force is (m_c is added to make the final equation look simpler):

$$F = m_c K_D(\dot{x}_c - \dot{x}_w) + m_c K_P(x_c - x_w)$$
$$where$$
$$x_w = y, \dot{x}_w = \dot{y}$$

The transfer function of the system is:

$$\frac{X_c}{Y} = \frac{\left(2\zeta\omega_n + K_D\right)s + \left(\omega_n^2 + K_P\right)}{s^2 + \left(2\zeta\omega_n + K_D\right)s + \left(2\zeta\omega_n + K_D\right)}$$

The rest is a standard PD controller design.

9-45) (a) Forward-path Transfer Function: (*N* = 10)

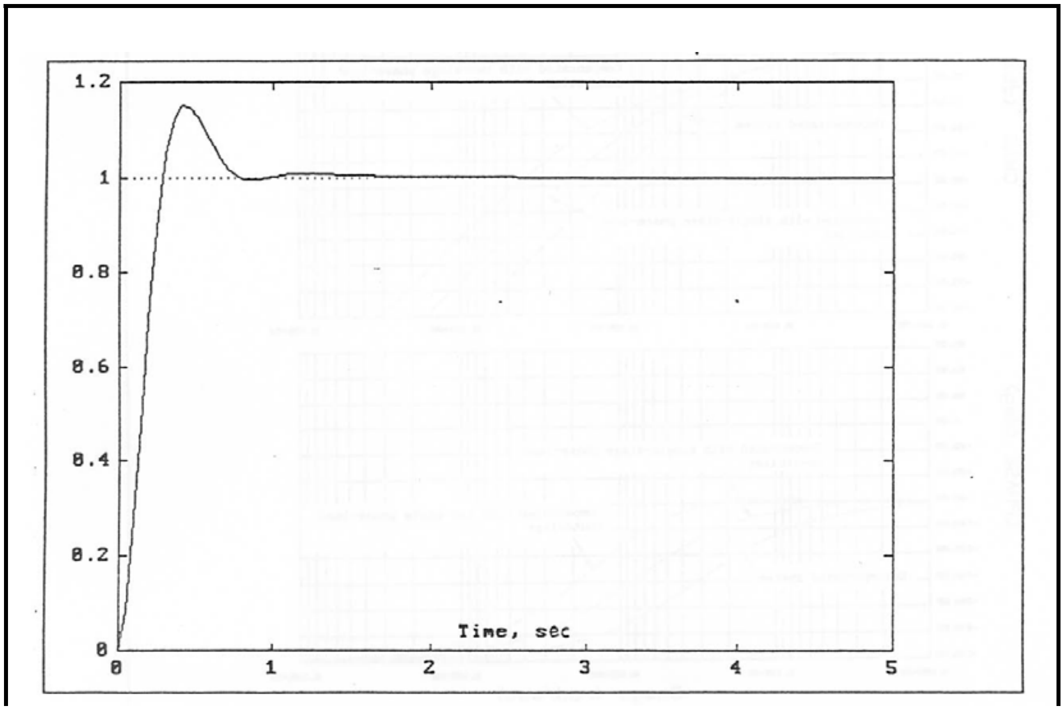
$$G(s) = G_c(s)G_p(s) = \frac{200\left(1+aTs\right)}{s\left(s+1\right)\left(s+10\right)\left(1+Ts\right)}$$

Starting with *a* = 1000, we vary *T* first to stabilize the system. The following time-domain attributes are obtained by varying the value of *T*.

T	**Max Overshoot (%)**	t_r	t_s
0.0001	59.4	0.370	5.205
0.0002	41.5	0.293	2.911
0.0003	29.9	0.315	1.83
0.0004	22.7	0.282	1.178
0.0005	18.5	0.254	1.013
0.0006	16.3	0.230	0.844
0.0007	15.4	0.210	0.699
0.0008	15.4	0.192	0.620
0.0009	15.5	0.182	0.533
0.0010	16.7	0.163	0.525

The maximum overshoot is at a minimum when *T* = 0.0007 or *T* = 0.0008. The maximum overshoot is 15.4%.

Unit-step Response. (*T* = 0.0008 sec *a* = 1000)



(b) Frequency-domain Design.

Similar to the design in part (a), we set a = 1000, and vary the value of T between 0.0001 and 0.001. The attributes of the frequency-domain characteristics are given below.

T	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.0001	17.95	60.00	3.194	4.849
0.0002	31.99	63.53	1.854	5.285
0.0003	42.77	58.62	1.448	5.941
0.0004	49.78	54.53	1.272	6.821
0.0005	53.39	51.16	1.183	7.817
0.0006	54.69	48.32	1.138	8.869
0.0007	54.62	45.87	1.121	9.913
0.0008	53.83	43.72	1.125	10.92
0.0009	52.68	41.81	1.140	11.88
0.0010	51.38	40.09	1.162	12.79

The phase margin is at a maximum of 54.69 deg when T = 0.0006. The performance worsens if the value of a is less than 1000.

9-46 (a) Bode Plot.

The attributes of the frequency response are:

PM = 4.07 deg GM = 1.34 dB M_r = 23.24 BW = 4.4 rad/sec

(b) Single-stage Phase-lead Controller.

$$G(s) = \frac{6(1+aTs)}{s(1+0.2s)(1+0.5s)(1+Ts)}$$

We first set a = 1000, and vary T. The following attributes of the frequency-domain characteristics are obtained.

T	**PM (deg)**	M_r
0.0050	17.77	3.21
0.0010	43.70	1.34
0.0007	47.53	1.24
0.0006	48.27	1.22
0.0005	48.06	1.23
0.0004	46.01	1.29
0.0002	32.08	1.81
0.0001	19.57	2.97

The phase margin is maximum at 48.27 deg when T = 0.0006.

Next, we set T = 0.0006 and reduce a from 1000. We can show that the phase margin is not very sensitive to the variation of a when a is near 1000. The optimal value of a is around 980, and the corresponding phase margin is 48.34 deg.

With a = 980 and T = 0.0006, the attributes of the unit-step response are:

Maximum overshoot = 18.8% t_r = 0.262 sec t_s = 0.851 sec

(c) Two-stage Phase-lead Controller. (a = 980, T = 0.0006)

$$G(s) = \frac{6(1+0.588s)(1+bT_2s)}{s(1+0.2s)(1+0.5s)(1+0.0006s)(1+T_2s)}$$

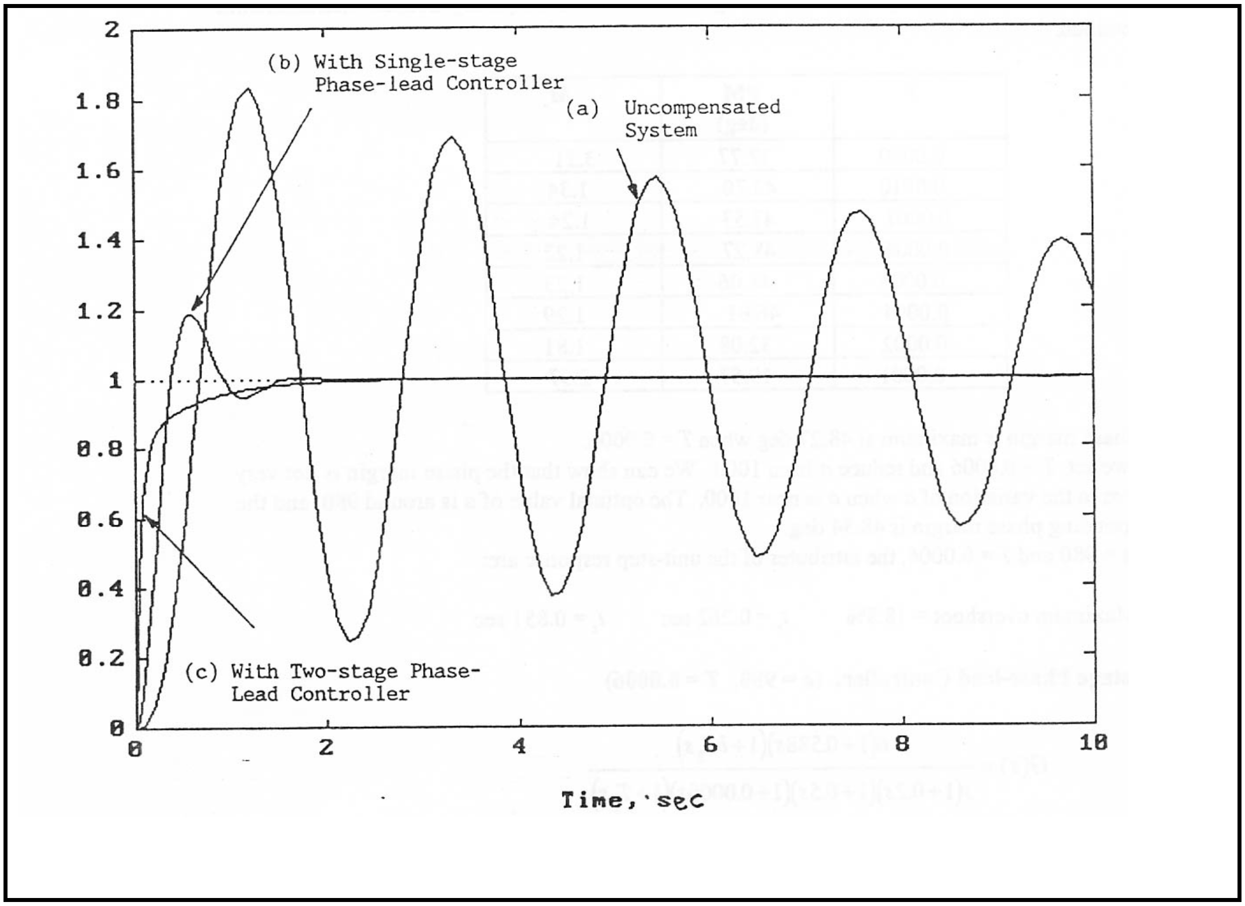
Again, let b = 1000, and vary T_2. The following results are obtained in the frequency domain.

T_2	PM (deg)	M_r
0.0010	93.81	1.00
0.0009	94.89	1.00
0.0008	96.02	1.00
0.0007	97.21	1.00
0.0006	98.43	1.00
0.0005	99.61	1.00
0.0004	100.40	1.00
0.0003	99.34	1.00
0.0002	91.98	1.00
0.0001	73.86	1.00

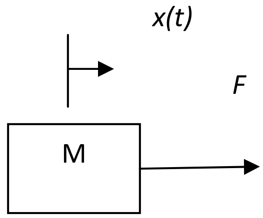
Reducing the value of b from 1000 reduces the phase margin. Thus, the maximum phase margin of 100.4 deg is obtained with b = 1000 and T_2 = 0.0004. The transfer function of the two-stage phase-lead controller is

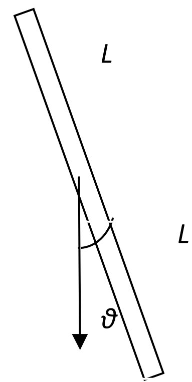
$$G_c(s) = \frac{(1+0.588s)(1+0.4s)}{(1+0.0006s)(1+0.0004s)}$$

(c) Unit-step Responses.



9-47) Also see derivations in 4-9.





Here is an alternative representation including friction (damping) μ. In this case the angle θ is measured differently.

Let's find the dynamic model of the system:

1) $(M+m)\ddot{x} + \mu\dot{x} - ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$
2) $(I+mI^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$

Let $\theta = \pi + \Phi$. If Φ is small enough then $\cos\Phi \to 1$ and $\sin\Phi \to \Phi$, therefore

$$\begin{cases}(M+m)\ddot{x} + \mu\dot{x} - ml\ddot{\Phi} = F \\ (l+ml^2)\ddot{\Phi} - mgl\Phi = ml\ddot{x}\end{cases}$$

which gives:

$$\frac{\Phi(s)}{F(s)} = \frac{mls^2}{[(M+m)(l+ml^2)-(ml)^2]s^3 + \mu(l+ml^2)s^2 - (M+m)mgls - \mu mgl}$$

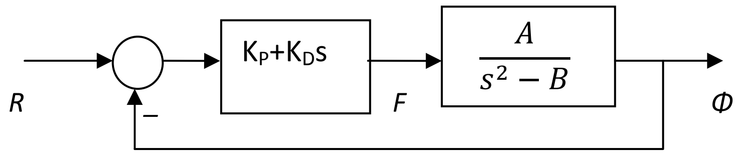
Ignoring friction $\mu = 0$.

$$\frac{\Phi(s)}{F(s)} = \frac{ml}{[(M+m)(l+ml^2)-(ml)^2]s^2 - (M+m)mgl} = \frac{A}{s^2 - B}$$

where

$$A = \frac{ml}{[(M+m)(I+ml^2)-(ml)^2]}; B = \frac{(M+m)mgl}{[(M+m)(l+ml^2)-(ml)^2]}$$

Ignoring actuator dynamics (DC motor equations), we can incorporate feedback control using a series PD compensator and unity feedback. Hence,



$$F(s) = K_p\left(R(s) - \Phi\right) - K_D s\left(R(s) - \Phi\right)$$

The system transfer function is:

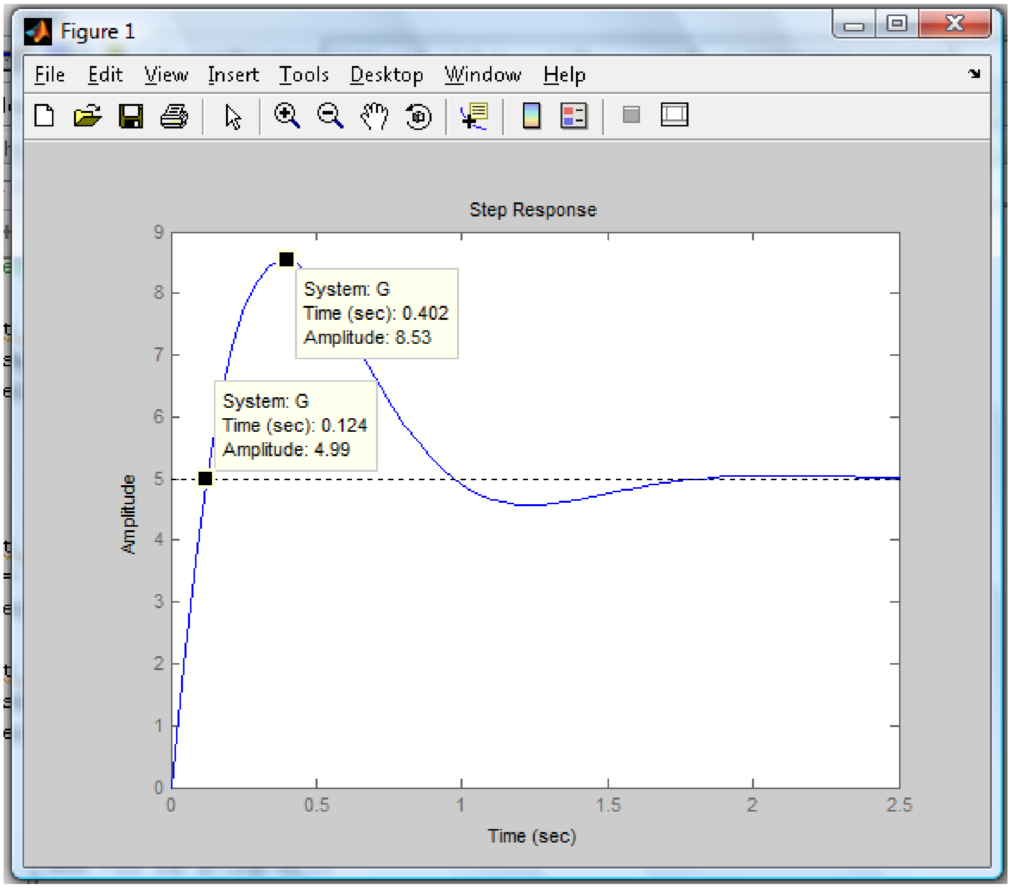
$$\frac{\Phi}{R} = \frac{A\left(K_p + K_D s\right)}{(s^2 + K_D s + A\left(K_p - B\right)}$$

Control is achieved by ensuring stability (K_p>B)

Use Routh Hurwitz to establish stability first. Use Acsys to do that as demonstrated in this chapter problems. Also Chapter 2 has many examples.

Use MATLAB to simulate response:

```
clear all
Kp=10;
Kd=5;
A=10;
B=8;
num = [A*Kd A*Kp];
den =[1 Kd A*(Kp-B)];
G=tf(num,den)
step(G)

Transfer function:
  50 s + 100
--------------
s^2 + 5 s + 20
```



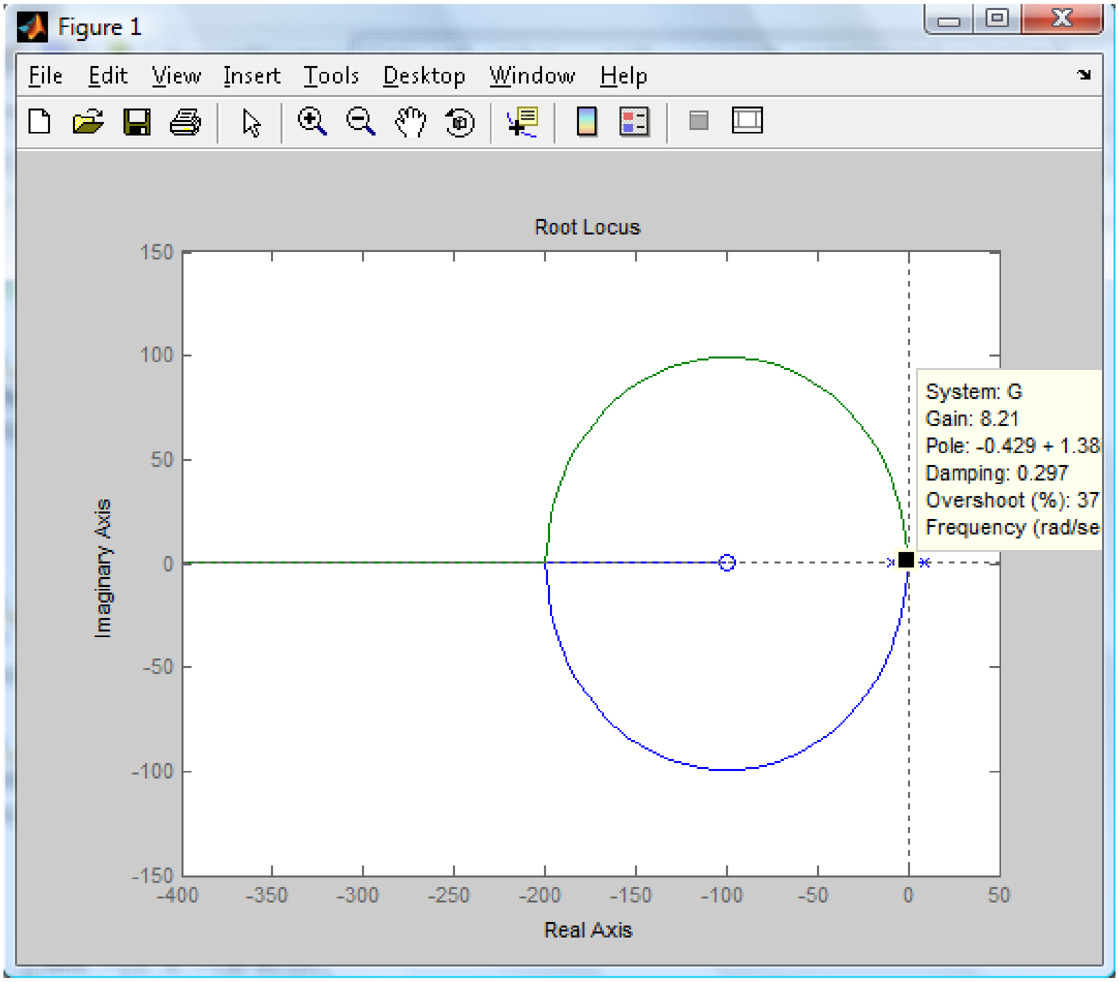
Adjust parameters to achieve desired response. Use THE PROCEDURE in Example 5-11-1.

You may look at the root locus of the forward path transfer function to get a better perspective.

$$\frac{\Phi}{E} = \frac{A\left(K_p + K_D s\right)}{s^2 - AB} = \frac{AK_D\left(z+s\right)}{s^2 - AB}$$

fix z and vary K_D.

```
clear all
z=100;
Kd=0.01;
A=10;
B=8;
num = [A*Kd A*Kd*z];
den =[1 0 -(A*B)];
G=tf(num,den)
rlocus(G)

Transfer function:
0.1 s + 10
----------
 s^2 - 80
```

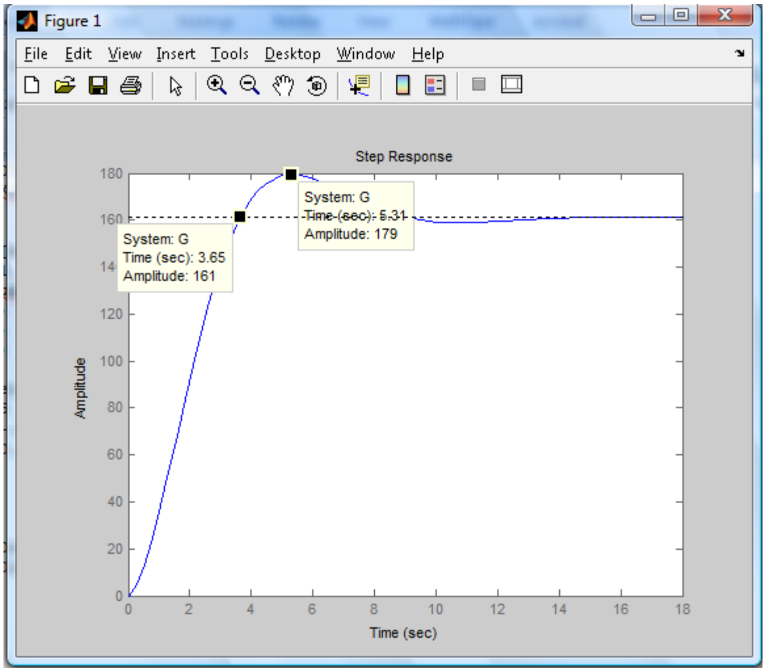


For z=10, a large K_D=0.805 results in:

```
clear all
Kd=0.805;
Kp=10*Kd;
A=10;
B=8;
num = [A*Kd A*Kp];
den =[1 Kd A*(Kp-B)];
G=tf(num,den)
pole(G)
zero(G)
step(G)

Transfer function:
   8.05 s + 80.5
-------------------
s^2 + 0.805 s + 0.5


ans =
```

```
  -0.4025 + 0.5814i
  -0.4025 - 0.5814i

ans =

   -10
```

Looking at dominant poles we expect to see an oscillatory response with overshoot close to desired values.



For a better design, and to meet rise time criterion, use Example 5-11-1 and Chapter 9 PD design examples.

9-48) (a) The loop transfer function of the system is

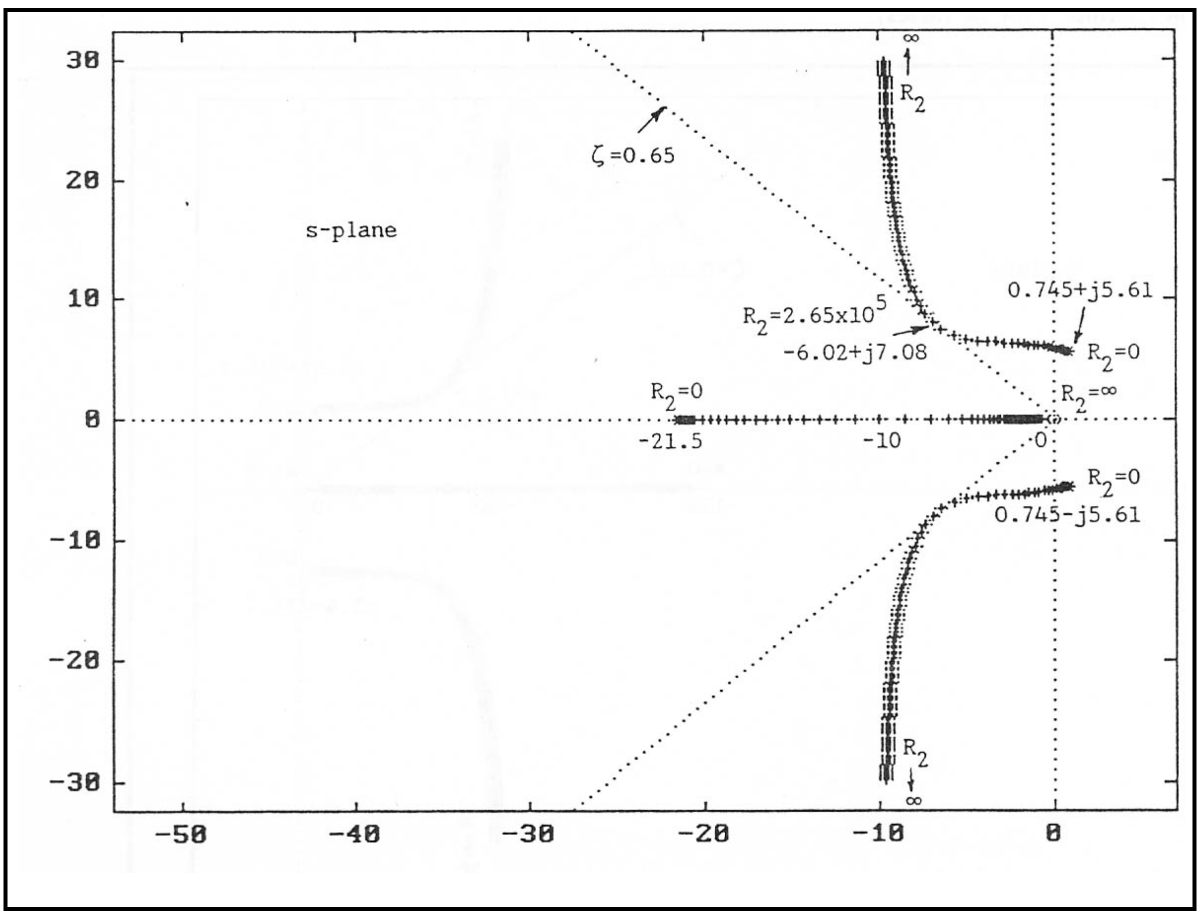
$$G(s)H(s) = \frac{10K_pK_aK_e}{Ns(1+0.05s)}\frac{1+R_2Cs}{R_1Cs} = \frac{68.76}{s(1+0.05s)}\frac{1+R_2\times 10^{-6}s}{2s}$$

The characteristic equation is $s^3 + 20s^2 + 6.876\times 10^{-4}R_2 s + 687.6 = 0$

For root locus plot with R_2 as the variable parameter, we have

$$G_{eq}(s) = \frac{6.876\times 10^{-4}R_2 s}{s^3+20s^2+687.6} = \frac{6.876\times 10^{-4}R_2 s}{(s+21.5)(s-0.745+j5.61)(s-0.745-j5.61)}$$

Root Locus Plot.



When $R_2 = 2.65\times10^5$, the roots are at $-6.02 \pm j7.08$, and the relative damping ratio is 0.65 which is maximum. The unit-step response is plotted at the end together with those of parts (b) and (c).

(b) Phase-lead Controller.

$$G(s)H(s) = \frac{68.76(1+aTs)}{s(1+0.05s)(1+Ts)}$$

Characteristic Equation: $Ts^3 + (1+20T)s^2 + (20+1375.2aT)s + 1375.2 = 0$

With T = 0.01, the characteristic equation becomes

$$s^3 + 120s^2 + (2000 + 1375.2a)s + 137520 = 0$$

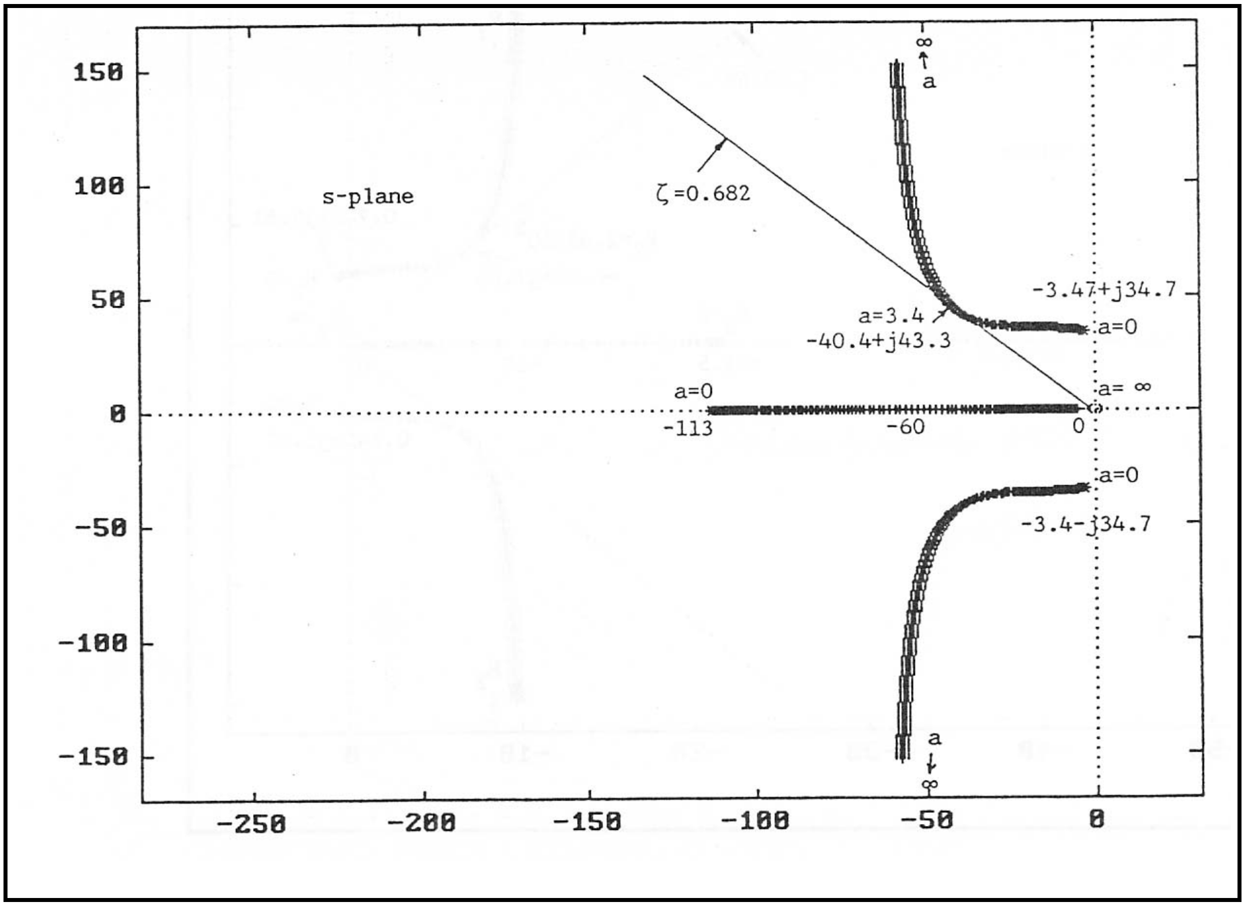
The last equation is conditioned for a root contour plot with *a* as the variable parameter.

Thus

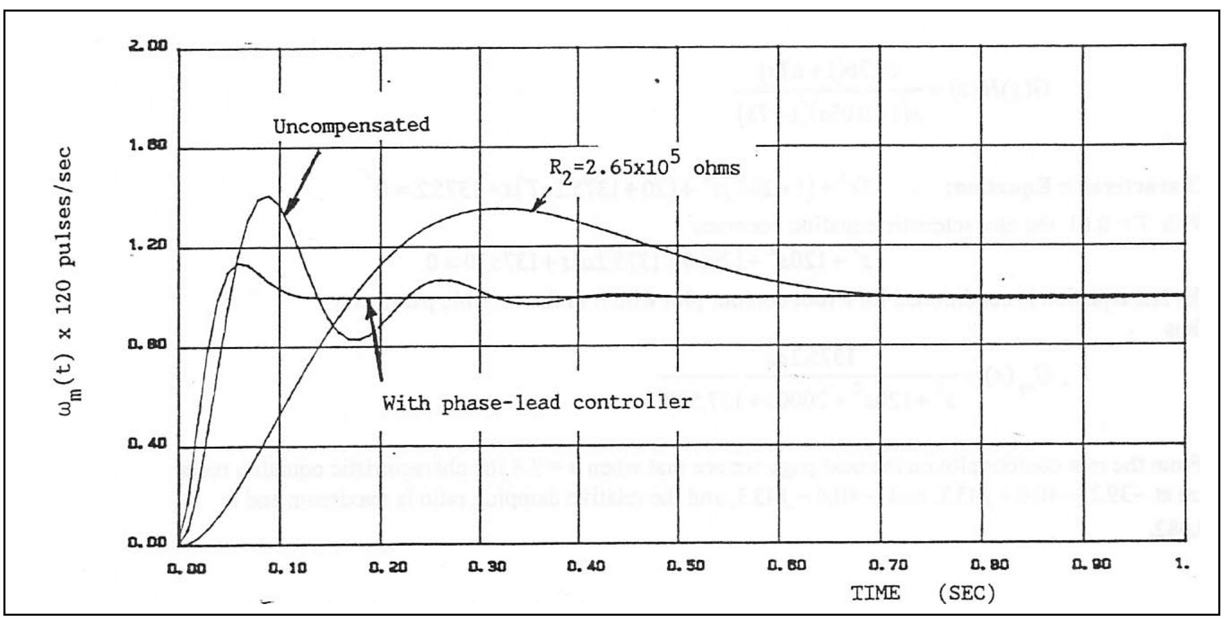
$$G_{eq}(s) = \frac{1375.2as}{s^3 + 120s^2 + 2000s + 137{,}520}$$

From the root contour plot on the next page we see that when *a* = 3.4 the characteristic equation roots are at $-39.2,\ -40.4 + j43.3,\ \text{and}\ -40.4 - j43.3$, and the relative damping ratio is maximum and is 0.682.

Root Contour Plot (*a* varies).



Unit-step Responses.



(c) Frequency-domain Design of Phase-lead Controller.

For a phase margin of 60 deg, *a* = 4.373 and *T* = 0.00923. The transfer function of the controller is

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.04036s}{1+0.00923s}$$

9-49 (a) Time-domain Design of Phase-lag Controller.

Process Transfer Function:

$$G_p(s) = \frac{200}{s(s+1)(s+10)}$$

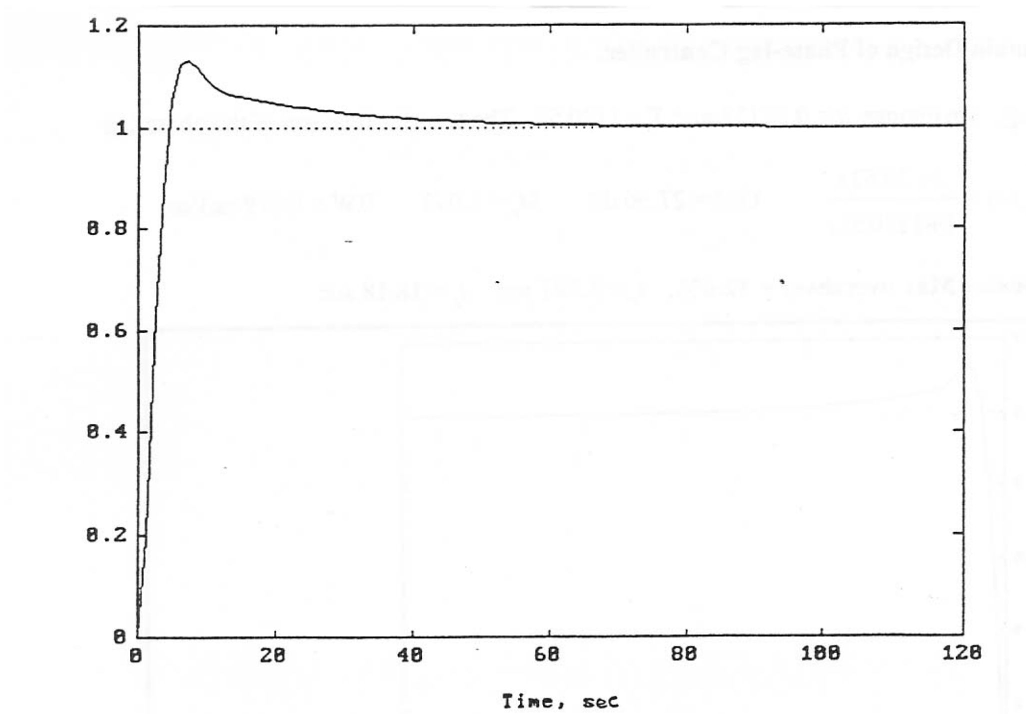
For the uncompensated system, the two complex characteristic equation roots are at $s = -0.475 + j0.471$ and $-0.475 - j0.471$ which correspond to a relative damping ratio of 0.707, when the forward path gain is 4.5 (as against 200). Thus, the value of *a* of the phase-lag controller is chosen to be

$$a = \frac{4.5}{200} = 0.0225$$ Select *T* = 1000 which is a large number.

Then

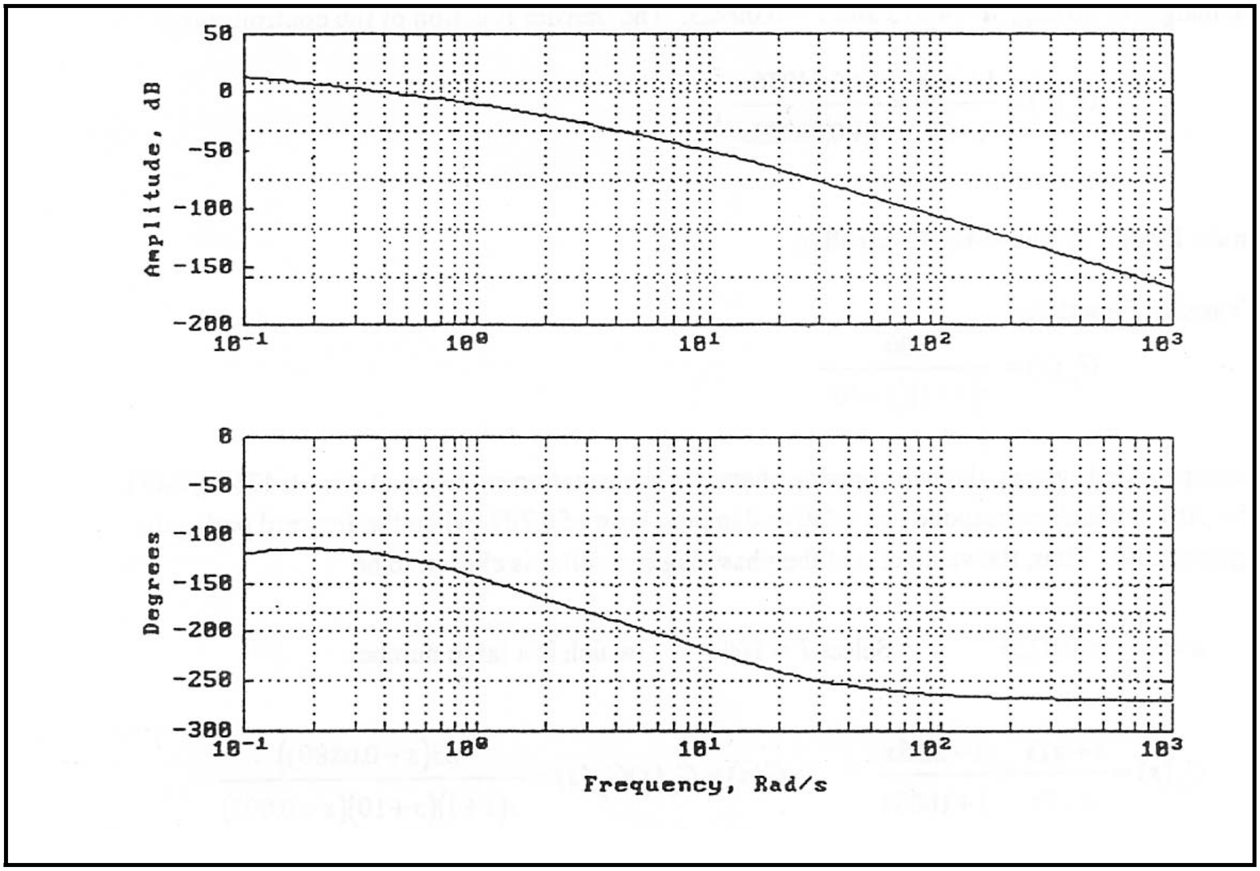
$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+22.5s}{1+1000s} \qquad G(s) = G_c(s)G_p(s) = \frac{4.5(s+0.0889))}{s(s+1)(s+10)(s+0.001)}$$

Unit-step Response.



Maximum overshoot = 13.6 t_r = 3.238 sec t_s = 18.86 sec

Bode Plot (with phase-lag controller, *a* = 0.0225, *T* = 1000)



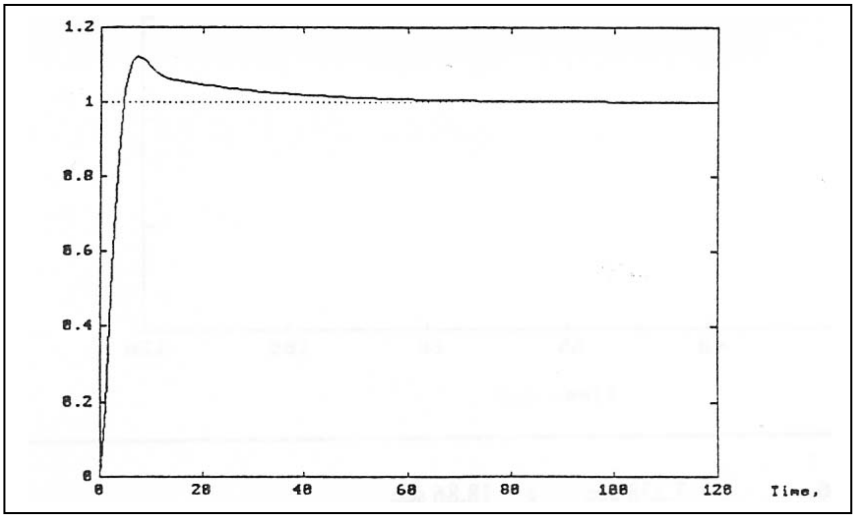
PM = 59 deg. GM = 27.34 dB M_r = 1.1 BW = 0.6414 rad/sec

(b) **Frequency-domain Design of Phase-lag Controller.**

For PM = 60 deg, we choose a = 0.02178 and T = 1130.55. The transfer function of the phase-lag controller is

$$G_c(s) = \frac{1+24.62s}{1+1130.55s}$$ GM = 27.66 dB M_r = 1.093 BW = 0.619 rad/sec

Unit-step Response. Max overshoot = 12.6%, t_r = 3.297 sec t_s = 18.18 sec



9-50 (a) Time-domain Design of Phase-lead Controller

Forward-path Transfer Function.

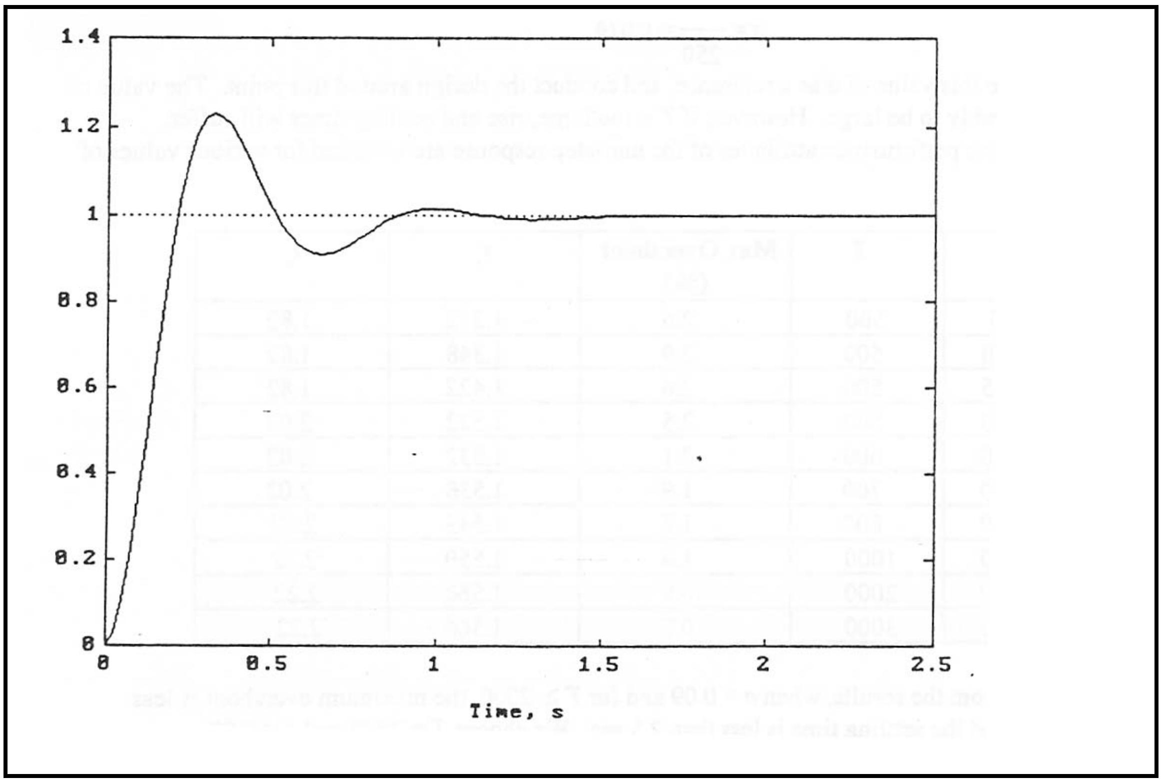
$$G(s) = G_c(s)G_p(s) = \frac{K(1+aTs)}{s(s+5)^2(1+Ts)} \qquad K_v = \lim_{s \to 0} sG(s) = \frac{K}{25} = 10 \qquad \text{Thus} \quad K = 250$$

With *K* = 250, the system without compensation is marginally stable. For *a* > 1, select a small value for *T* and a large value for *a*. Let *a* = 1000. The following results are obtained for various values of *T* ranging from 0.0001 to 0.001. When *T* = 0.0004, the maximum overshoot is near minimum at 23%.

T	**Max Overshoot (%)**	**t_r (sec)**	**t_s (sec)**
0.0010	33.5	0.0905	0.808
0.0005	23.8	0.1295	0.6869
0.0004	23.0	0.1471	0.7711

0.0003	24.4	0.1689	0.8765
0.0002	30.6	0.1981	1.096
0.0001	47.8	0.2326	2.399

As it turns out *a* = 1000 is near optimal. A higher or lower value for *a* will give larger overshoot.



Unit-step Response.

(b) Frequency-domain Design of Phase-lead Controller

$$G(s) = \frac{250(1+aTs)}{s^2(s+5)^2(1+Ts)}$$

Setting *a* = 1000, and varying *T*, the following attributes are obtained.

T	**PM (deg)**	M_r	**BW (rad/sec)**
0.00050	41.15	1.418	16.05
0.00040	42.85	1.369	14.15
0.00035	43.30	1.355	13.16
0.00030	43.10	1.361	12.12
0.00020	38.60	1.513	10.04

When *a* = 1000, the best value of *T* for a maximum phase margin is 0.00035, and PM = 43.3 deg. As it turns out varying the value of *a* from 1000 does not improve the phase margin. Thus the transfer function of the controller is

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.35s}{1+0.00035s} \quad \text{and} \quad G(s) = \frac{250(1+0.35s)}{s(s+5)^2(1+0.00035s)}$$

(c) Time-domain Design of Phase-lag Controller

Without compensation, the relative damping is critical when *K* = 18.5. Then, the value of *a* is chosen to be

$$a = \frac{18.5}{250} = 0.074$$

We can use this value of *a* as a reference, and conduct the design around this point. The value of *T* is preferrably to be large. However, if *T* is too large, rise and settling times will suffer. The following performance attributes of the unit-step response are obtained for various values of *a* and *T*.

a	***T***	**Max Overshoot (%)**	t_r	t_s
0.105	500	2.6	1.272	1.82
0.100	500	2.9	1.348	1.82
0.095	500	2.6	1.422	1.82
0.090	500	2.5	1.522	2.02
0.090	600	2.1	1.532	2.02
0.090	700	1.9	1.538	2.02
0.090	800	1.7	1.543	2.02
0.090	1000	1.4	1.550	2.22
0.090	2000	0.9	1.560	2.22
0.090	3000	0.7	1.566	2.22

As seen from the results, when a = 0.09 and for $T \geq$ 2000, the maximum overshoot is less than 1% and the settling time is less than 2.5 sec. We choose T = 2000 and a = 0.09.

The corresponding frequency-domain characteristics are:

PM = 69.84 deg GM = 20.9 dB M_r = 1.004 BW = 1.363 rad/sec

(d) Frequency-domain Design of Phase-lag Controller

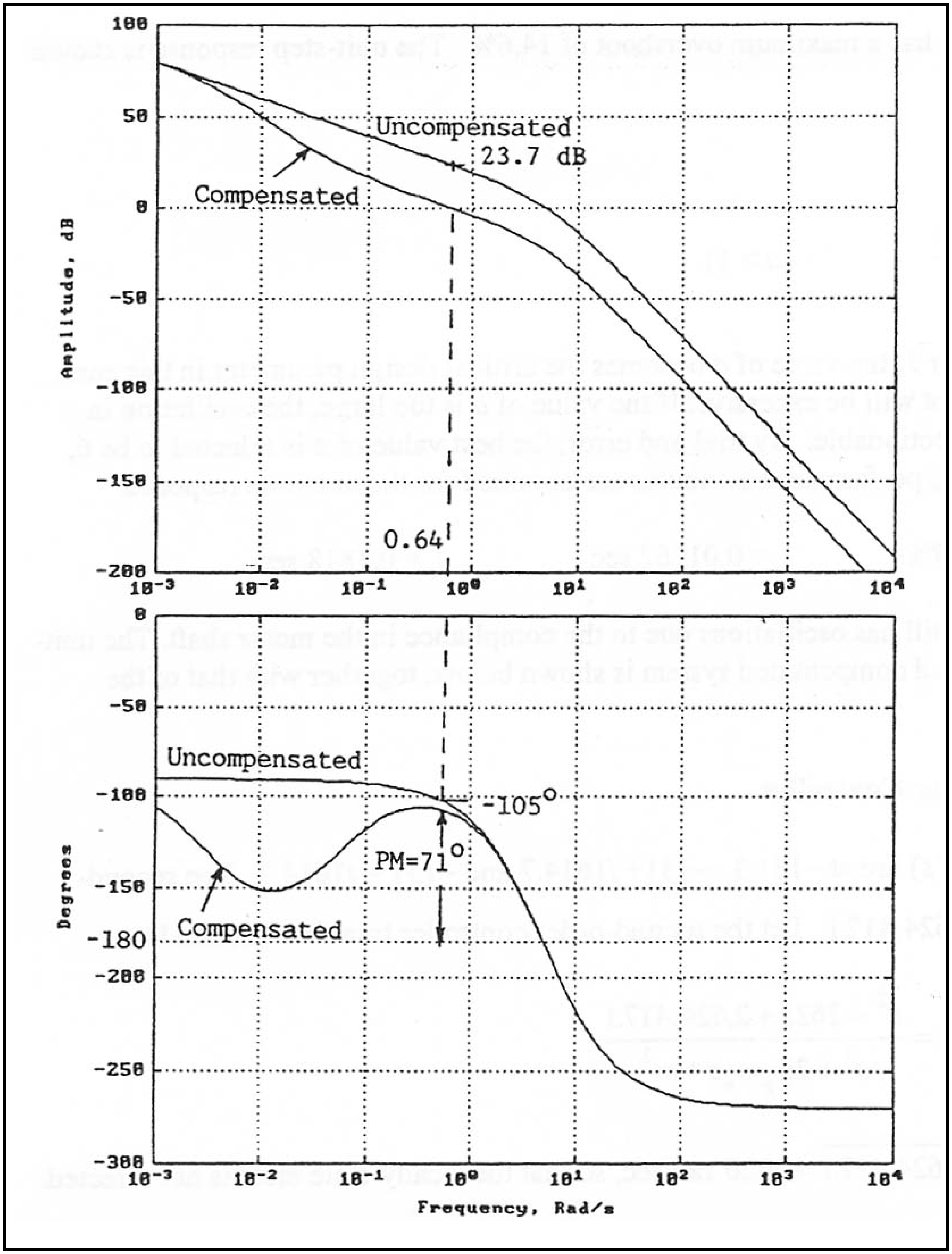
$$G(s) = \frac{250(1 + aTs)}{s(s+5)^2(1+Ts)} \qquad a < 1$$

The Bode plot of the uncompensated system is shown below. Let us add a safety factor by requiring that the desired phase margin is 75 degrees. We see that a phase margin of 75 degrees can be realized if the gain crossover is moved to 0.64 rad/sec. The magnitude of $G(j\omega)$ at this frequency is 23.7 dB. Thus the phase-lag controller must provide an attenuation of –23.7 dB at the new gain crossover frequency. Setting

$20\log_{10} a = -23.7$ dB we have $a = 0.065$

We can set the value of $1/aT$ to be at least one decade below 0.64 rad/sec, or 0.064 rad/sec. Thus, we get $T = 236$. Let us choose $T = 300$. The transfer function of the phase-lag controller becomes

$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 19.5s}{1 + 300s}$$



100
50
0
-50
-100
-150
-200
Amplitude, dB
Uncompensated
23.7 dB
Compensated
0.64
10-3
10-2
10-1
100
101
102
103
104
0
-50
-100
-150
-180
-200
-250
-300
Degrees
Uncompensated
-105°
PM=71°
Compensated
Frequency, Rad/s

The attributes of the frequency response of the compensated system are:

PM = 71 deg GM = 23.6 dB M_r = 1.065 BW = 0.937 rad/sec

The attributes of the unit-step response are:

Maximum overshoot = 6% t_r = 2.437 sec t_s = 11.11 sec

Comparing with the phase-lag controller designed in part (a) which has a = 0.09 and T = 2000, the time response attributes are:

Maximum overshoot = 0.9% t_r = 1.56 sec t_s = 2.22 sec

The main difference is in the large value of T used in part (c) which resulted in less overshoot, rise and settling times.

9-51)

9-52) **Forward-path Transfer Function (No compensation)**

$$G(s) = G_p(s) = \frac{6.087\times10^7}{s\left(s^3 + 423.42s^2 + 2.6667\times10^6 s + 4.2342\times10^8\right)}$$

The uncompensated system has a maximum overshoot of 14.6%. The unit-step response is shown below.

(a) **Phase-lead Controller**

$$G_c(s) = \frac{1+aTs}{1+Ts} \qquad (a > 1)$$

By selecting a small value for *T*, the value of *a* becomes the critical design parameter in this case. If *a* is too small, the overshoot will be excessive. If the value of *a* is too large, the oscillation in the step response will be objectionable. By trial and error, the best value of *a* is selected to be 6, and *T* = 0.001. The following performance attributes are obtained for the unit-step response.

Maximum overshoot = 0% t_r = 0.01262 sec t_s = 0.1818 sec

However, the step response still has oscillations due to the compliance in the motor shaft. The unit-step response of the phase-lead compensated system is shown below, together with that of the uncompensated system.

(b) **Phase-lead and Second-order Controller**

The poles of the process $G_p(s)$ are at –161.3, –131+ j1614.7 and –131 – j1614.7. The second-order term is $s^2 + 262s + 2{,}624{,}417.1$. Let the second-order controller transfer function be

$$G_{c1}(s) = \frac{s^2 + 262s + 2{,}624{,}417.1}{s^2 + 2\zeta_p \omega_n s + \omega_n^2}$$

The value of ω_n is set to $\sqrt{2{,}624{,}417.1} = 1620$ rad/sec, so that the steady-state error is not affected.

Let the two poles of $G_{c1}(s)$ be at $s = -1620$ and -1620. Then, $\zeta_p = 405$.

$$G_{c1}(s) = \frac{s^2 + 262s + 2{,}624{,}417.1}{s^2 + 3240s + 2{,}624{,}417.1}$$

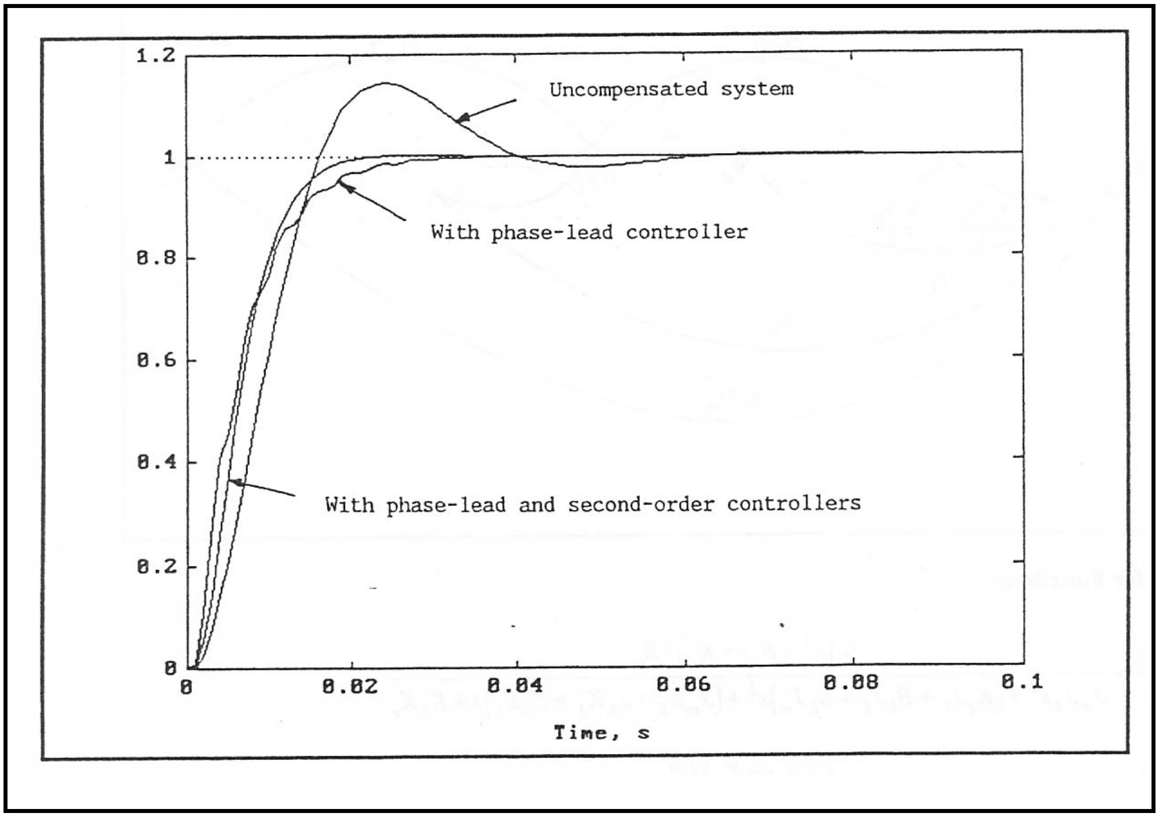
$$G(s) = G_c(s)G_{c1}(s)G_p(s) = \frac{6.087 \times 10^{10}\left(1 + 0.006s\right)}{s\left(s + 161.3\right)\left(s^2 + 3240s + 2,624,417.1\right)\left(1 + 0.001s\right)}$$

The unit-step response is shown below, and the attributes are:

Maximum overshoot = 0.2 t_r = 0.01012 sec t_s = 0.01414 sec

The step response does not have any ripples.

Unit-step Responses



9-53 (a) **System Equations.**

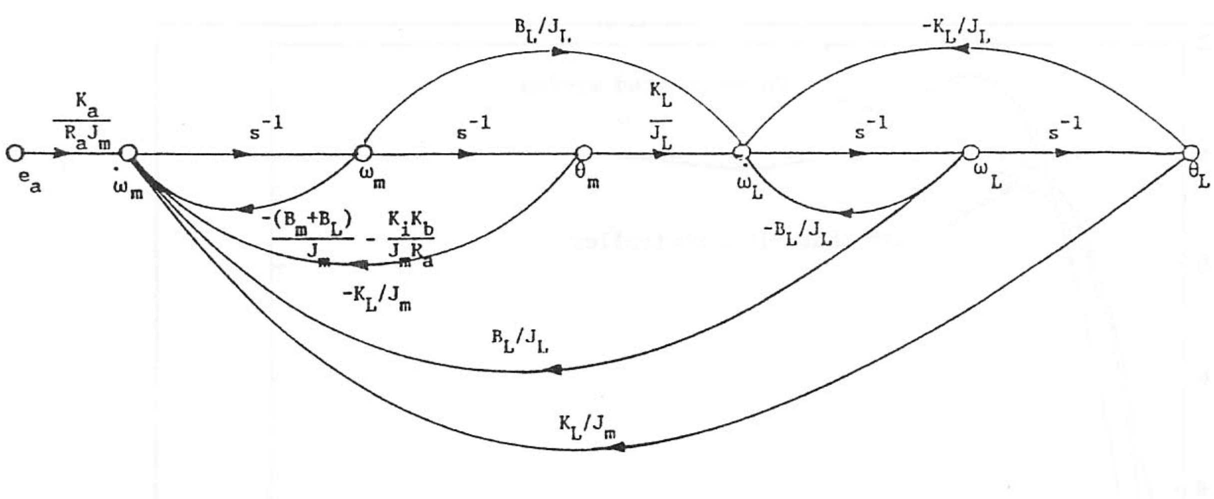
$$e_a = R_a i_a + K_b \omega_m \qquad T_m = K_i i_a \qquad T_m = J_m \frac{d\omega_m}{dt} + B_m \omega_m + K_L\left(\theta_m - \theta_L\right) + B_L\left(\omega_m - \omega_L\right)$$

$$K_L\left(\theta_m - \theta_L\right) + B_L\left(\omega_m - \omega_L\right) = J_L \frac{d\omega_L}{dt}$$

State Equations in Vector-matrix Form:

$$\begin{bmatrix} \frac{d\theta_L}{dt} \\ \frac{d\omega_L}{dt} \\ \frac{d\theta_m}{dt} \\ \frac{d\omega_m}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_L}{J_L} & -\frac{B_L}{J_L} & \frac{K_L}{J_L} & \frac{B_L}{J_L} \\ 0 & 0 & 0 & 1 \\ \frac{K_L}{J_m} & \frac{B_L}{J_m} & -\frac{K_L}{J_m} & -\frac{B_m + B_l}{J_m} - \frac{K_i K_b}{J_m R_a} \end{bmatrix} \begin{bmatrix} \theta_L \\ \omega_L \\ \theta_m \\ \omega_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_a}{R_a J_m} \end{bmatrix} e_a$$

State Diagram:



Transfer Functions:

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i\left(s^2 + B_L s + K_L\right)/R_a}{J_m J_L s^2 + \left(K_e J_L + B_L J_L + B_L J_m\right)s^2 + \left(J_m K_L + J_L K_L + K_e B_L\right)s + K_L K_e}$$

$$\frac{\Omega_L(s)}{E_a(s)} = \frac{K_i\left(B_L s + K_L\right)/R_a}{J_m J_L s^3 + \left(K_e J_L + B_L J_L + B_L J_m\right)s^2 + \left(J_m K_L + J_L K_L + K_e B_L\right)s + K_L K_e}$$

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{133.33\left(s^2+10s+3000\right)}{s^3+318.15s^2+60694.13s+58240} = \frac{133.33\left(s^2+10s+3000\right)}{(s+0.9644)(s+158.59+j187.71)(s+158.59-j187.71)}$$

$$\frac{\Omega_L(s)}{E_a(s)} = \frac{1333.33(s+300)}{(s+0.9644)(s+158.59+j187.71)(s+158.59-j187.71)}$$

(b) Design of PI Controller.

$$G(s) = \frac{\Omega_L(s)}{E(s)} = \frac{1333.33K_P\left(s+\dfrac{K_I}{K_P}\right)(s+300)}{s(s+0.9644)\left(s^2+317.186s+60388.23\right)}$$

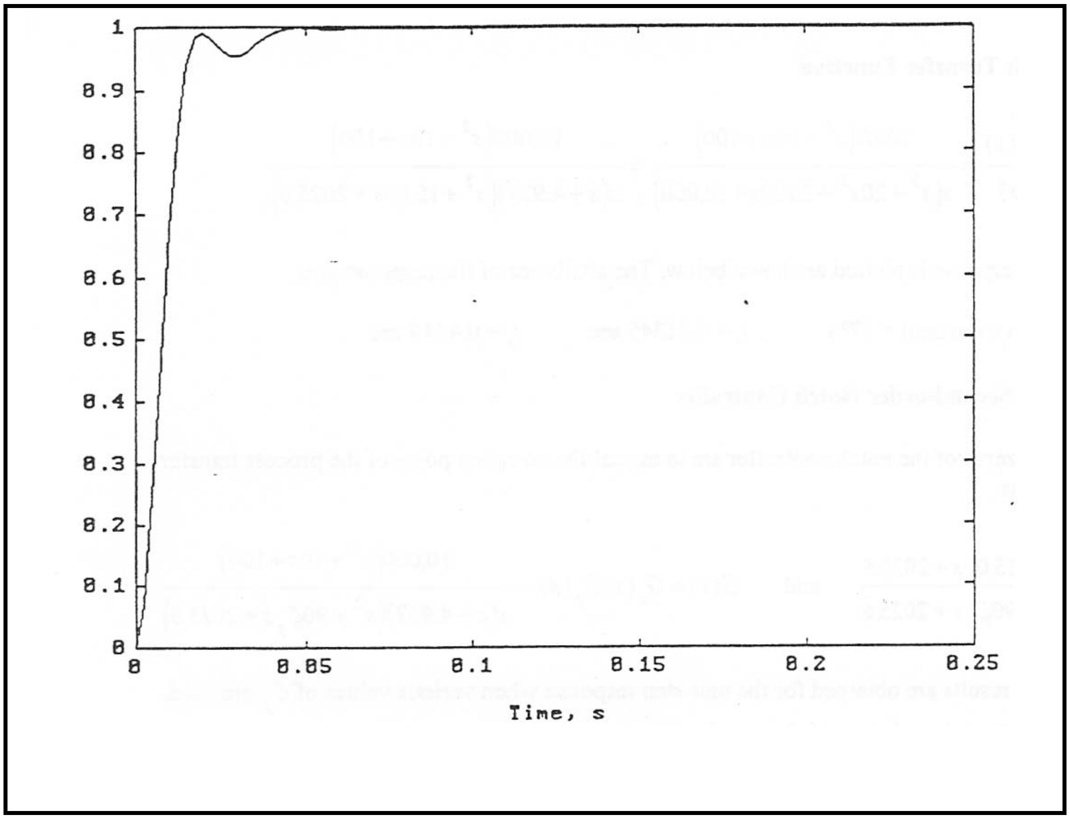
$$K_v = \lim_{s\to 0} sG(s) = \frac{1333.33\times 300K_I}{0.9644\times 60388.23} = 6.87K_I = 100 \qquad \text{Thus} \quad K_I = 14.56$$

With $K_I = 14.56$, we study the effects of varying K_P. The following results are obtained.

K_P	t_r (sec)	t_s (sec)	Max Overshoot (%)
20	0.00932	0.02778	4.2
18	0.01041	0.01263	0.7
17	0.01113	0.01515	0
16	0.01184	0.01515	0
15	0.01303	0.01768	0
10	0.02756	0.04040	0.6

With $K_I = 14.56$ and K_P ranging from 15 to 17, the design specifications are satisfied.

Unit-step Response:



(c) Frequency-domain Design of PI Controller (K_I = 14.56)

$$G(s) = \frac{1333.33\left(K_P s + 14.56\right)\left(s + 300\right)}{s\left(s^3 + 318.15s^2 + 60694.13s + 58240\right)}$$

The following results are obtained by setting $K_I = 14.56$ and varying the value of K_P.

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)	Max Overshoot (%)	t_r (sec)	t_s (sec)
20	65.93	$\propto$	1.000	266.1	4.2	0.00932	0.02778
18	69.76	$\propto$	1.000	243	0.7	0.01041	0.01263
17	71.54	$\propto$	1.000	229	0	0.01113	0.01515
16	73.26	$\propto$	1.000	211.6	0	0.01184	0.01515
15	74.89	$\propto$	1.000	190.3	0	0.01313	0.01768
10	81.11	$\propto$	1.005	84.92	0.6	0.0294	0.0404
8	82.66	$\propto$	1.012	63.33	1.3	0.04848	0.03492
7	83.14	$\propto$	1.017	54.19	1.9	0.03952	0.05253
6	83.29	$\propto$	1.025	45.81	2.7	0.04697	0.0606
5	82.88	$\propto$	1.038	38.12	4.1	0.05457	0.0606

From these results we see that the phase margin is at a maximum of 83.29 degrees when $K_P = 6$. However, the maximum overshoot of the unit-step response is 2.7%, and M_r is slightly greater than one. In part (b), the optimal value of K_P from the standpoint of minimum value of the maximum overshoot is between 15 and 17. Thus, the phase margin criterion is not a good indicator in the present case.

9-54 (a) Forward-path Transfer Function

$$G_p(s) = \frac{K\Theta_m(s)}{T_m(s)} = \frac{100K\left(s^2+10s+100\right)}{s\left(s^3+20s^2+2100s+10,000\right)} = \frac{10,000\left(s^2+10s+100\right)}{s\left(s+4.937\right)\left(s^2+15.06s+2025.6\right)}$$

The unit-step response is plotted as shown below. The attributes of the response are:

Maximum overshoot = 57% t_r = 0.01345 sec t_s = 0.4949 sec

(b) Design of the Second-order Notch Controller

The complex zeros of the notch controller are to cancel the complex poles of the process transfer function. Thus

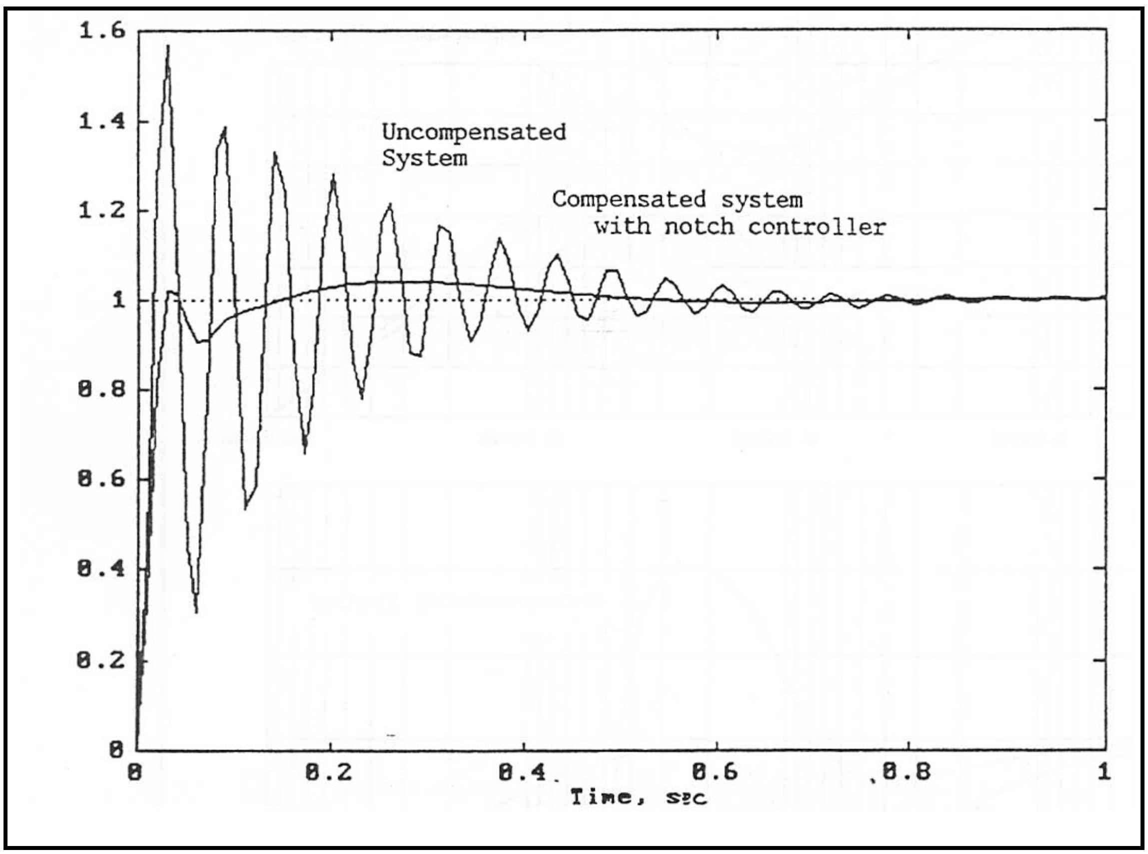
$$G_c(s) = \frac{s^2+15.06s+2025.6}{s^2+90\zeta_p s+2025.6} \quad \text{and} \quad G(s) = G_c(s)G_p(s) = \frac{10,000\left(s^2+10s+100\right)}{s\left(s+4.937\right)\left(s^2+90\zeta_p s+2025.6\right)}$$

The following results are obtained for the unit-step response when various values of ζ_p are used. The maximum overshoot is at a minimum of 4.1% when $\zeta_p = 1.222$. The unit-step response is plotted below, along with that of the uncompensated system.

ζ_p	$2\zeta\omega_n$	**Max Overshoot (%)**
2.444	200	7.3
2.333	210	6.9
2.222	200	6.5

1.667	150	4.9
1.333	120	4.3
1.222	110	4.1
1.111	100	5.8
1.000	90	9.8

Unit-step Response



(c) Frequency-domain Design of the Notch Controller

The forward-path transfer function of the uncompensated system is

$$G(s) = \frac{10000\left(s^2 + 10s + 100\right)}{s\left(s + 4.937\right)\left(s^2 + 15.06s + 2025.6\right)}$$

The Bode plot of $G(j\omega)$ is constructed in the following. We see that the peak value of $\left|G(j\omega)\right|$ is approximately 22 dB. Thus, the notch controller should provide an attentuation of –22 dB or 0.0794 at the resonant frequency of 45 rad/sec. Using Eq. (10-155), we have

$$\left|G_c(j45)\right| = \frac{\zeta_z}{\zeta_p} = \frac{0.167}{\zeta_p} = 0.0794 \qquad \text{Thus} \qquad \zeta_p = 2.1024$$

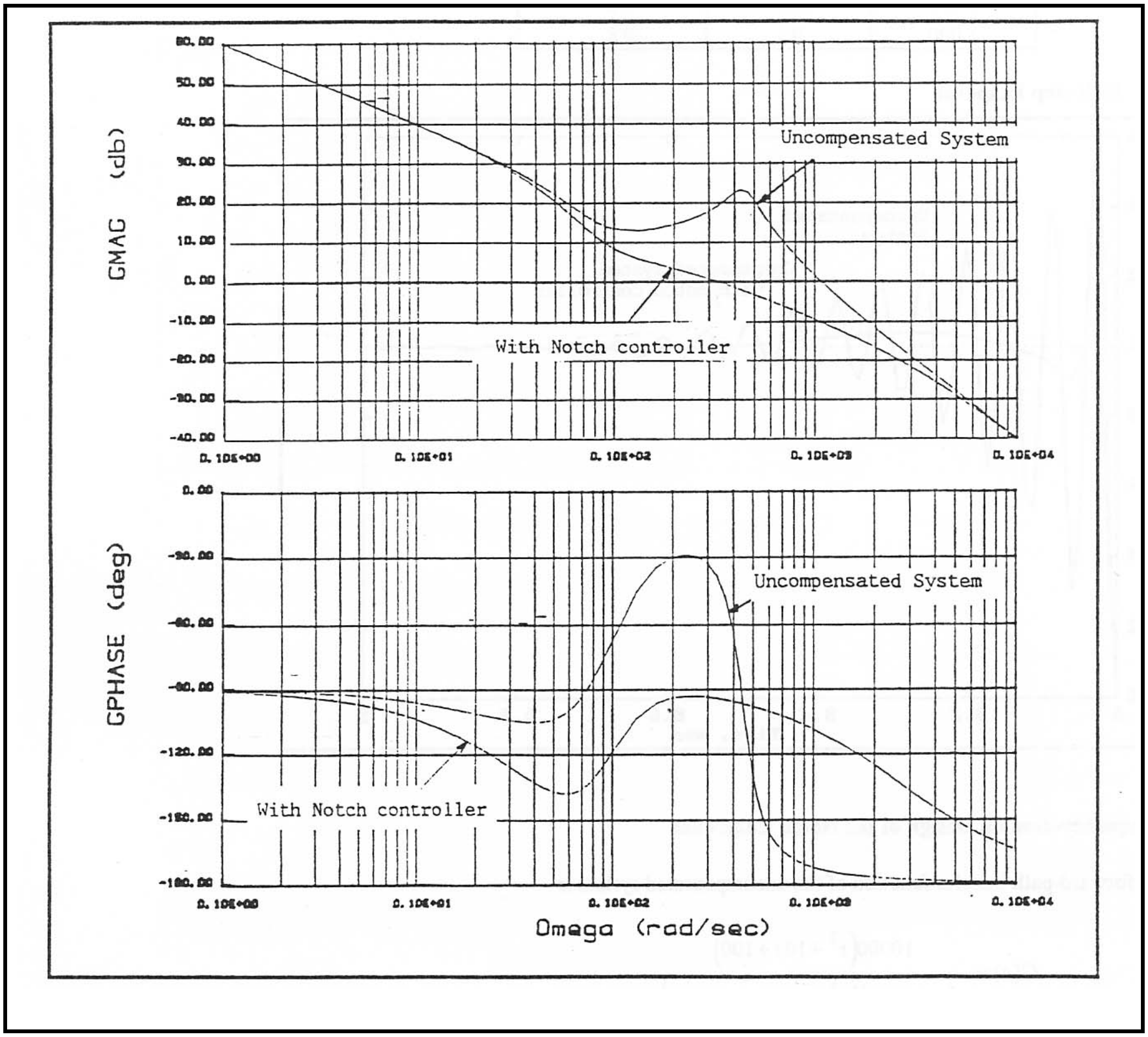
Notch Controller Transfer Function

$$G_c(s) = \frac{s^2 + 15.06s + 2025.6}{s^2 + 189.216s + 2025.6}$$

Forward-path Transfer Function

$$G(s) = \frac{10,000\left(s^2 + 10s + 100\right)}{s\left(s + 4.937\right)\left(s^2 + 189.22s + 2025.6\right)}$$

Bode Plots



Attributes of the frequency response: PM = 80.37 deg GM = infinite $M_r = 1.097$ BW = 66.4 rad/sec

Attributes of the frequency response of the system designed in part (b):

PM = 59.64 deg GM = infinite $M_r = 1.048$ BW = 126.5 rad/sec

9-55 (a) Process Transfer Function

$$G_p(s) = \frac{500(s+10)}{s(s^2+10s+1000)}$$

The Bode plot is constructed below. The frequency-domain attributes of the uncompensated system are:

PM = 30 deg GM = infinite $M_r = 1.86$ and BW = 3.95 rad/sec

The unit-step response is oscillatory.

(b) Design of the Notch Controller

For the uncompensated process, the complex poles have the following constants:

$$\omega_n = \sqrt{1000} = 31.6 \text{ rad/sec} \qquad 2\zeta\omega_\text{n} = 10 \qquad \text{Thus} \qquad \zeta = 0.158$$

The transfer function of the notch controller is

$$G_c(s) = \frac{s^2 + 2\zeta_z\omega_n s + \omega_n^2}{s^2 + 2\zeta_p s + \omega_n^2}$$

For the zeros of $G_c(s)$ to cancel the complex poles of $G_p(s)$, $\zeta_z = \zeta = 0.158$.

From the Bode plot, we see that to bring down the peak resonance of $|G(j\omega_n)|$ in order to smooth out the magnitude curve, the notch controller should provide approximately

–26 dB of attenuation. Thus, using Eq. (10-155),

$$\frac{\zeta_z}{\zeta_p} = 10^{\frac{-26}{20}} = 0.05 \qquad \text{Thus} \qquad \zeta_p = \frac{0.158}{0.05} = 3.1525$$

The transfer function of the notch controller is

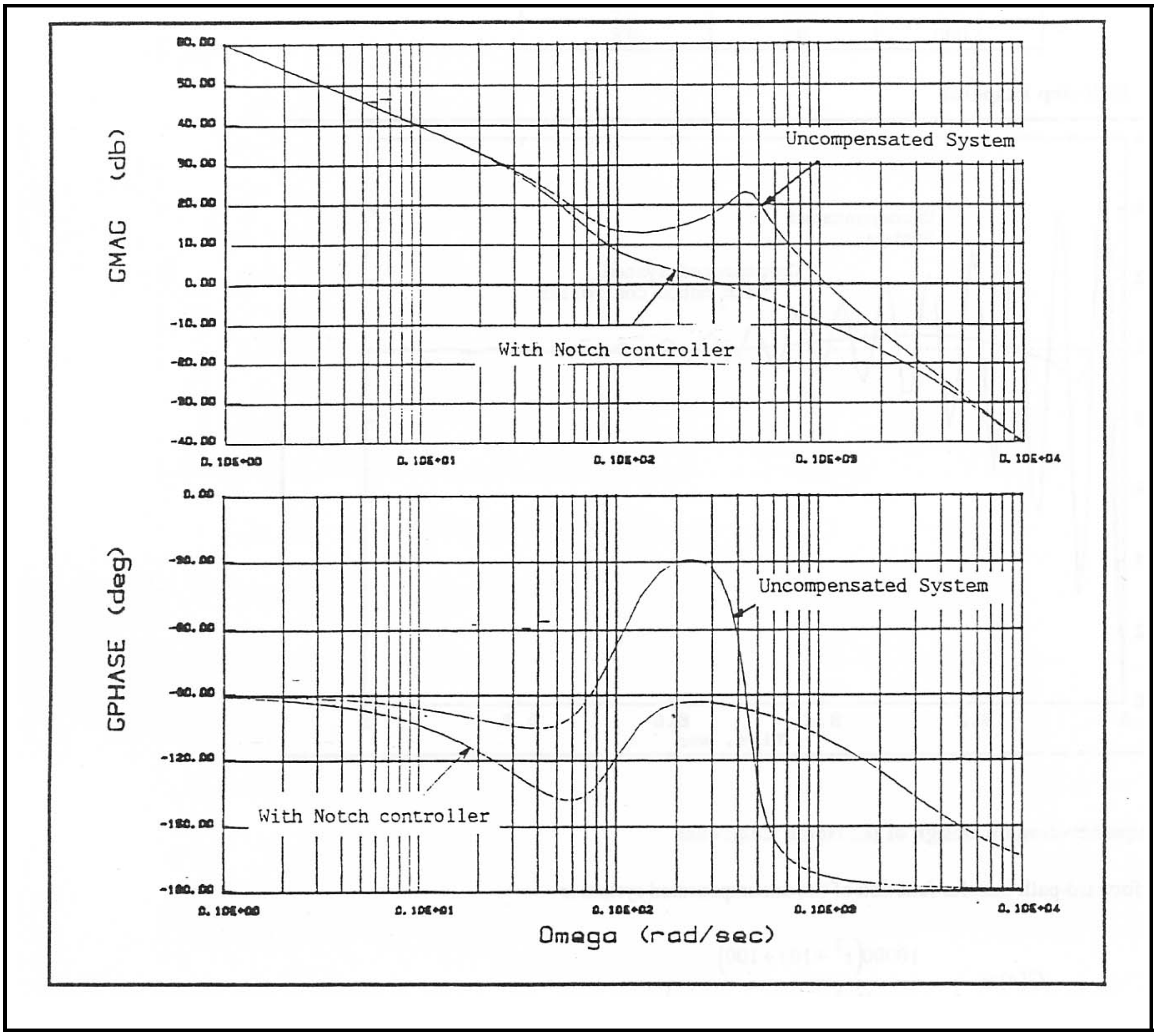
$$G_c(s) = \frac{s^2 + 10s + 1000}{s^2 + 199.08s + 1000} \qquad G(s) = G_c(s)G_p(s) = \frac{500(s+10)}{s(s^2 + 199.08s + 1000)}$$

The attributes of the compensated system are:

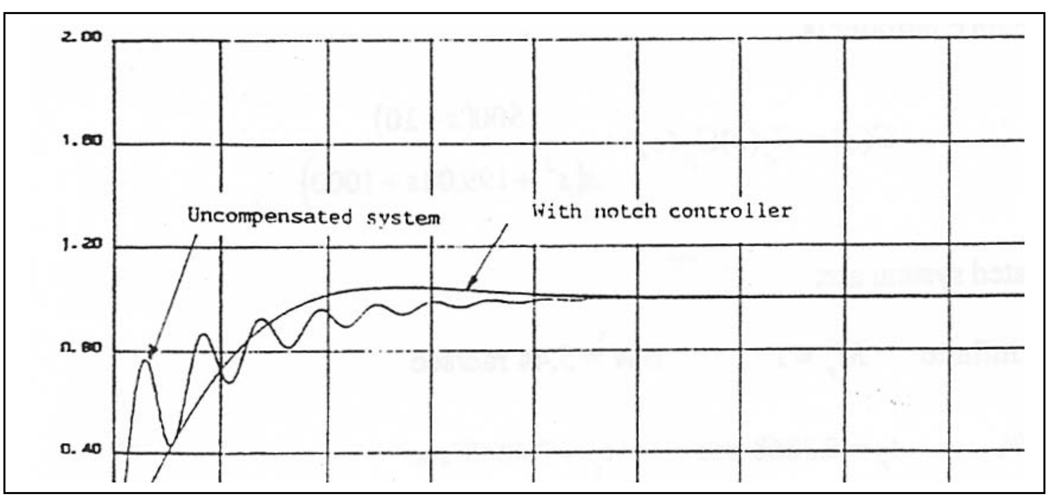
PM = 72.38 deg GM = infinite $M_r = 1$ BW = 5.44 rad/sec

Maximum overshoot = 3.4% $t_r = 0.3868$ sec $t_s = 0.4848$ sec

Bode Plots



Step Responses



(c) Time-domain design of the Notch Controller

With $\zeta_z = 0.158$ and $\omega_n = 31.6$, the forward-path transfer function of the compensated system is

$$G(s) = G_c(s)G_p(s) = \frac{500(s+10)}{s(s^2 + 63.2\zeta_p s + 1000)}$$

The following attributes of the unit-step response are obtained by varying the value of ζ_p.

ζ_p	$2\zeta\omega_n$	**Max Overshoot (%)**	t_r **(sec)**	t_s **(sec)**
1.582	100	0	0.4292	0.5859
1.741	110	0	0.4172	0.5657
1.899	120	0	0.4074	0.5455
2.057	130	0	0.3998	0.5253
2.215	140	0.2	0.3941	0.5152
2.500	158.25	0.9	0.3879	0.4840
3.318	209.7	4.1	0.3884	0.4848

When $\zeta_p = 2.5$ the maximum overshoot is 0.9%, the rise time is 0.3879 sec and the setting time is 0.4840 sec. These performance attributes are within the required specifications.

9-56 Let the transfer function of the controller be

$$G_c(s) = \frac{20,000\left(s^2 + 10s + 50\right)}{\left(s + 1000\right)^2}$$

Then, the forward-path transfer function becomes

$$G(s) = G_c(s)G_p(s) = \frac{20,000K\left(s^2 + 10s + 50\right)}{s\left(s^2 + 10s + 100\right)\left(s + 1000\right)^2}$$

For $G_{cf}(s) = 1,\qquad K_v = \lim_{s\to 0} sG(s) = \frac{10^6 K}{10^8} = 50\qquad$ Thus the nominal $K = 5000$

For ± 20% variation in *K*, $K_{\min} = 4000$ and $K_{\max} = 6000$. To cancel the complex closed-loop poles, we let

$$G_{cf}(s) = \frac{50\left(s + 1\right)}{s^2 + 10s + 50}$$ where the (*s* + 1) term is added to reduce the rise time.

Closed-loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{10^6 K\left(s + 1\right)}{s\left(s^2 + 10s + 100\right)\left(s + 1000^2\right) + 20,000K\left(s^2 + 10s + 50\right)}$$

Characteristic Equation:

***K* = 4000:** $s^5 + 2010s^4 + 1{,}020{,}100s^3 + 9.02\times 10^7 s^2 + 9\times 10^8 + 4\times 10^9 = 0$

Roots: $-97.7,\quad -648.9,\quad -1252.7,\quad -5.35 + j4.6635,\quad -5.35 - j4.6635$

Max overshoot ≅ 6.7% **Rise time < 0.04 sec**

K = 5000: $s^5 + 2010s^4 + 1{,}020100s^3 + 1.1\times10^8 s^2 + 1.1\times10^9 s + 5\times10^9 = 0$

Roots: $-132.46,\quad 587.44,\quad -1279.6,\quad -5.272 + j4.7353,\quad -5.272 - j4.7353$

Max overshoot ≅ 4% **Rise time < 0.04 sec**

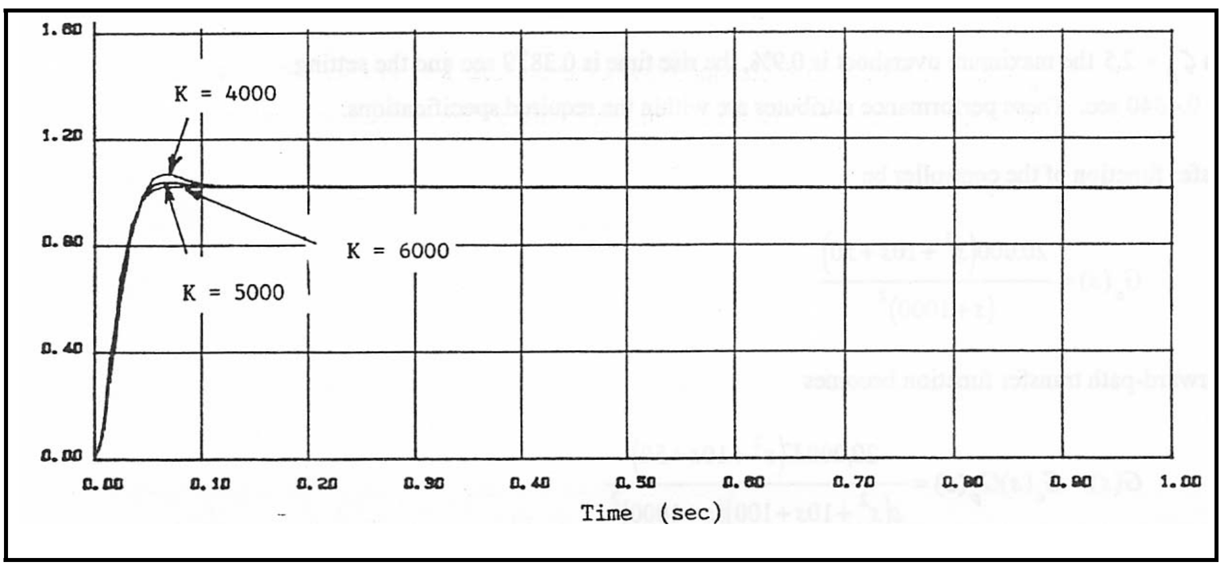
K = 6000 $s^5 + 2010s^4 + 1{,}020{,}100s^3 + 1.3\times10^8 s^2 + 1.3\times10^9 s + 6\times10^9 = 0$

Roots: $-176.77,\quad -519.37,\quad -1303.4,\quad -5.223 + j4.7818,\quad -5.223 - j4.7818$

Max overshoot ≅ 2.5% **Rise time < 0.04 sec**

Thus all the required specifications stay within the required tolerances when the value of K varies by plus and minus 20%.

Unit-step Responses



9-57 Let the transfer function of the controller be

$$G_c(s) = \frac{200\left(s^2 + 10s + 50\right)}{\left(s + 100\right)^2}$$

The forward-path transfer function becomes

$$G(s) = G_c(s)G_p(s) = \frac{200,000K\left(s^2 + 10s + 50\right)}{s\left(s + a\right)\left(s + 100\right)^2}$$

For a = 10,

$$K_v = \lim_{s \to 0} sG(s) = \frac{10^7 K}{10^5} = 100K = 100 \qquad \text{Thus} \qquad K = 1$$

Characteristic Equations: (*K* = 1)

***a* = 10:** $s^4 + 210s^3 + 2.12 \times 10^5 s^2 + 2.1 \times 10^6 s + 10^7 = 0$

Roots: $-4.978 + j4.78, \quad -4.978 - j4.78, \quad -100 + j447.16 \quad -100 - j447.16$

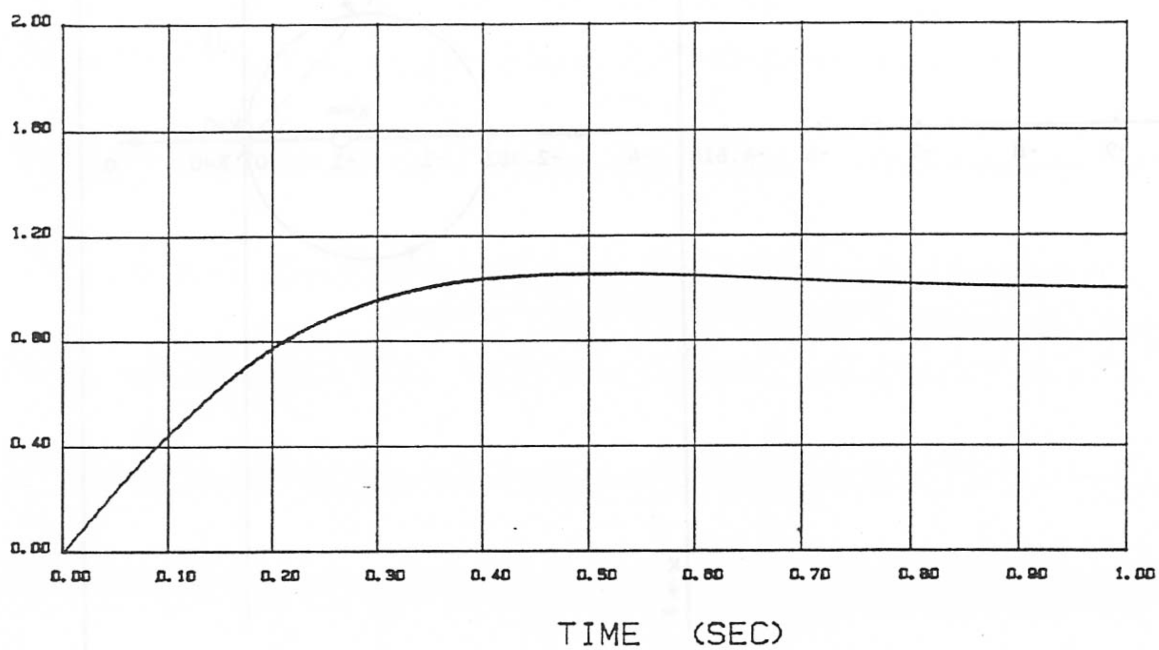
***a* = 8:** $s^4 + 208s^3 + 2.116 \times 10^5 s^2 + 2.08 \times 10^6 s + 10^7 = 0$

Roots: $-4.939 + j4.828, \quad -4.939 - j4.828, \quad -99.06 + j446.97, \quad -99.06 - j446.97$

***a* = 12:** $s^4 + 212s^3 + 2.124 \times 10^5 + 2.12 \times 10^6 s + 10^7 = 0$

Roots: $-5.017 + j4.73, \quad -5.017 - j4.73, \quad -100.98 + j447.36, \quad -100.98 - j447.36$

Unit-step Responses: All three responses for *a* = 8, *a* = 10, and 12 are similar.

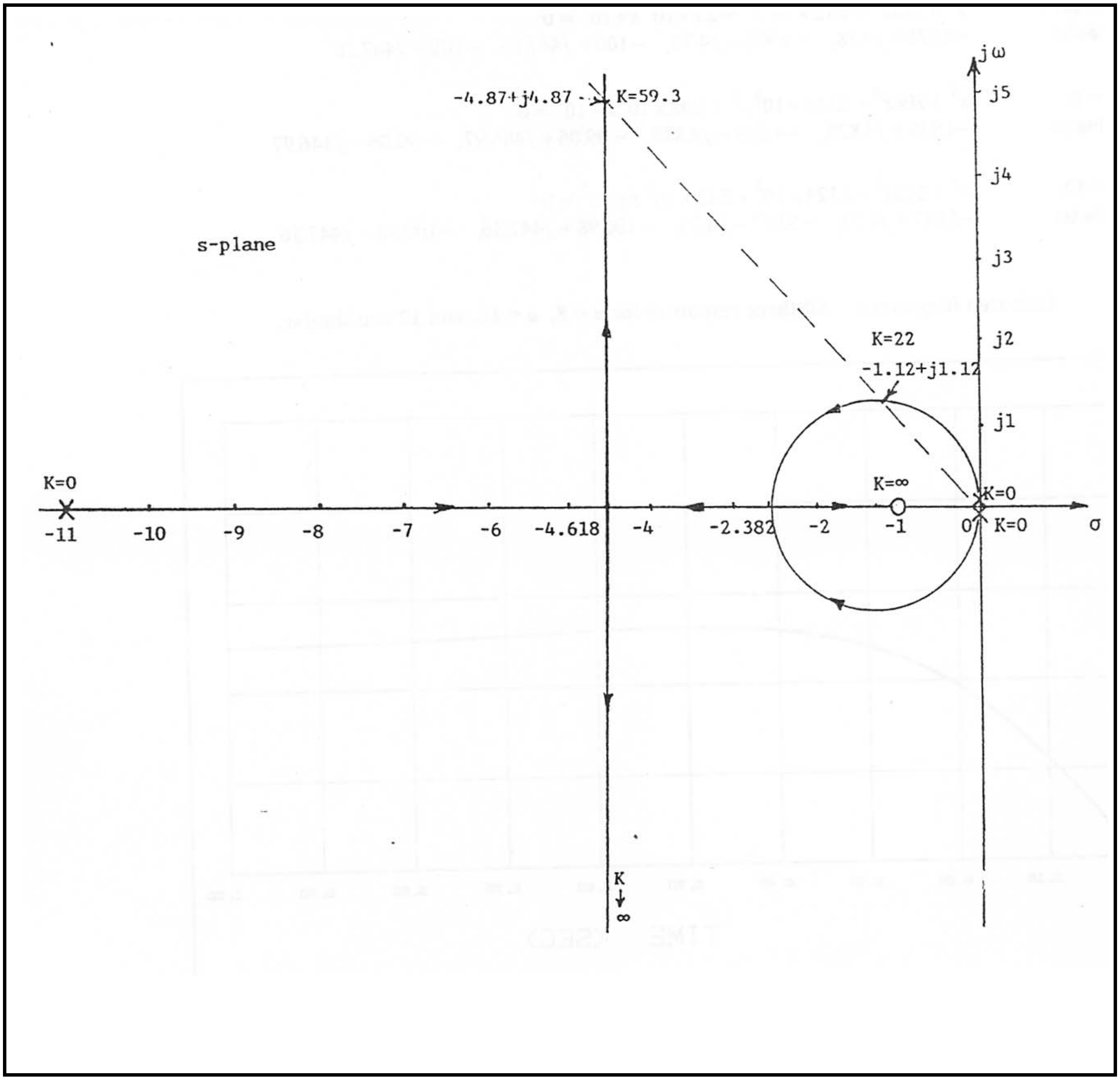


9-58 Forward-path Transfer Function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{K}{s(s+1)(s+10) + KK_t s} \qquad K_v = \lim_{s \to 0} sG(s) = \frac{K}{10 + KK_t} = 1$$

Characteristic Equation: $s^3 + 11s^2 + (10 + KK_t)s + K = s^3 + 11s^2 + Ks + K = 0$

For root loci, $G_{eq}(s) = \dfrac{K(s+1)}{s^2(s+11)}$



Root Locus Plot (*K* varies)

The root loci show that a relative damping ratio of 0.707 can be realized by two values of *K*. *K* = 22 and 59.3. As stipulated by the problem, we select *K* = 59.3.

9-59 Forward-path Transfer Function:

$$G(s) = \frac{10K}{s(s+1)(s+10)+10K_t s} \qquad K_v = \lim_{s \to 0} sG(s) = \frac{10K}{10+10K_t} = \frac{K}{1+K_t} = 1 \qquad \text{Thus} \quad K_t = K-1$$

Characteristic Equation: $s(s+1)(s+10)+10K_t+10K = s^3+11s^2+10Ks+10K = 0$

When $K = 5.93$ and $K_t = K-1 = 4.93$, the characteristic equation becomes

$$s^3+11s^2+10.046s+4.6=0$$

The roots are: -10.046, $-0.47723+j0.47976$, $-0.47723-j0.47976$

9-60 Forward-path Transfer Function:

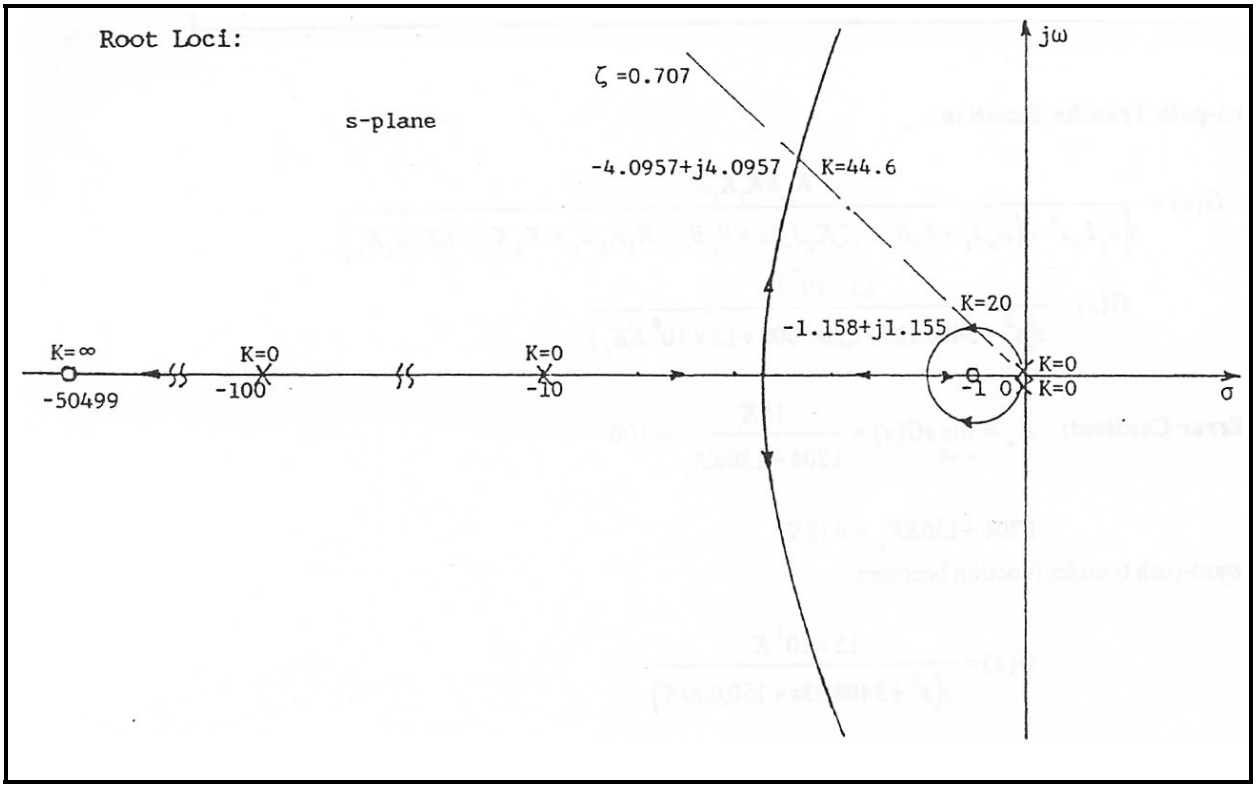
$$G(s) = \frac{K(1+aTs)}{s\left((1+Ts)\left(s^2+10s+KK_t\right)\right)} \qquad K_v = \lim_{s \to 0} sG(s) = \frac{1}{K_t} = 100 \qquad \text{Thus} \quad K_t = 0.01$$

Let $T = 0.01$ and $a = 100$. The characteristic equation of the system is written:

$$s^4+110s^3+1000s^2+K\left(0.001s^2+101s+100\right)=0$$

To construct the root contours as K varies, we form the following equivalent forward-path transfer function:

$$G_{eq}(s) = \frac{0.001K\left(s^2+101{,}000s+100{,}000\right)}{s^2(s+10)(s+100)} = \frac{0.001K(s+1)(s+50499)}{s^2(s+10)(s+100)}$$



From the root contour diagram we see that two sets of solutions exist for a damping ratio of 0.707.

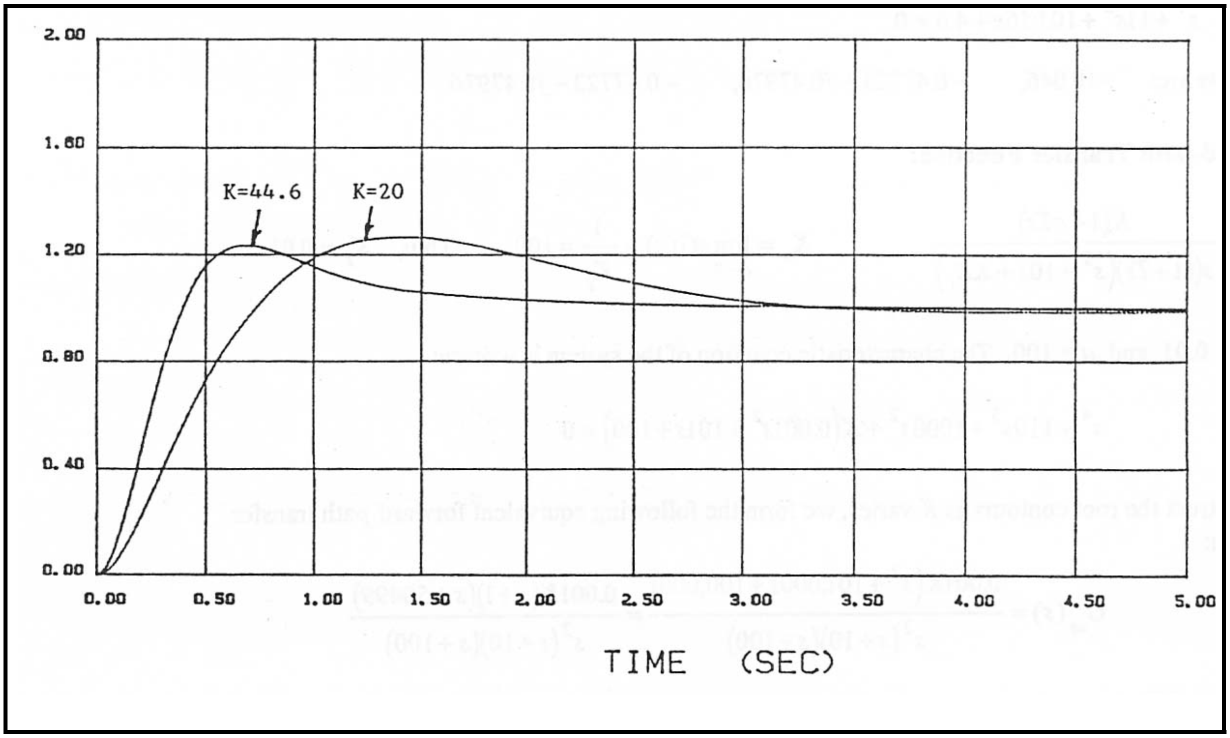
These are:

***K* = 20:** **Complex roots:** $-1.158 + j1.155, \quad -1.158 - j1.155$

***K* = 44.6:** **Complex roots:** $-4.0957 + j4.0957, \quad -4.0957 - j4.0957$

The unit-step responses of the system for *K* = 20 and 44.6 are shown below.

Unit-step Responses:



9-61 Forward-path Transfer Function:

$$G(s) = \frac{K_s K K_1 K_i N}{s\left[J_t L_a s^2 + \left(R_a J_t + L_a B_t + K_1 K_2 J_t\right)s + R_a B_t + K_1 K_2 B_t + K_b K_i + K K_1 K_i K_t\right]}$$

$$G(s) = \frac{1.5\times 10^7 K}{s\left(s^2 + 3408.33s + 1,204,000 + 1.5\times 10^8 K K_t\right)}$$

Ramp Error Constant: $K_v = \lim_{s\to 0} sG(s) = \frac{15K}{1.204 + 150KK_t} = 100$

Thus $1.204 + 150KK_t = 0.15K$

The forward-path transfer function becomes

$$G(s) = \frac{1.5 \times 10^7 K}{s\left(s^2 + 3408.33s + 150,000K\right)}$$

Characteristic Equation: $s^3 + 3408.33s + 150{,}000Ks + 1.5 \times 10^7 K = 0$

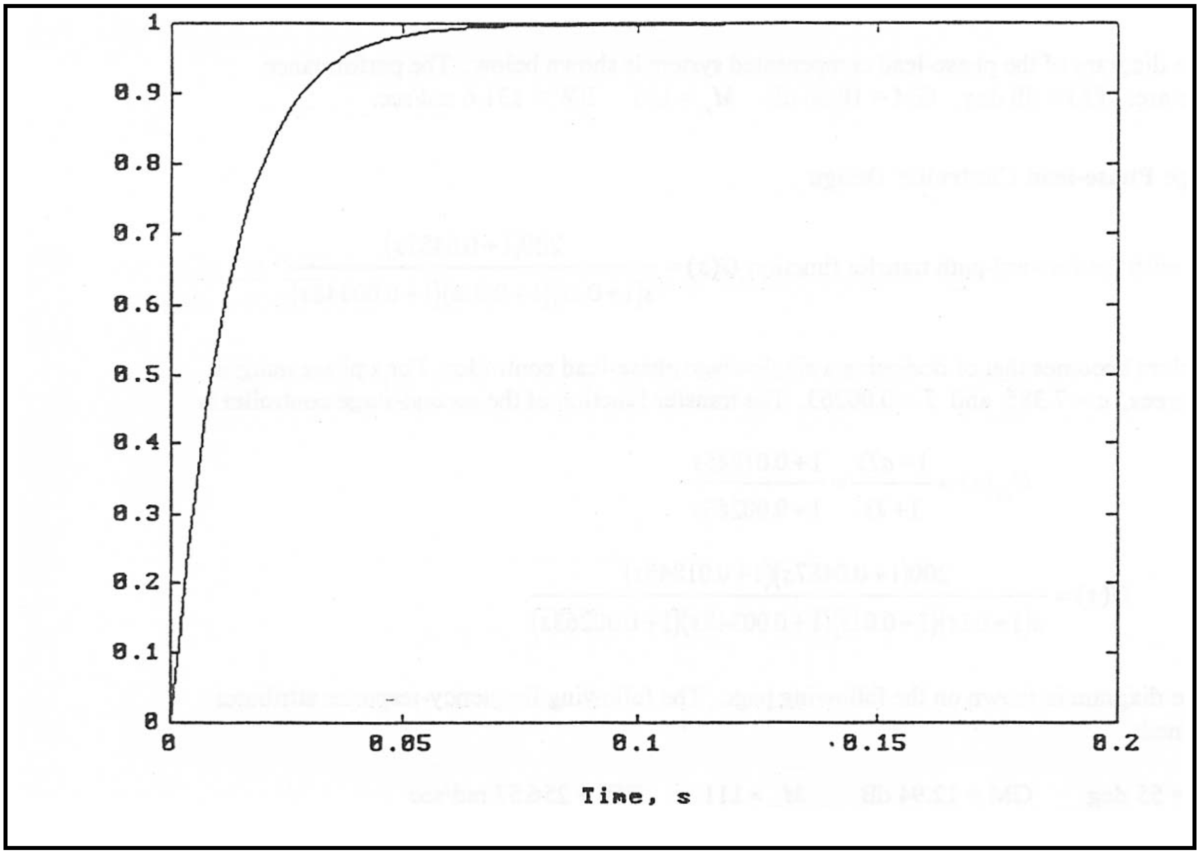
When $K = 38.667$ the roots of the characteristic equation are at

$-0.1065, \quad -1.651 + j1.65, \quad -1.651 - j1.65$ ($\zeta \cong 0.707$ for the complex roots)

The forward-path transfer function becomes

$$G(s) = \frac{5.8 \times 10^8}{s\left(s^2 + 3408.33s + 5.8 \times 10^6\right)}$$

Unit-step Response



Unit-step response attributes: Maximum overshoot = 0 Rise time = 0.0208 sec Settling time = 0.0283 sec

9-62 (a) Disturburnce-to-Output Transfer Function

$$\left.\frac{Y(s)}{T_L(s)}\right|_{r=0} == \frac{2\left(1+0.1s\right)}{s\left(1+0.01s)(1+0.1s\right)+20K} \qquad G_c(s)=1$$

For $T_L(s)=1/s$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{1}{10K} \le 0.01 \qquad \text{Thus} \qquad K \ge 10$$

(b) Performance of Uncompensated System. $K = 10$, $G_c(s)=1$

$$G(s) = \frac{200}{s(1+0.01s)(1+0.1s)}$$

The Bode diagram of $G(j\omega)$ is shown below. The system is unstable. The attributes of the frequency response are: PM = –9.65 deg GM = –5.19 dB.

(c) Single-stage Phase-lead Controller Design

To realize a phase margin of 30 degrees, *a* = 14 and *T* = 0.00348.

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.0487s}{1+0.00348s}$$

The Bode diagram of the phase-lead compensated system is shown below. The performance attributes are: PM = 30 deg GM = 10.66 dB $M_r = 1.95$ BW = 131.6 rad/sec.

(d) Two-stage Phase-lead Controller Design

Starting with the forward-path transfer function $G(s) = \dfrac{200(1+0.0487s)}{s(1+0.1s)(1+0.01s)(1+0.00348s)}$

The problem becomes that of designing a single-stage phase-lead controller. For a phase margin or 55 degrees, *a* = 7.385 and *T* = 0.00263. The transfer function of the second-stage controller is

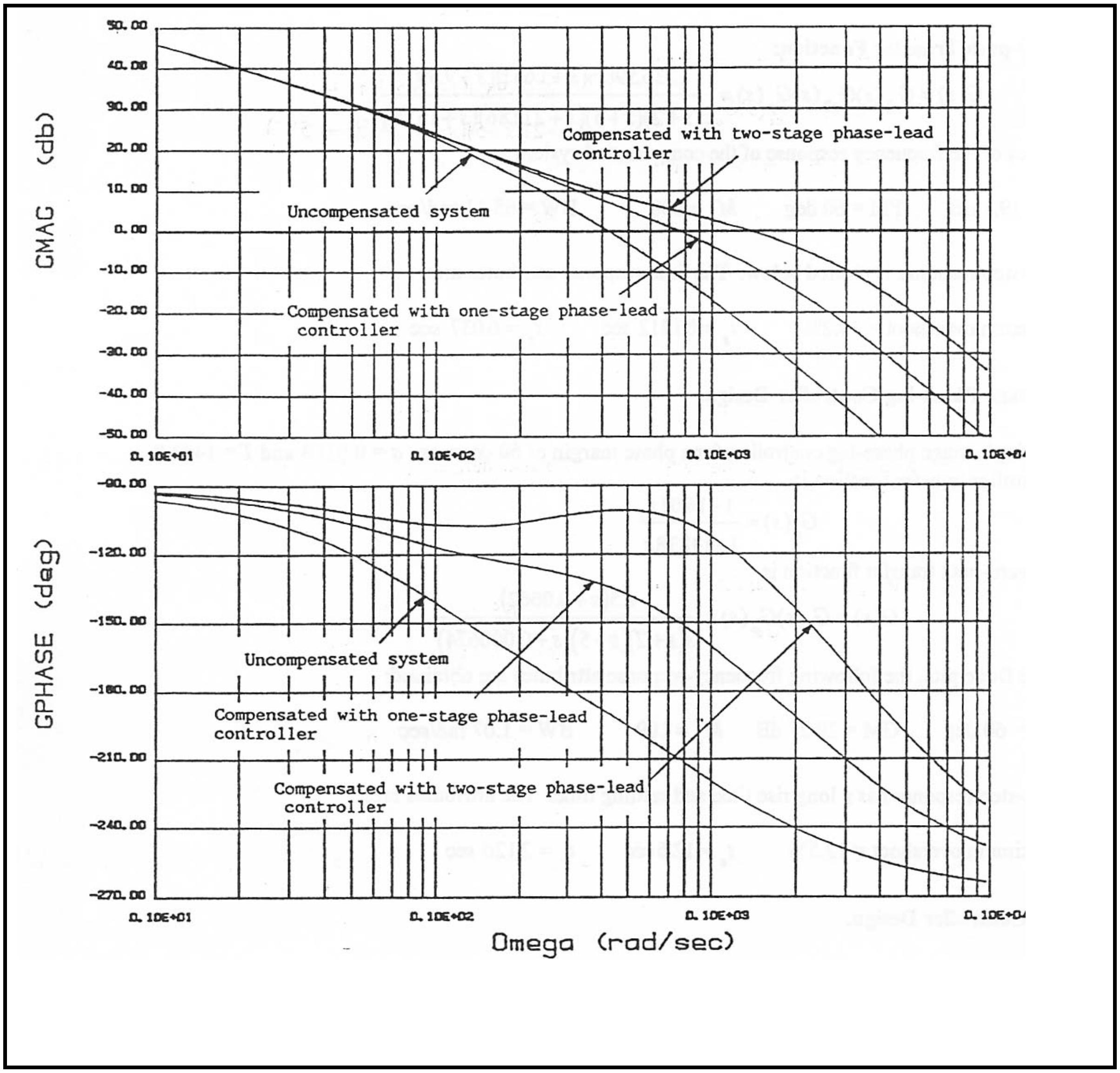
$$G_{c1}(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.01845s}{1+0.00263s}$$

Thus $$G(s)=\frac{200(1+0.0487s)(1+0.01845s)}{s(1+0.1s)(1+0.01s)(1+0.00348s)(1+0.00263s)}$$

The Bode diagram is shown on the following page. The following frequency-response attributes are obtained:

PM = 55 deg GM = 12.94 dB $M_r = 1.11$ BW = 256.57 rad/sec

Bode Plot [parts (b), (c), and (d)]


50.00
40.00
30.00
20.00
10.00
0.00
-10.00
-20.00
-30.00
-40.00
-50.00
GMAG (db)
Compensated with two-stage phase-lead controller
Uncompensated system
Compensated with one-stage phase-lead controller
0.10E+01
0.10E+02
0.10E+03
0.10E+04
-90.00
-120.00
-150.00
-180.00
-210.00
-240.00
-270.00
GPHASE (deg)
Uncompensated system
Compensated with one-stage phase-lead controller
Compensated with two-stage phase-lead controller
Omega (rad/sec)

9-63 (a) Two-stage Phase-lead Controller Design.

The uncompensated system is unstable. PM = –43.25 deg and GM = –18.66 dB.

With a single-stage phase-lead controller, the maximum phase margin that can be realized affectively is 12 degrees. Setting the desired PM at 11 deg, we have the parameters of the single-stage phase-lead controller as *a* = 128.2 and $T_1 = 0.00472$. The transfer function of the single-stage controller is

$$G_{c1}(s) = \frac{1 + aT_1 s}{1 + T_1 s} = \frac{1 + 0.6057s}{1 + 0.00472s}$$

Starting with the single-stage-controller compensated system, the second stage of the phase-lead controller is designed to realize a phase margin of 60 degrees. The parameters of the second-stage controller are: *b* = 16.1 and $T_2 = 0.0066$. Thus,

$$G_{c2}(s) = \frac{1 + bT_2 s}{1 + T_2 s} = \frac{1 + 0.106s}{1 + 0.0066s}$$

$$G_c(s) = G_{c1}(s)G_{c2}(s) = \frac{1 + 0.6057s}{1 + 0.00472s}\frac{1 + 0.106s}{1 + 0.0066s}$$

Forward-path Transfer Function:

$$G(s) = G_{c1}(s)G_{c2}(s)G_p(s) = \frac{1,236,598.6\left(s + 1.651\right)\left(s + 9.39\right)}{s\left(s + 2)(s + 5\right)\left(s + 211.86\right)\left(s + 151.5\right)}$$

Attributes of the frequency response of the compensated system are:

GM = 19.1 dB PM = 60 deg $M_r = 1.08$ BW = 65.11 rad/sec

The unit-step response is plotted below. The time-response attributes are:

Maximum overshoot = 10.2% $t_s = 0.1212$ sec $t_r = 0.037$ sec

(b) Single-stage Phase-lag Controller Design.

With a single-stage phase-lag controller, for a phase margin of 60 degrees, *a* = 0.0108 and *T* = 1483.8.

The controller transfer function is

$$G_c(s) = \frac{1 + 16.08s}{1 + 1483.8s}$$

The forward-path transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{6.5(s + 0.0662)}{s(s+2)(s+5)(s+0.000674)}$$

From the Bode plot, the following frequency-response attributes are obtained:

PM = 60 deg GM = 20.27 dB $M_r = 1.09$ BW = 1.07 rad/sec

The unit-step response has a long rise time and settling time. The attributes are:

Maximum overshoot = 12.5% $t_s = 12.6$ sec $t_r = 2.126$ sec

(c) Lead-lag Controller Design.

For the lead-lag controller, we first design the phase-lag portion for a 40-degree phase margin.

The result is *a* = 0.0238 and $T_1 = 350$. The transfer function of the controller is

$$G_{c1}(s) = \frac{1 + 8.333s}{1 + 350s}$$

The phase-lead portion is designed to yield a total phase margin of 60 degrees. The result is

b = 4.8 and $T_2 = 0.2245$. The transfer function of the phase-lead controller is

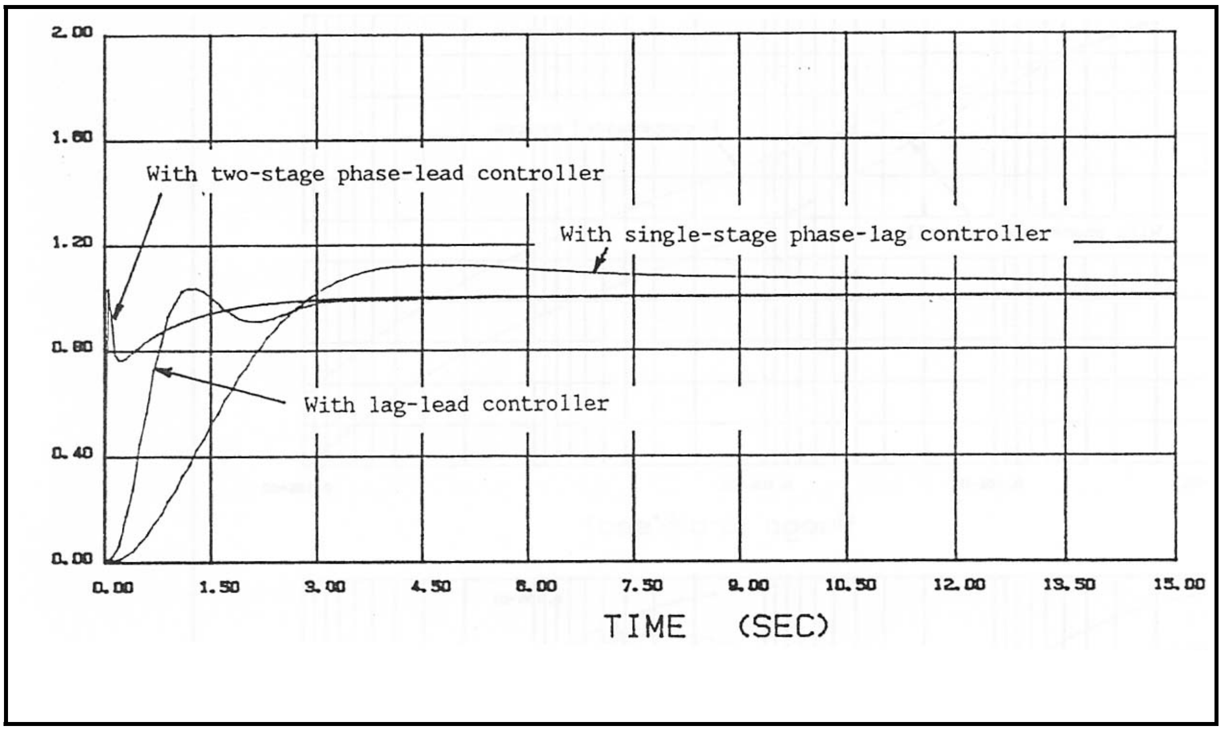
$$G_{c2}(s) = \frac{1 + 1.076s}{1 + 0.2245s}$$

The forward-path transfer function of the lead-lag compensated system is

$$G(s) = \frac{68.63(s+0.12)(s+0.929)}{s(s+2)(s+5)(s+0.00286)(s+4.454)}$$

Frequency-response attributes: PM = 60 deg GM = 13.07 dB $M_r = 1.05$ BW = 3.83 rad/sec

Unit-step response attributes: Maximum overshoot = 5.9% $t_s = 1.512$ sec $t_r = 0.7882$ sec



Unit-step Responses.

9-64 (a) The uncompensated system has the following frequency-domain attributes:

PM = 3.87 deg GM = 1 dB $M_r = 7.73$ BW = 4.35 rad/sec

The Bode plot of $G_p(j\omega)$ shows that the phase curve drops off sharply, so that the phase-lead controller would not be very effective. Consider a single-stage phase-lag controller. The phase margin of 60 degrees is realized if the gain crossover is moved from 2.8 rad/sec to 0.8 rad/sec.

The attenuation of the phase-lag controller at high frequencies is approximately –15 dB.

Choosing an attenuation of –17.5 dB, we calculate the value of *a* from

$$20\log_{10} a = -17.5 \text{ dB} \qquad \text{Thus} \qquad a = 0.1334$$

The upper corner frequency of the phase-lag controller is chosen to be at 1/*aT* = 0.064 rad/sec.

Thus, 1/*T* = 0.00854 or *T* = 117.13. The transfer function of the phase-lag controller is

$$G_c(s) = \frac{1+15.63s}{1+117.13}$$

The forward-path transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{5(1+15.63s)(1-0.05s)}{s(1+0.1s)(1+0.5s)(1+117.13s)(1+0.05s)}$$

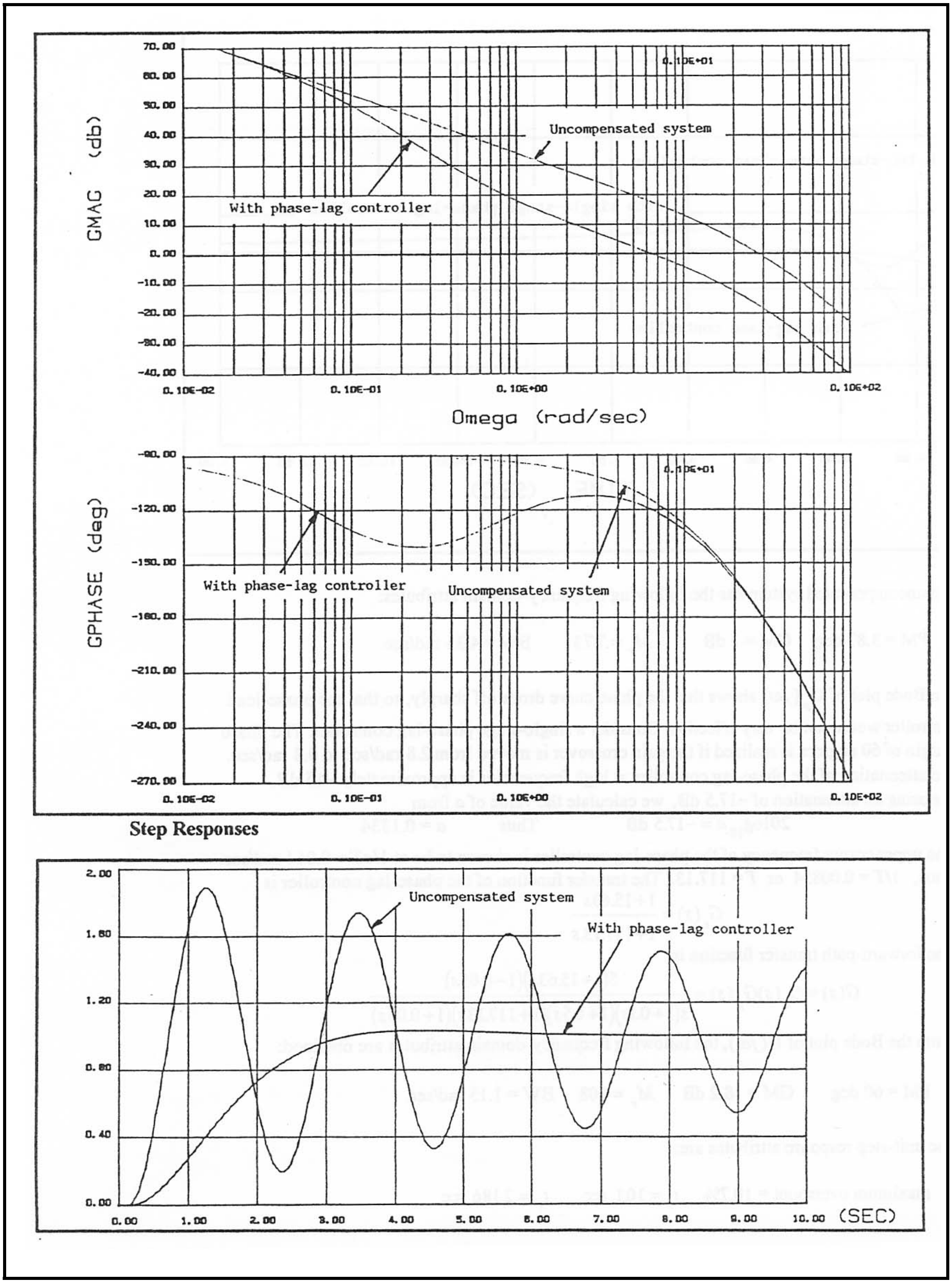
From the Bode plot of $G(j\omega)$, the following frequency-domain attributes are obtained:

PM = 60 deg GM = 18.2 dB $M_r = 1.08$ BW = 1.13 rad/sec

The unit-step response attributes are:

maximum overshoot = 10.7% $t_s = 10.1$ sec $t_r = 2.186$ sec

Bode Plots


GMAG (db)
Omega (rad/sec)
Uncompensated system
With phase-lag controller
GPHASE (deg)
With phase-lag controller
Uncompensated system
Step Responses
Uncompensated system
With phase-lag controller
(SEC)

9-64 (b) Using the exact expression of the time delay, the same design holds. The time and frequency domain attributes are not much affected.

9-65 (a) Uncompensated System.

Forward-path Transfer Function: $G(s) = \dfrac{10}{(1+s)(1+10s)(1+2s)(1+5s)}$

The Bode plot of $G(j\omega)$ is shown below.

The performance attributes are: PM = –10.64 deg GM = –2.26 dB

The uncompensated system is unstable.

(b) PI Controller Design.

Forward-path Transfer Function: $G(s) = \dfrac{10(K_p s + K_I)}{s(1+s)(1+10s)(1+2s)(1+5s)}$

Ramp-error Constant: $K_v = \lim_{s \to 0} sG(s) = 10K_I = 0.1$ Thus $K_I = 0.01$

$$G(s) = \frac{0.1(1+100K_p s)}{s(1+s)(1+10s)(1+2s)(1+5s)}$$

The following frequency-domain attributes are obtained for various values of K_P.

K_P	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.01	24.5	5.92	2.54	0.13
0.02	28.24	7.43	2.15	0.13
0.05	38.84	11.76	1.52	0.14

0.10	50.63	12.80	1.17	0.17
0.12	52.87	12.23	1.13	0.18
0.15	53.28	11.22	1.14	0.21
0.16	52.83	10.88	1.16	0.22
0.17	51.75	10.38	1.18	0.23
0.20	49.08	9.58	1.29	0.25

The phase margin is maximum at 53.28 degrees when $K_P = 0.15$.

The forward-path transfer function of the compensated system is

$$G(s) = \frac{0.1(1+15s)}{s(1+s)(1+10s)(1+5s)(1+2s)}$$

The attributes of the frequency response are:

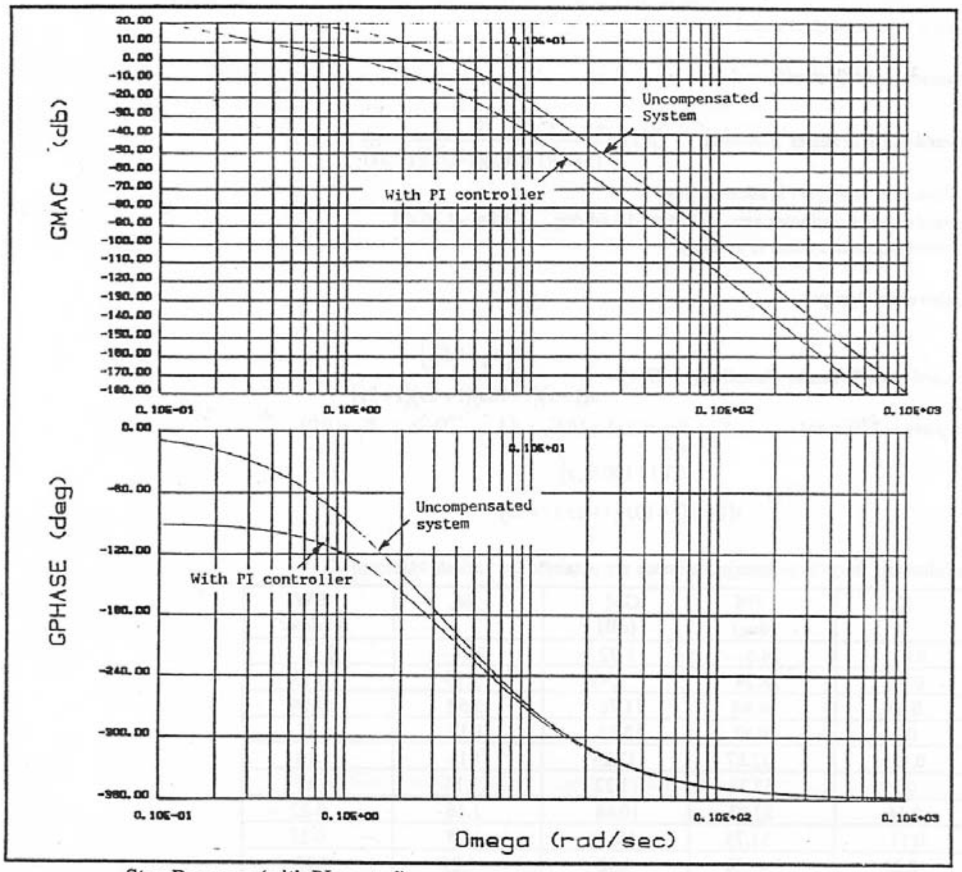
PM = 53.28 deg GM = 11.22 dB $M_r = 1.14$ BW = 0.21 rad/sec

The attributes of the unit-step response are:

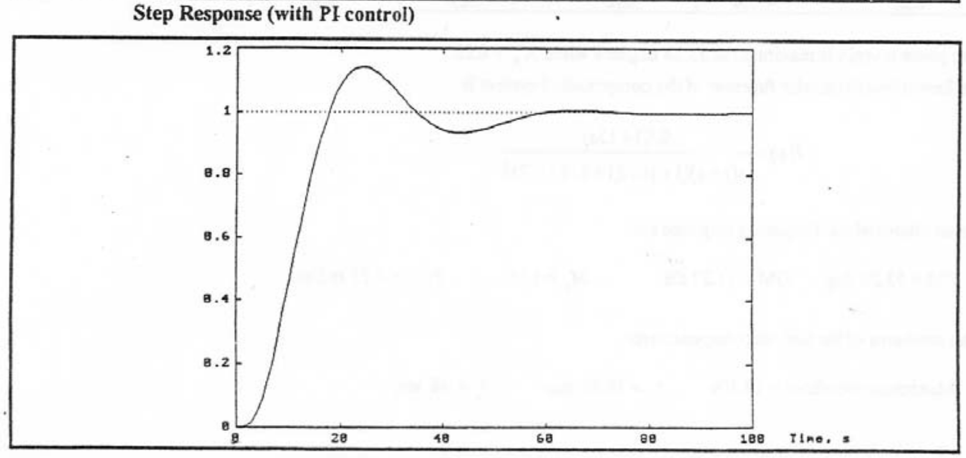
Maximum overshoot = 14.1% $t_r = 10.68$ sec $t_s = 48$ sec

Bode Plots

Bode Plots



Step Response (with PI control)



9-65 (c) Time-domain Design of PI Controller.

By setting $K_I = 0.01$ and varying K_P we found that the value of K_P that minimizes the maximum overshoot of the unit-step response is also 0.15. Thus, the unit-step response obtained in part (b) is still applicable for this case.

9-66 Closed-loop System Transfer Function.

$$\frac{Y(s)}{R(s)} = \frac{1}{s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + k_1}$$

For zero steady-state error to a step input, $k_1 = 1$. For the complex roots to be located at $-1 + j$ and $-1 - j$, we divide the characteristic polynomial by $s^2 + 2s + 2$ and solve for zero remainder.

$$\begin{array}{r|l} & s + (2 + k_2) \\ \hline s^2 + 2s + 2 & s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + 1 \\ & s^3 + \quad 2s^2 \quad + 2s \\ \hline & (2 + k_3)s^2 + (1 + k_2 + k_3)s + 1 \\ & (2 + k_3)s^2 + (4 + 2k_3)s \quad + 4 + 2k_3 \\ \hline & (-3 + k_2 - k_3)s - 3 - 2k_3 \end{array}$$

For zero remainder, $-3 - 2k_3 = 0$ Thus $k_3 = -1.5$

$-3 + k_2 - k_3 = 0$ Thus $k_2 = 1.5$

The third root is at –0.5. Not all the roots can be arbitrarily assigned, due to the requirement on the steady-state error.

9-67 (a) Open-loop Transfer Function.

$$G(s) = \frac{X_1(s)}{E(s)} = \frac{k_3}{s\left[s^2 + (4 + k_2)s + 3 + k_1 + k_2\right]}$$

Since the system is type 1, the steady-state error due to a step input is zero for all values of $k_1, k_2,$ and k_3 that correspond to a stable system. The characteristic equation of the closed-loop system is

$$s^3 + (4 + k_2)s^2 + (3 + k_1 + k_2)s + k_3 = 0$$

For the roots to be at $-1 + j$, $-1 - j$, and -10, the equation should be:

$$s^3 + 12s^2 + 22s + 20 = 0$$

Equating like coefficients of the last two equations, we have

$$4 + k_2 = 12 \qquad \text{Thus} \qquad k_2 = 8$$

$$3 + k_1 + k_2 = 22 \qquad \text{Thus} \qquad k_1 = 11$$

$$k_3 = 20 \qquad \text{Thus} \qquad k_3 = 20$$

(b) Open-loop Transfer Function.

$$\frac{Y(s)}{E(s)} = \frac{G_c(s)}{(s+1)(s+3)} = \frac{20}{s\left(s^2 + 12s + 22\right)} \qquad \text{Thus} \qquad G_c(s) = \frac{20(s+1)(s+3)}{s\left(s^2 + 12s + 22\right)}$$

Chapter 10

10-1) (a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$$

(b) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1$$

(c) State variables: $x_1 = \int_0^t y(\tau)d\tau, \quad x_2 = \frac{dx_1}{dt}, \quad x_3 = \frac{dy}{dt}, \quad x_4 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

(d) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}, \quad x_4 = \frac{d^3y}{dt^3}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2.5 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

10-2) a) $G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$

⇨ $(s^2 + 3s + 2)Y(s) = (s + 3)U(s)$

⇨ $sY(s) + 3Y(s) = -\frac{2}{s}Y(s) + \frac{3}{5}U(s) + V(s)$

Let $X(s) = -\frac{2}{s}Y(s) + \frac{3}{5}U(s)$

Then $\begin{cases} sY(s) = X(s) + U(s) + 3Y(s) \\ sX(s) = -2Y(s) + 3U(s) \end{cases}$ ⇨ $\begin{cases} \dot{y} = -3y + x + u \\ \dot{x} = -2y + 3u \end{cases}$

If $y = x_1$ and $x = x_2$, then

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 + u \\ \dot{x}_2 = -2x_1 + 3u \end{cases}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & +1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3+6s^2+11s+6}$

⇨ $Y(s)(s^3 + 6s^2 + 11s + 6) = 6U(s)$

⇨$sY(s) + 6Y(s) = -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s^2}U(s)$

Let X(s) $= -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s}U(s)$, therefore $sX(s) = -\frac{6}{s}Y(s) - 11Y(s) + \frac{6}{s}U(s)$

and Let $Z(s) = -\frac{6}{s}Y(s) + \frac{6}{s}U(s)$, then $sZ(s) = -6Y(s) + 6U(s)$. As a result:

$$\begin{cases} sY(s) = -6Y(s) + X(s) \\ sX(s) = -11Y(s) + Z(s) \\ sZ(s) = -6Y(s) + 6U(s) \end{cases}$$

or

$$\begin{cases} \dot{y} = -6y + x \\ \dot{x} = -11y + z \\ \dot{z} = -6y + 6u \end{cases}$$

If $y = x_1,\ x = x_2\ and\ z = x_3$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \text{u}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c) $G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$

$\Rightarrow Y(s)(s^2 + 7s + 12) = (s + 2)U(s)$

$\Rightarrow sY(s) = -7Y(s) - \frac{12}{s}Y(s) + U(s) + \frac{2}{s}U(s)$

Let $sX(s) = -\frac{12}{s}Y(s) + \frac{2}{s}U(s)$, then $sX(s) = -12Y(s) + 2U(s)$. As a result:

$$\begin{cases} \dot{y} = -7y + x + u \\ \dot{x} = -12y + 2u \end{cases}$$

Let $y = x_1$ and $x = x_2$, then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

d) $G(s) = \frac{Y(s)}{U(s)} = \frac{s^3+11s^2+35s+250}{s^2(s^3+4s^2+39s+108)}$

$\Rightarrow \ (s^3 + 4s + 39s + 108)Y(s) = \left[s + 11 + \frac{35}{s} + \frac{250}{s^2}\right]u(s)$

$\Rightarrow \ sY(s) = -4Y(s) + \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4}\right]U(s)$

Let $X_2(s) = \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4}\right]U(s)$, then

$sX_2(s) = 39Y(s) + \frac{108}{s}Y(s) + U(s) + \left[\frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4}\right]U(s)$

Now, let $X_3(s) = \frac{108}{s}Y(s) + \frac{11}{s^2}U(s) + \frac{35}{s^3}U(s) + \frac{250}{s^4}U(s)$, therefore

$$\begin{cases} sX_2(s) = 39Y(s) + X_3(s) + U(s) \\ sX_3(s) = 108Y(s) + \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s) \end{cases}$$

Let $X_4(s) = \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s)$, then $sX_4(s) = 11U(s) + \frac{35}{s}U(s) + \frac{250}{s^2}U(s)$

Let $X_5(s) = \frac{35}{s}U(s) + \frac{250}{s^2}U(s)$, or $sX_5(s) = 35U(s) + \frac{250}{s}U(s)$

Let $X_6(s) = \frac{250}{s}U(s)$, then $sX_6(s) = 250U(s)$. If $Y(s) = X_1(s)$, then:

$$\begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = 39x_1 + x_2 + u \\ \dot{x}_3 = 108x_1 + x_4 \\ \dot{x}_4 = 11u + x_5 \\ \dot{x}_5 = 35u + 36x_6 \\ \dot{x}_6 = 250u \end{cases}$$

or

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\\\dot{x}_5\\\dot{x}_6\end{bmatrix}=\begin{bmatrix}-4&1&0&0&0&0\\39&0&1&0&0&0\\108&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}+\begin{bmatrix}0\\1\\0\\11\\35\\250\end{bmatrix}u$$

10-3) (a) Alternatively use equations (10-225), (10-232) and (10-233)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{s+3}{s^2+3s+2}$$

The state variables are defined as

$$x_1(t)=y(t)$$
$$x_2(t)=\frac{dy(t)}{dt}$$

Then the state equations are represented by the vector-matrix equation

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)$$

where **x**(t) is the 2 × 1 state vector, $u(t)$ the scalar input, and

$$\mathbf{A}=\begin{bmatrix}0&1\\-2&-3\end{bmatrix}\qquad \mathbf{B}=\begin{bmatrix}3\\1\end{bmatrix}$$ (Also see section 2-3-3 or 10-6)

$$\mathbf{C}=\begin{bmatrix}1&0\end{bmatrix}\qquad \mathbf{D}=0$$

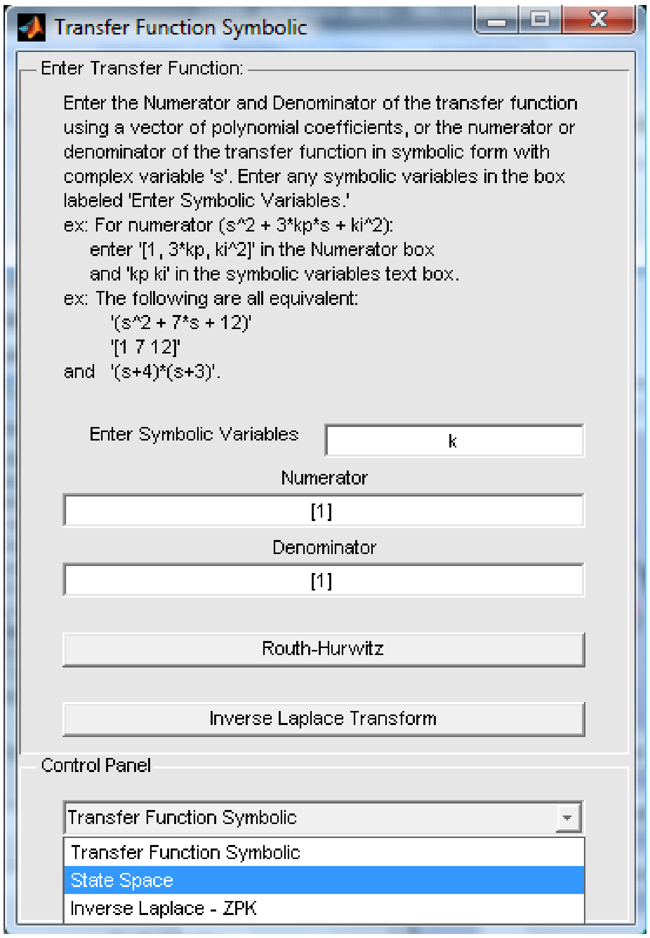
$$\mathbf{G}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

MATLAB

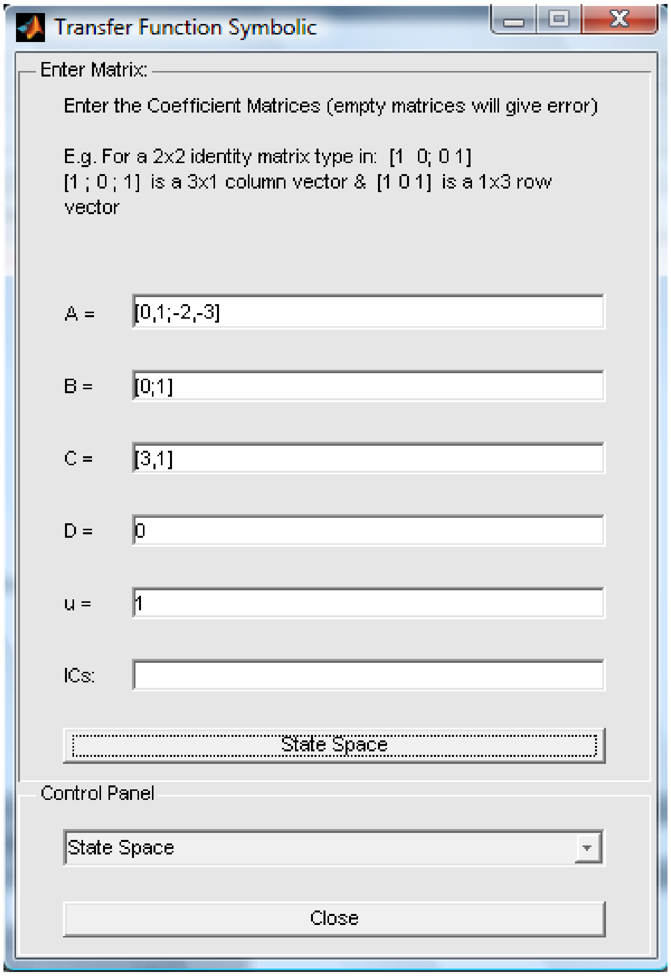
```
>> clear all

>> syms s

>> A=[0,1;-2,-3]

A =

     0     1

    -2    -3

>> B=[0;1]

B =

     0

     1

>> C=[3,1]

C =

     3     1

>> s*eye(2)-A

 ans =

 [   s,  -1]

[   2, s+3]

  >> inv(ans)

 ans =

 [ (s+3)/(s^2+3*s+2),     1/(s^2+3*s+2)]

[    -2/(s^2+3*s+2),     s/(s^2+3*s+2)]

  >> C*ans*B

 ans =

 3/(s^2+3*s+2)+s/(s^2+3*s+2)
```

Use ACSYS as demonstrate in section 10-19-2

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the "Transfer Function Symbolic" pushbutton
5) Enter the transfer function
6) Use the "State Space" option as shown below:



You get the next window. Enter the A,B,C, and D values.



State Space Analysis

Inputs:

A=| 0 1| B=|0|

|-2 -3| |1|

C=|3 1| D=|0|

State Space Representation:

Dx = | 0 1|x + |0|u

|-2 -3| |1|

```
y = |3  1|x + |0|u
```

Determinant of (s*I-A):

```
 2
s  + 3 s + 2
```

Characteristic Equation of the Transfer Function:

```
 2
s  + 3 s + 2
```

The eigen values of A and poles of the Transfer Function are:

```
-1
-2
```

Inverse of (s*I-A) is:

```
[    s + 3              1      ]
[ ------------     ------------]
[  2               2           ]
[ s  + 3 s + 2     s  + 3 s + 2]
[                              ]
[       2              s       ]
[- ------------    ------------]
[   2               2          ]
[  s  + 3 s + 2    s  + 3 s + 2]
```

State transition matrix (phi) of A:

```
[ 2 exp(-t) - exp(-2 t)       exp(-t) - exp(-2 t)  ]
[                                                  ]
[-2 exp(-t) + 2 exp(-2 t)    -exp(-t) + 2 exp(-2 t)]
```

Transfer function between u(t)and y(t) is:

```
      s + 3
    ------------
     2
    s  + 3 s + 2
```

No Initial Conditions Specified

States (X) in Laplace Domain:

```
[       1       ]
[---------------]
[(s + 2) (s + 1)]
[               ]
[       s       ]
[---------------]
[(s + 2) (s + 1)]
```

Inverse Laplace x(t):

```
[ exp(-t) - exp(-2 t)  ]
[                      ]
[-exp(-t) + 2 exp(-2 t)]
```

Output Y(s):

```
      s + 3
  ---------------
  (s + 2) (s + 1)
```

Inverse Laplace y(t):

```
2 exp(-t) - exp(-2 t)
```

Use the same procedure for parts b, c and d.

10-4) a) $x_1 = \frac{-x_4+u}{s+2}$ ➔ $(s+2)X_1 = -X_4 + U$, ➔ $\dot{x}_1 = -x_4 - 2x_1 + u$

$x_4 = \frac{x_3+x_1}{s}$ ➔ $sX_4 = X_3 + X_1$ ➔ $\dot{x}_4 = x_1 + x_3$

$x_2 = \frac{0.5}{s}x_1$ ➔ $sX_2 = 0.5X_1$ ➔ $\dot{x}_2 = 0.5x_1$

$x_3 = \frac{x_2}{s}$ ➔ $sX_3 = X_2$ ➔ $\dot{x}_3 = x_2$

$y = x_1 + x_2 + x_3$

As a result:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0 & -1\\0.5 & 0 & 0 & 0\\0 & 1 & 0 & 0\\1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}1\\0\\0\\0\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

b) $X_1(s) = \frac{1}{s+2}U(s)$ ➔ $sX_1(s) = -2X_1(s) + U(s)$ ➔ $\dot{x}_1 = -2x_1 + u$

$X_2 = \frac{s+4}{s+3}X_1$ ➔ $sX_2(s) = sX_1 + 4X_1 - 3X_2$ ➔ $\dot{x}_2 = \dot{x}_1 + 4x_1 - 3x_2 = 2x_1 - 3x_2 + u$

$X_3 = \frac{X_2+X_1-6X_3}{s}$ ➔ $sX_3(s) = X_2 + X_1 - 6X_3$ ➔ $\dot{x}_3 = x_2 + x_1 - x_3$

$y = x_3 + x_1$

As a result:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\2 & -3 & 0\\1 & 1 & -6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

c) $X_1(s) = \frac{1}{s+5}U(s)$ ➔ $sX_1 = -5X_1 + U$ ➔ $\dot{x}_1 = -5x_1 + u$

$X_2 = \frac{X_1+U-X_3}{s+2}$ ➔ $sX_2 = X_1 - 2X_2 - X_3 + U$ ➔ $\dot{x}_2 = x_1 - 2x_2 - x_3 + u$

$X_3 = \frac{X_2}{s+4}$ ➔ $sX_3 = X_2 - 4X_3$ ➔ $\dot{x}_3 = x_2 - 4x_3$

$X_4 = \frac{2X_3}{s}$ ➔ $sX_4 = 2X_3$ ➔ $\dot{x}_4 = 2x_3$

$y = x_2 + x_4$

As a result:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}-5 & 0 & 0 & 0\\ 1 & -2 & -1 & 0\\ 0 & 1 & -4 & 0\\ 0 & 0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 0\\ 0\end{bmatrix}\text{u}$$

$$y = \begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

10-5) We shall first show that

$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{1}{2!}\frac{\mathbf{A}^2}{s^2} + \cdots$$

We multiply both sides of the equation by $\left(s\mathbf{I} - \mathbf{A}\right)$, and we get **I** = **I**. Taking the inverse Laplace transform on both sides of the equation gives the desired relationship for $\phi(t)$.

10-6)

(a) USE MATLAB

```
Amat=[0 1;-2 -1]
[mA,nA]=size(Amat);
rankA=rank(Amat);
disp(' Characteristic Polynomial: ')
chareq=poly(Amat);
[mchareq,nchareq]=size(chareq);
syms s;
poly2sym(chareq,s)
[evecss,eigss]=eig(Amat);
disp(' Eigenvalues of A = Diagonal Canonical Form of A is:');
Abar=eigss,
disp('Eigen Vectors are  ')
T=evecss
% state transition matrix
ilaplace(inv([s 0;0 s]-Amat))
```

Results in MATLAB COMMAND LINE

Amat =

0 1

-2 -1

Characteristic Polynomial:

ans =

s^2+s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-0.5000 + 1.3229i 0

0 -0.5000 - 1.3229i

Eigen Vectors are

T =

-0.2041 - 0.5401i -0.2041 + 0.5401i

0.8165 0.8165

phi=ilaplace(inv([s 0;0 s]-Amat))

phi =

[1/7*exp(-1/2*t)*(7*cos(1/2*7^(1/2)*t)+7^(1/2)*sin(1/2*7^(1/2)*t)), 2/7*7^(1/2)*exp(-1/2*t)*sin(1/2*7^(1/2)*t)]

[-4/7*7^(1/2)*exp(-1/2*t)*sin(1/2*7^(1/2)*t), 1/7*exp(-1/2*t)*(7*cos(1/2*7^(1/2)*t)-7^(1/2)*sin(1/2*7^(1/2)*t))]

% use vpa to convert to digital format. Use digit(#) to adjust level of precision if necessary.

vpa(phi)

ans =

[.1428571*exp(.5000000*t)*(7.*cos(1.322876*t)+2.645751*sin(1.322876*t)),

.7559289*exp(-.5000000*t)*sin(1.322876*t)]

[-1.511858*exp(-.5000000*t)*sin(1.322876*t),

.1428571*exp(-.5000000*t)*(7.*cos(1.322876*t)-2.645751*sin(1.322876*t))]

ANALYTICAL SOLUTION:

Characteristic equation: $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + s + 2 = 0$

Eigenvalues: $s = -0.5 + j1.323, \quad -0.5 - j1.323$

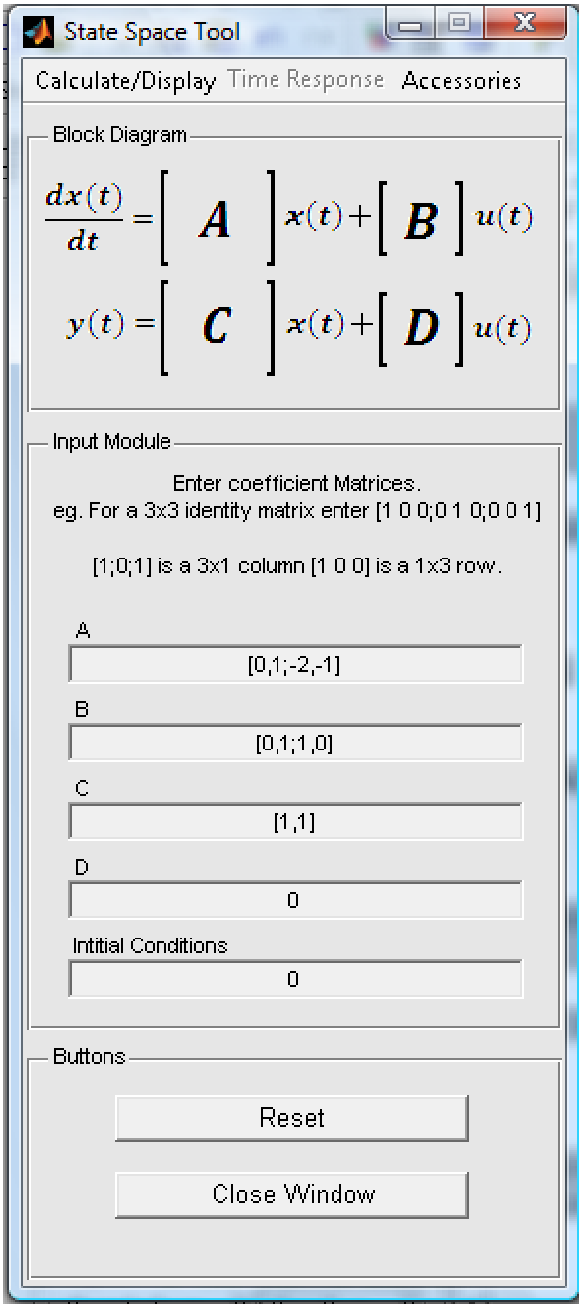
State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 1.323t + 0.378 \sin 1.323t & 0.756 \sin 1.323t \\ -1.512 \sin 1.323t & -1.069 \sin\left(1.323t - 69.3^\circ\right) \end{bmatrix} e^{-0.5t}$$

Alternatively

USE ACSYS as illustrated in section 10-19-1

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the “State Space” pushbutton
5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
6) Use the “Calculate/Display” menu and find the eigenvalues.



From MATLAB Command Window:

The A matrix is:

Amat =

0 1

-2 -1

Characteristic Polynomial:

ans =

s^2+s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-0.5000 + 1.3229i 0

0 -0.5000 - 1.3229i

Eigen Vectors are

T =

-0.2041 - 0.5401i -0.2041 + 0.5401i

0.8165 0.8165

THE REST ARE SAME AS PART A.

(b) **Characteristic equation:** $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 4 = 0$ **Eigenvalues:** $s = -4, \quad -1$

State transition matrix:

$$\phi(t) = \begin{bmatrix} 1.333e^{-t} - 0.333e^{-4t} & 0.333e^{-t} - 0.333e^{-4t} \\ -1.333e^{-t} - 1.333e^{-4t} & -0.333e^{-t} + 1.333e^{-4t} \end{bmatrix}$$

(c) **Characteristic equation:** $\Delta(s) = (s+3)^2 = 0$ **Eigenvalues:** $s = -3, \quad -3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(d) **Characteristic equation:** $\Delta(s) = s^2 - 9 = 0$ **Eigenvalues:** $s = -3, \quad 3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(e) **Characteristic equation:** $\Delta(s) = s^2 + 4 = 0$ **Eigenvalues:** $s = j2, \quad -j2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

(f) **Characteristic equation:** $\Delta(s) = s^3 + 5s^2 + 8s + 4 = 0$ **Eigenvalues:** $s = -1, \quad -2, \quad -2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

(g) **Characteristic equation:** $\Delta(s) = s^3 + 15s^2 + 75s + 125 = 0$ **Eigenvalues:** $s = -5, \quad -5, \quad -5$

$$\phi(t) = \begin{bmatrix} e^{-5t} & te^{-5t} & 0 \\ 0 & e^{-5t} & te^{-5t} \\ 0 & 0 & e^{-5t} \end{bmatrix}$$

State transition equation: $\mathbf{x}(t) = \phi(t)\mathbf{x}(t) + \int_0^t \phi(t-\tau)\mathbf{Br}(\tau)d\tau$ $\phi(t)$ for each part is given in Problem 5-3.

10-7)In MATLAB USE ilaplace to find $\mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right]$ see previous problem for codes.

(a)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+2}{s\left(s^2+s+2\right)} \\ \dfrac{s-2}{s\left(s^2+s+2\right)} \end{bmatrix} = \begin{bmatrix} 1 + 0.378\sin 1.323t - \cos 1.323t \\ -1 + 1.134\sin 1.323t + \cos 1.323t \end{bmatrix} \qquad t \geq 0$$

(b)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+6}{s(s+1)(s+2)} \\ \dfrac{s-4}{s(s+1)(s+4)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{1.5}{s} - \dfrac{1.67}{s+1} + \dfrac{0.167}{s+4} \\ \dfrac{-1}{s} + \dfrac{1.67}{s+1} - \dfrac{0.667}{s+4} \end{bmatrix} = \begin{bmatrix} 1.5 - 1.67e^{-t} + 0.167e^{-4t} \\ -1 + 1.67e^{-t} - 0.667e - 4t \end{bmatrix} \qquad t \ge 0$$

(c)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \ge 0$$

(d)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s-3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \ge 0$$

(e)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s^2+4} & 2 \\ \dfrac{-2}{s^2+4} & \dfrac{s}{s^2+4} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{2}{s} \\ \dfrac{1}{\left(s^2+4\right)} \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5\sin 2t \end{bmatrix} \qquad t \ge 0$$

(f)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{1}{s(s+2)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5\left(1-e^{-2t}\right) \\ 0 \end{bmatrix} \qquad t \ge 0$$

(g)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+5} & \frac{1}{(s+5)^2} & 0 \\ 0 & \frac{1}{s+5} & \frac{1}{(s+5)^2} \\ 0 & 0 & \frac{1}{s+5}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{1}{s(s+5)^2} \\ \frac{1}{s(s+5)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{0.04}{s} - \frac{0.04}{s+5} - \frac{0.2}{(s+5)^2} \\ \frac{0.2}{s} - \frac{0.2}{s+5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.04 - 0.04e^{-5t} - 0.2te^{-5t} \\ 0.2 - 0.2e^{-5t} \end{bmatrix} u_s(t)$$

10-8) State transition equation: $\mathbf{x}(t) = \phi(t)\mathbf{x}(t) + \int_0^t \phi(t-\tau)\mathbf{B}\mathbf{r}(\tau)d\tau$ $\phi(t)$ for each part is given in Problem 5-3.

(a)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathsf{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathsf{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathsf{L}^{-1}\begin{bmatrix} \dfrac{s+2}{s\left(s^2+s+2\right)} \\ \dfrac{s-2}{s\left(s^2+s+2\right)} \end{bmatrix} = \begin{bmatrix} 1+0.378\sin 1.323t-\cos 1.323t \\ -1+1.134\sin 1.323t+\cos 1.323t \end{bmatrix} \qquad t \ge 0$$

(b)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+6}{s(s+1)(s+2)} \\ \dfrac{s-4}{s(s+1)(s+4)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{1.5}{s}-\dfrac{1.67}{s+1}+\dfrac{0.167}{s+4} \\ \dfrac{-1}{s}+\dfrac{1.67}{s+1}-\dfrac{0.667}{s+4} \end{bmatrix} = \begin{bmatrix} 1.5-1.67e^{-t}+0.167e^{-4t} \\ -1+1.67e^{-t}-0.667e-4t \end{bmatrix} \qquad t \ge 0$$

(c)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \ge 0$$

(d)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s-3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \ge 0$$

(e)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s^2+4} & 2 \\ \dfrac{-2}{s^2+4} & \dfrac{s}{s^2+4} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{2}{s} \\ \dfrac{1}{\left(s^2+4\right)} \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5\sin 2t \end{bmatrix} \qquad t \ge 0$$

(f)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+1} & 0 & 0 \\ 0 & \dfrac{1}{s+2} & \dfrac{1}{\left(s+2\right)^2} \\ 0 & 0 & \dfrac{1}{s+2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+2\right)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5\left(1-e^{-2t}\right) \\ 0 \end{bmatrix} \qquad t \ge 0$$

(g)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+5} & \dfrac{1}{\left(s+5\right)^2} & 0 \\ 0 & \dfrac{1}{s+5} & \dfrac{1}{\left(s+5\right)^2} \\ 0 & 0 & \dfrac{1}{s+5} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+5\right)^2} \\ \dfrac{1}{s\left(s+5\right)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{0.04}{s} - \dfrac{0.04}{s+5} - \dfrac{0.2}{\left(s+5\right)^2} \\ \dfrac{0.2}{s} - \dfrac{0.2}{s+5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.04 - 0.04e^{-5t} - 0.2te^{-5t} \\ 0.2 - 0.2e^{-5t} \end{bmatrix} u_s(t)$$

10-9) (a) Not a state transition matrix, since $\phi(0) \neq \mathbf{I}$ (identity matrix).

(b) Not a state transition matrix, since $\phi(0) \neq \mathbf{I}$ (identity matrix).

(c) $\phi(t)$ is a state transition matrix, since $\phi(0) = \mathbf{I}$ and

$$\left[\phi(t)\right]^{-1} = \begin{bmatrix} 1 & 0 \\ 1-e^{-t} & e^{-t} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1-e^{t} & e^{t} \end{bmatrix} = \phi(-t)$$

(d) $\phi(t)$ is a state transition matrix, since $\phi(0) = \mathbf{I}$, and

$$\left[\phi(t)\right]^{-1} = \begin{bmatrix} e^{2t} & -te^{2t} & t^2e^{2t}/2 \\ 0 & e^{2t} & -te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} = \phi(-t)$$

10-10) a) $\dot{x} = Ax + Bu$ ➔$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$ and $(sI-A)^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

Therefore:

$$\Phi(t) = L^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

If $x(0) = 0$, then $x(t) = \int_0^t \Phi(t-\tau)Bu(t)d\,\tau = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$

b) $\Phi(t) = L^{-1}\{(sI-A)^{-1}\}$

$$= L^{-1}\left\{\frac{1}{s^2+s+0.5}\begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} e^{-0.5t}(cos\,0.5t - sin\,0.5t) & e^{-0.5t}\,sin\,0.5t \\ 2e^{-0.5t}\,sin\,0.5t & e^{-0.5t}(cos\,5t + sin\,0.5t) \end{bmatrix}$$

If $x(0) = 0$, then

$$x(t) = A^{-1}(e^{At} - I)B = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 0.5e^{-0.5t}(cos\,0.5t\ - sin\,0.5t) - 0.5 \\ e^{-0.5t} - sin\,0.5t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-0.5t}\,sin\,5t \\ -e^{-0.5t}(cos\,0.5t + sin 0.5t) + 1 \end{bmatrix}$$

and

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t}\sin 0.5t$$

10-11) (a) (1) Eigenvalues of A: $2.325, \quad -0.3376 + j0.5623, \quad -0.3376 - j0.5623$

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2s-1 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}U(s)$$

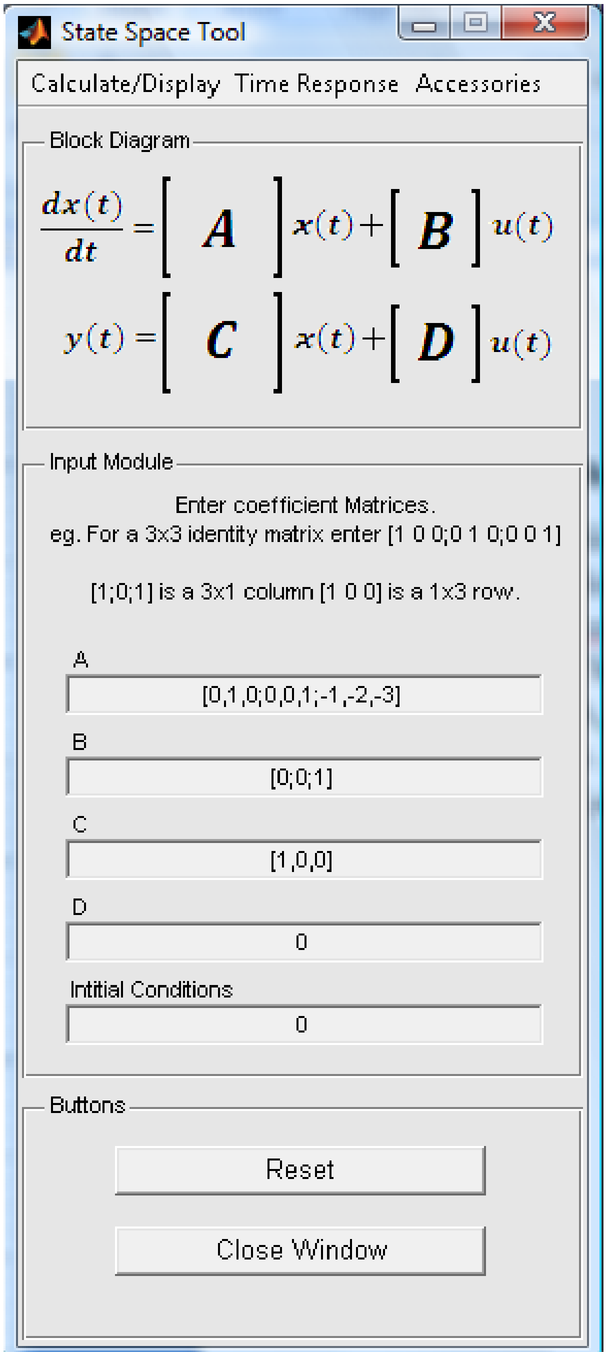
$$\Delta(s) = s^3 + 3s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{1}{s^3 + 3s^2 + 2s + 1}$$

USE ACSYS as illustrated in section 10-19-1

7) Activate MATLAB
8) Go to the folder containing ACSYS
9) Type in Acsys
10) Click the “State Space” pushbutton
11) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
12) Use the “Calculate/Display” menu and find the eigenvalues and other State space calculations.



The A matrix is:

Amat =

0	**1**	**0**
0	**0**	**1**
-1	**-2**	**-3**

Characteristic Polynomial:

ans =

s^3+3*s^2+2*s+1

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

```
 -2.3247          0                 0
      0      -0.3376 + 0.5623i      0
      0           0            -0.3376 - 0.5623i
```

Eigen Vectors are

T =

```
  0.1676      0.7868              0.7868
 -0.3896     -0.2657 + 0.4424i  -0.2657 - 0.4424i
  0.9056     -0.1591 - 0.2988i  -0.1591 + 0.2988i
```

State-Space Model is:

```
a =

      x1  x2  x3
  x1   0   1   0
  x2   0   0   1
  x3  -1  -2  -3


b =

       u1
  x1   0
  x2   0
  x3   1
```

```
c =

      x1  x2  x3

  y1   1   0   0


d =

      u1

  y1   0


Continuous-time model.

 Characteristic Polynomial:

 ans =

 s^3+3*s^2+2*s+1


  Equivalent Transfer Function Model is:

 Transfer function:

1.776e-015 s^2 + 6.661e-016 s + 1

---------------------------------

     s^3 + 3 s^2 + 2 s + 1


 Pole, Zero Form:


Zero/pole/gain:

1.7764e-015 (s^2  + 0.375s + 5.629e014)

---------------------------------------

  (s+2.325) (s^2  + 0.6753s + 0.4302)
```

The numerator is basically equal to 1

Use the same procedure for other parts.

(b) (1) Eigenvalues of A: $-1, \quad -1.$

(2) Transfer function relation:

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}U(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} \\ \dfrac{1}{(s+1)} \end{bmatrix}U(s) \qquad \Delta(s) = s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{(s+1)^2} \\ \dfrac{1}{s+1} \end{bmatrix} = \frac{1}{(s+1)^2} + \frac{1}{s+1} = \frac{s+2}{(s+1)^2}$$

(c) (1) Eigenvalues of A: $0, \quad -1, \quad -1.$

(2) Transfer function relation:

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2+2s=1 & s+2 & 1 \\ 0 & s(s+2)) & s \\ 0 & -s & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}U(s) \qquad \Delta(s) = s\left(s^2 + 2s + 1\right)$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{s+1}{s(s+1)^2} = \frac{1}{s(s+1)}$$

10-12) We write $$\frac{dy}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} = x_2 + x_3 \qquad \frac{d^2y}{dt} = \frac{dx_2}{dt} + \frac{dx_3}{dt} = -x_1 - 2x_2 - 2x_3 + u$$

$$\frac{d\overline{\mathbf{x}}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (1)$$

$$\overline{\mathbf{x}} = \begin{bmatrix} x_1 \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} \qquad \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \overline{\mathbf{x}} \qquad (2)$$

Substitute Eq. (2) into Eq. (1), we have

$$\frac{d\overline{\mathbf{x}}}{dt} = \mathbf{A}_1\overline{\mathbf{x}} + \mathbf{B}_1 u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

10-13) For MATLAB Codes see 10-15

(a)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -2 & 0 \\ -1 & s-2 & 0 \\ 1 & 0 & s-1 \end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 2, \quad a_1 = 0, \quad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 6 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}$$

(b)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s & -2 & 0 \\ -1 & s-1 & 0 \\ 1 & -1 & s-1 \end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 2, \qquad a_1 = 0, \qquad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+2 & -1 & 0 \\ 0 & s+2 & 0 \\ 1 & 2 & s+3 \end{vmatrix} = s^3 + 7s^2 + 16s + 12 = s^3 + a_2 s^2 + a_1 s + a_0 \quad a_0 = 12, \quad a_1 = 16, \quad a_2 = 7$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 4 \\ 1 & -6 & 23 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 9 & 6 & 1 \\ 6 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

(d)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+1 & -1 & 0 \\ 0 & s=1 & -1 \\ 0 & 0 & s+1 \end{vmatrix} = s^3 + 3s^2 + 3s + 1 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 1, \quad a_1 = 3, \quad a_2 = 3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

(e)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s-1 & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 2s - 1 = s^2 + a_1 s + a_0 \qquad a_0 = -1, \quad a_1 = 2$$

$$\mathbf{M} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

10-14) For MATLAB codes see 10-15

(a) From Problem 10-13(a),

$$\mathbf{M} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.5 & 1 & 3 \\ 0.5 & 1.5 & 4 \\ -0.5 & -1 & -2 \end{bmatrix}$$

(b) From Problem 10-13(b),

$$\mathbf{M} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 1 \\ 2 & 5 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.2308 & 0.3077 & 1.0769 \\ 0.1538 & 0.5385 & 1.3846 \\ -0.2308 & -0.3077 & -0.0769 \end{bmatrix}$$

(c) From Problem 10-13(c),

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

Since **V** is singular, the OCF transformation cannot be conducted.

(d) From Problem 10-13(d),

$$\mathbf{M} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -2 & 2 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

(e) From Problem 10-13(e),

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$ Then, $$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

10-15) (a) Eigenvalues of **A**: 1, 2.7321, –0.7321

$$\mathbf{T} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0.5591 & 0.8255 \\ 0 & 0.7637 & -0.3022 \\ 1 & -0.3228 & 0.4766 \end{bmatrix}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are the eigenvectors.

(b) Eigenvalues of **A**: 1, 2.7321, –0.7321

$$\mathbf{T} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 0 & 0.5861 & 0.7546 \\ 0 & 0.8007 & -0.2762 \\ 1 & 0.1239 & 0.5952 \end{bmatrix}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are the eigenvalues.

(c) Eigenvalues of **A**: -3, -2, -2. A nonsingular DF transformation matrix **T** cannot be found.

(d) Eigenvalues of **A**: –1, –1, –1

The matrix **A** is already in Jordan canonical form. Thus, the DF transformation matrix **T** is the identity matrix **I**.

(e) Eigenvalues of **A**: 0.4142, –2.4142

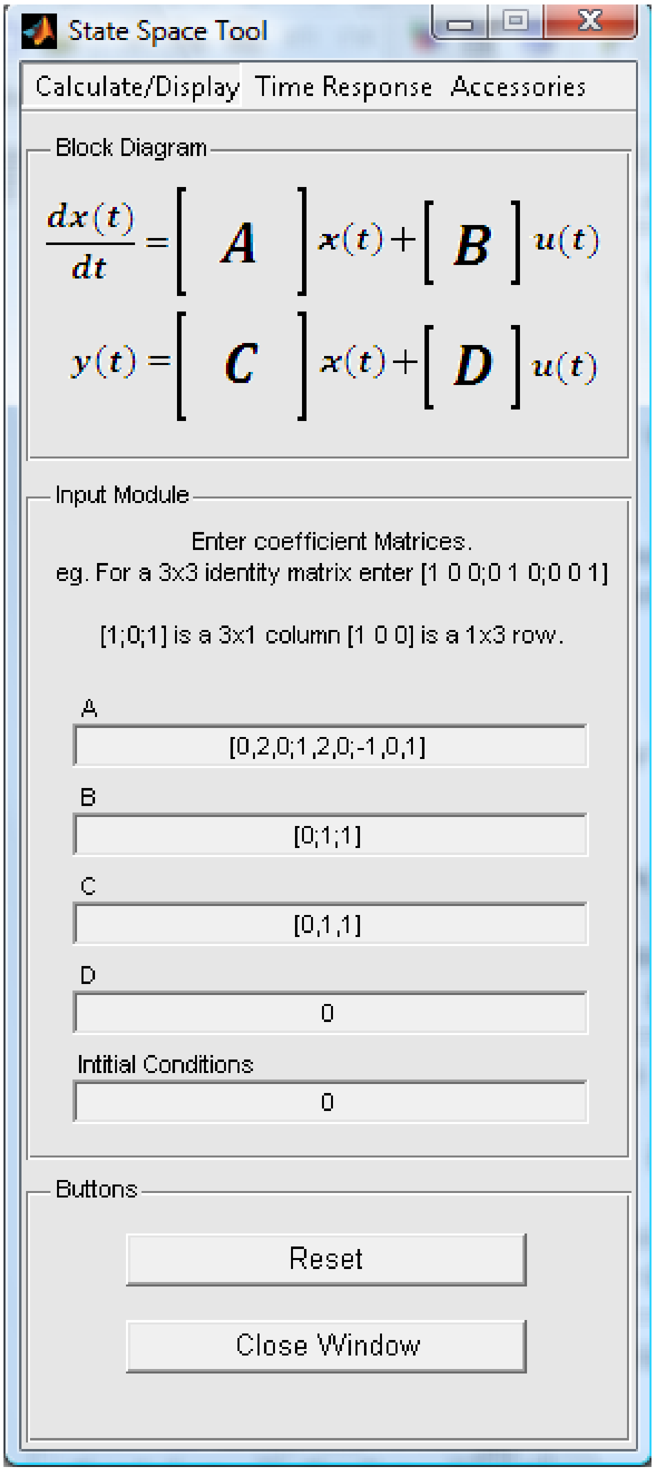
$$\mathbf{T} = [\mathbf{p}_2 \quad \mathbf{p}_2] = \begin{bmatrix} 0.8629 & -0.2811 \\ -0.5054 & 0.9597 \end{bmatrix}$$

USE ACSYS as illustrated in section 10-19-1

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the "State Space" pushbutton
5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
6) Use the "Calculate/Display" menu and find the eigenvalues.
7) Next use the "Calculate/Display" menu and conduct State space calculations.
8) Next use the "Calculate/Display" menu and conduct Controlability calculations.

NOTE: the above order of calculations MUST BE followed in the order stated, otherwise you will get an error.

SOLVE PART (a)



The A matrix is:

Amat =

0	**2**	**0**
1	**2**	**0**
-1	**0**	**1**

Characteristic Polynomial:

ans =

s^3-3*s^2+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

1.0000	**0**	**0**
0	**2.7321**	**0**
0	**0**	**-0.7321**

Eigen Vectors are

T =

0	**0.5591**	**0.8255**
0	**0.7637**	**-0.3022**
1.0000	**-0.3228**	**0.4766**

State-Space Model is:

a =

	x1	x2	x3
x1	**0**	**2**	**0**
x2	**1**	**2**	**0**
x3	**-1**	**0**	**1**

b =

	u1
x1	**0**
x2	**1**
x3	**1**

c =

	x1	x2	x3
y1	**0**	**1**	**1**

```
d =

      u1

   y1   0


Continuous-time model.

 Characteristic Polynomial:


ans =


s^3-3*s^2+2




 Equivalent Transfer Function Model is:


Transfer function:

      2 s^2 - 3 s - 4

------------------------------

s^3 - 3 s^2 + 8.882e-016 s + 2


 Pole, Zero Form:


Zero/pole/gain:

  2 (s-2.351) (s+0.8508)

--------------------------
```

(s-2.732) (s-1) (s+0.7321)

The Controllability Matrix [B AB A^2B ...] is =

Smat =

0	**2**	**4**
1	**2**	**6**
1	**1**	**-1**

The system is therefore Controllable, rank of S Matrix is =

rankS =

3

Mmat =

0	**-3**	**1**
-3	**1**	**0**
1	**0**	**0**

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

```
   -2     2     0
    0    -1     1
   -4    -2     1
```

The transformed matrices using CCF are:

Abar =

```
         0    1.0000    0.0000
         0   -0.0000    1.0000
   -2.0000    0.0000    3.0000
```

Bbar =

```
    0
    0
    1
```

Cbar =

```
   -4    -3     2
```

Dbar =

```
    0
```

10-16) a) $\frac{Y(s)}{U(s)} = \frac{s^2-1}{s^2(s^2-2)}$

Consider:

$$Y(s) = (s^{-2} - s^{-4})X(s)$$
$$X(s) = U(s) — 2s^{-2}X(s) = U(s) + 2s^{-2}X(s)$$

Therefore:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

As $\frac{Y(s)}{U(s)} = \frac{s^2-1}{s^2(s^2-2)}$, therefore $\mathrm{sY(s)} = \frac{2}{\mathrm{s}}\mathrm{Y(s)} + \frac{\mathrm{U(s)}}{\mathrm{s}} - \frac{\mathrm{U(s)}}{\mathrm{s}^3}$

Let $\mathrm{X_2(s)} = \frac{2}{\mathrm{s}}\mathrm{Y_2(s)} + \frac{\mathrm{U(s)}}{\mathrm{s}} - \frac{\mathrm{U(s)}}{\mathrm{s}^3}$. If $\mathrm{y} = \mathrm{x_1}$, then $\mathrm{sY(s)} = \mathrm{sX_1(s)} = \mathrm{X_2}$, or $\dot{\mathrm{x}}_1 = \mathrm{x_2}$. As a result:

$$sX_2 = 2X_1 + U(s) - \frac{U(s)}{s^2}$$

Now consider $\mathrm{X_3} = -\frac{U(s)}{s^2}$, and $sX_3 = \frac{U}{s} = X_4$, then

$$\dot{x}_2 = 2x_1 - x_3 + u$$

$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = u$$

Therefore:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 2 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\\ 1\end{bmatrix}\mathrm{u}$$

b) $\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+4s+4}$

Consider:

$$Y(s) = (2s^{-1} + s^{-2})X(s)$$

$$X(s) = U(s) - (4s^{-1} + 4s^{-2})X(s)$$

Therefore:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -4 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$y = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$As\ \frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+4s+4}$, therefore $sY(s) = -4Y(s) + \frac{4}{s}Y(s) + 2U(s) + \frac{U(s)}{s}$. As a result:

$$\begin{cases} y = x_1 \quad \rightarrow \quad \dot{x}_1 = -4x_1 + 2u + x_2 \\ X_2 = \frac{21}{s}Y(s) + \frac{U(s)}{s} \dashrightarrow sX_2 = 4Y(s) + U(s) \rightarrow \dot{x}_2 = 4x_1 + u \end{cases}$$

10-17) For MATLAB codes see 10-15

(a)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} = \begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$$ **S** is singular.

(b)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\ 2 & -2 & 2\\ 3 & -3 & 3\end{bmatrix}$$ **S** is singular.

(c)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} = \begin{bmatrix}2 & 2+2\sqrt{2}\\ \sqrt{2} & 2+\sqrt{2}\end{bmatrix}$$ **S** is singular.

(d)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}1 & -2 & 4\\ 0 & 0 & 0\\ 1 & -4 & 14\end{bmatrix}$$ **S** is singular.

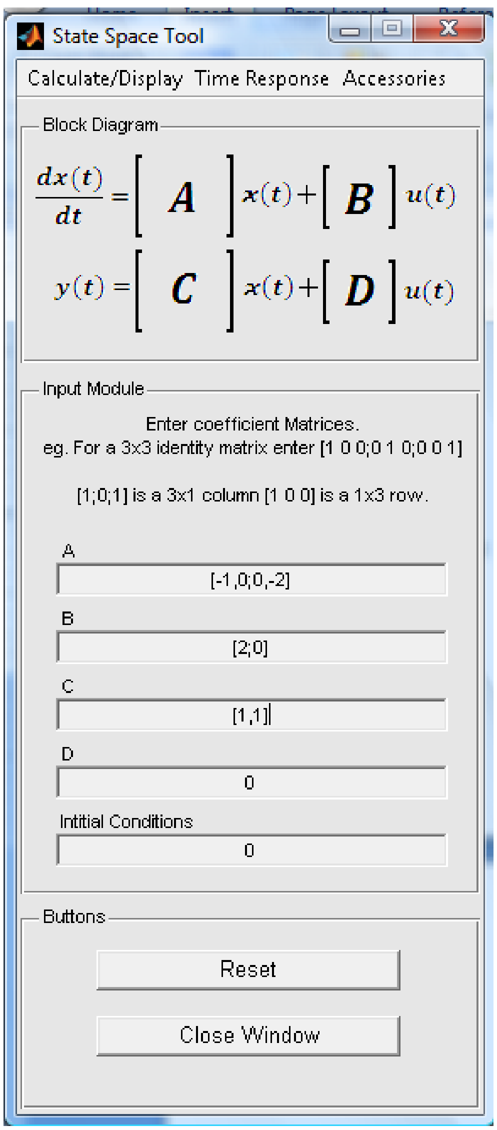
10-18) a, d and e are controllable

b, c, and f are not controllable

USE ACSYS as illustrated in section 10-19-1

9) Activate MATLAB
10) Go to the folder containing ACSYS
11) Type in Acsys
12) Click the "State Space" pushbutton
13) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
14) Use the "Calculate/Display" menu and find the eigenvalues.
15) Next use the "Calculate/Display" menu and conduct State space calculations.
16) Next use the "Calculate/Display" menu and conduct Controlability calculations.

NOTE: the above order of calculations MUST BE followed in the order stated, otherwise you will get an error.



For part b, the system is not Controllable because [B AB] is singular (rank is less than 2):

The A matrix is:

Amat =

```
   -1     0
    0    -2
```

Characteristic Polynomial:

ans =

s^2+3*s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

```
   -2     0
    0    -1
```

Eigen Vectors are

T =

```
    0     1
    1     0
```

Characteristic Polynomial:

ans =

s^2+3*s+2

Equivalent Transfer Function Model is:

Transfer function:

```
  2
-----
s + 1
```

Pole, Zero Form:

Zero/pole/gain:

```
  2
-----
(s+1)
```

The Controllability Matrix [B AB A^2B ...] is =

Smat =

2	**-2**
0	**0**

← Rank is 1, and this is a singular matrix

The system is therefore Not Controllable, rank of S Matrix is =

rankS =

1

Mmat =

3	1
1	0

The Controllability Canonical Form (CCF) Transformation matrix is:

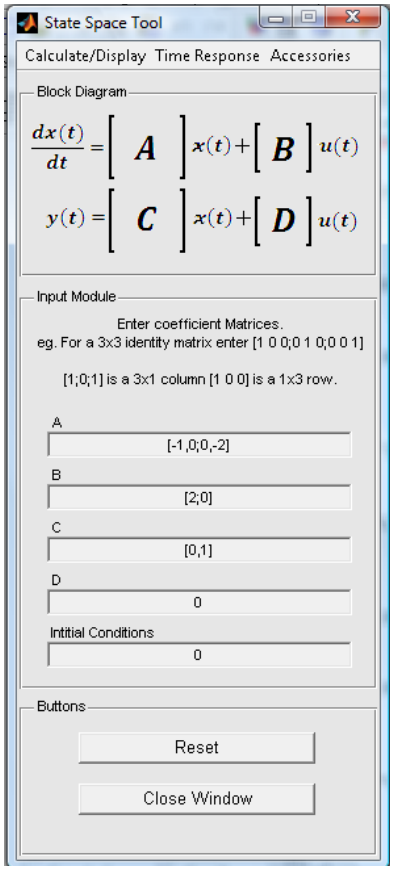
Ptran =

4	2
0	0

10-19) a, d, and e are observable

b, c, and f are not observable

Using ACSYS (also see the previous problem for more details):



For part b, the system is not observable. Note: you must choose a B matrix arbitrarily.

The A matrix is:

Amat =

-1 0

0 -2

Characteristic Polynomial:

ans =

s^2+3*s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-2 0

0 -1

Eigen Vectors are

T =

0 1

1 0

Characteristic Polynomial:

ans =

s^2+3*s+2

Equivalent Transfer Function Model is:

Transfer function:

0

Pole, Zero Form:

Zero/pole/gain:

0

The Observability Matrix (transpose:[C CA CA^2 ...]) is =

Vmat =

0 1

0 -2

The System is therefore Not Observable, rank of V Matrix is =

rankV =

1

Mmat =

3 1

1 0

10-20) (a) Rewrite the differential equations as:

$$\frac{d^2\theta_m}{dt^2} = -\frac{B}{J}\frac{d^2\theta_m}{dt^2} - \frac{K}{J}\theta_m + \frac{K_i}{J}i_a \qquad \frac{di_a}{dt} = -\frac{K_b}{L_a}\frac{d\theta_m}{dt} - \frac{R_a}{L_a}i_a + \frac{K_a K_s}{L_a}\left(\theta_r - \theta_m\right)$$

State variables: $x_1 = \theta_m, \quad x_2 = \frac{d\theta_m}{dt}, \quad x_3 = i_a$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K}{J} & -\frac{B}{J} & \frac{K_i}{J} \\ -\frac{K_a K_s}{L_a} & -\frac{K_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_a K_s}{L_a} \end{bmatrix} \theta_r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} = x_1$$

(b) Forward-path transfer function:

$$G(s) = \frac{\Theta_m(s)}{E(s)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ \frac{K}{J} & s+\frac{B}{J} & -\frac{K_i}{J} \\ 0 & \frac{K_b}{L_a} & s+\frac{R_a}{L_a} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \frac{K_a}{L_a} \end{bmatrix} = \frac{K_i K_a}{\Delta_o(s)}$$

$$\Delta_o(s) = JL_a s^3 + \left(BL_a + R_a J\right)s^2 + \left(KL_a + K_i K_b + R_a B\right)s + KR_a = 0$$

Closed-loop transfer function:

$$M(s)=\frac{\Theta_m(s)}{\Theta_r(s)}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix} s & -1 & 0 \\ \frac{K}{J} & s+\frac{B}{J} & -\frac{K_i}{J} \\ \frac{K_a K_s}{L_a} & \frac{K_b}{L_a} & s+\frac{R_a}{L_a}\end{bmatrix}^{-1}\begin{bmatrix}0 \\ 0 \\ \frac{K_a K_s}{L_a}\end{bmatrix}=\frac{K_s G(s)}{1+K_s(s)}$$

$$=\frac{K_i K_a K_s}{JL_a s^3+\left(BL_a+R_a J\right)s^2+\left(KL_a+K_i K_b+R_a B\right)s+K_i K_a K_s+KR_a}$$

10-21) (a)

$$\mathbf{A}=\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} \qquad \mathbf{A}^2=\begin{bmatrix}-1 & 0 \\ 0 & -1\end{bmatrix} \qquad \mathbf{A}^3=\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix} \qquad \mathbf{A}^4=\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t)=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^2t^2+\cdots=\begin{bmatrix}1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots & t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots \\ -t+\frac{t^3}{3!}-\frac{t^5}{5!}+\cdots & 1-\frac{t^2}{2!}+\frac{t^4}{4!}\cdots\end{bmatrix}=\begin{bmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s)=\left(s\mathbf{I}-\mathbf{A}\right)^{-1}=\begin{bmatrix}s & -1 \\ 1 & s\end{bmatrix}^{-1}=\frac{1}{s^2+1}\begin{bmatrix}s & 1 \\ -1 & s\end{bmatrix} \qquad \phi(t)=\begin{bmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{bmatrix}$$

(b)

$$\mathbf{A}=\begin{bmatrix}-1 & 0 \\ 0 & -2\end{bmatrix} \qquad \mathbf{A}^2=\begin{bmatrix}1 & 0 \\ 0 & 4\end{bmatrix} \qquad \mathbf{A}^3=\begin{bmatrix}-1 & 0 \\ 0 & -8\end{bmatrix} \qquad \mathbf{A}^4=\begin{bmatrix}-1 & 0 \\ 0 & 16\end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t)=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^2t^2+\cdots=\begin{bmatrix}1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots & 0 \\ 0 & 1-2t+\frac{4t^2}{2!}-\frac{8t^3}{3!}+\cdots\end{bmatrix}=\begin{bmatrix}e^{-t} & 0 \\ 0 & e^{-2t}\end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s) = (s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \qquad \phi(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{A}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots = \begin{bmatrix} 1+\dfrac{t^2}{2!}+\dfrac{t^4}{4!}+\cdots & t+\dfrac{t^3}{3!}+\dfrac{t^5}{5!}\cdots \\ t+\dfrac{t^3}{3!}+\dfrac{t^5}{5!}\cdots & 1+\dfrac{t^2}{2!}+\dfrac{t^4}{4!}+\cdots \end{bmatrix} = \begin{bmatrix} e^{-t}+e^{t} & -e^{-t}+e^{t} \\ -e^{-t}+e^{t} & e^{-t}+e^{t} \end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s) = (s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2-1}\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} \dfrac{0.5}{s+1}-\dfrac{0.5}{s-1} & \dfrac{-0.5}{s+1}+\dfrac{0.5}{s-1} \\ \dfrac{-0.5}{s+1}+\dfrac{0.5}{s-1} & \dfrac{0.5}{s+1}+\dfrac{0.5}{s-1} \end{bmatrix}$$

$$\phi(t) = 0.5\begin{bmatrix} e^{-t}+e^{t} & -e^{-t}+e^{t} \\ -e^{-t}+e^{t} & e^{-t}+e^{t} \end{bmatrix}$$

10-22) (a) $e = K_s(\theta_r - \theta_y) \qquad e_a = e - e_s \qquad e_s = R_s i_a \qquad e_u = Ke_a$

$$i_a = \frac{e_u - e_b}{R_a + R_s} \qquad e_b = K_b\frac{d\theta_y}{dt} \qquad T_m = K_i i_a = (J_m + J_L)\frac{d^2\theta_y}{dt^2}$$

Solve for i_a in terms of θ_y and $\dfrac{\mathrm{d}\theta_\mathrm{y}}{dt}$, we have

$$i_a = \frac{KK_s(\theta_r - \theta_y) - K_b\dfrac{d\theta_y}{dt}}{R_s + R_s + KR_s}$$

Differential equation:

$$\frac{d^2\theta_y}{dt^2} = \frac{K_i i_a}{J_m + J_L} = \frac{K_i}{(J_m + J_L)(R_a + R_s + KR_s)}\left(-K_b \frac{d\theta_y}{dt} - KK_s\theta_y + KK_s\theta_y\right)$$

State variables: $x_1 = \theta_y, \quad x_2 = \frac{d\theta_y}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-KK_sK_i}{(J_m + J_L)(R_a + R_s + KR_s)} & \frac{-K_bK_i}{(J_m + J_L)(R_a + R_s + KR_s)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-KK_sK_i}{(J_m + J_L)(R_a + R_s + KR_s)} \end{bmatrix} \theta_r$$

$$= \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} \theta_r$$

We can let $v(t) = 322.58\theta_r$, then the state equations are in the form of CCF.

(b)

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 322.58 & s+80.65 \end{bmatrix}^{-1} = \frac{1}{s^2 + 80.65s + 322.58} \begin{bmatrix} s+80.65 & 1 \\ -322.58 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-0.06}{s+76.42} - \frac{1.059}{s+4.22} & \frac{-0.014}{s+76.42} + \frac{0.014}{s+4.22} \\ \frac{4.468}{s+76.42} - \frac{4.468}{s+4.22} & \frac{1.0622}{s+76.42} - \frac{0.0587}{s+4.22} \end{bmatrix}$$

For a unit-step function input, $u_s(t) = 1/s$.

$$(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \frac{1}{s} = \begin{bmatrix} \frac{322.2}{s(s+76.42)(s+4.22)} \\ \frac{322.2}{s(s+76.42)(s+4.22)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} + \frac{0.0584}{s+76.42} - \frac{1.058}{s+4.22} \\ \frac{-4.479}{s+76.42} + \frac{4.479}{s+4.22} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} -0.06e^{-76.42t} - 1.059e^{-4.22t} & -0.014e^{-76.42t} + 0.01e^{-4.22t} \\ 4.468e^{-76.42t} - 4.468e^{-4.22t} & 1.0622e^{-76.42t} - 0.0587e^{-4.22t} \end{bmatrix} \mathbf{x}(0)$$

$$= \begin{bmatrix} 1 + 0.0584e^{-76.42t} - 1.058e^{-4.22t} \\ -4.479e^{-76.42t} + 4.479e^{-4.22t} \end{bmatrix} \qquad t \ge 0$$

(c) **Characteristic equation:** $\Delta(s) = s^2 + 80.65s + 322.58 = 0$

(d) From the state equations we see that whenever there is R_a there is $(1+K)R_s$. Thus, the purpose of R_s is to increase the effective value of R_a by $(1+K)R_s$. This improves the time constant of the system.

10-23) **(a)** **State equations:**

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-KK_sK_i}{J(R+R_s+KR_s)} & \dfrac{-K_bK_i}{J(R+R_s+KR_s)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{KK_sK_i}{J(R+R_s+KR_s)} \end{bmatrix} \theta_r$$

$$= \begin{bmatrix} 0 & 1 \\ -818.18 & -90.91 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 818.18 \end{bmatrix} \theta_r$$

Let $v = 818.18\theta_r$. The equations are in the form of CCF with *v* as the input.

(b)

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 818.18 & s+90.91 \end{bmatrix}^{-1} = \frac{1}{(s+10.128)(s+80.782)} \begin{bmatrix} s+90.91 & 1 \\ -818.18 & s \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 1.143e^{-10.128t} - 0.142e^{-80.78t} & 0.01415e^{-10.128t} - 0.0141e^{-80.78t} \\ -11.58e^{-10.128t} + 0.1433e^{-80.78t} & -0.1433e^{-10.128t} + 1.143e^{-80.78t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$+ \begin{bmatrix} 11.58e^{-10.128t} - 11.58e^{-80.78t} \\ 1 - 1.1434e^{-10.128t} + 0.1433e^{-80.78t} \end{bmatrix} \qquad t \geq 0$$

(c) **Characteristic equation:** $\Delta(s) = s^2 + 90.91s + 818.18 = 0$

Eigenvalues: –10.128, –80.782

(d) Same remark as in part (d) of Problem 5-14.

10-24) If $\dot{x} = Ax$ and P diagonalizing A, let consider $x = P\hat{x}$, therefore $\dot{x} = P\dot{\hat{x}}$ or $\dot{\hat{x}} = P^{-1}AP\hat{x} = D\hat{x}$

The solution for $\hat{x}$ is $\hat{x} = e^{Dt}\hat{x}(0)$, therefore

$$x(t) = P\hat{x}(t) = Pe^{Dt}P^{-1}x(0) \qquad (1)$$

on the other hand

$$x(t) = e^{At}x(0) \qquad (2)$$

From equation (1) and (2):

$$e^{At} = Pe^{Dt}P^{-1}$$

10-25) Consider $\dot{x} = Ax$ and $s^{-1}As = J$. If $x = S\hat{x}$, then $\dot{x} = S\dot{\hat{x}}$ or $\dot{\hat{x}} = s^{-1}As\hat{x} = J\hat{x}$

The solution for $\hat{x}$ is $\hat{x}(t) = e^{Jt}\hat{x}(0)$, therefore:

$$x(t) = s\hat{x}(t) = se^{Jt}s^{-1}x(0) \qquad (1)$$

On the other hand:

$$x(t) = e^{At}x(0) \qquad (2)$$

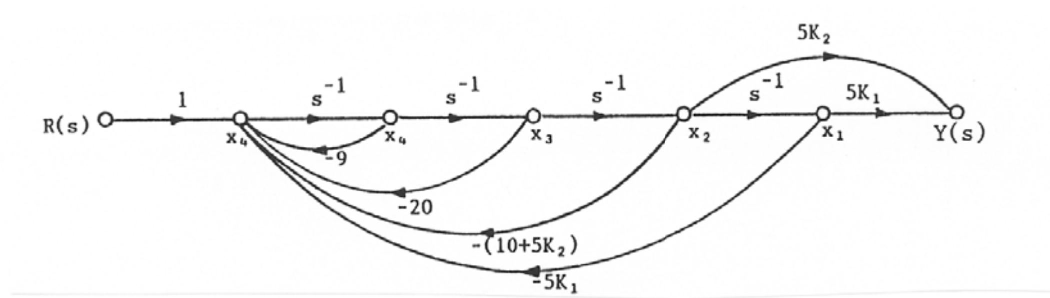
From equation (1) and (2):

$$e^{At} = se^{Jt}s^{-1}$$

10-26 (a) Forward-path transfer function: **Closed-loop transfer function:**

$$G(s) = \frac{Y(s)}{E(s)} = \frac{5(K_1 + K_2 s)}{s\left[s(s+4)(s+5)+10\right]} \qquad M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{5(K_1 + K_2 s)}{s^4 + 9s^3 + 20s^2 + (10+5K_2)s + 5K_1}$$

(b) State diagram by direct decomposition:



State equations: **Output equation:**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -5K & -(10+5K_2) & -20 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix} r \qquad y = \begin{bmatrix}5K_1 & 5K_2 & 0\end{bmatrix}\mathbf{x}$$

(c) Final value: $r(t) = u_s(t), \quad R(s) = \frac{1}{s}.$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{5\left(K_1 + K_2 s\right)}{s^4 + 9s^3 + 20s^2 + \left(10 + 5K_2\right)s + 5K_1} = 1$$

10-27 In CCF form,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 & 0 & 0 & \cdots & 0 \\ 0 & s & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & s + a_n \end{bmatrix}$$

$$\left| s\mathbf{I} - \mathbf{A} \right| = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$$

Since **B** has only one nonzero element which is in the last row, only the last column of $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)$ is going to contribute to $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)B$. The last column of $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)$ is obtained from the cofactors of the last row of $\left(s\mathbf{I} - \mathbf{A}\right)$. Thus, the last column of $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)B$ is $\left[1 \quad s \quad s^2 \quad \cdots \quad s^{n-1}\right]'$.

10-28 (a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2 y}{dt^2}$

State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) State transition matrix:

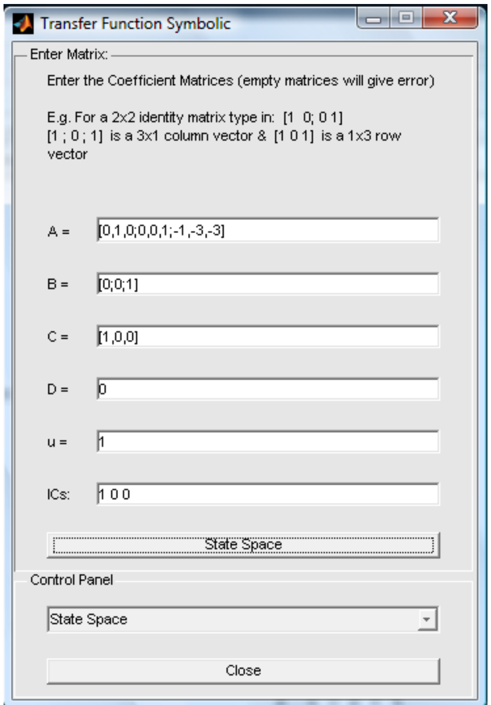
$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 3 & s+3 \end{bmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s^2+3s+3 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -3s-1 & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} & \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} & \frac{1}{(s+1)^3} \\ \frac{-1}{(s+1)^3} & \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{2}{(s+1)^3} & \frac{s}{(s+1)^3} \\ \frac{-s}{(s+1)^3} & \frac{-3}{(s+1)^2} + \frac{2}{(s+1)^3} & \frac{s^2}{(s+1)^3} \end{bmatrix}$$

$$\Delta(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3$$

$$\phi(t) = \begin{bmatrix} \left(1+t+t^2/2\right)e^{-t} & \left(t+t^2\right)e^{-t} & t^2e^{-t}/2 \\ -t^2e^{-t}/2 & \left(1+t-t^2\right)e^{-t} & \left(t-t^2/2\right)e^{-t} \\ \left(-t+t^2/2\right)e^{-t} & t^2e^{-t} & \left(1-2t+t^2/2\right)e^{-t} \end{bmatrix}$$

(c) Use ACSYS or MATLAB and follow the procedure shown in solution to 10-3.




```
-------------------------------------------------------------
      State Space Analysis
-------------------------------------------------------------
Inputs:
A=| 0  1  0|     B=|0|
  | 0  0  1|       |0|
  |-1 -3 -3|       |1|

C=|1  0  0|     D=|0|

State Space Representation:
     | 0  1  0|    |0|
Dx = | 0  0  1|x + |0|u
     |-1 -3 -3|    |1|

 y = |1  0  0|x + |0|u

Determinant of (s*I-A):

                         3      2
                        s  + 3 s  + 3 s + 1
Characteristic Equation of the Transfer Function:

                         3      2
```

```
                    s  + 3 s  + 3 s + 1
```

The eigen values of A and poles of the Transfer Function are:

```
                    -1
                    -1
                    -1
```

Inverse of (s*I-A) is:

```
              [ 2                       ]
              [s  + 3 s + 3     s + 3      1  ]
              [------------     -----      ----]
              [    %1           %1         %1 ]
              [                         ]
              [     1        s (s + 3)     s  ]
              [   - ----       ---------    ----]
              [     %1          %1         %1 ]
              [                         ]
              [                        2 ]
              [     s          3 s + 1     s  ]
              [   - ----       - -------    ----]
              [     %1           %1        %1 ]

                       3     2
                 %1 := s  + 3 s  + 3 s + 1
```

State transition matrix (phi) of A:

```
   [                  2        2               2       ]
   [1/2 exp(-t) (2 + 2 t + t ) , (t + t ) exp(-t) , 1/2 t  exp(-t)]

   [      2                  2                           2
   [- 1/2 t  exp(-t) , -(-t - 1 + t ) exp(-t) , - 1/2 exp(-t) (-2 t + t )

   ]
   ]

   [                2                  2
   [1/2 exp(-t) (-2 t + t ) , exp(-t) (-3 t + t ) ,

                     2 ]
   1/2 exp(-t) (2 - 4 t + t )]
```

Transfer function between u(t)and y(t) is:

```
                        1
                 -------------------
                   3     2
                 s  + 3 s  + 3 s + 1
```

No Initial Conditions Specified

States (X) in Laplace Domain:

```
[   1    ]
[--------]
[       3]
[(s + 1) ]
[        ]
[   s    ]
[--------]
[       3]
[(s + 1) ]
[        ]
[    2   ]
[   s    ]
[--------]
[       3]
[(s + 1) ]
```

Inverse Laplace x(t):

```
[          2              ]
[     1/2 t  exp(-t)      ]
[                         ]
[                      2  ]
[- 1/2 exp(-t) (-2 t + t ) ]
[                         ]
[                       2 ]
[1/2 exp(-t) (2 - 4 t + t )]
```

Output Y(s):

```
      1
   --------
          3
   (s + 1)
```

Inverse Laplace y(t):

```
      2
 1/2 t  exp(-t)
```

State Space Analysis

Inputs:

```
A=| 0  1  0|     B=|0|
  | 0  0  1|       |0|
  |-1 -3 -3|       |1|

C=|1  0  0|     D=|0|
```

State Space Representation:

```
     | 0  1  0|    |0|
Dx = | 0  0  1|x + |0|u
     |-1 -3 -3|    |1|

 y = |1  0  0|x + |0|u
```

Determinant of (s*I-A):

```
                     3      2
                    s  + 3 s  + 3 s + 1
```

Characteristic Equation of the Transfer Function:

```
                     3      2
                    s  + 3 s  + 3 s + 1
```

The eigen values of A and poles of the Transfer Function are:

```
                           -1
                           -1
                           -1
```

Inverse of (s*I-A) is:

```
               [ 2                            ]
               [s  + 3 s + 3      s + 3      1  ]
               [------------      -----      ----]
               [     %1            %1         %1 ]
               [                              ]
               [      1        s (s + 3)     s  ]
               [   - ----      ---------    ----]
               [      %1          %1         %1 ]
               [                              ]
               [                             2 ]
               [      s         3 s + 1      s  ]
               [   - ----      - -------    ----]
               [      %1          %1         %1 ]

                        3      2
                  %1 := s  + 3 s  + 3 s + 1
```

State transition matrix (phi) of A:

```
     [                   2          2                2        ]
     [1/2 exp(-t) (2 + 2 t + t ) , (t + t ) exp(-t) , 1/2 t  exp(-t)]

     [       2                     2                              2
     [- 1/2 t  exp(-t) , -(t - 1 + t ) exp(-t) , - 1/2 exp(-t) (-2 t + t )

     ]
     ]
```

```
[                  2                  2
[1/2 exp(-t) (-2 t + t ) , exp(-t) (-3 t + t ) ,

                  2 ]
1/2 exp(-t) (2 - 4 t + t )]
```

Transfer function between u(t)and y(t) is:

```
         1
  ------------------
   3     2
  s  + 3 s  + 3 s + 1
```

Initial Conditions:

```
x(0)= 1
      0
      0
```

States (X) in Laplace Domain:

```
[ 2          ]
[s  + 3 s + 4]
[------------]
[        3   ]
[  (s + 1)   ]
[            ]
[   s - 1    ]
[  --------  ]
[        3   ]
[  (s + 1)   ]
[            ]
[ s (s - 1)  ]
[ ---------  ]
[        3   ]
[ (s + 1)    ]
```

Inverse Laplace x(t):

```
[           2          ]
[ (t + 1 + t ) exp(-t)  ]
[                      ]
[         2            ]
[  -(-t + t ) exp(-t)   ]
[                      ]
[           2          ]
[(-3 t + 1 + t ) exp(-t)]
```

Output Y(s) (with initial conditions):

```
 2
s  + 3 s + 4
------------
```

3
(s + 1)
Inverse Laplace y(t):

2
(t + 1 + t) exp(-t)

(d) Characteristic equation: $\Delta(s) = s^3 + 3s^2 + 3s + 1 = 0$

Eigenvalues: $-1,\ -1,\ -1$

10-29 (a) State variables: $x_1 = y, \quad x_2 = \dfrac{dy}{dt}$

State equations:

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

State transition matrix:

$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+1)^2} & \dfrac{1}{(s+1)^2} \\ \dfrac{-1}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix} \qquad \phi(t) = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Characteristic equation: $\Delta(s) = (s+1)^2 = 0$

(b) State variables: $x_1 = y, \quad x_2 = y + \dfrac{dy}{dt}$

State equations:

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2 - y = x_2 - x_1 \qquad \frac{dx_2}{dt} = \frac{d^2y}{dt^2} + \frac{dy}{dt} = -y - \frac{dy}{dt} + r = -x_2 + r$$

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

State transition matrix:

$$\Phi(s) = \begin{bmatrix} s+1 & -2 \\ 0 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{-2}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} \qquad \phi(t) = \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

(c) Characteristic equation: $\Delta(s) = (s+1)^2 = 0$ which is the same as in part (a).

10-30 (a) State transition matrix:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix} \qquad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix} \qquad \Delta(s) = s^2 - 2\sigma + (\sigma^2 + \omega^2)$$

$$\phi(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} e^{\sigma t}$$

(b) Eigenvalues of A: $\sigma + j\omega, \quad \sigma - j\omega$

10-31 (a)

$$\frac{Y_1(s)}{U_1(s)} = \frac{s^{-3}}{1+s^{-1}+2s^{-2}+3s^{-3}} = \frac{1}{s^3+s^2+2s+3}$$

$$\frac{Y_2(s)}{U_2(s)} = \frac{s^{-3}}{1+s^{-1}+2s^{-2}+3s^{-3}} = \frac{1}{s^3+s^2+2s+3} = \frac{Y_1(s)}{U_1(s)}$$

(b) State equations [Fig. 5-21(a)]: $\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1 u_1$ **Output equation:** $\mathbf{y}_1 = \mathbf{C}_1\mathbf{x}$

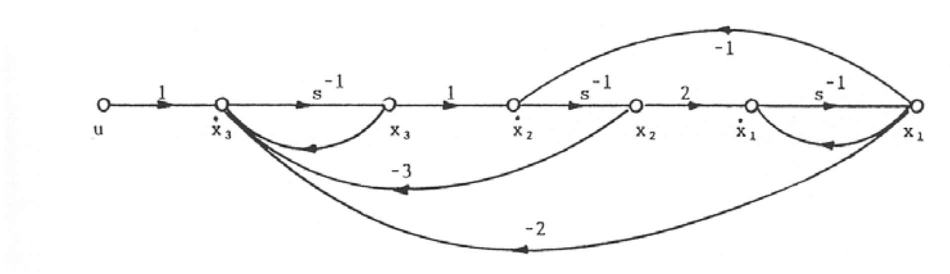
$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{bmatrix} \qquad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

State equations [Fig. 5-21(b)]: $\dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2 u_2$ **Output equation:** $y_2 = \mathbf{C}_2\mathbf{x}$

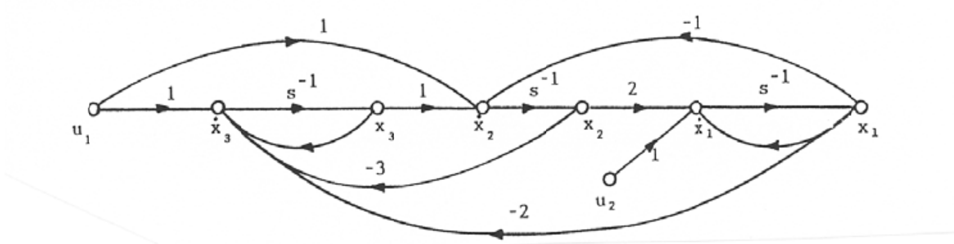
$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Thus, $\mathbf{A}_2 = \mathbf{A}_1'$

10-32 (a) **State diagram:**



(b) **State diagram:**

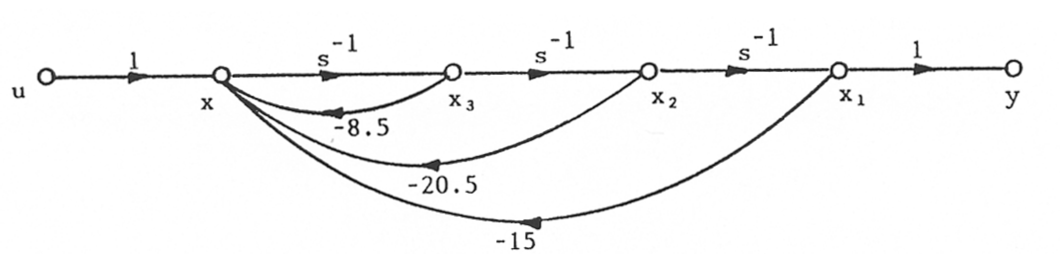


10-33 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-3}}{1 + 8.5s^{-1} + 20.5s^{-2} + 15s^{-3}} \frac{X(s)}{X(s)} \qquad Y(s) = 10X(s)$$

$$X(s) = U(s) - 8.5s^{-1}X(s) - 20.5s^{-2}X(s) - 15s^{-3}X(s)$$

State diagram:



State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -20.5 & -8.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

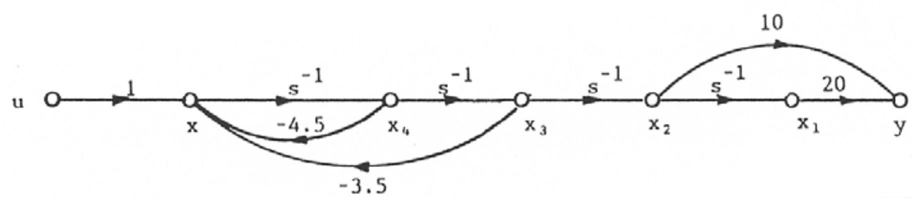
A and **B** are in CCF

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-3} + 20s^{-4}}{1 + 4.5s^{-1} + 3.5s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 10s^{-3}X(s) + 20s^{-4}X(s) \qquad X(s) = -4.5s^{-1}X(s) - 3.5s^{-2}X(s) + U(s)$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3.5 & -4.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

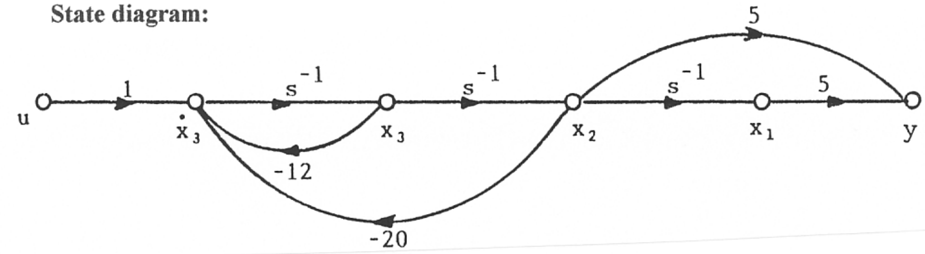
A and **B** are in CCF

(c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{5s^{-2} + 5s^{-3}}{1 + 12s^{-1} + 20s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 5s^{-2}X(s) + 5s^{-3}X(s) \qquad X(s) = U(s) - 12s^{-1}X(s) - 20s^{-2}X(s)$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -20 & -12 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

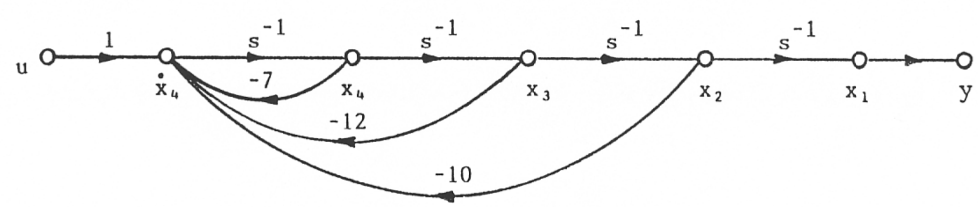
A and **B** are in CCF

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)\left(s^2+2s+2\right)} = \frac{s^{-4}}{1+7s^{-1}+12s^{-2}+10s^{-3}}\frac{X(s)}{X(s)}$$

$$Y(s) = s^{-4}X(s) \qquad X(s) = U(s) - 7s^{-1}X(s) - 12s^{-2}X(s) - 10s^{-3}$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

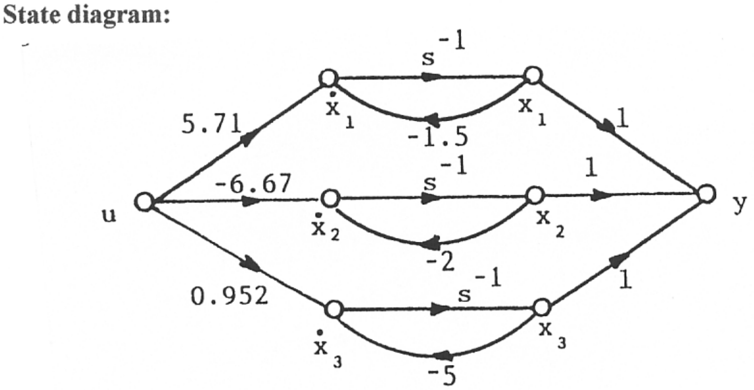
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -10 & -12 & -7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A and **B** are in CCF

10-34 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^3 + 8.5s^2 + 20.5s + 15} = \frac{5.71}{s+15} - \frac{6.67}{s+2} + \frac{0.952}{s+5}$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

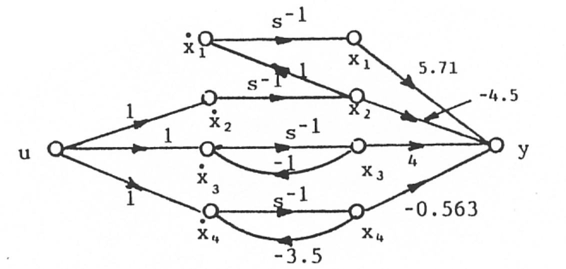
$$\mathbf{A} = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5.71 \\ -6.67 \\ 0.952 \end{bmatrix}$$

The matrix **B** is not unique. It depends on how the input and the output branches are allocated.

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+2)}{s^2(s=1)(s+3.5)} = \frac{-4.5}{s} + \frac{0.49}{s+3.5} + \frac{4}{s+1} + \frac{5.71}{s^2}$$

State diagram:

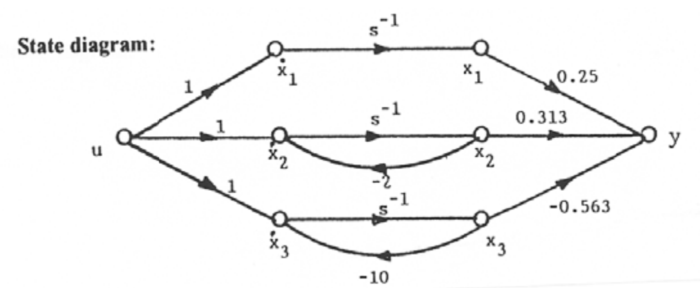


State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{2.5}{s} + \frac{0.313}{s+2} - \frac{0.563}{s+10}$$

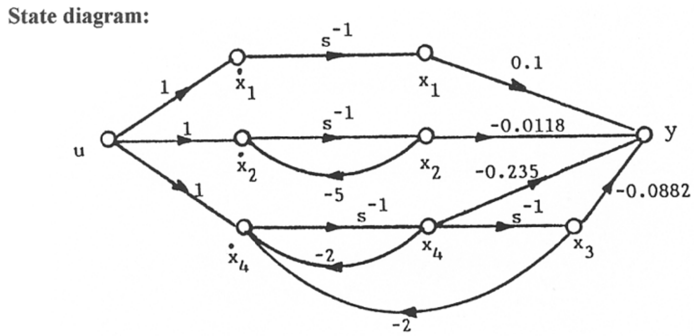


State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)} = \frac{0.1}{s} - \frac{0.0118}{s+5} - \frac{0.0882s+0.235}{s^2+2s+2}$$



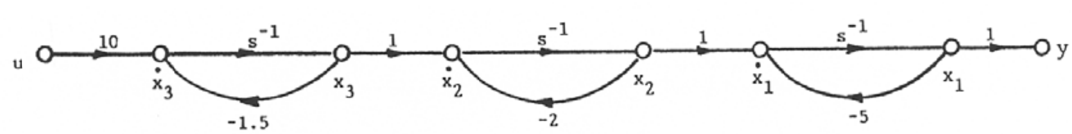
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

10-35 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1.5)(s+2)(s+5)}$$

State diagram:



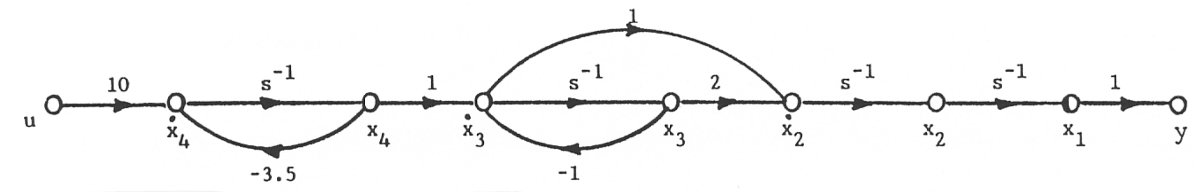
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+2)}{s^2(s+1)(s+3.5)} = \left(\frac{10}{s^2}\right)\left(\frac{s+2}{s+1}\right)\left(\frac{1}{s+3.5}\right)$$

State diagram:



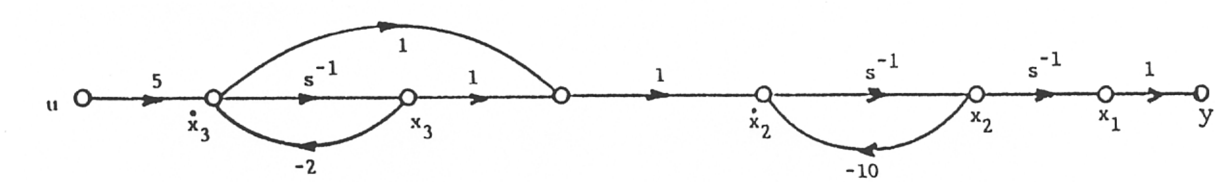
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

(c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{59s+1)}{s(s+2)(s+10)} = \left(\frac{5}{s}\right)\left(\frac{s+1}{s+2}\right)\left(\frac{1}{s+10}\right)$$

State diagram:



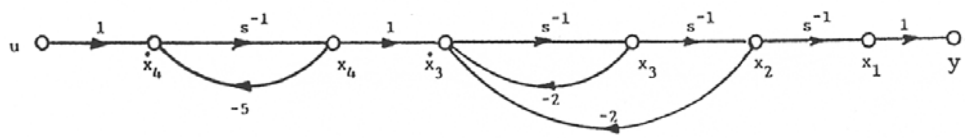
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & -1 \\ 0 & 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

(d)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{s\left(s+5\right)\left(s^2+2s+2\right)}$$

State diagram:



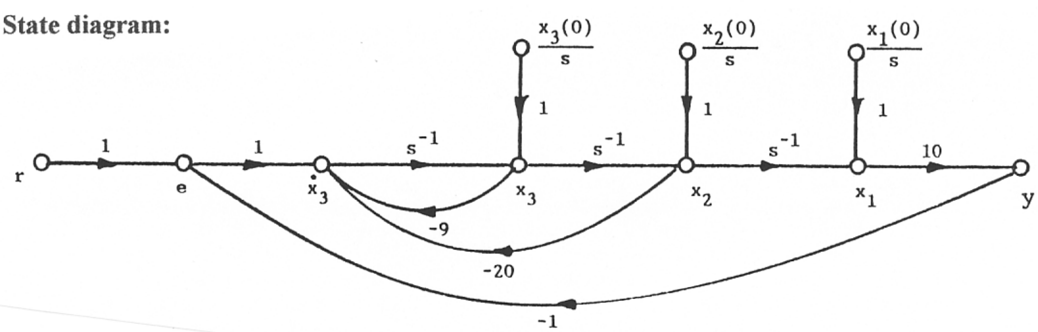
State equations: $\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)$

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -2 & -2 & 1\\ 0 & 0 & 0 & -5\end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

10-36 (a)

$$G(s)=\frac{Y(s)}{E(s)}=\frac{10}{s(s+4)(s+5)}=\frac{10s^{-3}}{1+9s^{-1}+20s^{-2}}\frac{X(s)}{X(s)}$$

State diagram:



(b) Dynamic equations:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -10 & -20 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}r \qquad y=\begin{bmatrix}10 & 0 & 0\end{bmatrix}\mathbf{x}$$

(c) State transition equation:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} s^{-1}\left(1+9s^{-1}+20s^{-2}\right) & s^{-2}\left(1+9s^{-1}\right) & s^{-3} \\ -10s^{-3} & s^{-1}\left(1+9s^{-1}\right) & s^{-2} \\ -10s^{-2} & -20s^{-2} & s^{-1} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta(s)} \begin{bmatrix} s^{-3} \\ s^{-2} \\ s^{-1} \end{bmatrix} \frac{1}{s}$$

$$= \frac{1}{\Delta_c(s)} \begin{bmatrix} s^2+9s+20 & s+9 & 1 \\ -10 & s(s+9) & s \\ -10s & -10(2s+1) & s^2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta_c(s)} \begin{bmatrix} \frac{1}{s} \\ 1 \\ s \end{bmatrix}$$

$$\Delta(s) = 1+9s^{-1}+20s^{-2}+10s^{-3} \qquad \Delta_c(s) = s^3+9s^2+20s+10$$

$$\mathbf{x}(t) = \left\{ \begin{bmatrix} 1.612 & 0.946 & 0.114 \\ -1.14 & -0.669 & -0.081 \\ 0.807 & 0.474 & 0.057 \end{bmatrix} e^{-0.708t} + \begin{bmatrix} -0.706 & -1.117 & -0.169 \\ 1.692 & 2.678 & 4.056 \\ -4.056 & -6.420 & -0.972 \end{bmatrix} e^{-2.397t} + \begin{bmatrix} 0.0935 & 0.171 & 0.055 \\ -0.551 & -1.009 & -0.325 \\ 3.249 & 5.947 & 1.915 \end{bmatrix} e^{-5.895t} \right\} \mathbf{x}(0)$$

$$+ \begin{bmatrix} 0.1-0.161e^{-0.708t}+0.0706e^{-2.397t}-0.00935e^{-5.895t} \\ 0.114e^{-0.708t}-0.169e^{-2.397t}+0.055e^{-5.895t} \\ -0.087e^{-0.708t}+0.406e^{-2.397t}-0.325e^{-5.895t} \end{bmatrix} \qquad t \geq 0$$

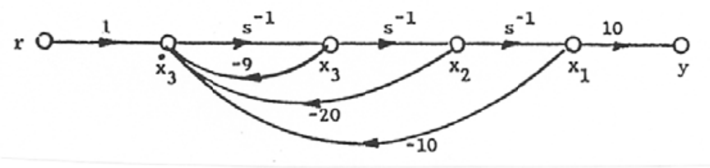
(d) Output:

$$y(t) = 10x_1(t) = 10\left(1.612e^{-0.708t}-0.706e^{-2.397t}+0.0935e^{-5.897t}\right)x_1(0) + 10\left(0.946e^{-0.708t}-1.117e^{-2.397t}+0.1711e^{-5.895t}\right)x_2(0)$$
$$+10\left(1.141e^{-0.708t}-0.169e^{-2.397t}+0.0550e^{-5.895t}\right)x_3(0)+1-1.61e^{-0.708t}+0.706e^{-2.397t}-0.0935e^{-5.895t} \qquad t \geq 0$$

10-37(a) Closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{10}{s^3+9s^2+20s+10}$$

(b) State diagram:



(c) **State equations:**

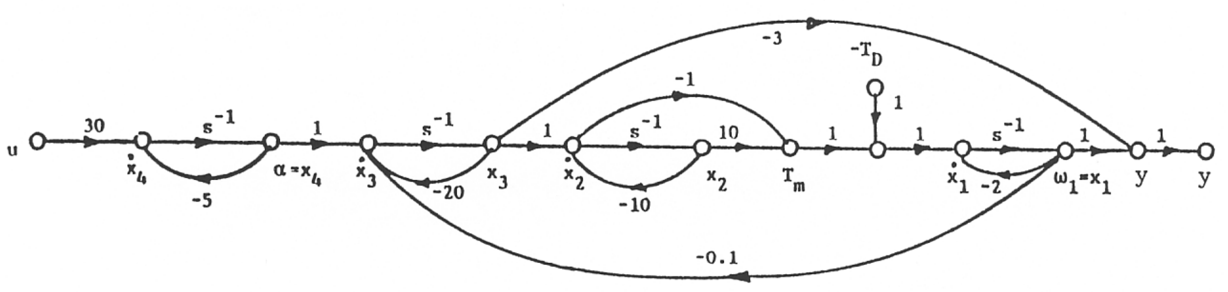
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

(d) **State transition equations:**

[Same answers as Problem 5-26(d)]

(e) **Output:** [Same answer as Problem 5-26(e)]

10-38 (a) **State diagram:**



(b) **State equations:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -2 & 20 & -1 & 0 \\ 0 & -10 & 1 & 0 \\ -0.1 & 0 & -20 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \end{bmatrix} \begin{bmatrix} u \\ T_D \end{bmatrix}$$

(c) **Transfer function relations:**

From the system block diagram,

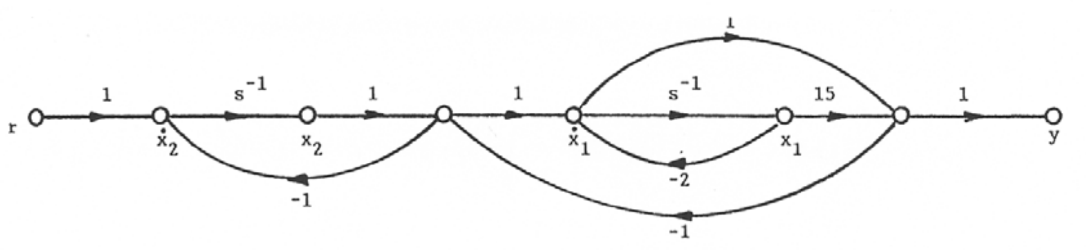
$$Y(s) = \frac{1}{\Delta(s)} \left(\frac{-1}{s+2} T_D(s) + \frac{0.3}{(s+2)(s+20)} T_D(s) + \frac{30e^{-0.2s}U(s)}{(s+2)(s+5)(s+20)} + \frac{90U(s)}{(s+5)(s+20)} \right)$$

$$\Delta(s) = 1 + \frac{0.1e^{-0.2s}}{(s+2)(s+20)} = \frac{(s+2)(s+20)+0.1e^{-0.2s}}{(s+2)(s+20)}$$

$$Y(s) = \frac{-(s+19.7)}{(s+2)(s+20)+0.1e^{-0.2s}}T_D(s) + \frac{30e^{-0.2s}+90(s+2)U(s)}{(s+5)\left[(s+2)(s+20)+0.1e^{-0.2s}\right]}$$

$$\Omega(s) = \frac{-(s+20)}{(s+2)(s+20)+0.1e^{-0.2s}}T_D(s) + \frac{30e^{-0.2s}U(s)}{(s+5)\left[(s+2)(s+20)+0.1e^{-0.2s}\right]}$$

10-39 (a) There should not be any incoming branches to a state variable node other than the s^{-1} branch. Thus, we should create a new node as shown in the following state diagram.



(b) **State equations:** Notice that there is a loop with gain –1 after all the s^{-1} branches are deleted, so Δ = 2.

$$\frac{dx_1}{dt} = \frac{17}{2}x_1 + \frac{1}{2}x_2 \qquad \frac{dx_2}{dt} = \frac{15}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}r$$

Output equation: $y = 6.5x_1 + 0.5x_2$

10-40 (a) **Transfer function:**

$$\frac{Y(s)}{R(s)} = \frac{Ks^2+5s+1}{(s+1)(s^2+11s+2)}$$

(b) **Characteristic equation:**

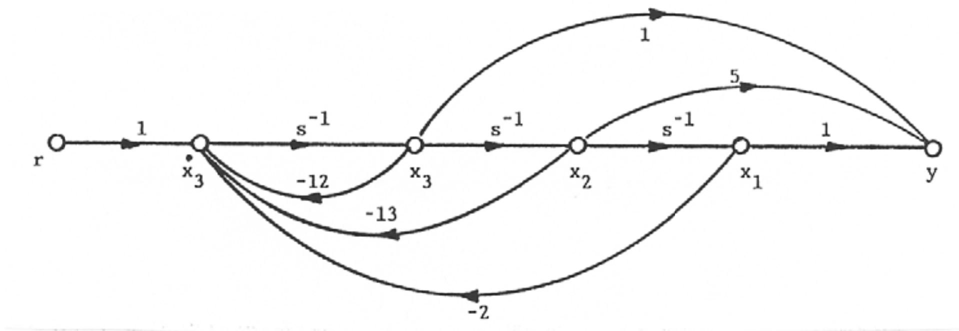
$$(s+1)(s^2+11s+2) = 0$$

Roots of characteristic equation: –1, –0.185, –-10.82. These are not functions of *K*.

(c) **When *K* = 1:**

$$\frac{Y(s)}{R(s)} = \frac{s^2+5s+1}{s^3+12s^2+13s+2}$$

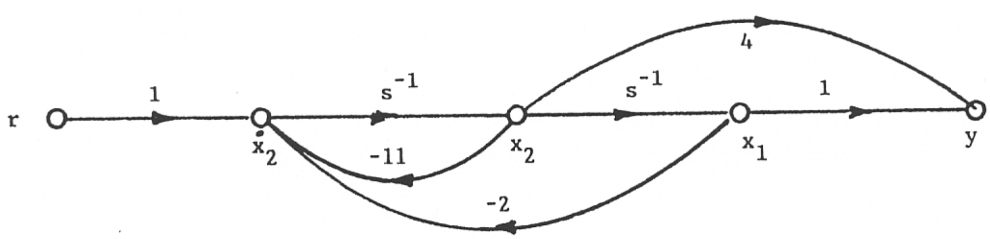
State diagram:



(d) **When *K* = 4:**

$$\frac{Y(s)}{R(s)} = \frac{4s^2+5s+1}{(s+1)(s^2+11s+2)} = \frac{(s+1)(4s+1)}{(s+1)(s^2+11s+2)} = \frac{4s+1}{s^2+11s+2}$$

State diagram:



(e)

$$\frac{Y(s)}{R(s)} = \frac{Ks^2+5s+1}{(s+1)(s^2+11s+2)} \qquad (s+1)(s^2+11s+2) = 0$$

MATLAB

```
solve(s^2+11*s+2)
 ans =  -11/2+1/2*113^(1/2)
 -11/2-1/2*113^(1/2)
 >> vpa(ans)
 ans =
   -.20
 -10.8
```

$$\frac{Y(s)}{R(s)} = \frac{Ks^2+5s+1}{(s+1)(s+0.2)(s+10.82)}$$

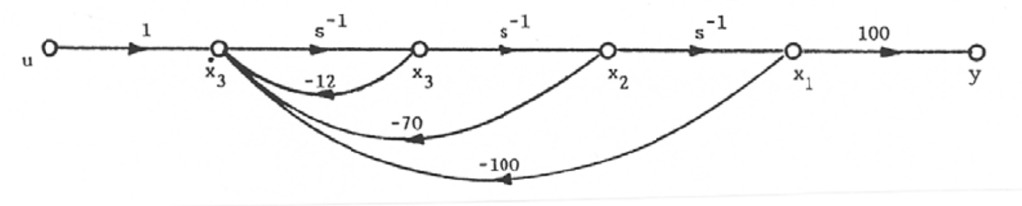
$$K = 4, 2.1914, 0.4536$$

Pole zero cancelation occurs for the given values of K.

10-41 (a)

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{(1+0.5s)(1+0.2s+0.02s^2)} = \frac{100}{s^3+12s^2+70s+100}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -100 & -70 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

(b) **Characteristic equation of closed-loop system:**

$$s^3 + 12s^2 + 70s + 200 = 0$$

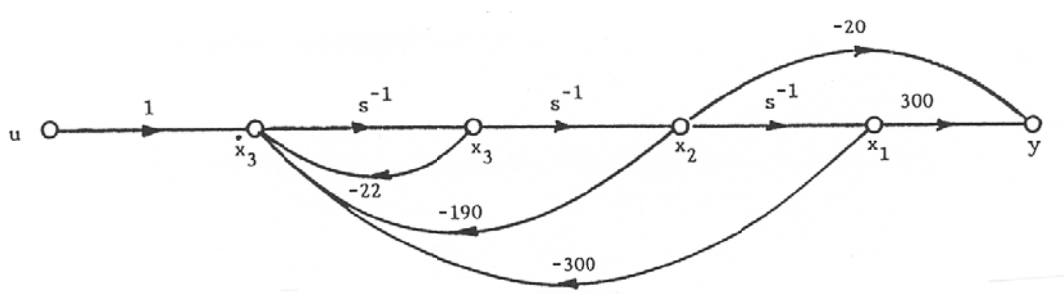
Roots of characteristic equation:

$$-5.88, \quad -3.06 + j4.965, \quad -3.06 - j4.965$$

10-42 (a)

$$G_p(s) = \frac{Y(s)}{U(s)} \cong \frac{1-0.066s}{(1+0.5s)(1+0.133s+0.0067s^2)} = \frac{-20(s-15)}{s^3+22s^2+190s+300}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -300 & -190 & -22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Characteristic equation of closed-loop system:

$$s^3 + 22s^2 + 170s + 600 = 0$$

Roots of characteristic equation:

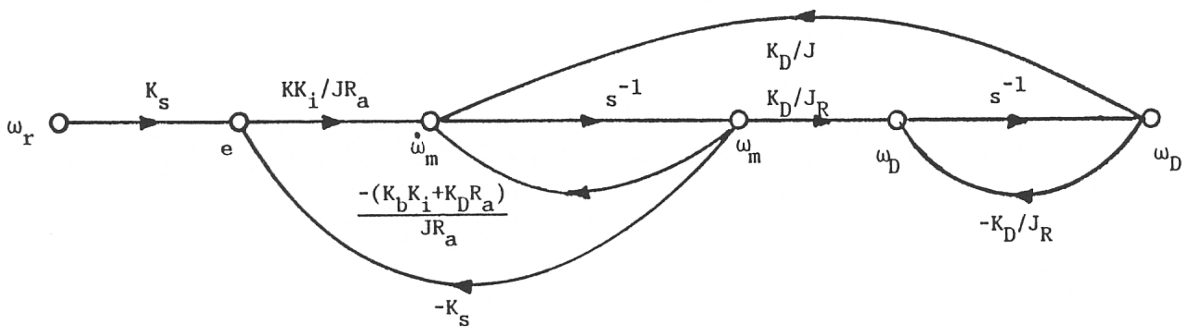
$-12,\ -5 + j5,\ -5 - j5$

10-43 (a) **State variables:** $x_1 = \omega_m$ and $x_2 = \omega_D$

State equations:

$$\frac{d\omega_m}{dt} = -\frac{K_b K_i + K_b R_a}{JR_a} + \frac{K_D}{J}\omega_D + \frac{KK_i}{JR_a}e \qquad \frac{d\omega_D}{dt} = \frac{K_D}{J_R}\omega_m - \frac{K_D}{J_R}\omega_D$$

(b) **State diagram:**



(c) **Open-loop transfer function:**

$$\frac{\Omega_m(s)}{E(s)} = \frac{KK_i\left(J_R s + K_D\right)}{JJ_R R_a s^2 + \left(K_b J_R K_i + K_D R_a J_R + K_D J R_a\right)s + K_D K_b K_i}$$

Closed-loop transfer function:

$$\frac{\Omega_m(s)}{\Omega_r(s)} = \frac{K_s K K_i \left(J_R s + K_D\right)}{J J_R R_a s^2 + \left(K_b J_R K_i + K_D R_a J_R + K_D J R_a + K_s K K_i J_R\right) s + K_D K_b K_i + K_s K K_i K_D}$$

(d) **Characteristic equation of closed-loop system:**

$$\Delta(s) = J J_R R_a s^2 + \left(K_D J_R K_i + K_D R_a J_R + K_D J R_a + K_s K K_i J_R\right) s + K_D K_b K_i + K_s K K_i K_D = 0$$

$$\Delta(s) = s^2 + 1037 s + 20131.2 = 0$$

Characteristic equation roots: –19.8, –1017.2

10-44 (a) **State equations:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} -b & d \\ c & -a \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since **S** is nonsingular, the system is controllable.

(b)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & d \\ 1 & -a \end{bmatrix}$$

The system is controllable for $d \neq 0$.

10-45 (a)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

S is singular. The system is uncontrollable.

(b)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix}$$

S is nonsingular. The system is controllable.

10-46 (a) **State equations:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

S is singular. The system is uncontrollable.

Output equation: $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} = \mathbf{Cx}$ $\qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

V is nonsingular. The system is observable.

(b) Transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+2s-3} = \frac{1}{s-1}$$

Since there is pole-zero cancellation in the input-output transfer function, the system is either uncontrollable or unobservable or both. In this case, the state variables are already defined, and the system is uncontrollable as found out in part (a).

10-47 (a) $\alpha = 1, 2, \text{or } 4$. These values of α will cause pole-zero cancellation in the transfer function.

(b) The transfer function is expanded by partial fraction expansion,

$$\frac{Y(s)}{R(s)} = \frac{\alpha-1}{3(s+1)} - \frac{\alpha-2}{2(s+2)} + \frac{\alpha-4}{6(s+4)}$$

By parallel decomposition, the state equations are: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$, output equation: $y(t) = \mathbf{C}\mathbf{x}(t)$.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \alpha-1 \\ \alpha-2 \\ \alpha-4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

The system is uncontrollable for $\alpha = 1$, or $\alpha = 2$, or $\alpha = 4$.

(c) Define the state variables so that

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{2} \\ \frac{-1}{6} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \alpha - 1 & \alpha - 2 & \alpha - 4 \end{bmatrix}$$

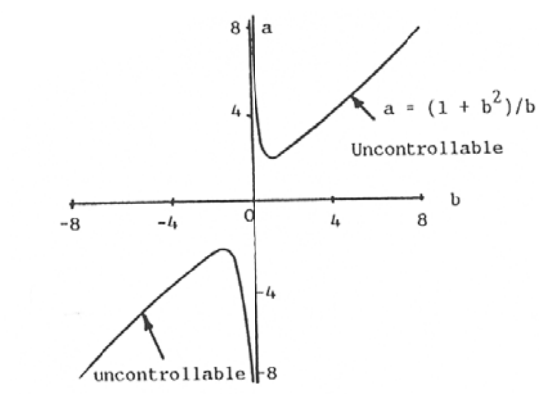
The system is unobservable for α = 1, or α = 2, or α = 4.

10-48

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & ab-1 \end{bmatrix} \qquad |\mathbf{S}| = ab - 1 - b^2 \neq 0$$

The boundary of the region of controllability is described by $ab - 1 - b^2 = 0.$

Regions of controllability:



10-49

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} b_1 & b_1 + b_2 \\ b_2 & b_2 \end{bmatrix} \qquad |\mathbf{S}| = 0 \text{ when } b_1 b_2 - b_1 b_2 - b_2^2 = 0, \text{ or } b_2 = 0$$

The system is completely controllable when $b_2 \neq 0.$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 + d_2 \end{bmatrix} \qquad |\mathbf{V}| = 0 \text{ when } d_1 \neq 0.$$

The system is completely observable when $d_2 \neq 0.$

10-50 (a) State equations:

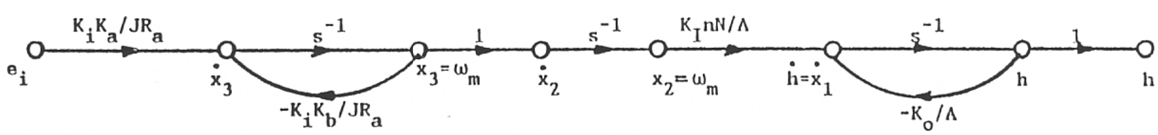
$$\frac{dh}{dt} = \frac{1}{A}\left(q_i - q_o\right) = \frac{K_I nN}{A}\theta_m - \frac{K_o}{A}h \qquad \frac{d\theta_m}{dt} = \omega_m \qquad \frac{d\omega_m}{dt} = -\frac{K_i K_b}{JR_a}\omega_m + \frac{K_i K_a}{JR_a}e_i \qquad J = J_m + n^2 J_L$$

State variable: $x_1 = h, \quad x_2 = \theta_m, \quad x_3 = \frac{d\theta_m}{dt} = \omega_m$

State equations: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}e_i$

$$\mathbf{A} = \begin{bmatrix} \frac{-K_o}{A} & \frac{K_I nN}{A} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{K_i K_b}{JR_a} \end{bmatrix} = \begin{bmatrix} -1 & 0.016 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -11.767 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{K_i K_a}{JR_a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8333.33 \end{bmatrix}$$

State diagram:



(b) Characteristic equation of A:

$$\left|s\mathbf{I} - \mathbf{A}\right| = \begin{vmatrix} s + \frac{K_o}{A} & \frac{-K_I nN}{A} & 0 \\ 0 & s & -1 \\ 0 & 0 & s + \frac{K_i K_b}{JR_a} \end{vmatrix} = s\left(s + \frac{K_o}{A}\right)\left(s + \frac{K_i K_b}{JR_a}\right) = s(s+1)(s+11.767)$$

Eigenvalues of A: 0, –1, –11.767.

(c) Controllability:

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 133.33 \\ 0 & 8333.33 & -98058 \\ 8333.33 & -98058 & 1153848 \end{bmatrix}$$

$|\mathbf{S}| \neq 0.$ The system is controllable.

(d) Observability:

(1) $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$**:**

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0.016 & -0.016 \\ 0 & 0 & 0.016 \end{bmatrix}$$

V is nonsingular. The system is observable.

(2) $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$**:**

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -11.767 \end{bmatrix}$$

V is singular. The system is unobservable.

(3) $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$**:**

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -11.767 & 138.46 \end{bmatrix}$$

V is singular. The system is unobservable.

10-51 (a) Characteristic equation: $\Delta(s) = \left| s\mathbf{I} - \mathbf{A}^* \right| = s^4 - 25.92s^2 = 0$

Roots of characteristic equation: $-5.0912, \quad 5.0912, \; 0, \; 0$

(b) Controllability:

$$\mathbf{S} = \begin{bmatrix} \mathbf{B}^* & \mathbf{A}^*\mathbf{B}^* & \mathbf{A}^{*2}\mathbf{B}^* & \mathbf{A}^{*3}\mathbf{B}^* \end{bmatrix} = \begin{bmatrix} 0 & -0.0732 & 0 & -1.8973 \\ -0.0732 & 0 & -1.8973 & 0 \\ 0 & 0.0976 & 0 & 0.1728 \\ 0.0976 & 0 & 0.1728 & 0 \end{bmatrix}$$

S is nonsingular. Thus, $\begin{bmatrix} \mathbf{A}^*, & \mathbf{B}^* \end{bmatrix}$ is controllable.

(c) Observability:

(1) $\mathbf{C}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 25.92 & 0 \\ 0 & 1 & 0 & 25.92 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

S is singular. The system is unobservable.

(2) $\mathbf{C}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 0 & 25.92 & 0 & 671.85 \\ 1 & 0 & 25.92 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

S is singular. The system is unobservable.

(3) $\mathbf{C}^* = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & -2.36 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

S is nonsingular. The system is observable.

(4) $\mathbf{C}^* = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 0 & -2.36 & 0 & -61.17 \\ 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

S is singular. The system is unobservable.

10-52 The controllability matrix is

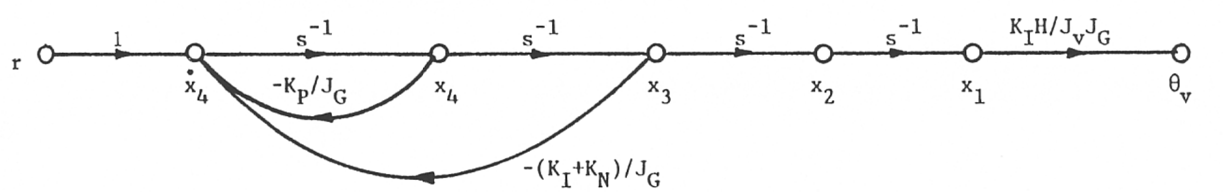
$$\mathbf{S} = \begin{bmatrix} 0 & -1 & 0 & -16 & 0 & -384 \\ -1 & 0 & -16 & 0 & -384 & 0 \\ 0 & 0 & 0 & 16 & 0 & 512 \\ 0 & 0 & 16 & 0 & 512 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of **S** is 6. The system is controllable.

10-53 (a) Transfer function:

$$\frac{\Theta_v(s)}{R(s)} = \frac{K_I H}{J_v s^2 \left(J_G s^2 + K_P s + K_I + K_N \right)}$$

State diagram by direct decomposition:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-\left(K_I + K_N\right)}{J_G} & \dfrac{-K_P}{J_G} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Characteristic equation: $J_v s^2 \left(J_G s^2 + K_P s + K_I + K_N \right) = 0$

10-54 (a) State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

S is nonsingular. $[\mathbf{A}, \ \mathbf{B}]$ is controllable.

Output equation: $y_2 = \mathbf{Cx}$ $\qquad \mathbf{C} = \begin{bmatrix} -1 & 1 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

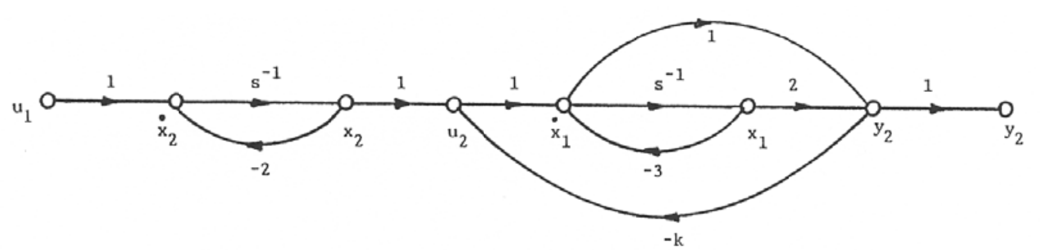
V is singular. The system is unobservable.

(b) With feedback, $u_2 = -kc_2$, the state equation is: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$.

$$\mathbf{A} = \begin{bmatrix} \frac{-3-2k}{1+g} & \frac{1}{1+k} \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 0 & \frac{1}{1+k} \\ 1 & -2 \end{bmatrix}$$

S is nonsingular for all finite values of k. The system is controllable.

State diagram:



Output equation: $y_2 = \mathbf{Cx}$ $\qquad \mathbf{C} = \begin{bmatrix} \frac{-1}{1+k} & \frac{1}{1+k} \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{D}' & \mathbf{A}'\mathbf{D}' \end{bmatrix} = \begin{bmatrix} \frac{-1}{1+K} & \frac{3+2k}{(1+k)^2} \\ \frac{1}{1+k} & -\frac{3+2k}{(1+k)^2} \end{bmatrix}$$

V is singular for any *k*. The system with feedback is unobservable.

10-55 (a)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -7 \end{bmatrix}$$ **S** is nonsingular. System is controllable.

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$ **V** is nonsingular. System is observable.

(b) $u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\mathbf{x}$

$$\mathbf{A}_c = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \\ -1-2k_1 & -3-2k_2 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{A}_c\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -k_1 - 2k_2 + 2 \\ 2 & -7 - 2k_1 - 4k_2 \end{bmatrix} \qquad |\mathbf{S}| = -11 - 2k_2 \neq 0$$

For controllabillity, $k_2 \neq -\dfrac{11}{2}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}_c'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} -1 & -1-3k_1 \\ 1 & -2-3k_2 \end{bmatrix}$$

For observability, $|\mathbf{V}| = -1 + 3k_1 - 3k_2 \neq 0$

10-56

Same as 10-21 (a)

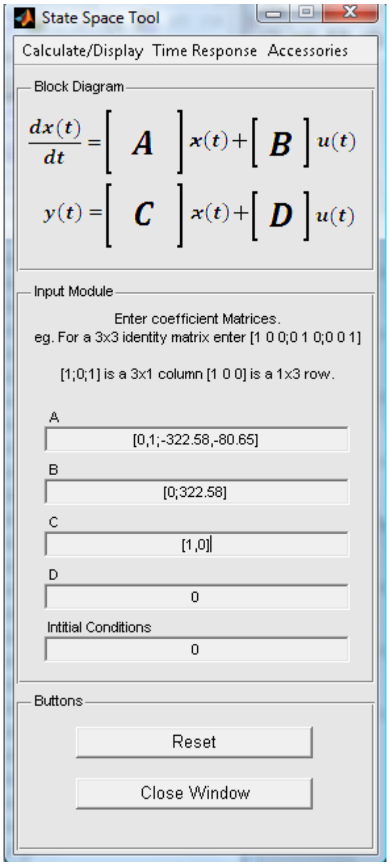
10-57

$$= \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} \theta_r$$

From 10-22 $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} \theta_r$

$$A = \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 322.58 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$D = 0$$

Use the state space tool of ACSYS



The A matrix is:

Amat =

0 1.0000

-322.5800 -80.6500

Characteristic Polynomial:

ans =

s^2+1613/20*s+16129/50

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-4.2206 0

0 -76.4294

Eigen Vectors are

T =

0.2305 -0.0131

-0.9731 0.9999

State-Space Model is:

```
a =
        x1      x2
  x1      0       1
  x2  -322.6  -80.65

b =
        u1
  x1      0
  x2  322.6

c =
      x1  x2
  y1   1   0

d =
      u1
  y1   0
```

Continuous-time model.

Characteristic Polynomial:

```
ans =
s^2+1613/20*s+16129/50
```

Equivalent Transfer Function Model is:

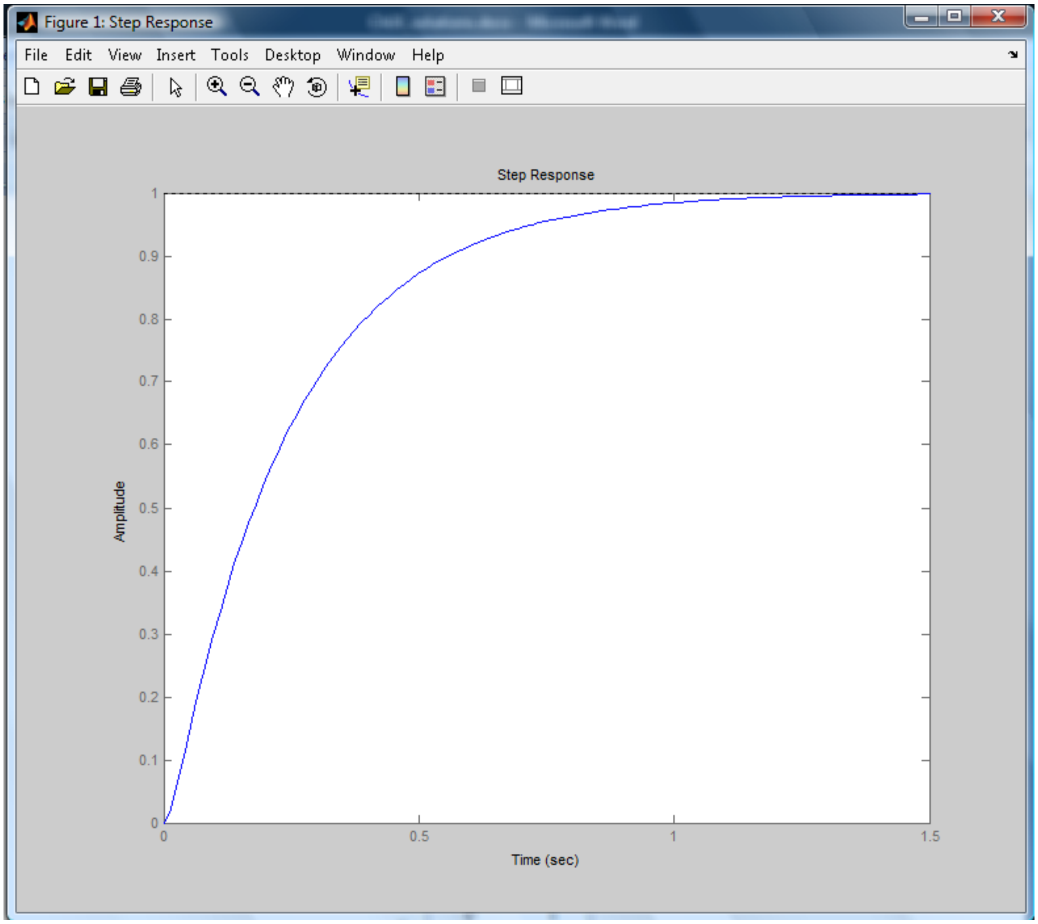
Transfer function:

```
       322.6
---------------------
s^2 + 80.65 s + 322.6
```

Pole, Zero Form:

Zero/pole/gain:

```
     322.58
-------------------
(s+76.43) (s+4.221)
```



10-58

Closed-loop System Transfer Function.

$$\frac{Y(s)}{R(s)} = \frac{1}{s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + k_1}$$

For zero steady-state error to a step input, $k_1 = 1$. For the complex roots to be located at $-1 + j$ and $-1 - j$,

we divide the characteristic polynomial by $s^2 + 2s + 2$ and solve for zero remainder.

$$\begin{array}{r|l} & s + (2 + k_2) \\ \hline s^2 + 2s + 2 & s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + 1 \\ & s^3 + \quad 2s^2 \quad\quad + 2s \\ \cline{2-2} & \quad (2 + k_3)s^2 + (1 + k_2 + k_3)s + 1 \end{array}$$

$$\frac{(2+k_3)s^2+(4+2k_3)s \quad +4+2k_3}{(-3+k_2-k_3)s-3-2k_3}$$

For zero remainder, $-3-2k_3=0$ Thus $k_3=-1.5$

$-3+k_2-k_3=0$ Thus $k_2=1.5$

The third root is at –0.5. Not all the roots can be arbitrarily assigned, due to the requirement on the steady-state error.

10-59 (a) Open-loop Transfer Function.

$$G(s)=\frac{X_1(s)}{E(s)}=\frac{k_3}{s\left[s^2+(4+k_2)s+3+k_1+k_2\right]}$$

Since the system is type 1, the steady-state error due to a step input is zero for all values of k_1, k_2, and k_3 that correspond to a stable system. The characteristic equation of the closed-loop system is

$$s^3+(4+k_2)s^2+(3+k_1+k_2)s+k_3=0$$

For the roots to be at $-1+j$, $-1-j$, and -10, the equation should be:

$$s^3+12s^2+22s+20=0$$

Equating like coefficients of the last two equations, we have

$4+k_2=12$ Thus $k_2=8$

$3+k_1+k_2=22$ Thus $k_1=11$

$k_3=20$ Thus $k_3=20$

(b) Open-loop Transfer Function.

$$\frac{Y(s)}{E(s)} = \frac{G_c(s)}{(s+1)(s+3)} = \frac{20}{s(s^2+12s+22)} \qquad \text{Thus} \qquad G_c(s) = \frac{20(s+1)(s+3)}{s(s^2+12s+22)}$$

10-60 (a)

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B}^* = \begin{bmatrix} 0 \\ -0.0732 \\ 0 \\ 0.0976 \end{bmatrix}$$

The feedback gains, from k_1 to k_4:

–2.4071E+03	–4.3631E+02	–8.4852E+01	–1.0182E+02

The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
–1.5028E+02	–3.1938E+01	–6.2112E+00	–7.4534E+00
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
2/3258E+02	4.2584E+01	8.2816E+00	9.9379E+00

The **B** vector

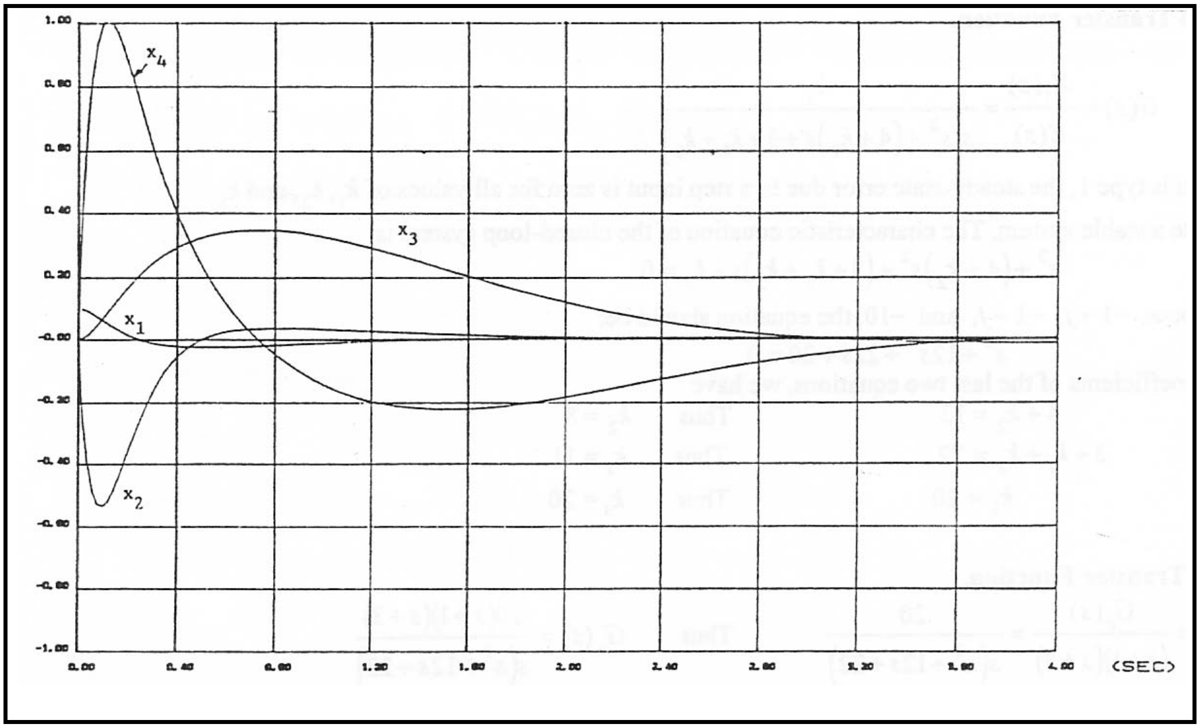
0.0000E+00

–7.3200E–02

0.0000E+00

9.7600E–02

Time Responses:



10-60 (b) The feedback gains, from k_1 to k_2:

–9.9238E+03	–1.6872E+03	–1.3576E+03	–8.1458E+02

The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
–7.0051E+02	–1.2350E+02	–9.9379E+01	–5.9627E+01
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
9.6621E+02	1.6467E+02	1.3251E+02	7.9503E+01

The **B** vector

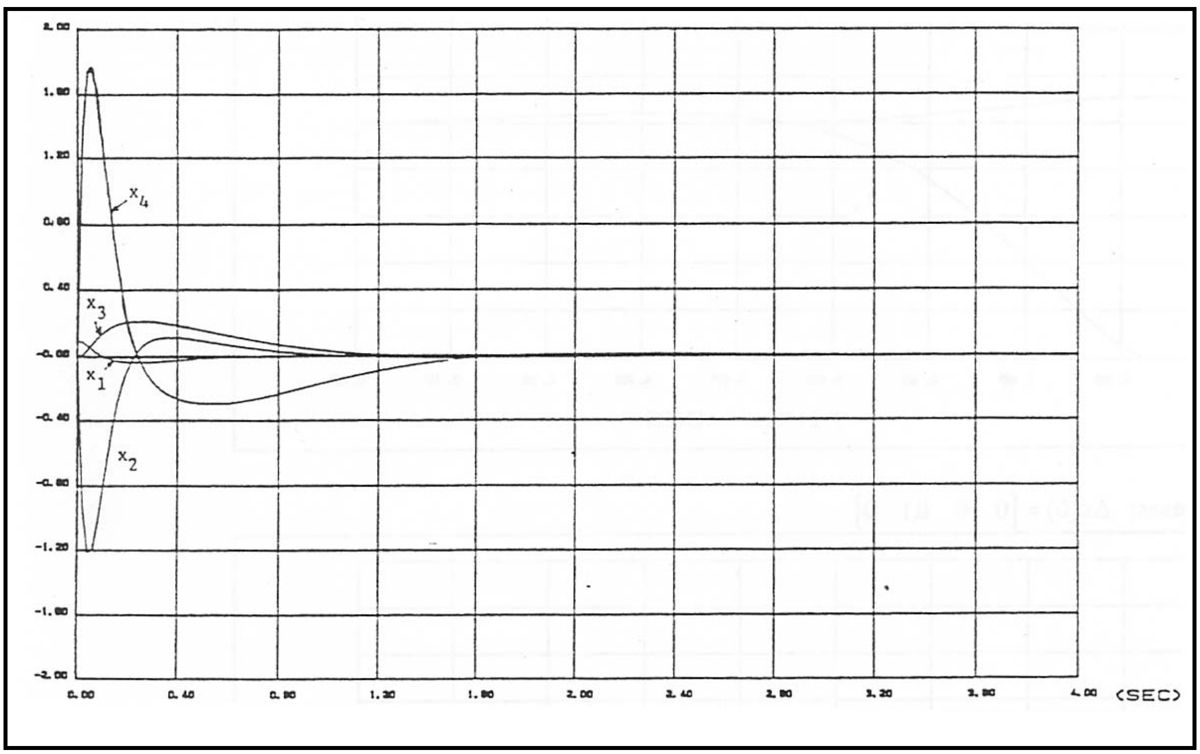
0.0000E+00

–7.3200E–02

0.0000E+00

9.7600E–02

Time Responses:



10-61 The solutions using MATLAB

(a) The feedback gains, from k_1 to k_2:

–6.4840E+01	–5.6067E+00	2.0341E+01	2.2708E+00

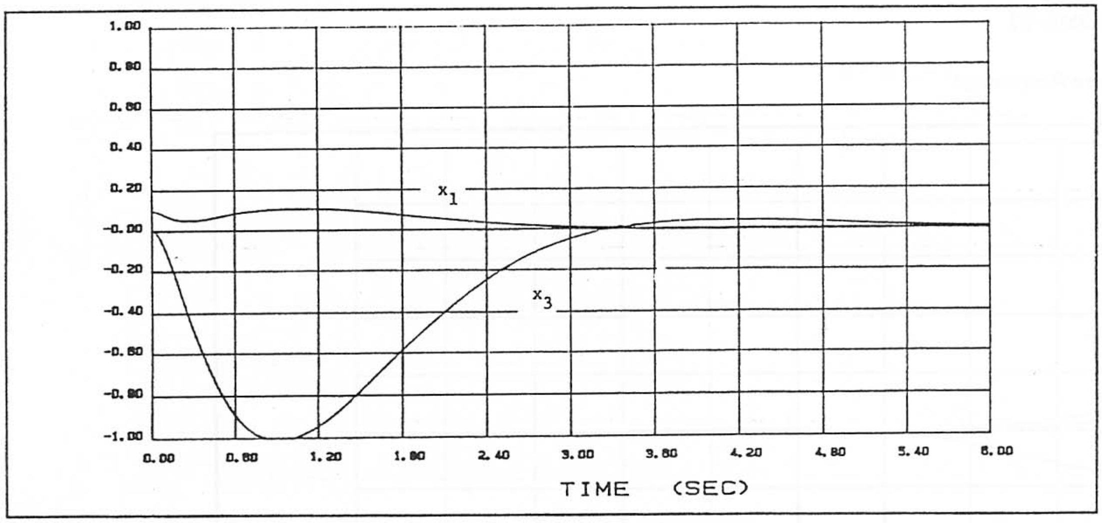
The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
–3.0950E+02	–3.6774E+01	1.1463E+02	1.4874E+01
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
–4.6190E+02	–3.6724E+01	1.7043E+02	1.477eE+01

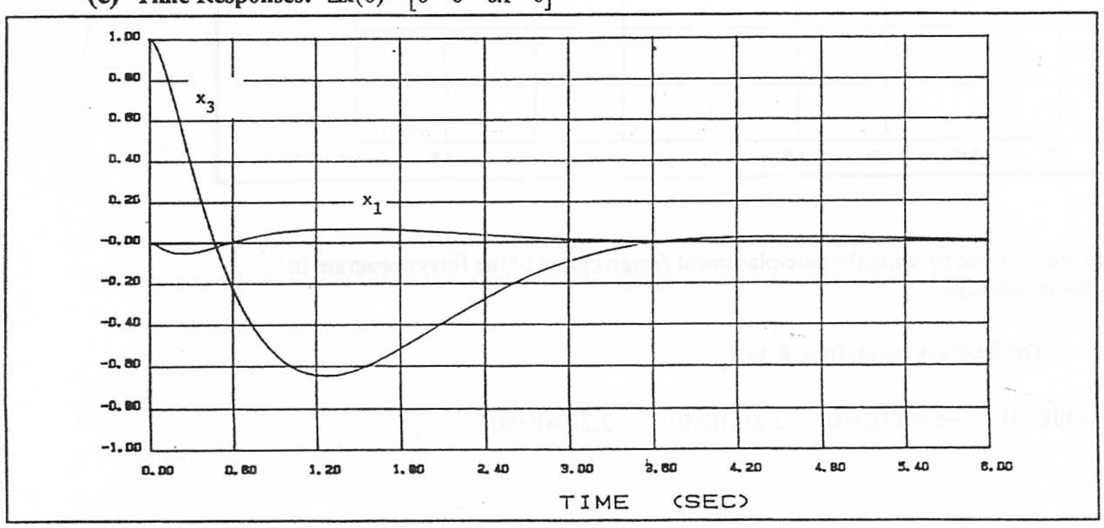
The **B** vector

0.0000E+00
–6.5500E+00
0.0000E+00
–6.5500E+00

(b) **Time Responses:** $\Delta \mathbf{x}(0) = \begin{bmatrix} 0.1 & 0 & 0 & 0 \end{bmatrix}'$



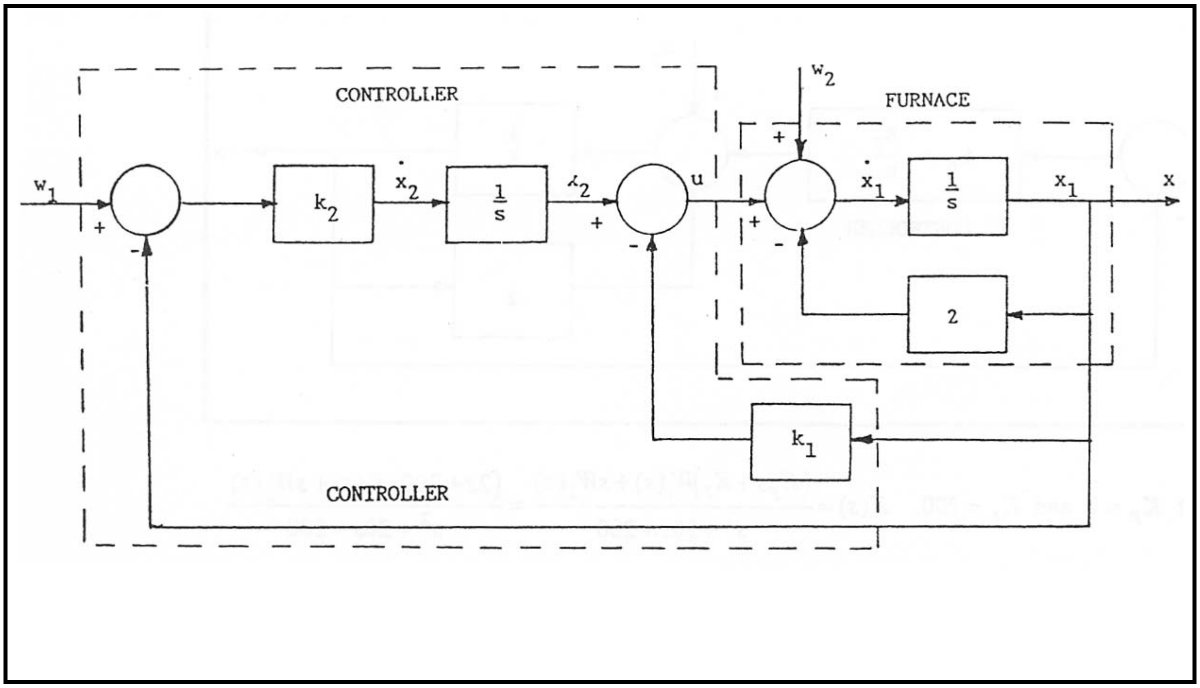
(c) Time Responses: $\Delta\mathbf{x}(0) = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}'$



With the initial states $\Delta\mathbf{x}(0) = \begin{bmatrix} 0.1 & 0 & 0 & 0 \end{bmatrix}'$, the initial position of Δx_1 or Δy_1 is preturbed downward from its stable equilibrium position. The steel ball is initially pulled toward the magnet, so $\Delta x_3 = \Delta y_2$ is negative at first. Finally, the feedback control pulls both bodies back to the equilibrium position.

With the initial states $\Delta\mathbf{x}(0) = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}'$, the initial position of Δx_3 or Δy_2 is preturbed downward from its stable equilibrium. For $t > 0$, the ball is going to be attracted up by the magnet toward the equilibrium position. The magnet will initially be attracted toward the fixed iron plate, and then settles to the stable equilibrium position. Since the steel ball has a small mass, it will move more actively.

10-62 (a) Block Diagram of System.



$$u = -k_1 x_1 + k_2 \int \left(-x_1 + w_1\right) dt$$

State Equations of Closed-loop System:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2-k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ k_2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Characteristic Equation:

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+2+k_1 & -1 \\ k_2 & s \end{vmatrix} = s^2 + (2+k_1)s + k_2 = 0$$

For $s = -10, -10$, $|s\mathbf{I}-\mathbf{A}| = s^2 + 20s + 200 = 0$ Thus $k_1 = 18$ and $k_2 = 200$

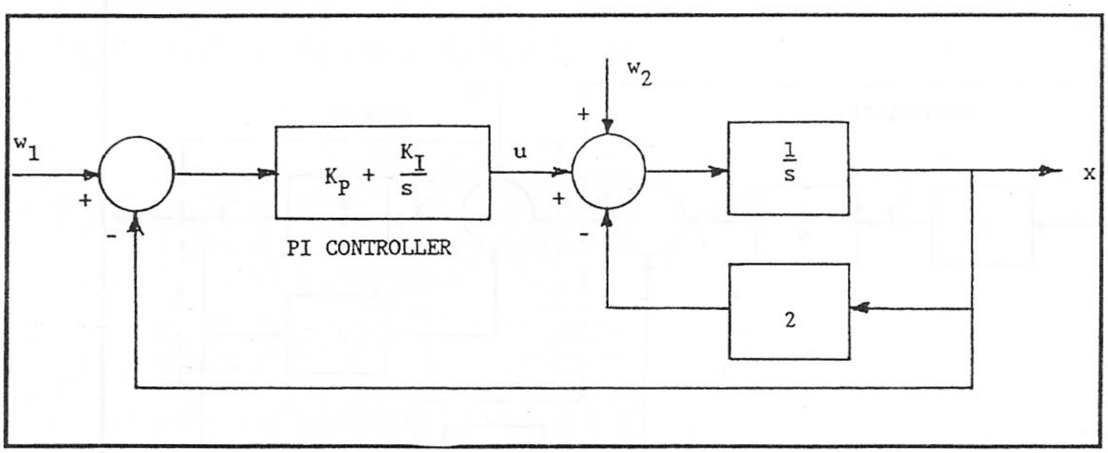
$$X(s) = X_1(s) = \frac{200W_1(s)s^{-2} + s^{-1}W_2(s)}{1 + 2s^{-1} + 18s^{-1} + 200s^{-2}} = \frac{200W_1(s) + sW_2(s)}{s^2 + 20s + 200}$$

$$W_1(s) = \frac{1}{s} \qquad W_2(s) = \frac{W_2}{s} \qquad W_2 = \text{ constant}$$

$$X(s) = \frac{200 + W_2 s}{s\left(s^2 + 20s + 200\right)} \qquad \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = 1$$

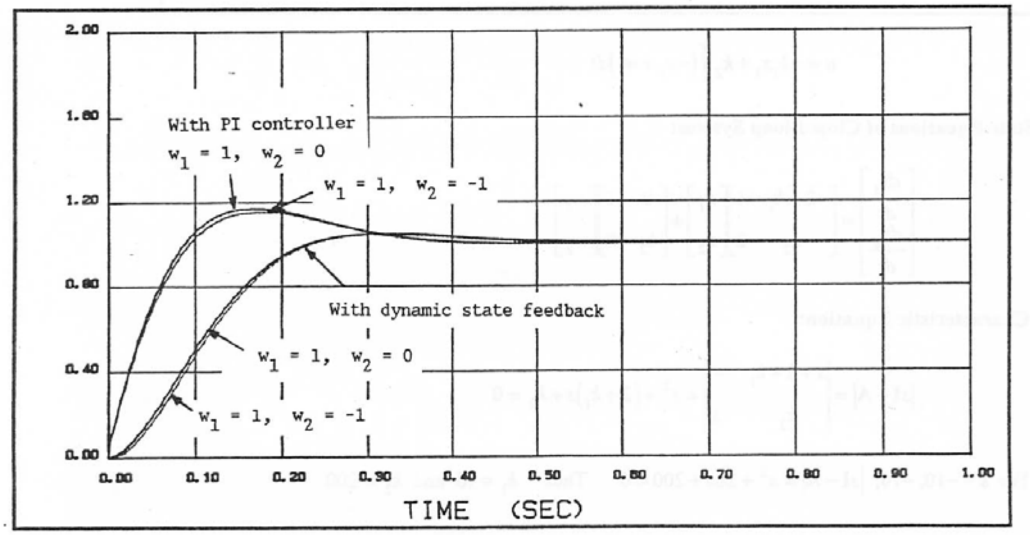
10-62 (b) With PI Controller:

Block Diagram of System:



Set $K_P = 2$ and $K_I = 200$. $X(s) = \dfrac{(K_P s + K_I)W_1(s) + sW_2(s)}{s^2 + 20s + 200} = \dfrac{(2s + 200)W_1(s) + sW_2(s)}{s^2 + 20s + 200}$

Time Responses:



10-63)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)} = \frac{10}{s^3 + 6s^2 + 11s + 6}$$

Consider:

$$\begin{cases} Y(s) = s^{-3}X(s) \\ X(s) = 10U(s) - (6s^{-1} + 11s^{-2} + 6s^{-3})X(s) \end{cases}$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

As a result:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = [0]$$

Using MATLAB, we'll find:

$$K = \begin{bmatrix} 15.4 & 4.5 & 0.8 \end{bmatrix}$$

10-64)

Inverted Pendulum on a cart

The equations of motion from Problem 4-21 are obtained (by ignoring all the pendulum inertia term):

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = f$$
$$ml(-g\sin\theta - \ddot{x}\cos\theta + l\ddot{\theta}) = 0$$

These equations are nonlinear, but they can be linearized. Hence

$$\theta \approx 0$$
$$\cos\theta \approx 1$$
$$\sin\theta \approx \theta$$

$$(M+m)\ddot{x} + ml\ddot{\theta} = f$$
$$ml(-g\theta - \ddot{x} + l\ddot{\theta}) = 0$$

Or

$$\begin{bmatrix} (M+m) & ml \\ -ml & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} f \\ mlg\theta \end{bmatrix}$$

Pre-multiply by inverse of the coefficient matrix

inv([(M+m),m*l;-m*l,m*l^2])

ans =

[1/(M+2*m), -1/l/(M+2*m)]

[1/l/(M+2*m), (M+m)/m/l^2/(M+2*m)]

For values of M=2, m=0.5, l=1, g=9.8

ans =

0.3333 -0.3333

0.3333 1.6667

Hence

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 5/3 \end{bmatrix} \begin{bmatrix} f \\ 49/10\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \text{1/3*f-49/30}\theta \\ \text{1/3*f+49/6}\theta \end{bmatrix}$$

The state space model is:

$$\begin{bmatrix} \dot{x}_4 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \text{1/3*f-49/30}x_1 \\ \text{1/3*f+49/6}x_1 \end{bmatrix}$$

Or:

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1/3\text{*f+}49/6x_1 \\ 1/3\text{*f-}49/30x_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 49/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -49/30 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix} f$$

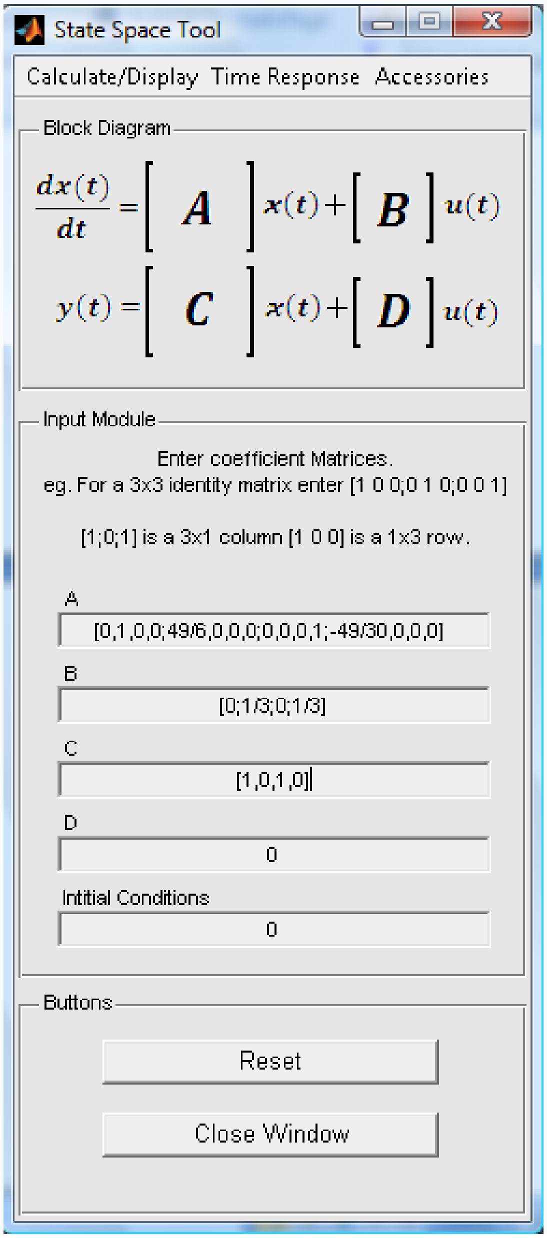
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 49/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -49/30 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

$$D = 0$$

Use ACSYS State tool and follow the design process stated in Example 10-17-1:



The A matrix is:

Amat =

```
         0    1.0000         0         0
    8.1667         0         0         0
         0         0         0    1.0000
   -1.6333         0         0         0
```

Characteristic Polynomial:

ans =

s^4-49/6*s^2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

```
         0         0         0         0
         0         0         0         0
         0         0    2.8577         0
         0         0         0   -2.8577
```

Eigen Vectors are

T =

```
         0         0    0.3239   -0.3239
         0         0    0.9256    0.9256
    1.0000   -1.0000   -0.0648    0.0648
         0    0.0000   -0.1851   -0.1851
```

State-Space Model is:

a =

```
           x1      x2      x3      x4
   x1       0       1       0       0
   x2   8.167       0       0       0
   x3       0       0       0       1
   x4  -1.633       0       0       0
```

b =

```
           u1
   x1       0
   x2  0.3333
```

```
   x3      0
   x4  0.3333

c =
       x1  x2  x3  x4
   y1   1   0   1   0

d =
       u1
   y1   0

Continuous-time model.
 Characteristic Polynomial:

ans =

s^4-49/6*s^2


 Equivalent Transfer Function Model is:

Transfer function:
4.441e-016 s^3 + 0.6667 s^2 - 2.22e-016 s - 3.267
-------------------------------------------------
               s^4 - 8.167 s^2

 Pole, Zero Form:

Zero/pole/gain:
4.4409e-016 (s+1.501e015) (s+2.214) (s-2.214)
---------------------------------------------
          s^2 (s-2.858) (s+2.858)

 The Controllability Matrix [B AB A^2B ...] is =

Smat =

         0    0.3333         0    2.7222
    0.3333         0    2.7222         0
         0    0.3333         0   -0.5444
    0.3333         0   -0.5444         0

 The system is therefore Not Controllable, rank of S Matrix is =

rankS =

     4
```

Mmat =

0	-8.1667	0	1.0000
-8.1667	0	1.0000	0
0	1.0000	0	0
1.0000	0	0	0

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

0	0	0.3333	0
0	0	0	0.3333
-3.2667	0	0.3333	0
0	-3.2667	0	0.3333

The transformed matrices using CCF are:

Abar =

0	1.0000	0	0
0	0	1.0000	0
0	0	0	1.0000
0	0	8.1667	0

Bbar =

0
0
0
1

Cbar =

-3.2667	0	0.6667	0

Dbar =

0

Note incorporating **–K** in Abar:

Abar K=

```
0    1.0000       0            0
0         0  1.0000            0
0         0       0       1.0000
-k1     -k2  8.1667-k3       -k4
```

System Characteristic equation is:
-k4*s^4+(8.1667-k3)*s^3-k2*s-k1=0

From desired poles we have:

```
>> collect((s-210)*(s-210)*(s+20)*(s-12))

ans =

-10584000+s^4-412*s^3+40500*s^2+453600*s
```

Hence: k1=10584000, k2=40500, k3=412+8.1667and k4=1

10-65) If $t_p = 3$ and $\xi = 0.707$, then $\omega_n = 1.414$. The 2nd order desired characteristic equation of the system is:

$$s^2 + 2s + 2 = 0 \qquad (1)$$

On the other hand:

$$\dot{X} = (A - BK)x = \begin{bmatrix} 0 & 1 \\ -6 - K_1 & -5 - K_2 \end{bmatrix} x$$

where the characteristic equation would be:

$$s^2 + (5 + K_2)s + (6 + K_1) = 0 \qquad (2)$$

Comparing equation (1) and (2) gives:

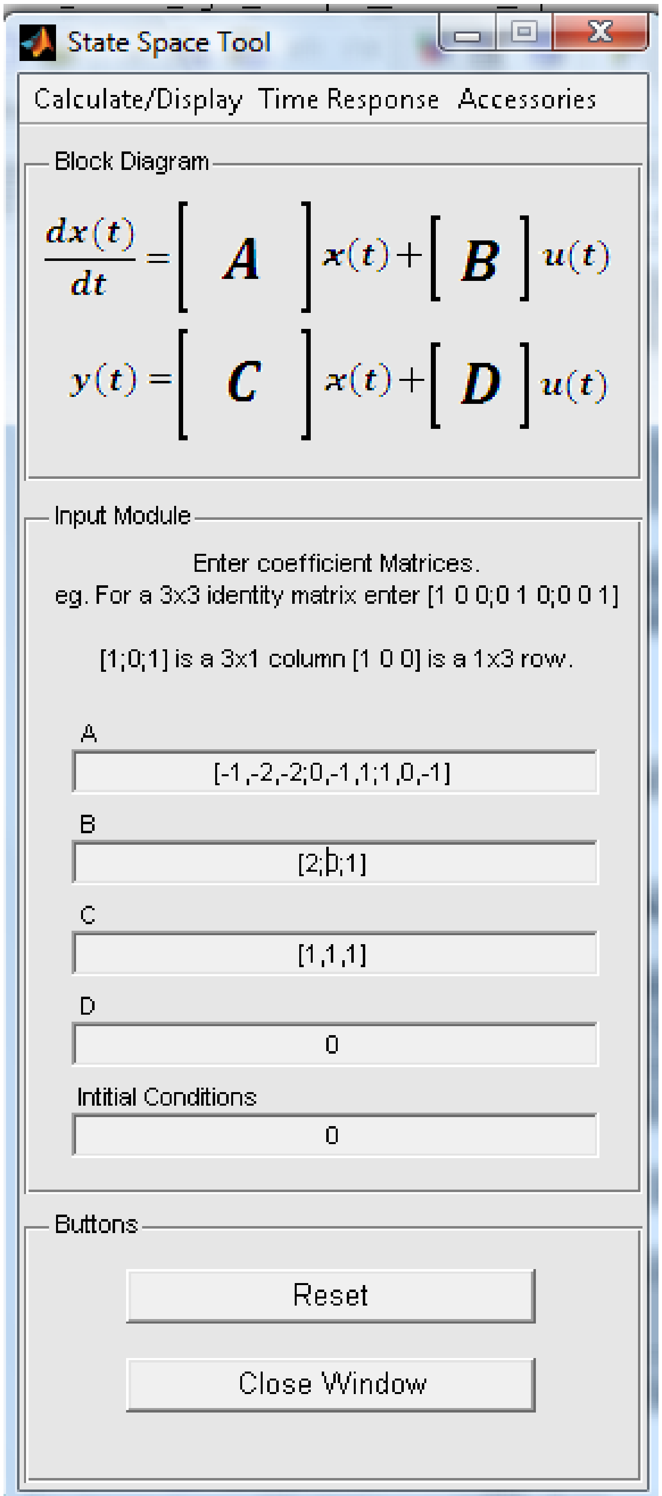
$$\begin{cases} 5 + K_2 = 2 \\ 6 + K_1 = 2 \end{cases}$$

which means $K_1 = -4$ and $K_2 = -3$

10-66) Using ACSYS we can convert the system into transfer function form.

USE ACSYS as illustrated in section 10-19-1

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the "State Space" pushbutton
5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
6) Use the "Calculate/Display" menu and find the eigenvalues.
7) Next use the "Calculate/Display" menu and conduct State space calculations.
8) Next verify Controlability and find the $\overline{A}$ matrix
9) Follow the design procedures in section 10-17 (pole placement)


State Space Tool
Calculate/Display
Time Response
Accessories
Block Diagram
dx(t)/dt = [A] x(t) + [B] u(t)
y(t) = [C] x(t) + [D] u(t)
Input Module
Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0;0 1 0;0 0 1]
[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.
A
[-1,-2,-2;0,-1,1;1,0,-1]
B
[2;0;1]
C
[1,1,1]
D
0
Intitial Conditions
0
Buttons
Reset
Close Window

The A matrix is:

Amat =

```
    -1    -2    -2
     0    -1     1
     1     0    -1
```

Characteristic Polynomial:

ans =

s^3+3*s^2+5*s+5

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

```
  -0.6145 + 1.5639i        0                  0
        0            -0.6145 - 1.5639i        0
        0                  0            -1.7709
```

Eigen Vectors are

T =

```
  -0.8074             -0.8074             -0.4259
   0.2756 + 0.1446i    0.2756 - 0.1446i   -0.7166
  -0.1200 + 0.4867i   -0.1200 - 0.4867i    0.5524
```

State-Space Model is:

a =

```
       x1  x2  x3
   x1  -1  -2  -2
   x2   0  -1   1
   x3   1   0  -1
```

```
b =

      u1

   x1   2

   x2   0

   x3   1

c =

      x1  x2  x3

   y1   1   1   1

d =

      u1

   y1   0

Continuous-time model.

 Characteristic Polynomial:

ans =

s^3+3*s^2+5*s+5

 Equivalent Transfer Function Model is:

Transfer function:

   3 s^2 + 7 s + 4

---------------------

s^3 + 3 s^2 + 5 s + 5

 Pole, Zero Form:

Zero/pole/gain:

        3 (s+1.333) (s+1)

---------------------------------

(s+1.771) (s^2  + 1.229s + 2.823)
```

The Controllability Matrix [B AB A^2B ...] is =

Smat =

```
     2    -4     0
     0     1     0
     1     1    -5
```

The system is therefore Controllable, rank of S Matrix is =

rankS =

```
     3
```

Mmat =

```
     5     3     1
     3     1     0
     1     0     0
```

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

```
    -2     2     2
     3     1     0
     3     4     1
```

The transformed matrices using CCF are:

Abar =

```
         0    1.0000         0
         0         0    1.0000
   -5.0000   -5.0000   -3.0000
```

Bbar =

0

0

1

Cbar =

4 7 3

Dbar =

0

Using Equation (10-324) we get:

$$\left|sI-(A-BK)\right| = s^3+(3+k_3)s^2+(5+k_2)s+(5+k_1)=0$$

Using a 2nd order prototype system, for $t_s \le 5$, then $\xi\omega_n = 1$. For overshoot of 4.33%, $\xi = 0.707$. Then the desired 2nd order system will have a characteristic equation:

$$s^2+2\zeta\omega_n s+\omega_n{}^2 = s^2+2s+2=0$$

The above system poles are: $s_{1,2} = -1 \pm j$

One approach is to pick K=[k1 k2 k3] values so that two poles of the system are close to the desired second order poles and the third pole reduces the effect of the two system zeros that are at z=-1.333 and z=-1. Let's set the third pole at s=-1.333. Hence

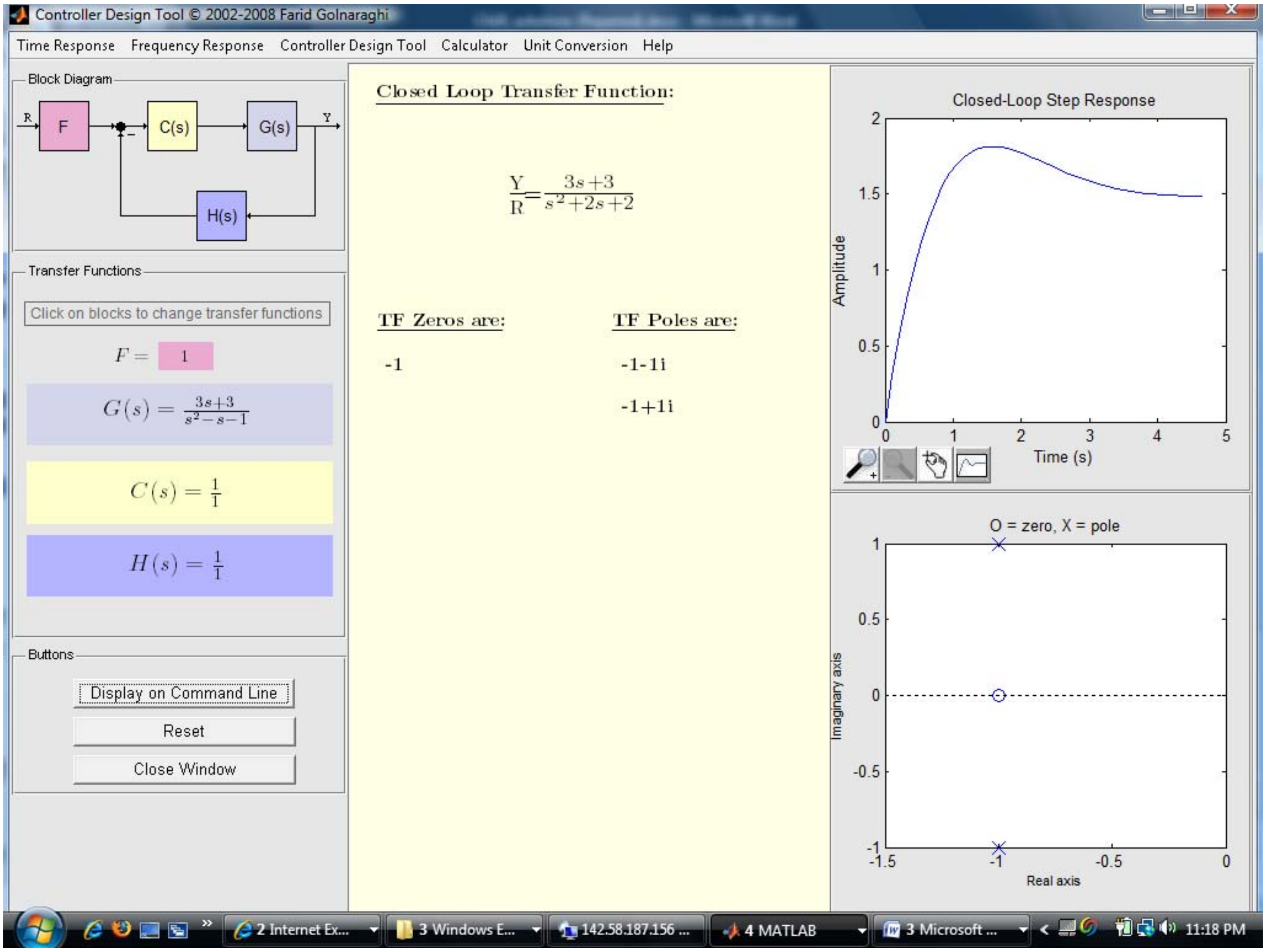
(s+1.333)*(s^2+2*s+2)= s^3+3.33*s^2+4.67*s+2.67

and K=[-2.37 -0.37 0.33].

$$\frac{Y}{R} = \frac{3(s+1)}{s^2+2s+2}$$

Use ACSYS control tool to find the time response. First convert the transfer function to a unity feedback system to make compatible to the format used in the Control toolbox.

$$G = \frac{3(s+1)}{s^2-s-1}$$



Overshoot is about 2%. You can adjust K values to obtain alternative results by repeating this process.

10-67) a) According to the circuit:

$$\frac{Ldi_2}{dt} = R_2 i_2 = v_c + R_1 C\frac{dv_c}{dt}$$

$$\frac{dv_c}{dt} = i(t) - i_2$$

$$y = (i(t) - i_2)R_2$$

If $i_2 = x_1, v_c = x_2 \; and \; i(t) = u$, then

$$\begin{cases} L\dot{x}_1 + R_2 x_1 = x_2 + R_1 C\dot{x}_2 \\ \dot{x}_2 = \frac{1}{C}(u - x_1) \\ y = (u - x_1)R_2 \end{cases}$$

or

$$\begin{cases} \dot{x}_1 = -\frac{2R_2}{L}x_1 + \frac{1}{L}x_2 + \frac{R_1}{L}(u - x_1) \\ \dot{x}_2 = \frac{1}{C}(u - x_1) \\ y = (u - x_1)R_2 \end{cases}$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2R_2}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{R_1}{L} \\ \frac{1}{C} \end{bmatrix} u$$

$$y = [-R_2 \quad 0]x + R_2 u$$

b) Uncontrollability condition is:

$$det[B \;\; AB] = det(C) = 0$$

According to the state-space of the system, C is calculated as:

$$C = \begin{bmatrix} \frac{R_1}{L} & -\frac{2R_1R_2}{L^2} + \frac{1}{LC} \\ \frac{1}{C} & -\frac{R_1}{LC} \end{bmatrix}$$

$$det(C) = \frac{R_1R_2}{L^2C} - \frac{1}{LC^2}$$

As $det(C) \neq 0$, because $R_1R_2 \neq RC$, then the system is controllable

c) Unobservability condition is:

$$det\begin{bmatrix} C \\ CA \end{bmatrix} = det(H) = 0$$

According to the state-space of the system, C is calculated as:

$$H = \begin{bmatrix} -R_2 & 0 \\ \frac{2R_2}{L} & -\frac{R_2}{L} \end{bmatrix}$$

$$det(H) = \frac{R_2^2}{L}$$

Since det(H) $\neq$ 0, because R $\neq$ 0 or L $\neq \infty$, then the system is observable.

d) The same as part (a)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{R_1C} & 0\\ 0 & \frac{R_2}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{R_1C}\\ \frac{1}{L}\end{bmatrix}u$$

$$y = \begin{bmatrix}-\frac{1}{R_1} & 1\end{bmatrix}x + \frac{1}{R_1}u$$

For controllability, we define G as:

$$G = [B\ \ AB] = \begin{bmatrix}\frac{1}{R_1C} - \frac{1}{(R_1C)^2}\\ \frac{1}{L} - \frac{R_2}{L^2}\end{bmatrix}$$

$$det(G) = -\frac{R_2}{R_1CL^2} + \frac{1}{L(R_1C)^2}$$

If $R_1R_2C = L$,and then $det(G) = 0$, which means the system is not controllable.

For observability, we define H as:

$$H = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}-\frac{1}{R_1} & 1\\ \frac{1}{R_1^2C} & -\frac{R_2}{L}\end{bmatrix}$$

$$det(H) = \frac{R_2}{R_1L} - \frac{1}{R_1^2C}$$

If $R_1R_2C = L$, then $det(H) = 0$, which means the system is not observable.